Quantum Yang-Mills 4d Theory and Time-Reversal Symmetric 5d Higher-Gauge TQFT: Anyonic-String/Brane Braiding Statistics to Topological Link Invariants

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Abstract

We explore various 4d Yang-Mills gauge theories (YM) living as boundary conditions of 5d gapped short/long-range entangled (SRE/LRE) topological states. Specifically, we explore 4d time-reversal symmetric pure YM of an SU(2) gauge group with a second-Chern-class topological term at $\theta = \pi$ ($\mathrm{SU}(2)_{\theta=\pi}$ YM). Its higher 't Hooft anomalies of generalized global symmetries indicate that the 4d $\mathrm{SU}(2)_{\theta=\pi}$ YM, in order to realize all global symmetries locally, necessarily couples to a 5d higher symmetry-protected topological state (SPTs, as an invertible TQFT, or as a 5d 1-form-center-symmetry-protected interacting "topological superconductor" in condensed matter). We revisit the 4d $\mathrm{SU}(2)_{\theta=\pi}$ YM-5d SRE-higher-SPTs coupled systems in [arXiv:1812.11968] and find their "Fantastic Four Siblings" with four sets of new higher anomalies associated with the Kramers singlet/doublet and bosonic/fermionic properties of Wilson lines. Following Weyl's gauge principle, by dynamically gauging the 1-form center symmetry, we transform a 5d bulk SRE SPTs into an LRE symmetry-enriched topologically ordered state (SETs); thus we obtain the 4d $\mathrm{SO}(3)_{\theta=\pi}$ YM-5d LRE-higher-SETs coupled system with dynamical higher-form gauge fields. Apply the tool introduced in [arXiv:1612.09298], we derive new exotic anyonic statistics of extended objects such as 2-worldsheet of strings and 3-worldvolume of branes, which physically characterize the 5d SETs. We discover new triple and quadruple link invariants potentially associated with the underlying 5d higher-gauge TQFTs, hinting a new intrinsic relation between non-supersymmetric 4d pure YM and topological links in 5d. We provide lattice simplicial complex regularizations and "condensed matter" realizations.

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1 Introduction and Summary

The world where we reside, to our best present understanding, can be described by quantum theory and the underlying long-range entanglement. Quantum gauge field theory, under the spell of "Gauge Principle" following the insights since Maxwell, Hilbert, Weyl, Pauli, and others (See a historical review [1]) embodies the quantum, special relativity and long-range entanglement into a systematic framework. Yang-Mills (YM) gauge theory [2], generalizing the U(1) gauge group to a non-abelian Lie group, has been proven to be powerful to describe the Standard Model physics.

A pure YM theory with an SU(N) gauge group (i.e., SU(N)-YM) in 4-dimensional spacetime (i.e., 4d),[1] without additional matter fields, without supersymmetry and without a Chern-class topological term ($\theta = 0$), is believed to be confined and trivially gapped in Euclidean spacetime $\mathbb{R}^4$ [3]. Formally, YM Euclidean path integral (or partition function) $\mathbf{Z}^{\text{4d}}_{\text{YM}}$ of a non-abelian Lie group G is

$$\mathbf{Z}^{\text{4d}}_{\text{YM}} \equiv \int [\mathcal{D}a] \exp\big(-S_{\text{YM}+\theta}[a]\big) \equiv$$
$$\int [\mathcal{D}a] \exp\big(-S_{\text{YM}}[a]\big)\exp\big(-S_{\theta}[a]\big) \equiv \int [\mathcal{D}a] \exp\Bigg(-\int_{M^4} \frac{1}{g^2}\mathrm{Tr}\, F_a \wedge \star F_a + \int_{M^4} \frac{\mathrm{i}\theta}{8\pi^2}\mathrm{Tr}\, F_a \wedge F_a\Bigg), \quad (1.1)$$

with the standard notations, where readers who are unfamiliar about the notations can access this information from our footnote.[2]

[1] We denote nd for an n-dimensional spacetime. We denote $m+1$D for an m-dimensional space and 1-dimensional time. We denote mD for an m-dimensional spatial object.

[2] Here a is locally the 1-form SU(N)-gauge field connection obtained from parallel transporting the principal-G bundle over the spacetime manifold M^4. Locally $a = a_\mu \mathrm{d}x^\mu = a_\mu^\alpha T^\alpha \mathrm{d}x^\mu$ with T^α is the generator of Lie algebra $\mathbf{g}$ for the gauge group (G), constrained by the commutator $[T^\alpha, T^\beta] = \mathrm{i} f^{\alpha\beta\gamma} T^\gamma$, where $f^{\alpha\beta\gamma}$ is a fully anti-symmetric structure constant. Locally $\mathrm{d}x^\mu$ is a differential 1-form, the μ runs through the indices of coordinate of M^4. Then $a_\mu = a_\mu^\alpha T^\alpha$ is the Lie algebra valued gauge field, valued in the adjoint representation of the Lie algebra. The $[\mathcal{D}a]$ is the path integral measure, for a certain configuration of the gauge field $a(t,x)$ over the spacetime (t,x). The path integral measures $\int[\mathcal{D}a]$ integrated over all allowed gauge inequivalent configurations, where gauge redundancy is removed or mod out later. The integration is under a weight factor $\exp\big(-S_{\text{YM}+\theta}[a]\big)$. The g is YM coupling constant. The $F_a = \mathrm{d}a - \mathrm{i}a \wedge a$ is the G-gauge field strength or curvature, while d is the exterior derivative and $\wedge$ is the wedge product; the $\star F_a$ is F_a's Hodge dual. The $\mathrm{Tr}\,(F_a \wedge \star F_a)$ is the Yang-Mills Lagrangian [2] (a non-abelian generalization of Maxwell U(1) gauge theory). The Tr denotes the trace as an invariant quadratic form of the Lie algebra of gauge group G. Under the variational principle, YM theory's classical equation of motion (EOM) is non-linear. The θ-topological term in physics $(\frac{\theta}{8\pi^2}\mathrm{Tr}\, F_a \wedge F_a) = \theta(-c_2 + \frac{1}{2}c_1^2)$ is formally related to the second and first Chern classes, with $c_2(E) = -\frac{1}{8\pi^2}\mathrm{Tr}(F_a \wedge F_a) + \frac{1}{8\pi^2}(\mathrm{Tr}F_a)\wedge(\mathrm{Tr}F_a)$ for the E-gauge (complex vector) bundle, where $c_1(E) = \frac{\mathrm{Tr}F_a}{2\pi} = 0$ if $G = \mathrm{SU(N)}$. This path integral is sensible for physicists, but may not be precisely mathematically well-defined. We will also point out how to grasp the meaning of YM path integral on unorientable manifolds in Sec. 2.

The YM without any Chern-class topological term, say the 4d $\mathrm{SU(N)}_{\theta=0}$ theory ($\theta = 0$ in Eq. (1.1) and footnote 2), has a trivially energy gap and is in the confinement phase [3], with no 't Hooft anomaly [4]. Recently, Ref. [5] discovers that for SU(N)-YM with a second Chern-class topological term at $\theta = \pi$, denoted the 4d $\mathrm{SU(N)}_{\theta=\pi}$ theory, there is a subtle 't Hooft anomaly [4] of the generalized global symmetries [6] of the mixed types at an even integer N, mixing between a *linear* 0-form time-reversal global symmetry (say $\mathbb{Z}_2^T$) transformation and *quadratic* 1-form $\mathbb{Z}_\mathrm{N}$-center global symmetry (say $\mathbb{Z}_{\mathrm{N},[1]}^e$) transformations.[3] Intuitively speaking, since the 4d 't Hooft anomaly is captured by a 5d topological term through the anomaly inflow [7], schematically, Ref. [5] suggests an analogous 5d form of topological term:

$$\sim \mathcal{T}BB. \tag{1.2}$$

Where $\mathcal{T}$ implies a "1-form background field" for time-reversal symmetry $\mathbb{Z}_2^T$ transformation and $B \equiv B_2$ implies a 2-form background field for 1-form center symmetry $\mathbb{Z}_{\mathrm{N},[1]}^e$ transformation.

Further recently, Ref. [8] suggests that there are additional new higher 't Hooft anomalies for some 4d $\mathrm{SU(N)}_{\theta=\pi}$ theories at even N: From one perspective, Ref. [8] suggests that, at $\mathrm{N} = 2$, there is a mixed anomaly of a *cubic* power of 0-form $\mathbb{Z}_2^T$ time-reversal symmetry transformation and a *linear* 1-form $\mathbb{Z}_{\mathrm{N},[1]}^e$-center symmetry transformation. Schematically and intuitively, Ref. [8] suggests an analogous 5d topological term to capture a new higher 't Hooft anomaly:

$$\sim \mathcal{T}\mathcal{T}\mathcal{T}B. \tag{1.3}$$

From another perspective, Ref. [8] suggests that 4d $\mathrm{SU(N)}_{\theta=\pi}$ at an even integer $\mathrm{N} \geq 4$ contains new mixed anomalies mixing between $\mathbb{Z}_2^T$, $\mathbb{Z}_{\mathrm{N},[1]}^e$ and a 0-form charge conjugation (a $\mathbb{Z}_2$ outer-automorphism) symmetry. Schematically, Ref. [8] suggests new analogous 5d topological terms to capture new higher 't Hooft anomalies:

$$\sim \mathcal{T}AAB, \tag{1.4}$$

$$\sim \mathcal{T}\mathcal{T}AB. \tag{1.5}$$

Here A implies a "1-form background field" for 0-form $\mathbb{Z}_2^C$ charge conjugation or outer-automorphism transformation. Other notations follow earlier statements. In the following, we will make the above schematic 5d topological terms Eq. (1.2), Eq. (1.3), Eq. (1.4), and Eq. (1.5) mathematically precise, by following the setup in Ref. [8] and Ref. [9].

The above 5d topological terms can be regarded as the semi-classical partition functions (definable on closed 5-manifolds with appropriate structures) whose functional values depend on the couplings to global symmetry-background probed fields. In the present work, we will further dynamically gauge the higher 1-form symmetry $\mathbb{Z}_{\mathrm{N},[1]}^e$ associated to the coupled systems of 4d YM and 5d topological terms above, in order to transform these 5d "short-range entangled" topological terms into a 5d "long-range entangled" topological quantum field theory (TQFT). Then, to the punchline of our work, we apply the methods developed in Refs. [10,11] and [12] to analytically compute the physical observables of the higher-gauge 5d TQFTs with dynamical 2-form gauge fields. The physical observables of 5d TQFTs include, for example, (i) the partition functions $\mathbf{Z}[M^5]$ on closed manifolds M^5, (b) braiding statistics of anyonic strings and anyonic branes (whose spacetime trajectories forming 2-worldsheets and 3-worldvolumes, respectively) and link invariants of these 2-surfaces and 3-surfaces in a spacetime 5-manifold. We uncover new spacetime braiding process and link invariants, including a triple-linking, its quadratic enhancement, and a quadruple linking analogous to previous works in [10,11,13–15], except that we are now studying the phenomena in a higher dimensional spacetime in 5d.[4]

[3] When we say "symmetry" in this article, we always mean "global symmetry," unless we state explicitly otherwise. (We hardly mean gauge symmetry, since it is only a redundancy.)

[4] Here we comment on the physical and mathematical meanings of fractional statistics or non-abelian statistics associated

to the spacetime braiding process for 0D anyonic particles, 1D anyonic strings or 2D anyonic branes, or other extended objects, etc. Note that in the below discussions, we take a generalized definition of "anyonic."

• In a more *restricted definition*, "anyonic" means the self-exchange statistics can go beyond bosons or fermions [16].

• In our *generalized definition*, "anyonic" means that either self-exchange statistics (of identical objects) or the mutual statistics (of multiple n distinguishable objects, where n can be 2, 3, 4, or more) can go beyond bosonic or fermionic statistics.

— In 3d (2+1D) spacetime M^3, braiding statistics of particles can be fractional (such as the exchange statistics of two identical particles, or mutual statistics of two different particles) which are called anyonic particles (see an excellent historical overview [16]). As an example, this can be understood from a 3d TQFT Chern-Simons action with local 1-form gauge field a integrated over a spacetime 3-manifold M^3

$$\sim \int_{M^3} a_I \, \mathrm{d}a_J$$

which modifies the quantum statistics of particle worldline whose open ends host the anyonic particles.

— In 4d (3+1D) spacetime M^4, braiding statistics of particles *cannot* be fractional as the two 1-worldlines cannot be intrinsically linked in 4d. Thus there is no anyonic particle and no fractional particle statistics (beyond bosons or fermions) in 4d. However, braiding statistics of strings *can* be fractional which we may call anyonic strings. As an example, this can be understood from a 4d TQFT term with local 1-form gauge field a and 2-form gauge field b, as

$$\sim \int_{M^4} b \, \mathrm{d}a$$

which modifies the mutual quantum statistics of a 0D particle from 1-worldline a linked with a 1D string from 2-worldsheet b in 4d spacetime.

anyonic strings and anyonic branes. Since particle cannot carry fractional charge in 4d, we can interpret as the anyonic string carry fractional flux in 4d. Another way to interpret the fractional statistics of anyonic strings, is through the dimensional reduction picture from 4d to 3d, where we can see that the anyonic strings can become anyonic particles in the dimensionally reduced 3d through an S^1 compactification, where the closed anyonic strings wrap around the compact S^1 — see demonstrations in the earlier work [17–19] on such 4d-to-3d dimensional reduction interpretation on braiding statistics. From the field theory side, these additions of 4d TQFT terms with local 1-form gauge field a, as

$$\sim \int_{M^4} a_I a_J \, \mathrm{d}a_K, \quad \sim \int_{M^4} a_I a_J a_K a_L$$

can modify the braiding statistics of strings, see the formulations in [10, 11, 20–24]. See the relations between Dijkgraaf-Witten's group cohomology gauge theory [25] and these TQFTs discussed in [10,11,20]. Besides, a 4d TQFT term with local 2-form gauge field b can be still made gauge invariant with

$$\sim \int_{M^4} b_I b_J,$$

which can restrict the particle (worldline) must be attached to strings (worldsheet), see the formulations in [6, 11, 26].

— In 5d (4+1D) spacetime M^5, for example, we can have a self or mutual coupling type of 5d TQFT term with local 2-form gauge fields b, b_I, b_J, etc.,

$$\sim \int_{M^5} b \, \mathrm{d}b, \quad \sim \int_{M^5} b_I \, \mathrm{d}b_J.$$

The self coupling term $\int_{M^5} b \, \mathrm{d}b$ actually follows the *restricted definition* [16] to introduce new "anyonic" string, which means the self-exchange statistics of string can go beyond bosonic or fermionic statistics. The mutual coupling term $\int_{M^5} b_I \, \mathrm{d}b_J$ obeys our *generalized definition*, "anyonic" means that mutual statistics (of distinguishable 1D strings) can go beyond bosonic or fermionic statistics. Both terms modify the quantum statistics of string worldsheet whose open ends host the 1D anyonic string.

We can have another Aharonov-Bohm like topological term with local 1-form gauge field a and 3-form gauge field c.

$$\sim \int_{M^5} c \, \mathrm{d}a,$$

which we interpret that the 0D particle from 1-worldline a is not anyonic (with an integrally quantized charge), but the 2D brane from 3-worldvolume (with a fractional "generalized flux"), can be anyonic branes. Again we can also let an anyonic brane become anyonic particles in the dimensionally reduced 5d to 3d through an T^2 compactification, where a closed anyonic brane wraps around the compact T^2 generalizing the idea of [17–19].

The analogous phenomena happen in various dimensions.[5]

Now let us take a step back to digest the physical meanings of these 5d topological terms Eq. (1.2)-Eq. (1.5). The dd 't Hooft anomaly of ordinary 0-form global symmetries is known to be captured by a $(d+1)$d invertible topological field theory[6] (i.e., iTQFT, or the so-called $(d+1)$d Symmetry-Protected Topological State [SPTs] in condensed matter physics, see recent reviews [27–30]). For a short account of the recent development on the relations between SPT terms and response probed field theories/partition functions, we should mention that these have been *systematically* studied, selectively, in [20, 31–36] (and References therein), and climax to the hint of cobordism formulation/classification of SPTs pointed out by [37, 38].

Recently the iTQFTs and SPTs are found to be systematically classified and computed by a powerful cobordism theory framework of Freed-Hopkins [39], following the earlier work of Thom-Madsen-Tillmann spectra [40, 41]. Further recently, Ref. [9] generalizes the above Thom-Madsen-Tillmann-Freed-Hopkins cobordism theory [39–41] to include the higher-form and generalized higher global symmetries [6]. So, the generalized cobordism group computation of Ref. [9], which involves the bordism group of higher classifying spaces and their fibrations, e.g. $\mathrm{B}\mathbb{G}$, can capture the dd higher 't Hooft anomaly of generalized global symmetries $\mathbb{G}$ by $(d+1)$d bordism invariants (again, certain more general iTQFTs). Similar earlier or recent pursuits on a systematic framework to obtain *higher-SPTs*, *higher-anomalies* and *higher-gauge theory* through cobordism theories or cohomology theories include the pioneers and the recent works of [42–51] and citations therein.

In other words, we should be able to identify the 4d anomalies of Eq. (1.2)-Eq. (1.5) and their corresponding 5d topological terms as mathematically precise 5d bordism invariants,[7] or equivalently 5d higher-SPTs in condensed matter terminology. The goal of this Introduction Sec. 1 and Sec. 2 are first to summarize some of the results obtained in Refs. [8] and [9], then introduce additional new results obtained in this work.

1.1 The Outline and The Plan

Here are the *outlines* of the goals of our present work and the plan of our article.

• Sec. 2 — By identifying these 5d bordism invariants and 5d higher-SPTs that couple to 4d $\mathrm{SU(N)}_{\theta=\pi}$ YM theory (especially at $\mathrm{N}=2$) thanks to higher-anomaly matching, as illustrated in Figure. The anomaly matching is of course done in a non-perturbative exact analytical way. This issue is addressed in Sec. 2.

[5]There are many other terms allowed to be added in 5d and in higher dimensional TQFTs, see [20].
Note that in the above case, when we have a Aharonov-Bohm like topological term of

$$\int c_m \,\mathrm{d}c_n \sim \int c_n \,\mathrm{d}c_m$$

(see [11]), say we have local n and m-differential forms with $n < m$, we always take the higher-dimensional object from the c_m-field to have fractional statistics (the analogs of fractional flux), while we take the lower-dimensional object from the c_n-field to have a regular statistics (the analogs of integrally quantized charge).

[6] By invertible topological field theory (iTQFT), physically it means that the absolute value of partition function $|\mathbf{Z}| = 1$ on any closed manifold. Thus whose $\mathbf{Z}$ can only be a complex phase $\mathbf{Z} = \mathrm{e}^{\,\mathrm{i}\,\theta}$, which can thus be inverted and cancelled by $\mathrm{e}^{-\,\mathrm{i}\,\theta}$ as another "inverse" iTQFT.

[7] For the mathematical terminology, we call:
• the *bordism group generators* as the manifolds or manifold generators, which generate finite Abelian groups, e.g., $\mathbb{Z}_n$.
• the *cobordism group generators* as the topological terms or iTQFTs, which generate Abelian groups, e.g., $\mathbb{Z}_n$ or $\mathbb{Z}$, etc.
• the *co/bordism invariants* (people call bordism invariants as cobordism invariants with the same meaning) mean that they are invariant under the bordism class of manifolds; thus co/bordism invariants mean the topological terms or iTQFTs, which again generate Abelian groups, $\mathbb{Z}_n$ or $\mathbb{Z}$, etc.

• Sec. 3 — Clarify and enumerate the possible distinct classes of 4d $\mathrm{SU}(2)_{\theta=\pi}$ YM theories. Here we focus on their high-energy UV (ultraviolet) completion (such as on a lattice, by quantum many-body or condensed matter systems) requires only the bosonic systems, instead of fermionic systems. These types of YM theories, we may call them the bosonic YM theories. As we will find later these bosonic YM theories still can allow Wilson line operators as worldlines of particles being (1) either bosonic or fermionic in quantum statistics, (2) either Kramers doublet or Kramers singlet under the time-reversal symmetry. We will see that this result supplements as a *partial classification* of 4d $\mathrm{SU}(2)_{\theta=\pi}$ bosonic YM theories. We apply the tools in Ref. [36] to understand the relation between gauge bundle constraint, the properties of line/surface operators towards the classification of gauge theories. This issue is addressed in Sec. 3.

In fact, from Sec. 2 and Sec. 3, we will see that there are at least four closely related 4d $\mathrm{SU}(2)_{\theta=\pi}$ non-supersymmetric pure YM theories (which we nickname the "Fantastic Four Siblings" of 4d $\mathrm{SU}(2)_{\theta=\pi}$ YM theories) with a bosonic UV completion say on a lattice. Each of them carries distinct 4d 't Hooft anomaly, thus they correspond to four distinct 5d higher-SPTs. The four distinct 5d higher-SPTs labeled by four distinct 5d bordism invariants, are actually the physical analogs of 5d (4+1D) one-form-center-symmetry-protected interacting topological superconductors in a condensed matter language. In short, there are also four distinct ("Fantastic Four Siblings") of 4d $\mathrm{SU}(2)_{\theta=\pi}$ YM-5d-higher-SPTs coupled systems.

• Sec. 4 — We dynamically gauge the 1-form center symmetry $\mathbb{Z}^e_{\mathrm{N},[1]}$, such that this procedure turns the 4d $\mathrm{SU(N)}_{\theta=\pi}$ YM-5d-higher-SPTs coupled systems in [8] into a 4d $\mathrm{PSU(N)}_{\theta=\pi}$ YM-5d-higher-SETs coupled systems. The SETs stands for the symmetry-enriched topologically ordered state (SETs), see the overview of such states in comparison with SPTs in [29,30] or the footnote.[8] In particular, we focus on N=2 case, this dynamically gauging procedure turns the 4d $\mathrm{SU}(2)_{\theta=\pi}$ YM-5d-higher-SPTs coupled systems in [8] into a 4d $\mathrm{SO}(3)_{\theta=\pi}$ YM-5d-higher-SETs coupled systems. This issue is addressed in Sec. 4.

• Sec. 4 and Sec. 5 — We then explore the detailed properties of various 5d higher-SETs obtained in Sec. 4. The 5d higher-SETs are actually 5d time-reversal symmetric higher-TQFTs with (emergent) 2-form dynamical gauge fields. Thus they are also 5d higher-gauge TQFTs (including at least 2-form gauge fields). We mainly focus on the "Fantastic Four Siblings" of 5d higher-SETs, although we also consider other highly relevant exotic 5d higher-SETs. To characterize these 5d time-reversal symmetric higher-gauge 2-form TQFTs, we compute and derive their properties:

1. Partition function $\mathbf{Z}[M^5]$ without extended operator (1-line, 2-surface, 3-submanifold) insertions on 5-manifold M^5. We compute $\mathbf{Z}[M^5]$ following the techniques and tools built from [11] and [12]. This issue is addressed in Sec. 4.

2. Topological ground state degeneracy (the so-called topological GSD) on a spatial M^4, obtained from computing $\mathbf{Z}[M^4 \times S^1]$. We compute $\mathbf{Z}[M^5]$ following the techniques and tools built from [12]. This issue is addressed in Sec. 4.

3. Braiding statistics of anyonic 1D string/2D branes, etc. And the associated link invariants of the spacetime 2-worldsheet/3-worldvolume, etc. To achieve this goal, we compute the path integral $\mathbf{Z}[M^5; W, U, \dots]$ with submanifold extended-operator insertions $(W, U, \dots)$, following the techniques and tools built from [10,11,14,15]. This issue is addressed in Sec. 5.

• Sec. 6 — We provide the exemplary spacetime braiding process of anyonic string/brane in 5d, and the

[8] Symmetry-Protected Topological State (SPTs) is a short-ranged entangled (SRE) quantum state defined on a lattice (UV complete such as on a triangulable manifold or a simplicial complex) — once we break global symmetry, SPTs can be deformed to a trivial product state under local unitary transformation. Symmetry-enriched topologically ordered state (SETs) is a long-ranged entangled (LRE) quantum state defined on a lattice (UV complete such as on a triangulable manifold or a simplicial complex) — even if we break all of global symmetries, SETs cannot be deformed to a trivial product state under local unitary transformation. SETs have the same LRE nature as topologically ordered states. See recent reviews [27–30].

link configurations of extended operators, which can be detected by the link invariants that we derived in Sec. 5.

• Sec. 7 — We come back to make more comments on the 4d $\mathrm{SO}(3)_{\theta=\pi}$ YM, which lives on the boundary of 5d-higher-SETs. In particular, we re-examine these 4d $\mathrm{SO}(3)_{\theta=\pi}$ YM-5d-higher-SETs coupled systems in Sec. 4.

• Sec. 8 — We construct the lattice regularization and UV completion of some of our systems. This includes a lattice realization of 5d higher-SPTs and higher-gauge SETs by implementing on 5d simplicial complex spacetime path integral, and a 4+1D "condensed matter" realization on the spatial Hamiltonian operator. We also provide a lattice regularization of (1) higher-symmetry-extended and (2) higher-symmetry-preserving anomalous 3+1D topologically ordered gapped boundaries by generalizing the method of [52].

• Sec. 9 — We conclude and make connections to physics and mathematics in other perspectives.

Before we proceed to the detailed discussions in the main text, we first give a quick overview on more colloquial and pedestrian summaries in terms of schematic descriptions and Table 1, in Sec. 1.2. Readers who are not familiar certain mathematical information or physical motivations may seek for additional helps from Refs. [36] (and its Appendices), [8] and [9].

1.2 Summaries and Tables

As we mention, in Sec. 2 and Sec. 3, we will see that there are at least four closely related 4d $\mathrm{SU}(2)_{\theta=\pi}$ non-supersymmetric pure YM theories (nicknamed the "Fantastic Four Siblings" of 4d $\mathrm{SU}(2)_{\theta=\pi}$ YM theories) with a bosonic UV completion. All of them carry distinct 4d higher 't Hooft anomaly, thus they correspond to four distinct 5d higher-SPTs labeled by four distinct 5d bordism invariants, (physical analogs of 5d (4+1D) one-form-center-symmetry-protected interacting "topological superconductors" in a condensed matter language.[9]) Here we advertise these results in a colloquial and pedestrian manner.

1. The 1st sibling of 4d $\mathrm{SU}(2)_{\theta=\pi}$ with Kramers singlet ($T^2 = +1$) bosonic Wilson line has the 4d anomaly/5d bordism invariant schematically as:

$$\sim w_1(TM)BB, \tag{1.6}$$

with $w_1(TM)$ the Stiefel-Whitney (SW) class of spacetime M's tangent bundle TM. Mathematically precisely, $w_1(TM)BB$ is given by $\frac{1}{2}\tilde{w}_1(TM) \cup \mathcal{P}(B)$, explained in [8] and later sections.

2. The 2nd sibling of 4d $\mathrm{SU}(2)_{\theta=\pi}$ with Kramers doublet ($T^2 = -1$) bosonic Wilson line has the 4d anomaly/5d bordism invariant schematically as:

$$\sim w_1(TM)BB + w_1(TM)^3 B. \tag{1.7}$$

3. The 3rd sibling of 4d $\mathrm{SU}(2)_{\theta=\pi}$ with Kramers singlet ($T^2 = +1$) fermionic Wilson line has the 4d anomaly/5d bordism invariant schematically as:

$$\sim w_1(TM)BB + w_3(TM)B. \tag{1.8}$$

[9] In condensed matter, "topological superconductors" refers to electronic systems with time-reversal symmetry but without U(1) electron charge conservation symmetry (see an overview [28, 29]), for example due to the Cooper pairing breaking U(1) down to a discrete subgroup or down to nothing.

4. The 4th sibling of 4d $\mathrm{SU}(2)_{\theta=\pi}$ with Kramers doublet ($T^2=-1$) fermionic Wilson line has the 4d anomaly/5d bordism invariant schematically as:

$$\sim w_1(TM)BB + w_1(TM)^3B + w_3(TM)B. \tag{1.9}$$

All these anomalies that we discussed above are the mod 2 non-perturbative global anomalies, like the SU(2) anomalies [53, 54]. We remark that our investigations on Kramers time reversal properties and bosonic/fermionic statistics of line operators (for non-abelian gauge theories here) give rise to a further refined "classification" of gauge theories somehow beyond the previous framework of Ref. [55] and [6]. (See Ref. [56, 57] for the case of abelian U(1) gauge theories. See also [58] and [59] for other examples of non-abelian gauge theories.)

A schematic illustration of 4d $\mathrm{SU}(2)_{\theta=\pi}$ Yang-Mills theory (YM)-5d short-ranged entangled (SRE)-higher-SPTs coupled systems is shown in Fig. 1. See Table 1 for a short summary for these "Fantastic Four Siblings" of 4d $\mathrm{SU}(2)_{\theta=\pi}$ YM theories and coupling to 5d systems, and their physical properties. See Table 2 for a summary of 5d TQFT's link invariants and link configurations, and references/hyperlinks to their Sections.

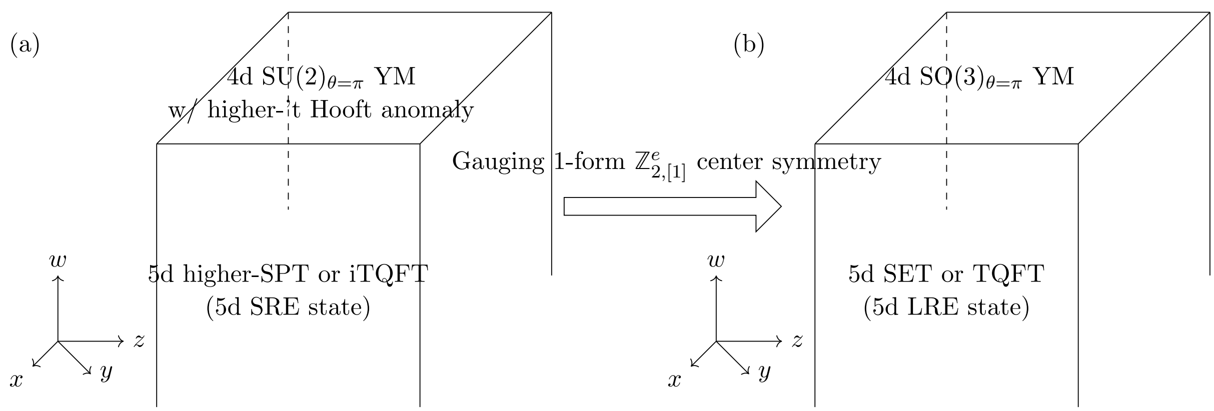


Figure 1:
(a) Schematic illustration of 4d-5d coupled system: $\mathrm{SU}(2)_{\theta=\pi}$ Yang-Mills theory (YM)-5d short-ranged entangled (SRE)-higher-SPTs (invertible TQFT) coupled systems studied in Ref. [5] and [8]. We revisit the system and follow the mathematically notations prescribed in [8]. We find "Fantastic Four Siblings" of such systems with bosonic UV completion, summarized in Table 1. Locally we use x, y, z (and the time t) to label the spacetime coordinates of 4d (3+1D) YM, and we introduce an extra w to label an extra spacetime coordinate of 5d higher SPTs.
(b) Schematic illustration of 4d-5d coupled system: 4d $\mathrm{SO}(3)_{\theta=\pi}$ Yang-Mills theory (YM)-5d long-ranged entangled (LRE)-higher-SETs (higher-gauge TQFT with 2-form gauge fields) coupled systems obtained via gauging 1-form $\mathbb{Z}^e_{2,[1]}$ center symmetry for the whole bulk-boundary system in Fig. 1 (a). We study "Fantastic Four Siblings" of such 5d SET systems with bosonic UV completion, summarized in Table 1. Locally we use x, y, z (and the time t) to label the spacetime coordinates of 4d (3+1D) YM, and we introduce an extra w to label an extra spacetime coordinate of 5d higher SETs.
See also Fig. 16.

"Fantastic Four Siblings" of 5d SRE-higher-SPTs-4d SU(2)$_{\theta=\pi}$ YM coupled systems and their gauged analogous "Fantastic Four Siblings" of 5d LRE-higher-SETs-4d SO(3)$_{\theta=\pi}$ YM coupled systems

(i). 5d higher-anomaly polynomial (5d bordism invariants of $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$) involving 1-form center $\mathbb{Z}_{2,[1]}^e$ time-reversal $\mathbb{Z}_2^T$-symmetries 5d iTQFT / SPT partition function : $\mathbf{Z}^{\text{5d}}_{\text{SPT}_{(K_1,K_2)}}[M^5]$. 5d TQFT / SET path integral : $\mathbf{Z}^{\text{5d}}_{\text{SET}_{(K_1,K_2)}}[M^5]$.	(ii). 4d SU(2)$_{\theta=\pi}$ YM obtained via dynamical gauging 4d SPTs (4d bordism invariants of $\Omega_4^{G'}$) G' for a group extension: $1 \to \mathrm{SU}(2) \to G' \to \mathrm{O}(d) \to 1$. Gauge and spacetime bundle/connection constraints Wilson line operator W properties	(iii). 5d-spacetime-braiding process of anyonic-1D-strings/2D-branes from 2-worldsheet and 3-worldvolume of 5d Higher-Gauge TQFTs/SETs: Path-integral $\mathbf{Z}[M^5, \text{Link}]/\mathbf{Z}[M^5] \equiv \langle \text{Link} \rangle$ New 5d Topological Link Invariants
1st system $(K_1 = 0, K_2 = 0)$: $\frac{1}{2}\tilde{w}_1(TM) \cup \mathcal{P}(B) = B\mathrm{Sq}^1 B + \mathrm{Sq}^2\mathrm{Sq}^1 B \sim w_1 BB$ iTQFT: $\mathbf{Z}^{\text{5d}}_{\text{SPT}_{(0,0)}}[M^5]$ of Eq. (2.19) TQFT: $\mathbf{Z}^{\text{5d}}_{\text{SET}_{(0,0)}}[M^5]$ of Eq. (4.1)	Eq. (3.2) $G' = \mathrm{O}(d) \times \mathrm{SU}(2)$ in Eq. (3.6) $w_2(V_{\mathrm{PSU}(2)}) = B$ Kramers singlet $(T^2 = +1)$ bosonic W	Eq. (5.25) $\#(V_X^4 \cap V^3_{U_{(\mathrm{i})}} \cap V^3_{U_{(\mathrm{ii})}}) \equiv \mathrm{Tlk}^{(5)}_{w_1 BB}(\Sigma^3_X, \Sigma^2_{U_{(\mathrm{i})}}, \Sigma^2_{U_{(\mathrm{ii})}})$
2nd system $(K_1 = 1, K_2 = 0)$: $\frac{1}{2}\tilde{w}_1(TM) \cup \mathcal{P}(B) + w_1(TM)^3 B = B\mathrm{Sq}^1 B + w_2(TM)\mathrm{Sq}^1 B \sim w_1 BB + (w_1)^3 B$ iTQFT: $\mathbf{Z}^{\text{5d}}_{\text{SPT}_{(1,0)}}[M^5]$ of Eq. (2.19) TQFT: $\mathbf{Z}^{\text{5d}}_{\text{SET}_{(1,0)}}[M^5]$ of Eq. (4.1)	Eq. (3.2) $G' = \mathrm{E}(d) \times_{\mathbb{Z}_2} \mathrm{SU}(2)$ in Eq. (3.8) $w_2(V_{\mathrm{PSU}(2)}) = B + w_1(TM)^2$ Kramers doublet $(T^2 = -1)$ bosonic W	Eq. (5.96) $\frac{1}{2}\#(V^3_{U_h} \cap \Sigma^2_{U_b}) \equiv \frac{1}{2}\mathrm{Lk}^{(5)}_{w_2\,\mathrm{d}B}(\Sigma^2_{U_h}, \Sigma^2_{U_b})$
3rd system $(K_1 = 0, K_2 = 1)$: $\frac{1}{2}\tilde{w}_1(TM) \cup \mathcal{P}(B) + w_3(TM)B = B\mathrm{Sq}^1 B + w_1(TM)^2\mathrm{Sq}^1 B \sim w_1 BB + w_3 B$ iTQFT: $\mathbf{Z}^{\text{5d}}_{\text{SPT}_{(0,1)}}[M^5]$ of Eq. (2.19) TQFT: $\mathbf{Z}^{\text{5d}}_{\text{SET}_{(0,1)}}[M^5]$ of Eq. (4.1)	Eq. (3.2) $G' = \mathrm{Pin}^+(d) \times_{\mathbb{Z}_2} \mathrm{SU}(2)$ in Eq. (3.10) $w_2(V_{\mathrm{PSU}(2)}) = B + w_2(TM)$ Kramers singlet $(T^2 = +1)$ fermionic W	Eq. (5.96) $\#(V^4_{X_{(\mathrm{i})}} \cap V^4_{X_{(\mathrm{ii})}} \cap \Sigma^2_U) \equiv \mathrm{Tlk}^{(5)}_{w_1 w_1\,\mathrm{d}B}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^2_U)$
4th system $(K_1 = 1, K_2 = 1)$: $\frac{1}{2}\tilde{w}_1(TM)\mathcal{P}(B) + w_1(TM)^3 B + w_3(TM)B = B\mathrm{Sq}^1 B \sim w_1 BB + (w_1)^3 B + w_3 B$ iTQFT: $\mathbf{Z}^{\text{5d}}_{\text{SPT}_{(1,1)}}[M^5]$ of Eq. (2.19) TQFT: $\mathbf{Z}^{\text{5d}}_{\text{SET}_{(1,1)}}[M^5]$ of Eq. (4.1)	Eq. (3.2) $G' = \mathrm{Pin}^-(d) \times_{\mathbb{Z}_2} \mathrm{SU}(2)$ in Eq. (3.12) $w_2(V_{\mathrm{PSU}(2)}) = \big(B + w_1(TM)^2 + w_2(TM)\big)$ Kramers doublet $(T^2 = -1)$ fermionic W	Eq. (5.96) $\#(V^3_{U_b} \cap \Sigma^2_{U_b}) \equiv \mathrm{Lk}^{(5)}_{B\,\mathrm{d}B}(\Sigma^2_{U_b}, \Sigma^2_{U_b})$

Table 1: A short summary of some results obtained in our work for the "Fantastic Four Siblings" of 4d pure non-supersymmetric SU(2)$_{\theta=\pi}$ YM theories or SO(3) YM theories, and for the 4d-5d-SPT coupled systems or 4d-5d-higher-SET coupled systems.

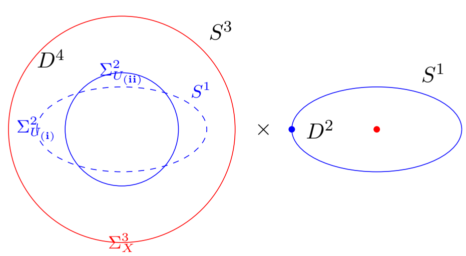
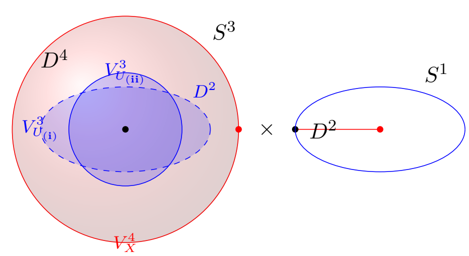
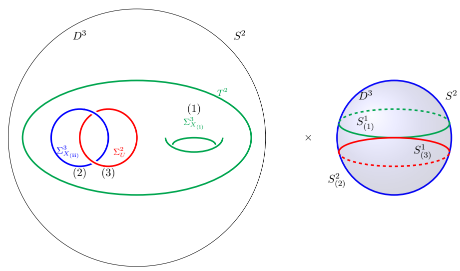
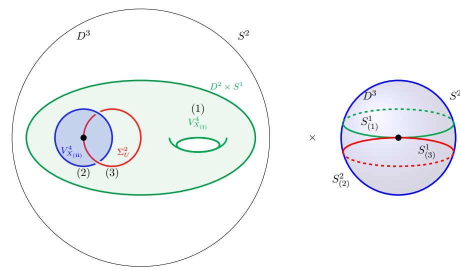
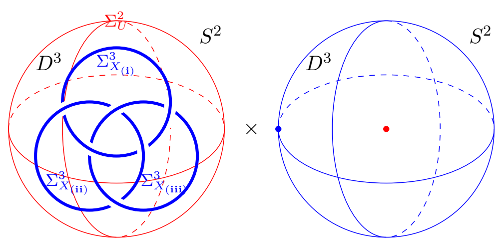
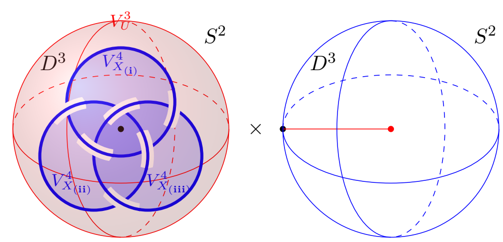
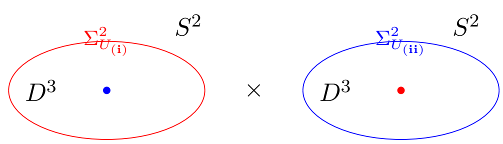
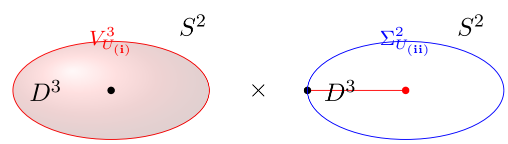
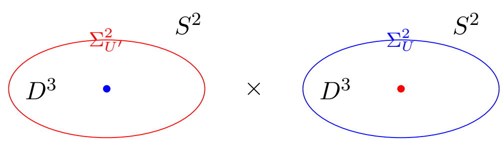
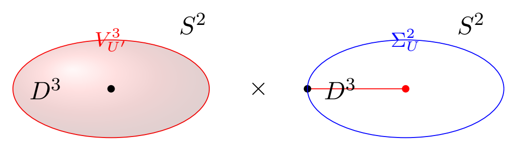
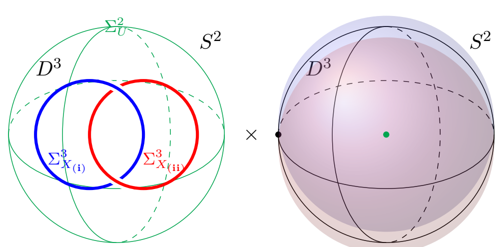
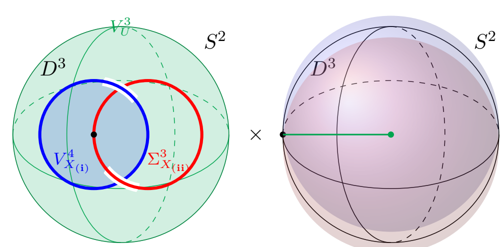
Section and Link Invariant	
Link Configuration	Intersecting Number Configuration
Sec. 5.1 and Sec. 6.2: $\#(V_X^4 \cap V_{U_{(\mathrm{i})}}^3 \cap V_{U_{(\mathrm{ii})}}^3) \equiv \mathrm{Tlk}_{w_1 BB}^{(5)}(\Sigma_X^3, \Sigma_{U_{(\mathrm{i})}}^2, \Sigma_{U_{(\mathrm{ii})}}^2)$	
	
Sec. 5.2.2, Sec. 5.4 and Sec. 6.3: $\#(V_{X_{(\mathrm{i})}}^4 \cap V_{X_{(\mathrm{ii})}}^4 \cap \Sigma_U^2) \equiv \mathrm{Tlk}_{w_1 w_1\, \mathrm{d}B}^{(5)}(\Sigma_{X_{(\mathrm{i})}}^3, \Sigma_{X_{(\mathrm{ii})}}^3, \Sigma_U^2)$	
	
Sec. 5.2.1 and Sec. 6.4: $\#(V_{X_{(\mathrm{i})}}^4 \cap V_{X_{(\mathrm{ii})}}^4 \cap V_{X_{(\mathrm{iii})}}^4 \cap V_U^3) \equiv \mathrm{Qlk}^{(5)}(\Sigma_{X_{(\mathrm{i})}}^3, \Sigma_{X_{(\mathrm{ii})}}^3, \Sigma_{X_{(\mathrm{iii})}}^3, \Sigma_U^2)$	
	
Sec. 5.4 and Sec. 6.5: $\#(V_{U_{(\mathrm{i})}}^3 \cap \Sigma_{U_{(\mathrm{ii})}}^2) \equiv \mathrm{Lk}_{B\,\mathrm{d}B}^{(5)}(\Sigma_{U_{(\mathrm{i})}}^2, \Sigma_{U_{(\mathrm{ii})}}^2)$	
	
Sec. 5.3, Sec. 5.4 and Sec. 6.6: $\#(V_{U'}^3 \cap \Sigma_U^2) \equiv \mathrm{Lk}_{w_2\,\mathrm{d}B}^{(5)}(\Sigma_U^2, \Sigma_{U'}^2)$	
	
Sec. 6.7: $\#(V_{X_{(\mathrm{i})}}^4 \cap \Sigma_{X_{(\mathrm{ii})}}^3 \cap V_U^3) \equiv \mathrm{Tlk}_{(A\,\mathrm{d}A)B}^{(5)}(\Sigma_{X_{(\mathrm{i})}}^3, \Sigma_{X_{(\mathrm{ii})}}^3, \Sigma_U^2)$	
	

Table 2: Link invariants and link configurations of 2-worldsheet and 3-worldvolume from the "anyonic"-1D-Strings/2D-Branes' spacetime braiding process in 5d TQFTs, here 5d higher-gauge time-reversal SETs in Sec. 5 and 6.

2 4d $\mathrm{SU}(2)_{\theta=\pi}$ Yang-Mills Gauge Theories coupled to the Boundary of 5d SPTs/Short-Range Entangled Invertible-TQFTs

2.1 Derivation of New Higher-Anomalies of SU(2) Yang-Mills at $\theta = \pi$

We start with SU(2) Yang-Mills theory with $\theta = \pi$, denoted $\mathrm{SU}(2)_{\theta=\pi}$. The Euclidean action $\mathbf{S}_E$ is

$$\mathbf{S}_E[M^4] = \frac{1}{g^2}\int_{M^4} \mathrm{Tr} F \wedge \star F - \frac{\mathrm{i}\pi}{8\pi^2}\int_{M^4} \mathrm{Tr} F \wedge F. \tag{2.1}$$

Since the anomaly is a renormalization group flow invariant, in the following discussion, the kinetic term which is proportional to the running coupling constant $1/g^2$ will not play a role. Hence we only consider the theta term (the second term involving $\theta \mathrm{Tr} F \wedge F$). To probe the anomaly, we turn on the background gauge fields $\mathcal{B}$ for the $\mathbb{Z}^e_{2,[1]}$ 1-form symmetry. Here $\mathcal{B}$ is a $\mathbb{Z}_2$ 2-form gauge field, with $\oint_\Sigma \mathcal{B} = \pi\mathbb{Z}$ for any closed surface Σ, and it is related to the 2-cochain B via $\mathcal{B} \sim \pi B$, and we also convert the wedge product "$\wedge$" to the cup product "$\cup$." To couple the SU(2) Yang-Mills to the background gauge field $\mathcal{B}$, one promotes the SU(2) gauge field b to a U(2) gauge field $\widehat{b}$, and the theta term at $\theta = \pi$ reads[10]

$$\frac{\pi}{8\pi^2}\int_X \mathrm{Tr}\left(\widehat{F} - \mathcal{B}\mathbb{I}_2\right) \wedge \left(\widehat{F} - \mathcal{B}\mathbb{I}_2\right) \tag{2.2}$$

where $\widehat{F} = \mathrm{d}\widehat{b} - \mathrm{i}\widehat{b} \wedge \widehat{b}$ is the U(2) field strength, and $\mathbb{I}_2$ is the two dimensional identity matrix. To restore the SU(2) gauge field, the U(2) field strength should satisfy the gauge bundle constraint

$$w_2(V_{\mathrm{PSU}(2)}) = w_2(V_{\mathrm{SO}(3)}) = \frac{\mathrm{Tr}\widehat{F}}{2\pi} = \frac{2\mathcal{B}}{2\pi} = B \mod 2. \tag{2.3}$$

Here $w_2(V_{\mathrm{PSU}(2)}) = w_2(V_{\mathrm{SO}(3)})$ is the Stiefel-Whitney class of the associated vector bundle of the $\mathrm{PSU}(2) = \mathrm{SO}(3)$ (the principal gauge bundle of $\mathrm{PSU}(2) = \mathrm{SO}(3)$).

To activate the background field for the time reversal symmetry, we formulate Eq. (2.2) on an unorientable manifold M^4. On an unorientable manifold, the top differential form is not well defined, due to the lack of the volume form whose definition needs an orientation. To make sense of Eq. (2.2) on an unorientable manifold, we apply the 1st and the 2nd Chern classes of the associated vector bundles of U(N) (which we denote as $c_j(V_{\mathrm{U(N)}})$ for the jth Chern class for the principal gauge bundle of U(N)):

$$\begin{aligned} c_1(V_{\mathrm{U(N)}}) &= \frac{\mathrm{Tr}\widehat{F}}{2\pi}, \\ c_2(V_{\mathrm{U(N)}}) &= -\frac{1}{8\pi^2}\mathrm{Tr}(\widehat{F} \wedge \widehat{F}) + \frac{1}{8\pi^2}(\mathrm{Tr}\widehat{F}) \wedge (\mathrm{Tr}\widehat{F}). \end{aligned} \tag{2.4}$$

Here $\mathcal{P} \equiv \mathcal{P}_2$ is the Pontryagin square. Replacing $\frac{1}{8\pi^2}\mathrm{Tr}(\widehat{F} \wedge \widehat{F})$ by $\frac{\mathcal{P}(c_1)}{2} - c_2$, Eq. (2.2) is rewritten as

$$\pi \int_{M^4} \left(\frac{\mathcal{P}(c_1(V_{\mathrm{U}(2)}))}{2} - c_2(V_{\mathrm{U}(2)}) - \frac{1}{2} B \cup c_1(V_{\mathrm{U}(2)}) + \frac{\mathcal{P}(B)}{4} \right). \tag{2.5}$$

On an unorientable manifold $M = M^4$, $w_1(TM)$ is nontrivial and one can consider it as the background gauge field for the time reversal symmetry. This allows us to modify the gauge bundle constraint Eq. (2.3) by an additional term $K_1 w_1(TM)^2$, with $K_1 = 0, 1$.

[10] The topological term for the Euclidean action $S_{\mathbf{E},topological}$ in the Euclidean partition function $\mathbf{Z} = \exp(-S_{\mathbf{E},topological})$ contains a factor of imaginary i, namely $\mathbf{S}_E = -\mathrm{i}(\dots)$ in Eq. (2.1). However, by converting $\exp(-\mathbf{S}_E) = \exp(\mathrm{i}\mathbf{S})$, we have the following "Minkowski" $\mathbf{S}$ in Eq. (2.2).

Furthermore, we also activate the term $K_2 w_2(TM)$ with $K_2 = 0,1$, since the underlying manifold does not necessarily allow a Spin/Pin structure. In summary, there are four choices of gauge bundle constraints

$$\boxed{w_2(V_{\mathrm{PSU}(2)}) = w_2(V_{\mathrm{SO}(3)}) = c_1(V_{\mathrm{U}(2)}) = B + K_1 w_1(TM)^2 + K_2 w_2(TM) \mod 2, \quad K_{1,2} \in \mathbb{Z}_2}. \tag{2.6}$$

The value of $K_{1,2}$ has physical consequences: when $K_1 = 0,1$, the SU(2) charge is Kramer singlet ($T^2 = +1$) or Kramer doublet ($T^2 = -1$) under time-reversal transformation; when $K_2 = 0,1$, the SU(2) charge is a boson (quantum spin as an integer) or a fermion (quantum spin as a half-integer). (More details about the Wilson line properties are derived in Sec. 3.)

Since the time reversal and 1-form $\mathbb{Z}^e_{2,[1]}$ symmetry background gauge fields are activated, we would like to check whether the action Eq. (2.5) is gauge invariant. Failure to be gauge invariant implies the existence of 't Hooft anomaly for the global symmetries. Under the 1-form background gauge transformation

$$B \to B + \delta\lambda, \tag{2.7}$$

c_1 also transforms $c_1 \to c_1 + \delta\lambda$ due to the gauge bundle constraint Eq. (2.6). The variation of the theta term Eq. (2.5) is

$$\pi \int_{M^4} \left(\frac{1}{2}\delta\lambda \cup c_1(V_{\mathrm{U}(2)}) + \frac{\mathcal{P}(\delta\lambda)}{4} \right) = \pi \int_{M^4} \left(\frac{1}{2}\delta\lambda \cup (B + \mathrm{Sq}^1\lambda + K_1 w_1(TM)^2 + K_2 w_2(TM)) \right). \tag{2.8}$$

The right hand side of Eq. (2.8) does not vanish.

To interpret such non-invariance as a 4d higher-anomaly (associated to a 5d bordism invariant, or physically to a 5d higher-SPTs, given in [8] and [9], more in the next subsection 2.2), we need to examine Eq. (2.8) can not be cancelled by adding 4d counter terms of the background gauge field. The 4d counter terms that we can add to the action are the topological terms characterizing 4d SPTs, whose general form are

$$\pi \int_{M^4} \left(\frac{L_1}{2}\mathcal{P}(B) + L_2 w_2(TM)^2 + L_3 w_1(TM)^4 + L_4 w_1(TM)^2 B + L_5 w_2(TM) B \right), \tag{2.9}$$

where $L_1 \in \mathbb{Z}_4$, and $L_j \in \mathbb{Z}_2$ for $j = 2,3,4,5$ characterize distinct 4d (higher-)SPT phases. These 4d (higher-)SPTs topological terms have been classified in Ref. [8, 9] via a generalized cobordism theory for higher global symmetries. Under the 1-form gauge transformation Eq. (2.7), the variation of Eq. (2.9) is

$$\pi \int_{M^4} \delta\lambda \cup \Big(L_1 B + L_1 \mathrm{Sq}^1\lambda + L_4 w_1(TM)^2 + L_5 w_2(TM) \Big) \tag{2.10}$$

which does not cancel Eq. (2.8). We conclude that there is indeed a higher 't Hooft anomaly involving the time reversal 0-form ordinary global symmetry and the $\mathbb{Z}^e_{2,[1]}$ 1-form center global symmetry.

2.2 Proof of Anomaly Matching of 5d-4d Inflow and Cobordism Group Data

In this section, we propose that the nontrivial variation Eq. (2.8) is cancelled by a 5d anomaly polynomial

$$\boxed{\pi \int_{M^5} \Big(B \cup \mathrm{Sq}^1 B + \mathrm{Sq}^2\mathrm{Sq}^1 B + K_1 w_1(TM)^3 \cup B + K_2 w_3(TM) \cup B \Big)}. \tag{2.11}$$

The last two term terms are *new higher 't Hooft anomalies* in 4d, which we will explain.

We prove and explain the 5d anomaly polynomials from two complimentary perspectives:

1. From the mathematical perspective, we compare Eq. (2.11) with the bordism group data given in [8] and [9]. Since the global symmetries of 4d $\mathrm{SU}(2)_{\theta=\pi}$ YM theory that we concern include only $\mathbb{Z}_2^T$ 0-form time-reversal and $\mathbb{Z}_{2,[1]}^e$ 1-form center symmetry, we apply the cobordism classification of 5dSPTs/4d anomaly via the 5d bordism group

$$\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2) = \mathbb{Z}_2^4. \tag{2.12}$$

Hence there are four independent generators of the bordism group $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$ of degree two, which we enumerate below

$$\begin{aligned} &B\mathrm{Sq}^1B,\\ &\mathrm{Sq}^2\mathrm{Sq}^1B = (w_2(TM)+w_1(TM)^2)\mathrm{Sq}^1B = (w_3(TM)+w_1(TM)^3)B,\\ &w_1(TM)^2\mathrm{Sq}^1B = w_1(TM)^3B,\\ &w_2(TM)w_3(TM). \end{aligned} \tag{2.13}$$

Clearly, our proposal Eq. (2.11) is a bordism invariant based on an appropriate linear combination of Eq. (2.13), which specifies a 5d higher-SPT and 4d anomaly by a 5d topological term:

$$\exp\left[\mathrm{i}\pi\int_{M^5}\Big(B\mathrm{Sq}^1B + (1+K_2)\mathrm{Sq}^2\mathrm{Sq}^1B + (K_1+K_2)w_1(TM)^3B\Big)\right] \tag{2.14}$$

2. From the quantum field theoretical perspective, we match the anomaly of the 4d $\mathrm{SU}(2)_{\theta=\pi}$ YM theory Eq. (2.8) with the anomaly inflow from the 5d SPTs Eq. (2.11) (or equivalently Eq. (2.14)). This will be demonstrated explicitly below.

To show that Eq. (2.11) is the correct 5d anomaly polynomial, we consider the gauge transformation Eq. (2.7) and examine, when M^5 has a boundary $\partial M^5 = M^4$, whether the variation of Eq. (2.11) is cancelled by the higher 't Hooft anomaly of the $\mathrm{SU}(2)_{\theta=\pi}$ YM theory. Under Eq. (2.7), the variation of Eq. (2.11) is

$$\pi\int_{M^5}\Big(\delta\lambda\mathrm{Sq}^1B_2 + B_2\mathrm{Sq}^1\delta\lambda + \delta\lambda\mathrm{Sq}^1\delta\lambda + \mathrm{Sq}^2\mathrm{Sq}^1\delta\lambda + K_1w_1(TM)^3\delta\lambda + K_2w_3(TM)\delta\lambda\Big). \tag{2.15}$$

Let us simplify the first four terms in Eq. (2.15).

$$\begin{aligned} &\pi\int_{M^5}\Big(\delta\lambda\mathrm{Sq}^1B + B\mathrm{Sq}^1\delta\lambda + \delta\lambda\mathrm{Sq}^1\delta\lambda + \mathrm{Sq}^2\mathrm{Sq}^1\delta\lambda\Big) = \pi\int_{M^5}\Big(\mathrm{Sq}^1(\delta\lambda B) + \frac{1}{2}\mathrm{Sq}^1(\delta\lambda\delta\lambda) + \mathrm{Sq}^2\mathrm{Sq}^1\delta\lambda\Big)\\ &= \pi\int_{M^5}\Big(\mathrm{Sq}^1(\delta\lambda B) + \frac{1}{2}\mathrm{Sq}^1(\delta\lambda\delta\lambda)\Big) = \pi\int_{M^4}\frac{1}{2}\delta\lambda\cup\Big(B + \mathrm{Sq}^1\lambda\Big), \end{aligned} \tag{2.16}$$

where in the second equality we used $\delta = 2\mathrm{Sq}^1$ or $\frac{1}{2}\delta = \mathrm{Sq}^1$, and discarded the last term on the first line since $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$. Therefore, we have shown the *first two* terms in Eq. (2.15) cancel the *first two* terms in Eq. (2.8).

To show that the *last two* terms in Eq. (2.15) cancel the corresponding *last two* terms in Eq. (2.8), we perform integration by parts which allows us to write the 5d terms as 4d terms:

$$\pi\int_{M^5}\Big(K_1w_1(TM)^3\delta\lambda + K_2w_3(TM)\delta\lambda\Big) = \pi\int_{M^4}\lambda\cup\Big(K_1w_1(TM)^3 + K_2w_3(TM)\Big). \tag{2.17}$$

To show Eq. (2.17) matches the anomaly in Eq. (2.8), we manipulate the 4d terms as follows

$$
\begin{aligned}
&\frac{\pi}{2}\int_{M^4}\delta\lambda\cup\Big(K_1 w_1(TM)^2+K_2 w_2(TM)\Big)=\pi\int_{M^4}\mathrm{Sq}^1\lambda\cup\Big(K_1 w_1(TM)^2+K_2 w_2(TM)\Big)\\
&=\pi\int_{M^4}\mathrm{Sq}^1\Big(\lambda\cup(K_1 w_1(TM)^2+K_2 w_2(TM))\Big)+\pi\int_{M^4}\lambda\cup\Big(K_2 w_1(TM)w_2(TM)+K_2 w_3(TM)\Big)\\
&=\pi\int_{M^4}\lambda\cup\Big(K_1 w_1(TM)^3+2K_2 w_1(TM)w_2(TM)+K_2 w_3(TM)\Big)\\
&=\pi\int_{M^4}\lambda\cup\Big(K_1 w_1(TM)^3+K_2 w_3(TM)\Big).
\end{aligned}
\tag{2.18}
$$

The last line precisely cancels Eq. (2.17). This demonstrates the matching of the anomaly. Namely, we have shown all four terms in Eq. (2.15) from the 5d higher-anomaly polynomial (or 5d higher-SPTs), matches the 4d higher-anomaly in Eq. (2.8) of the $\mathrm{SU}(2)_{\theta=\pi}$ YM theory.

2.3 5d SPTs/Bordism Invariants whose Boundary allows 4d $\mathrm{SU}(2)_{\theta=\pi}$ YM

Using the bordism group data and the identities given in Ref. [8] and [9], we can rewrite the 4d higher-anomalies and 5d higher-SPTs/bordism invariants/anomaly polynomials Eq. (2.11) and Eq. (2.14) in various equivalent ways

$$
\mathbf{Z}^{5\mathrm{d}}_{\mathrm{SPT}_{(K_1,K_2)}}[M^5]
$$

$$
\equiv \exp\Big(\mathrm{i}\pi\int_{M^5}\frac{1}{2}\tilde{w}_1(TM)\cup\mathcal{P}(B)+K_1 w_1(TM)^3 B+K_2 w_3(TM)B\Big) \tag{2.19}
$$

$$
\begin{aligned}
&= \exp\Big(\mathrm{i}\pi\int_{M^5}\frac{1}{4}\delta(\mathcal{P}_2(B_2))+K_1 w_1(TM)^2\mathrm{Sq}^1 B+K_2 w_2(TM)\mathrm{Sq}^1 B\Big)\\
&= \exp\Big(\mathrm{i}\pi\int_{M^5}B\mathrm{Sq}^1 B+\mathrm{Sq}^2\mathrm{Sq}^1 B+K_1 w_1(TM)^2\mathrm{Sq}^1 B+K_2 w_2(TM)\mathrm{Sq}^1 B\Big)\\
&= \exp\Big(\mathrm{i}\pi\int_{M^5}B\mathrm{Sq}^1 B+(w_2(TM)+w_1(TM)^2)\mathrm{Sq}^1 B+K_1 w_1(TM)^2\mathrm{Sq}^1 B+K_2 w_2(TM)\mathrm{Sq}^1 B\Big)\\
&= \exp\Big(\mathrm{i}\pi\int_{M^5}B\mathrm{Sq}^1 B+(1+K_1)w_1(TM)^2\mathrm{Sq}^1 B+(1+K_2)w_2(TM)\mathrm{Sq}^1 B\Big) \qquad (2.20)\\
&= \exp\Big(\mathrm{i}\pi\int_{M^5}B\mathrm{Sq}^1 B+(w_3(TM)+w_1(TM)^3)B+K_1 w_1(TM)^3 B+K_2 w_3(TM)B\Big)\\
&= \exp\Big(\mathrm{i}\pi\int_{M^5}B\mathrm{Sq}^1 B+(1+K_1)w_1(TM)^3 B+(1+K_2)w_3(TM)B\Big).
\end{aligned}
$$

3 Classification of 4d $\mathrm{SU}(2)_{\theta=\pi}$ Yang-Mills theories: Bosonic UV completions

In this section we aim to better digest the constraints between the Eq. (2.6), the gauge connection $w_2(V_{\mathrm{SO}(3)})$ and the spacetime connection $w_j(TM)$, i.e.,

$$w_2(V_{\mathrm{SO}(3)}) = B + K_1 w_1(TM)^2 + K_2 w_2(TM) \mod 2, \quad K_{1,2} \in \mathbb{Z}_2.$$

and discuss their physical consequences.

3.1 Kramers Time Reversal Even/Odd and Bosonic/Fermionic Wilson line

Below we provide some physical interpretations of the "Fantastic Four Siblings" of 4d SU(2) YM theories based on its 1d Wilson line properties.

First, we introduce the standard 4d SU(2) Yang-Mills path integral $\mathbf{Z}^{\text{4d}}_{\mathrm{SU(2)YM}}[B]$ with background 2-form B field coupling. Here $\mathbf{Z}^{\text{4d}}_{\mathrm{SU(2)YM}}[B]$ is the combination of Eq. (1.1)'s $\mathbf{Z}^{\text{4d}}_{\mathrm{YM}}$, with the field strength coupling $\hat{F} - B$ following in Eq. (2.2). The Stiefel-Whitney (SW) class of the associated vector bundle of the gauge bundle E for the SU(2) gauge theory is constrained as the SW class of the associated vector bundle of SO(3):

$$w_2(E) = w_2(V_{\mathrm{SO}(3)}) \tag{3.1}$$

Conventionally we have the 4d YM coupling to a background 2-form B as [6] (our notation follows [8])[11]

$$\int [D\Lambda]\, \mathbf{Z}^{\text{4d}}_{\mathrm{SU(2)YM}}[B] \exp\big(\mathrm{i}\pi \int \Lambda \cup (w_2(E) - B)\big),$$

• Electric 2-surface U_e: Mathematically, integrating out the Lagrange multiplier Λ, set $(w_2(E) - B) = 0$ mod 2. Physically, $\exp(\mathrm{i}\pi \int \Lambda)$ plays the role of an electric 2-surface $U_e = \exp(\mathrm{i}\pi \int \Lambda)$, which measures 1-form e-symmetry $\mathbb{Z}^e_{2,[1]}$. The magnetic 't Hooft line lives on the boundary of an electric 2-surface $U_e = \exp(\mathrm{i}\pi \int \Lambda)$. Since U_e is dynamical, 't Hooft line is not genuine thus not in the line spectrum for the SU(2) gauge theory [6].

• Magnetic 2-surface U_m is given by $\exp(\mathrm{i}\pi \int w_2(E))$. We can show from the fact that the 2-surface $w_2(E)$ defined by a 2-surface defect (where each small 1-loop of 't Hooft line linked with this $w_2(E)$ getting a nontrivial π-phase $e^{\mathrm{i}\pi}$). Thus, the $w_2(E)$ has its boundary with Wilson loop $W_e = \mathrm{Tr}(\mathrm{P}\exp(\mathrm{i} \oint a))$ such that $U_e U_m \sim \exp(\mathrm{i}\pi \int \Lambda \cup w_2(E))$ specifies that when a 2-surface U_e links with (i.e. wraps around) a 1-Wilson loop W_e, it yields a nontrivial statistical π-phase $e^{\mathrm{i}\pi} = -1$.

Now we propose to modify YM partition function following a different bundle/connection constraint Eq. (2.6), so we arrive at a new partition function:

$$\mathbf{Z}^{\text{4d}}_{\mathrm{SU(2)YM}_{(K_1,K_2)}}[B] \equiv \int [D\Lambda]\, \mathbf{Z}^{\text{4d}}_{\mathrm{SU(2)YM}}[B] \exp\Big(\mathrm{i}\pi \int \Lambda \cup \Big(w_2(E) - \Big(B + K_1 w_1(TM)^2 + K_2 w_2(TM)\Big)\Big)\Big). \tag{3.2}$$

As we just deduce that the magnetic 2-surface $U_m \sim \exp(\mathrm{i}\pi \int w_2(E))$ has its boundary as 1-Wilson loop $W_e = \mathrm{Tr}(\mathrm{P}\exp(\mathrm{i} \oint a))$, together with the modified YM partition function Eq. (3.2) and its constraint Eq. (2.6), now we can show that

[11] We can also introduce an additional Pontryagin square B term $\exp\big(\mathrm{i}\,\frac{\pi}{2} p \mathcal{P}(B)\big)$ with $p \in \mathbb{Z}_4$ into the path integral, as the pioneer works Ref. [55] and [6] do. However, this weight factor term only will result in shifting (thus relabeling) of the classification of 4d $\mathrm{SU}(2)_{\theta=\pi}$ theories that we are going to reveal.

1. $(K_1, K_2) = (0, 0)$: The gauge bundle constraint is $w_2(E) = B \mod 2$. The magnetic 2-surface $U_m \sim \exp(\mathrm{i}\pi \int w_2(E))$ has no decoration other than the 2-form background B field. Thus the magnetic 2-surface U_m's boundary 1-Wilson line W_e is Kramer singlet ($T^2 = +1$) and bosonic.

2. $(K_1, K_2) = (1, 0)$: The gauge bundle constraint becomes $w_2(E) = B + w_1(TM)^2 \mod 2$. The magnetic 2-surface $U_m \sim \exp(\mathrm{i}\pi \int w_2(E))$ has a decoration $\int w_1(TM)^2$ other than the 2-form B field. But $\int w_1(TM)^2$ is a topological term in a cohomology group $\mathrm{H}^2(\mathbb{Z}_2^T, \mathrm{U}(1))$ also in bordism group $\Omega_2^{\mathrm{O}}(pt)$, which is effectively a 2d Haldane's anti-ferromagnetic quantum spin-1 chain protected by time-reversal symmetry. It is well-known that the 2d Haldane's spin-1 chain's each open 1d boundary has two-fold degeneracy due to Kramer doublet ($T^2 = -1$). Thus due to $\int w_1(TM)^2$ decoration, the magnetic 2-surface U_m's boundary 1-Wilson line W_e is Kramer doublet ($T^2 = -1$) and bosonic.

3. $(K_1, K_2) = (0, 1)$: The gauge bundle constraint becomes $w_2(E) = B + w_2(TM) \mod 2$. The magnetic 2-surface $U_m \sim \exp(\mathrm{i}\pi \int w_2(E))$ has a decoration $\int w_2(TM)$ other than the 2-form B field. But $\int w_2(TM)$ is associated to a spin structure. The 2d $\int w_2(TM)$'s each open 1d boundary as a worldline of particle has fermionic statistics. Thus due to $\int w_2(TM)$ decoration, the magnetic 2-surface U_m's boundary 1-Wilson line W_e is Kramer singlet ($T^2 = +1$) and fermionic.

4. $(K_1, K_2) = (1, 1)$: The gauge bundle constraint is $w_2(E) = B + w_1(TM)^2 + w_2(TM) \mod 2$. The combined effects mean that the magnetic 2-surface U_m's boundary 1-Wilson line W_e is Kramer doublet ($T^2 = -1$) and fermionic.

In fact, our above discussions are universal applicable to more general SU(N) YM theories![12] This way of enumerating gauge theories (based on new gauge bundle constraints) guides us to obtain new classes of gauge theories beyond the frame work of Ref. [55]. The implications are not restricted to merely 4d $\mathrm{SU}(2)_{\theta=\pi}$ YM.

3.2 Enumeration of Gauge Theories from Dynamically Gauging 4d SPTs

We have discussed the "Fantastic Four Siblings" of $\mathrm{SU}(2)_{\theta=\pi}$ YM theories given by $\mathbf{Z}^{4\mathrm{d}}_{\mathrm{SU}(2)\mathrm{YM}_{(K_1,K_2)}}[B]$ in Eq. (3.2), with four distinct sets of new anomalies derived in Sec. 2, and with Kramer singlet/doublet ($T^2 = +1/-1$) or bosonic/fermionic Wilson lines in Sec. 3.1. With these properties shown, we are confident that they are really four distinct classes of $\mathrm{SU}(2)_{\theta=\pi}$ YM theories (at least at the UV high energy). The distinct 't Hooft anomalies of (K_1, K_2) also shows that the four classes of $\mathrm{SU}(2)_{\theta=\pi}$ YM theories are distinct.

In this subsection, we like to construct and enumerate these "Fantastic Four Siblings" of $\mathrm{SU}(2)_{\theta=\pi}$ YM theories by dynamically gauging the SU(2) symmetry from 4d time-reversal symmetric SU(2)-SPTs. To this end, we follow Freed-Hopkins [39] to consider a suitable group extension from the time-reversal symmetry (where the spacetime d-manifold requires the O(d)-structure) via a SU(2) extension:

$$1 \to \mathrm{SU}(2) \to G' \to \mathrm{O}(d) \to 1. \tag{3.3}$$

These 4d SPTs can be regarded as 4d co/bordism invariants of

$$\Omega^{G'}_{4,\mathrm{tor}}, \tag{3.4}$$

and the 4d SPTs are classified by this torsion subgroup $\Omega^{G'}_{4,\mathrm{tor}}$ of the bordism group $\Omega_4^{G'}$ for all the possible G' under the above group extension. The extension is classified by $\mathrm{H}^2(\mathrm{BO}(d), \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$ for $d > 1$, generated by $w_1^2(TM)$ and $w_2(TM)$.

[12] Related studies along this line of analysis have also appeared in [58] and [59].

The solution G' of this extension problem $1 \to \mathrm{SU}(2) \to G' \to \mathrm{O} \to 1$, is given in [39] with indeed four choices of $G' = \mathrm{O} \times \mathrm{SU}(2)$ or $\mathrm{E} \times_{\mathbb{Z}_2} \mathrm{SU}(2)$ or $\mathrm{Pin}^+ \times_{\mathbb{Z}_2} \mathrm{SU}(2)$ or $\mathrm{Pin}^- \times_{\mathbb{Z}_2} \mathrm{SU}(2)$.

Follow the similar study in Ref. [36], there is a correspondence between the element $\mathrm{b} = K_1 w_1(TM)^2 + K_2 w_2(TM)$ and $\mathrm{H}^2(\mathrm{BO}(d), \mathbb{Z}_2) = (\mathbb{Z}_2)^2$. It will soon become clear that b is related to $w_2(V_{\mathrm{SO}(3)}) - B$ (i.e., the difference of the gauge bundle $E = V_{\mathrm{SO}(3)}$ connection and the background gauge connection B). Then the 4 central extension choices labeled by b are:

1. $\mathrm{b} = 0 \Rightarrow G' = \mathrm{O}(d) \times \mathrm{SU}(2) \Rightarrow$ After gauging SU(2), we gain the gauge bundle constraint with $K_1 = K_2 = 0$,
$$w_2(V_{\mathrm{SO}(3)}) - B = 0.$$
One can compute the co/bordism group in Table 3 (also given in [36]), we obtain in 4d:
$$\boxed{\Omega_{4,\mathrm{tor}}^{\mathrm{O}(d)\times\mathrm{SU}(2)} = \mathbb{Z}_2^3}, \tag{3.5}$$
whose bordism invariants are generated by three generators of mod 2 classes:
$$\begin{cases} w_1^4(TM), \\ w_2^2(TM), \\ c_2 \mod 2. \end{cases} \tag{3.6}$$

2. $\mathrm{b} = w_1(TM)^2 \Rightarrow G' = \mathrm{E(d)} \times_{\mathbb{Z}_2} \mathrm{SU}(2) \Rightarrow$ After gauging SU(2), we gain the gauge bundle constraint with $K_1 = 1$ and $K_2 = 0$,
$$w_2(V_{\mathrm{SO}(3)}) - B = w_1(TM)^2.$$
We compute the co/bordism group in Table 4, we obtain in 4d:
$$\boxed{\Omega_{4,\mathrm{tor}}^{\mathrm{E(d)}\times_{\mathbb{Z}_2}\mathrm{SU}(2)} = \mathbb{Z}_2}, \tag{3.7}$$
whose bordism invariant is generated by one generator of mod 2 class:
$$\{\ \tilde{b} w_2(V_{\mathrm{SO}(3)}). \tag{3.8}$$
The $\mathrm{E}(d)$ is defined in [39] where $\mathrm{E}(d)$ is a subgroup of $\mathrm{O}(d) \times \mathbb{Z}_4$, described by two data $(M, j) \in (\mathrm{O}(d), \mathbb{Z}_4)$ where such that the $\det M = j^2$. We define $\tilde{b} = b \mod 2$ where b is the generator of $\mathrm{H}^2(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_4)$.

3. $\mathrm{b} = w_2(TM) \Rightarrow G' = \mathrm{Pin}^+ \times_{\mathbb{Z}_2} \mathrm{SU}(2) \Rightarrow$ After gauging SU(2), we gain the gauge bundle constraint with $K_1 = 0$ and $K_2 = 1$,
$$w_2(V_{\mathrm{SO}(3)}) - B = w_2(TM).$$
The co/bordism group is computed in [36, 39] and in Table 5, we obtain in 4d:
$$\boxed{\Omega_{4,\mathrm{tor}}^{\mathrm{Pin}^+\times_{\mathbb{Z}_2}\mathrm{SU}(2)}} = \mathbb{Z}_4 \times \mathbb{Z}_2, \tag{3.9}$$
whose bordism invariants are generated by generators of mod 4 and mod 2 classes:
$$\begin{cases} \nu \eta_{\mathrm{SU}(2)}, \text{with a } \nu \in \mathbb{Z}_4 \text{ class} \\ w_2^2(TM). \end{cases} \tag{3.10}$$
This is related to the interacting version of CI class topological superconductor in condensed matter physics ([60], [39], and [36]). Details of these topological terms are discussed in [36].

4. $\mathrm{b} = w_2(TM) + w_1(TM)^2 \Rightarrow G' = \mathrm{Pin}^- \times_{\mathbb{Z}_2} \mathrm{SU}(2) \Rightarrow$ After gauging SU(2), we gain the gauge bundle constraint with $K_1 = K_2 = 1$,
$$w_2(V_{\mathrm{SO}(3)}) - B = w_2(TM) + w_1(TM)^2.$$

The co/bordism group is computed in [36,39] and in Table 6, we obtain in 4d:

$$\boxed{\Omega_{4,\text{tor}}^{\text{Pin}^- \times_{\mathbb{Z}_2} \text{SU}(2)}} = (\mathbb{Z}_2)^3, \tag{3.11}$$

whose bordism invariants are generated by three generators of mod 2 classes:

$$\begin{cases} N_0' \mod 2, \\ w_1^4(TM), \\ w_2^2(TM). \end{cases} \tag{3.12}$$

This is related to the interacting version of CII class topological insulator in condensed matter physics ([60], [39], and [36]). Details of these topological terms are discussed in [36].

More information about these (co)bordism group calculations can be read from [36,39]. See Appendix of [36] for a quick background review. In particular, since the computation involve no odd torsion, we can use Adams spectral sequence to compute $\Omega_n^{G'} = \pi_n(MTG')$:

$$\text{Ext}_{\mathcal{A}_2}^{s,t}(\text{H}^*(MTG', \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(MTG')_2^\wedge. \tag{3.13}$$

Here $\pi_{t-s}(MTG')_2^\wedge$ is the 2-completion of the group $\pi_{t-s}(MTG')$. For example, $MT(\text{O} \times \text{SU}(2)) = M\text{O} \wedge \text{BSU}(2)_+$, $MT(\text{E} \times_{\mathbb{Z}_2} \text{SU}(2)) = M\text{SO} \wedge \Sigma^{-2} M\mathbb{Z}_4 \wedge \Sigma^{-3} M\text{SO}(3)$, $MT(\text{Pin}^+ \times_{\mathbb{Z}_2} \text{SU}(2)) = M\text{Spin} \wedge \Sigma^{-3} M\text{O}(3)$, $MT(\text{Pin}^- \times_{\mathbb{Z}_2} \text{SU}(2)) = M\text{Spin} \wedge \Sigma^3 MT\text{O}(3)$. $\text{BSU}(2)_+$ is the disjoint union of BSU(2) and a point, Σ is the suspension.

Let M be an n-manifold, $V_{\text{SO}(3)}$ be the associated vector bundle of the SO(3) gauge bundle. Below we compute the Stiefel-Whitney classes of $(TM - n) \otimes V_{\text{SO}(3)}$. They are used to express the cobordism invariants of $\Omega_d^{\text{Pin}^\pm \times_{\mathbb{Z}_2} \text{SU}(2)}$. Below w_i means the i-th Stiefel-Whitney class, w means the total Stiefel-Whitney class, namely, we have $w = 1 + w_1 + w_2 + w_3 + \cdots$. Denote $w_i' = w_i(V_{\text{SO}(3)})$, $w_i = w_i((TM - n) \otimes V_{\text{SO}(3)})$. In addition, the $w_i(TM)$ means specifically the i-th Stiefel-Whitney class of spacetime tangent bundle TM.

$$\begin{aligned} & w((TM-n) \otimes V_{\text{SO}(3)}) \\ = & \frac{w(TM \otimes V_{\text{SO}(3)})}{w(V_{\text{SO}(3)})^n} \\ = & \frac{1 + w_1(TM) + w_1(TM)^2 + w_2(TM) + nw_2' + w_1(TM)^3 + nw_1(TM)w_2' + w_3(TM) + nw_3' + \cdots}{(1 + w_2' + w_3' + \cdots)^n} \\ = & 1 + w_1(TM) + w_1(TM)^2 + w_2(TM) + w_1(TM)^3 + w_3(TM) + \cdots \end{aligned} \tag{3.14}$$

So $w_1 = w_1(TM)$, $w_2 = w_1(TM)^2 + w_2(TM)$, $w_3 = w_1(TM)^3 + w_3(TM)$, etc.

We also use the notation "TP" for the classification of topological phases defined in [39], such that

$$\text{TP}_{d,\text{tor}}(G') = \Omega_{d,\text{tor}}^{G'}. \tag{3.15}$$

Here are the list of tables summarizing the results in 4d and in 5d:

We conclude this section with some comments. The "Fantastic Four Siblings" of 4d $\text{SU}(2)_{\theta=\pi}$ YM theories are obtained, specifically, from summing over the SU(2) gauge connections of following four topological terms (i.e., gauging the SU(2) symmetry four distinct SPTs):

1. $(-1)^{c_2}$ in Eq. (3.6).

d	$\mathrm{TP}_{d,\mathrm{tor}}(\mathrm{O}(d)\times\mathrm{SU}(2))$	co/bordism invariants
4	$\mathbb{Z}_2^3$	$w_1^4(TM), w_2^2(TM), c_2 \mod 2$
5	$\mathbb{Z}_2$	$w_2(TM)w_3(TM)$

Table 3: Cobordism groups $\mathrm{TP}_i(\mathrm{O}(d)\times\mathrm{SU}(2))$ and co/bordism invariants. Here $w_i(TM)$ is the i-th Stiefel-Whitney class of the spacetime tangent bundle, c_2 is the second Chern class of the SU(2) gauge bundle.

d	$\mathrm{TP}_{d,\mathrm{tor}}(\mathrm{E}(d)\times_{\mathbb{Z}_2}\mathrm{SU}(2))$	cobordism invariants
4	$\mathbb{Z}_2$	$\tilde{b}w_2'$
5	$\mathbb{Z}_2^3\times\mathbb{Z}_4^2$	$w_2(TM)w_3(TM), w_2'w_3', \tilde{a}\tilde{b}w_2', ab^2, a\mathcal{P}_2(w_2(TM))$

Table 4: Cobordism groups $\mathrm{TP}_d(\mathrm{E}(d)\times_{\mathbb{Z}_2}\mathrm{SU}(2))$ and cobordism invariants. Here $\tilde{a}=a \mod 2$ where a is the generator of $\mathrm{H}^1(\mathrm{B}\mathbb{Z}_4,\mathbb{Z}_4)$, and $\tilde{b}=b \mod 2$ where b is the generator of $\mathrm{H}^2(\mathrm{B}\mathbb{Z}_4,\mathbb{Z}_4)$, $w_i(TM)$ is the i-th Stiefel-Whitney class of the spacetime tangent bundle, w_i' is the i-th Stiefel-Whitney class of the SO(3) gauge bundle. Note that there is a short exact sequence of groups: $1\to\mathrm{SO}(d)\to\mathrm{E}(d)\to\mathbb{Z}_4\to 1$, and $MT(\mathrm{E}(d)\times_{\mathbb{Z}_2}\mathrm{SU}(2)) = MT\mathrm{E}(d)\wedge\Sigma^{-3}M\mathrm{SO}(3) = M\mathrm{SO}(d)\wedge\Sigma^{-2}M\mathbb{Z}_4\wedge\Sigma^{-3}M\mathrm{SO}(3)$.

d	$\mathrm{TP}_{d,\mathrm{tor}}(\mathrm{Pin}^+(d)\times_{\mathbb{Z}_2}\mathrm{SU}(2))$	cobordism invariants
4	$\mathbb{Z}_2\times\mathbb{Z}_4$	$w_2^2, \eta_{\mathrm{SU}(2)}$
5	$\mathbb{Z}_2$	w_2w_3

Table 5: Cobordism groups $\mathrm{TP}_d(\mathrm{Pin}^+(d)\times_{\mathbb{Z}_2}\mathrm{SU}(2))$ and cobordism invariants. Here w_i is the i-th Stiefel-Whitney class of $(TM-n)\otimes V_{\mathrm{SO}(3)}$ where $V_{\mathrm{SO}(3)}$ is the associated vector bundle of the SO(3) gauge bundle. The w_i is computed in Eq. (3.14). The $\eta_{\mathrm{SU}(2)}$ is an eta invariant of Dirac operator defined in [36]. More details of computation can be read from [36,39].

d	$\mathrm{TP}_{d,\mathrm{tor}}(\mathrm{Pin}^-(d)\times_{\mathbb{Z}_2}\mathrm{SU}(2))$	cobordism invariants
4	$\mathbb{Z}_2^3$	w_2^2, w_1^4, $(N_0'^{(4)} \mod 2)\sim w_3\tilde{\eta}$
5	$\mathbb{Z}_2^2$	w_2w_3, $(N_0'^{(5)} \mod 2)\sim w_3\mathrm{Arf}$

Table 6: Cobordism groups $\mathrm{TP}_d(\mathrm{Pin}^-(d)\times_{\mathbb{Z}_2}\mathrm{SU}(2))$ and cobordism invariants. Here w_i is the i-th Stiefel-Whitney class of $(TM-n)\otimes V_{\mathrm{SO}(3)}$ where $V_{\mathrm{SO}(3)}$ is the associated vector bundle of the SO(3) gauge bundle. The w_i is computed in Eq. (3.14). The $N_0'^{(4)}$ is the number of the zero modes of the Dirac operator in 4d. Its value mod 2 is a spin-topological invariant known as the mod 2 index defined as N_0' mod 2 in [36]. More details of computation can be read from [36,39]. We find that the bordism invariant of $N_0'^{(4)}$ mod 2 read from Adams chart has the similar form related to $w_3\tilde{\eta}$, where $\tilde{\eta}$ is the eta invariant for 2d Dirac operator, given by the generator of the 2d spin bordism group $\Omega_{2,\mathrm{tor}}^{\mathrm{Spin}}(pt)=\mathbb{Z}_2$. The $N_0'^{(5)}$ is the number of the zero modes of the Dirac operator in 5d. Its value mod 2 is a spin-topological invariant known as the mod 2 index defined in [53,54]. We find that the bordism invariant of $N_0'^{(5)}$ mod 2 read from Adams chart has the similar form related to $w_3\mathrm{Arf}$, where Arf is an Arf invariant.

2. $(-1)^{\tilde{b}w_2(V_{\mathrm{SO}(3)})}$ in Eq. (3.8).
3. $\exp(2\pi \mathrm{i}\nu\eta_{\mathrm{SU}(2)})$ with an odd class of $\nu=1,3\in\mathbb{Z}_4$ in Eq. (3.10).
4. $(-1)^{N_0'}$ in Eq. (3.12).

These four theories exactly map to the enumeration of four gauge theories in Sec. 3.1. Adding other SPTs/bordism invariants such as $(-1)^{w_1^4(TM)}$ and $(-1)^{w_2^2(TM)}$ (and then dynamically gauging them), do not alter or gain new classes of gauge theories. They only affect a gauge theory to the same gauge theory tensor product with 4d SPTs, namely (4d $\mathrm{SU}(2)_{\theta=\pi}$ YM) $\otimes$ (4d SPTs).[13]

4 Time-Reversal Symmetry-Enriched 5d Higher-Gauge TQFTs

4.1 Partition Function of 5d Higher-Gauge TQFTs

Following the discussions of four classes of 5d time-reversal and 1-form (center) symmetry $\mathbb{Z}_2^T \times \mathbb{Z}_{2,[1]}^e$ higher-SPTs $\mathbf{Z}^{5\mathrm{d}}_{\mathrm{SPT}_{(K_1,K_2)}}[M^5]$ in Sec. 2.3 with their partition functions in Eq. (2.19), we proceed to dynamically gauge the 1-form (center) symmetry $\mathbb{Z}_{2,[1]}^e$. Then we obtain the 5d time-reversal symmetric SET with 2-form $\mathbb{Z}_2$-valued B gauge field. We can define the four classes of 5d partition functions $\mathbf{Z}^{5\mathrm{d}}_{\mathrm{SET}_{(K_1,K_2)}}[M^5]$ as:

$$
\begin{aligned}
\mathbf{Z}^{5\mathrm{d}}_{\mathrm{SET}_{(K_1,K_2)}}[M^5] &\equiv \frac{|\,\mathrm{H}^0(M,\mathbb{Z}_2)|}{|\,\mathrm{H}^1(M,\mathbb{Z}_2)|} \sum_{B\in \mathrm{H}^2(M^5,\mathbb{Z}_2)} \mathrm{e}^{\,\mathrm{i}\,\pi\int_{M^5} \frac{1}{2}\tilde{w}_1(TM)\cup\mathcal{P}(B)+K_1 w_1(TM)^3 B + K_2 w_3(TM) B} &(4.1)\\
&= \frac{|\,\mathrm{H}^0(M,\mathbb{Z}_2)|}{|\,\mathrm{H}^1(M,\mathbb{Z}_2)|} \sum_{B\in \mathrm{H}^2(M^5,\mathbb{Z}_2)} \mathrm{e}^{\,\mathrm{i}\,\pi\int_{M^5} B\mathrm{Sq}^1 B+(1+K_1)w_1(TM)^2\mathrm{Sq}^1 B+(1+K_2)w_2(TM)\mathrm{Sq}^1 B} &(4.2)\\
&= \frac{|\,\mathrm{H}^0(M,\mathbb{Z}_2)|}{|\,\mathrm{H}^1(M,\mathbb{Z}_2)|} \sum_{\substack{B,b,h\in \mathrm{C}^2(\mathrm{M}^5,\mathbb{Z}_2)\\ c\in \mathrm{C}^3(\mathrm{M}^5,\mathbb{Z}_2)}} \exp\Big(\mathrm{i}\pi \int_{M^5} \delta w_1(TM)\cup c + \delta w_2(TM)\cup h + b\cup \delta B\\
&\qquad + B\mathrm{Sq}^1 B + (1+K_1)w_1(TM)^2\mathrm{Sq}^1 B + (1+K_2)w_2(TM)\mathrm{Sq}^1 B\Big) &(4.3)\\
&\cong \int [\mathcal{D}B][\mathcal{D}b][\mathcal{D}h][\mathcal{D}c]\exp\Big(\mathrm{i}\pi \int_{M^5} (\mathrm{d}w_1(TM))c + (\mathrm{d}w_2(TM))h + b\mathrm{d}B\\
&\qquad + B\frac{1}{2}\mathrm{d}B + (1+K_1)w_1(TM)^2\frac{1}{2}\mathrm{d}B + (1+K_2)w_2(TM)\frac{1}{2}\mathrm{d}B\Big). &(4.4)
\end{aligned}
$$

In the last step, we have convert the 5d higher-cochain TQFT to 5d higher-form gauge field continuum TQFT for $\mathbf{Z}^{5\mathrm{d}}_{\mathrm{SET}_{(K_1,K_2)}}[M^5]$. Moreover, we can insert extended operators (say $U, X, Y, \dots$) into the path integral:

$$
\begin{aligned}
\mathbf{Z}^{5\mathrm{d}}_{\mathrm{SET}_{(K_1,K_2)}}[M^5; U, X, Y, \dots] \equiv \int [\mathcal{D}B][\mathcal{D}b][\mathcal{D}h][\mathcal{D}c]\; U\cdot X\cdot Y\dots\\
\exp\Big(\mathrm{i}\pi \int_{M^5} (\mathrm{d}w_1(TM))c + (\mathrm{d}w_2(TM))h + b\mathrm{d}B\\
+B\frac{1}{2}\mathrm{d}B + (1+K_1)w_1(TM)^2\frac{1}{2}\mathrm{d}B + (1+K_2)w_2(TM)\frac{1}{2}\mathrm{d}B\Big).
\end{aligned}
\tag{4.5}
$$

for the 5d higher-form continuum TQFT.

[13] For the classification of gauge theory, we identify the following phases

(gauge theory) $\otimes$ (SPTs) $\simeq$ (gauge theory).

For the classification of 4d $\mathrm{SU}(2)_{\theta=\pi}$ YM, we identify the following phases

(4d $\mathrm{SU}(2)_{\theta=\pi}$ YM) $\otimes$ (4d SPTs) $\simeq$ (4d $\mathrm{SU}(2)_{\theta=\pi}$ YM).

See more physically motivated discussions in [36] and References therein.

4.2 Partition Function and Topological Degeneracy

Below we compute the partition function $\mathbf{Z}(M^5)$ on closed manifolds $M = M^5$. When $M^5 = M^4 \times S^1$, we can interpret it as topological ground state degeneracy (GSD) of TQFT. Our computations follow the strategy in [12, 14], while we directly summarize the results in Tables 7, 8, and 9.

4.2.1 5d SPTs as Short-Range Entangled Invertible TQFTs

$\mathbf{Z}(M^5)$ with M^5:	$(\mathrm{W}, 0)$	$(S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2, \gamma\alpha_1)$	$(S^1 \times \mathbb{RP}^4, \gamma\zeta)$	$(\mathbb{RP}^2 \times \mathbb{RP}^3, \alpha\beta)$
$\mathbf{Z}_{\mathrm{SPT}}^{\mathrm{trivial}}(M^5)$	1	1	1	1
$\mathbf{Z}_{\mathrm{SPT}_{B\mathrm{Sq}^1 B}}(M^5)$	1	1	1	-1
$\mathbf{Z}_{\mathrm{SPT}_{\mathrm{Sq}^2\mathrm{Sq}^1 B}}(M^5)$	1	1	-1	1
$\mathbf{Z}_{\mathrm{SPT}_{w_1(TM)^2\mathrm{Sq}^1 B}}(M^5)$	1	-1	-1	1
$\mathbf{Z}_{\mathrm{SPT}_{w_2(TM)\mathrm{Sq}^1 B}}(M^5)$	1	-1	1	1

Table 7: Partition Function $\mathbf{Z}(M^5)$ and Topological Degeneracy (GSD) of 5d higher-SPTs, for example, $\mathbf{Z}_{\mathrm{SPT}_{B\mathrm{Sq}^1 B}}(M^5) := (-1)^{\int_{M^5} B\mathrm{Sq}^1 B}$. The notations $\alpha, \beta, \gamma, \zeta$ are explained in the computation below.

4.2.2 5d SETs, as Long-Range Entangled TQFTs

$\mathbf{Z}(M^5)$ with M^5:	T^5	$S^1 \times S^4$	$S^1 \times \mathbb{RP}^4$	$T^2 \times S^3$	$S^1 \times S^2 \times S^2$	$S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2$	$\mathbb{RP}^2 \times \mathbb{RP}^3$	S^5	W
$\mathbf{Z}_{\text{2-form } B}^{\mathrm{untwist}}(M^5)$	$\frac{2^{10}\cdot 2}{2^5} = 64$	$\frac{2^0\cdot 2}{2^1} = 1$	$\frac{2^2\cdot 2}{2^2} = 2$	$\frac{2^1\cdot 2}{2^2} = 1$	$\frac{2^2\cdot 2}{2^1} = 4$	$\frac{2^5\cdot 2}{2^3} = 8$	$\frac{2^3\cdot 2}{2^2} = 4$	$\frac{2^0\cdot 2}{2^0} = 2$	4
$\mathbf{Z}_{\mathrm{SET}_{(0,0)}}(M^5)$	64	1	1	1	4	2	2	2	4
$\mathbf{Z}_{\mathrm{SET}_{(1,0)}}(M^5)$	64	1	1	1	4	2	2	2	4
$\mathbf{Z}_{\mathrm{SET}_{(0,1)}}(M^5)$	64	1	1	1	4	2	2	2	0
$\mathbf{Z}_{\mathrm{SET}_{(1,1)}}(M^5)$	64	1	1	1	4	2	2	2	0

Table 8: Partition Function $\mathbf{Z}(M^5)$ and Topological Degeneracy (GSD) of 5d higher-SETs, $\mathbf{Z}_{\mathrm{SET}_{(K_1,K_2)}}(M^5) := \frac{|\mathrm{H}^0(M^5,\mathbb{Z}_2)|}{|\mathrm{H}^1(M^5,\mathbb{Z}_2)|} \sum_{B \in \mathrm{H}^2(M^5,\mathbb{Z}_2)} (-1)^{\int_{M^5} B\mathrm{Sq}^1 B + (1+K_1) w_1(TM)^2 \mathrm{Sq}^1 B + (1+K_2) w_2(TM)\mathrm{Sq}^1 B}$.

$\mathbf{Z}(M^5)$ with M^5:	W	$S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2$	$S^1 \times \mathbb{RP}^4$	$\mathbb{RP}^2 \times \mathbb{RP}^3$
$\mathbf{Z}^{\text{untwist}}_{\text{2-form } B}(M^5)$	4	8	2	4
$\mathbf{Z}_{\text{SET}_{B\text{Sq}^1 B}}(M^5)$	0	2	1	2
$\mathbf{Z}_{\text{SET}_{\text{Sq}^2\text{Sq}^1 B}}(M^5)$	0	8	0	4
$\mathbf{Z}_{\text{SET}_{w_1(TM)^2\text{Sq}^1 B}}(M^5)$	4	0	0	4
$\mathbf{Z}_{\text{SET}_{w_2(TM)\text{Sq}^1 B}}(M^5)$	0	0	2	4

Table 9: Partition Function $\mathbf{Z}(M^5)$ and Topological Degeneracy (GSD) of 5d higher-SETs, for example, $\mathbf{Z}_{\text{SET}_{B\text{Sq}^1 B}}(M^5) := \frac{|\operatorname{H}^0(M^5,\mathbb{Z}_2)|}{|\operatorname{H}^1(M^5,\mathbb{Z}_2)|}\sum_{B\in \operatorname{H}^2(M^5,\mathbb{Z}_2)}(-1)^{\int_{M^5} B\text{Sq}^1 B}$.

Now we illustrate our computation:

1. For $M = S^1\times\mathbb{RP}^4$, let γ be the generator of $\operatorname{H}^1(S^1,\mathbb{Z}_2)=\mathbb{Z}_2$ and ζ be the generator of $\operatorname{H}^1(\mathbb{RP}^4,\mathbb{Z}_2)=\mathbb{Z}_2$. Note that $w_1(TM)=\zeta$. The $\operatorname{H}^0(S^1\times\mathbb{RP}^4,\mathbb{Z}_2)=\mathbb{Z}_2$, $\operatorname{H}^1(S^1\times\mathbb{RP}^4,\mathbb{Z}_2)=\mathbb{Z}_2^2$, $\operatorname{H}^2(S^1\times\mathbb{RP}^4,\mathbb{Z}_2)=\mathbb{Z}_2^2$ whose two generators are $\gamma\zeta$ and ζ^2. If $B=\lambda_1\gamma\zeta+\lambda_2\zeta^2$, then $\text{Sq}^1 B=\lambda_1\gamma\zeta^2$. Hence
$$\int_{S^1\times\mathbb{RP}^4} B\text{Sq}^1 B = \lambda_1\lambda_2, \tag{4.6}$$
$$\int_{S^1\times\mathbb{RP}^4} B\text{Sq}^1 B + w_1(TM)^2\text{Sq}^1 B = \lambda_1\lambda_2+\lambda_1. \tag{4.7}$$
On the other hand, since $w_2(TM)=0$ for $S^1\times\mathbb{RP}^4$, we have
$$\mathbf{Z}_{\text{SET}_{(0,0)}}(S^1\times\mathbb{RP}^4) = \mathbf{Z}_{\text{SET}_{(0,1)}}(S^1\times\mathbb{RP}^4), \tag{4.8}$$
$$\mathbf{Z}_{\text{SET}_{(1,0)}}(S^1\times\mathbb{RP}^4) = \mathbf{Z}_{\text{SET}_{(1,1)}}(S^1\times\mathbb{RP}^4). \tag{4.9}$$
We have
$$\mathbf{Z}_{\text{SET}_{(0,0)}}(S^1\times\mathbb{RP}^4) = \frac{1}{2}\sum_{\lambda_1,\lambda_2\in\mathbb{Z}_2}(-1)^{\lambda_1(\lambda_2+1)}, \tag{4.10}$$
$$\mathbf{Z}_{\text{SET}_{(0,1)}}(S^1\times\mathbb{RP}^4) = \frac{1}{2}\sum_{\lambda_1,\lambda_2\in\mathbb{Z}_2}(-1)^{\lambda_1\lambda_2}. \tag{4.11}$$
Since the number of (λ_1,λ_2) satisfying the constraint $\lambda_1\lambda_2=1$ is only one:
$$\#\{(\lambda_1,\lambda_2)\in\mathbb{Z}_2^2|\lambda_1\lambda_2=1\}=1, \tag{4.12}$$
also note that changing λ_2 to λ_2+1 doesn't affect the sum, so
$$\begin{aligned}&\mathbf{Z}_{\text{SET}_{(0,0)}}(S^1\times\mathbb{RP}^4)=\mathbf{Z}_{\text{SET}_{(1,1)}}(S^1\times\mathbb{RP}^4)\\ =\ &\mathbf{Z}_{\text{SET}_{(0,1)}}(S^1\times\mathbb{RP}^4)=\mathbf{Z}_{\text{SET}_{(1,0)}}(S^1\times\mathbb{RP}^4)=\frac{1}{2}(3-1)=1.\end{aligned} \tag{4.13}$$

2. For $M=\mathbb{RP}^2\times\mathbb{RP}^3$, let α be the generator of $\operatorname{H}^1(\mathbb{RP}^2,\mathbb{Z}_2)=\mathbb{Z}_2$, β be the generator of $\operatorname{H}^1(\mathbb{RP}^3,\mathbb{Z}_2)=\mathbb{Z}_2$. Note that $w_1(TM)=\alpha$. $\operatorname{H}^0(\mathbb{RP}^2\times\mathbb{RP}^3,\mathbb{Z}_2)=\mathbb{Z}_2$, $\operatorname{H}^1(\mathbb{RP}^2\times\mathbb{RP}^3,\mathbb{Z}_2)=\mathbb{Z}_2^2$, $\operatorname{H}^2(\mathbb{RP}^2\times$

$\mathbb{RP}^3, \mathbb{Z}_2) = \mathbb{Z}_2^3$ whose three generators are α^2, β^2 and $\alpha\beta$. If $B = \lambda_1\alpha^2 + \lambda_2\beta^2 + \lambda_3\alpha\beta$, then $\mathrm{Sq}^1 B = \lambda_3\alpha^2\beta + \lambda_3\alpha\beta^2$. Hence

$$\int_{\mathbb{RP}^2\times\mathbb{RP}^3} B\mathrm{Sq}^1 B = \lambda_3^2 + \lambda_2\lambda_3, \tag{4.14}$$

$$\int_{\mathbb{RP}^2\times\mathbb{RP}^3} B\mathrm{Sq}^1 B + w_1(TM)^2\mathrm{Sq}^1 B = \lambda_3^2 + \lambda_2\lambda_3. \tag{4.15}$$

On the other hand, since $w_2(TM) + w_1(TM)^2 = 0$ for $\mathbb{RP}^2 \times \mathbb{RP}^3$, so

$$\mathbf{Z}_{\mathrm{SET}_{(0,0)}}(\mathbb{RP}^2 \times \mathbb{RP}^3) = \mathbf{Z}_{\mathrm{SET}_{(1,1)}}(\mathbb{RP}^2 \times \mathbb{RP}^3), \tag{4.16}$$

$$\mathbf{Z}_{\mathrm{SET}_{(0,1)}}(\mathbb{RP}^2 \times \mathbb{RP}^3) = \mathbf{Z}_{\mathrm{SET}_{(1,0)}}(\mathbb{RP}^2 \times \mathbb{RP}^3). \tag{4.17}$$

We have

$$\mathbf{Z}_{\mathrm{SET}_{(0,0)}}(\mathbb{RP}^2 \times \mathbb{RP}^3) = \frac{1}{2}\sum_{\lambda_1,\lambda_2,\lambda_3\in\mathbb{Z}_2}(-1)^{\lambda_3^2+\lambda_2\lambda_3}, \tag{4.18}$$

$$\mathbf{Z}_{\mathrm{SET}_{(0,1)}}(\mathbb{RP}^2 \times \mathbb{RP}^3) = \frac{1}{2}\sum_{\lambda_1,\lambda_2,\lambda_3\in\mathbb{Z}_2}(-1)^{\lambda_3^2+\lambda_2\lambda_3}. \tag{4.19}$$

Since

$$\#\{(\lambda_1,\lambda_2,\lambda_3) \in \mathbb{Z}_2^3 | \lambda_3^2 + \lambda_2\lambda_3 = 1\} = 2, \tag{4.20}$$

so

$$\begin{aligned} &\mathbf{Z}_{\mathrm{SET}_{(0,0)}}(\mathbb{RP}^2 \times \mathbb{RP}^3) = \mathbf{Z}_{\mathrm{SET}_{(1,1)}}(\mathbb{RP}^2 \times \mathbb{RP}^3) \\ = \ &\mathbf{Z}_{\mathrm{SET}_{(0,1)}}(\mathbb{RP}^2 \times \mathbb{RP}^3) = \mathbf{Z}_{\mathrm{SET}_{(1,0)}}(\mathbb{RP}^2 \times \mathbb{RP}^3) = \frac{1}{2}(6-2) = 2. \end{aligned} \tag{4.21}$$

3. For $M = S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2$, let γ be the generator of $\mathrm{H}^1(S^1, \mathbb{Z}_2) = \mathbb{Z}_2$ and α_i be the generator of $\mathrm{H}^1(\mathbb{RP}^2, \mathbb{Z}_2) = \mathbb{Z}_2$ of the i-th factor $\mathbb{RP}^2$ $(i = 1, 2)$. Note that $w_1(TM) = \alpha_1 + \alpha_2$. $\mathrm{H}^0(S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2, \mathbb{Z}_2) = \mathbb{Z}_2$, $\mathrm{H}^1(S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2, \mathbb{Z}_2) = \mathbb{Z}_2^3$, $\mathrm{H}^2(S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2, \mathbb{Z}_2) = \mathbb{Z}_2^5$ whose five generators are α_1^2, α_2^2, $\gamma\alpha_1$, $\gamma\alpha_2$ and $\alpha_1\alpha_2$. If $B = \lambda_1\alpha_1^2 + \lambda_2\alpha_2^2 + \lambda_3\gamma\alpha_1 + \lambda_4\gamma\alpha_2 + \lambda_5\alpha_1\alpha_2$, then $\mathrm{Sq}^1 B = \lambda_3\gamma\alpha_1^2 + \lambda_4\gamma\alpha_2^2 + \lambda_5\alpha_1^2\alpha_2 + \lambda_5\alpha_1\alpha_2^2$. Hence

$$\int_{S^1\times\mathbb{RP}^2\times\mathbb{RP}^2} B\mathrm{Sq}^1 B = \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_3\lambda_5 + \lambda_4\lambda_5, \tag{4.22}$$

$$\int_{S^1\times\mathbb{RP}^2\times\mathbb{RP}^2} B\mathrm{Sq}^1 B + w_1(TM)^2\mathrm{Sq}^1 B = \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_3\lambda_5 + \lambda_4\lambda_5 + \lambda_3 + \lambda_4. \tag{4.23}$$

On the other hand, since $w_2(TM) + w_1(TM)^2 = 0$ for $S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2$, so

$$\mathbf{Z}_{\mathrm{SET}_{(0,0)}}(S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2) = \mathbf{Z}_{\mathrm{SET}_{(1,1)}}(S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2), \tag{4.24}$$

$$\mathbf{Z}_{\mathrm{SET}_{(0,1)}}(S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2) = \mathbf{Z}_{\mathrm{SET}_{(1,0)}}(S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2). \tag{4.25}$$

We have

$$\mathbf{Z}_{\mathrm{SET}_{(0,0)}}(S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2) = \frac{1}{4}\sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5\in\mathbb{Z}_2}(-1)^{\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_3\lambda_5+\lambda_4\lambda_5}, \tag{4.26}$$

$$\mathbf{Z}_{\mathrm{SET}_{(0,1)}}(S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2) = \frac{1}{4}\sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5\in\mathbb{Z}_2}(-1)^{\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_3(\lambda_5+1)+\lambda_4(\lambda_5+1)}. \tag{4.27}$$

Since

$$\#\{(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)\in\mathbb{Z}_2^5|\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_3\lambda_5+\lambda_4\lambda_5=1\}=12, \tag{4.28}$$

also note that changing λ_5 to λ_5+1 doesn't affect the sum, so

$$\begin{aligned} & \mathbf{Z}_{\mathrm{SET}_{(0,0)}}(S^1\times\mathbb{RP}^2\times\mathbb{RP}^2)=\mathbf{Z}_{\mathrm{SET}_{(1,1)}}(S^1\times\mathbb{RP}^2\times\mathbb{RP}^2)\\ = \; & \mathbf{Z}_{\mathrm{SET}_{(0,1)}}(S^1\times\mathbb{RP}^2\times\mathbb{RP}^2)=\mathbf{Z}_{\mathrm{SET}_{(1,0)}}(S^1\times\mathbb{RP}^2\times\mathbb{RP}^2)=\frac{1}{4}(20-12)=2. \end{aligned} \tag{4.29}$$

4. We remark that, for a 5d Wu manifold $\mathrm{W}=\mathrm{SU}(3)/\mathrm{SO}(3)$, with $\mathrm{H}^0(\mathrm{W},\mathbb{Z}_2)=\mathbb{Z}_2$, $\mathrm{H}^1(\mathrm{W},\mathbb{Z}_2)=0$, note that $w_1(T\mathrm{W})=0$, so we can actually distinguish some of the four classes of 5d SETs:

$$\begin{aligned} \mathbf{Z}_{\mathrm{SET}_{(0,0)}}(\mathrm{W})&=\mathbf{Z}_{\mathrm{SET}_{(1,0)}}(\mathrm{W}),\\ \mathbf{Z}_{\mathrm{SET}_{(0,1)}}(\mathrm{W})&=\mathbf{Z}_{\mathrm{SET}_{(1,1)}}(\mathrm{W}). \end{aligned} \tag{4.30}$$

$\mathrm{H}^2(\mathrm{W},\mathbb{Z}_2)=\mathbb{Z}_2$ which is generated by $w_2(T\mathrm{W})$. $\mathrm{Sq}^1w_2(T\mathrm{W})=w_3(T\mathrm{W})$.

$$\mathbf{Z}_{\mathrm{SET}_{(0,0)}}(\mathrm{W}) = 2\sum_{B=0,w_2(T\mathrm{W})}(-1)^{B\mathrm{Sq}^1B+w_2(T\mathrm{W})\mathrm{Sq}^1B}=4, \tag{4.31}$$

$$\mathbf{Z}_{\mathrm{SET}_{(0,1)}}(\mathrm{W}) = 2\sum_{B=0,w_2(T\mathrm{W})}(-1)^{B\mathrm{Sq}^1B}=0. \tag{4.32}$$

In the next section, we will use the anyonic string/brane braiding statistics and the link invariants of 5d TQFTs to characterize and distinguish these 5d SETs.

5 Anyonic String/Brane Braiding Statistics and Link Invariants of 5d TQFTs

Now we compute the path integral Eq. (4.5) with extended operator insertions. To recall the general definitions, we have

• Partition or path integral w/out insertion is

$$\sum_{\substack{B\in C^2(M,\mathbb{Z}_2)\\ \dots}}(e^{\mathrm{i}S}).$$

• Physics vacuum expectation value (v.e.v) of a theory S is defined as

$$\boxed{\langle\mathcal{O}\rangle_{(\mathrm{v.e.v})}=\frac{\langle\mathcal{O}\rangle_{(\mathrm{v.e.v})}}{\langle 1\rangle_{(\mathrm{v.e.v})}}=\frac{\sum_{\substack{B\in C^2(M,\mathbb{Z}_2)\\ \dots}}(e^{\mathrm{i}S}\mathcal{O})}{\sum_{\substack{B\in C^2(M,\mathbb{Z}_2)\\ \dots}}(e^{\mathrm{i}S})}=\frac{\sum_{\substack{B\in C^2(M,\mathbb{Z}_2)\\ \dots}}(e^{\mathrm{i}S}\mathcal{O})}{\mathbf{Z}}}$$
$$=\frac{\text{path integral with insertions }\mathcal{O}}{\text{path integral without insertions}}. \tag{5.1}$$

For example, this includes the link invariant that we will focus on in this section:

$$\langle\exp(\mathrm{i}\dots(\text{Link invariants of }U,X,Y,\dots))\rangle_{(\mathrm{v.e.v})}=\frac{\mathbf{Z}^{5\mathrm{d}}_{\mathrm{SET}_{(K_1,K_2)}}[M^5;U,X,Y,\dots]}{\mathbf{Z}^{5\mathrm{d}}_{\mathrm{SET}_{(K_1,K_2)}}[M^5]}. \tag{5.2}$$

For conventions of our notations, we label the 1d Wilson line as W, the 2d surface operator as U, U', etc. We label the 3d membrane operator as X and the 4d operator as Y, etc. We label the "dd-hyper-surface" of general operators that we inserted as Σ^d, while we label this Σ^d's "$(d+1)$d-Seifert-hyper-volume" as V^{d+1}.

In this section 5, we focus on deriving the general link invariants for these 5d TQFTs/SETs.[14] In the next Sec. 6, we will provide explicit examples of the spacetime braiding process as the link configurations that can be detected by these link invariants derived here Sec. 5. The techniques for computing all these link invariants below are based on Ref. [11]. Below we simply apply the methods and notations introduced in Ref. [11].

Caveat: Note that while in the first section 5.1, we explicitly study the discrete cochain version of TQFT, the later sections instead we implement the continuum formulation of TQFT. The reason is related to a fact that the graded non-commutativity of cochain fields is much more complicated to be dealt with than the continuum differential form fields. The subtle fact will be commented further in footnotes 16 and 17. We also note that when we deal with the continuum differential form fields later in Sec. 5.2 to Sec. 5.4, we *choose* a normalization of differential form fields as $\oint B \in \mathbb{Z}$ with the periodicity $\oint B \sim \oint B + 2$ (thus more similar to the convention of discrete cochain fields), *instead of* the more conventional $\oint B \in \pi\mathbb{Z}$ with the periodicity $\oint B \sim \oint B + 2\pi$.

5.1 $\frac{1}{2}\tilde{w}_1(TM)\mathcal{P}(B)$ and a Triple Link Invariant $\mathbf{Tlk}^{(5)}_{w_1BB}(\Sigma^3_X, \Sigma^2_{U_{(\mathrm{i})}}, \Sigma^2_{U_{(\mathrm{ii})}})$

We start with a 5d TQFT obtained from summing over 2-form field B of $\frac{1}{2}\tilde{w}_1(TM)\mathcal{P}(B)$ (gauging 1-form $\mathbb{Z}_2$ of this 5d SPTs). This is equivalent to the $\mathbf{Z}_{\mathrm{SET}_{(K_1=0,K_2=0)}}$ example in Eq. (4.5).

$$\mathbf{Z} = \int [\mathcal{D}B][\mathcal{D}\tilde{c}][\mathcal{D}b] \exp(\mathrm{i}\mathbf{S}). \tag{5.3}$$

$$\mathbf{Z} = \sum_{\substack{B,b \in C^2(M^5,\mathbb{Z}_2)\\ \tilde{c}\in C^3(M^5,\mathbb{Z}_4)}} \exp(\mathrm{i}\pi \int_{M^5} \frac{1}{2}\delta\tilde{w}_1(TM)\cup\tilde{c} + b\cup\delta B + \frac{1}{2}\tilde{w}_1(TM)\cup\mathcal{P}(B)). \tag{5.4}$$

The action is (see footnote 10)

$$\mathbf{S} = \pi \int_{M^5} (\frac{1}{2}\delta\tilde{w}_1(TM)\cup\tilde{c} + b\cup\delta B + \frac{1}{2}\tilde{w}_1(TM)\cup\mathcal{P}(B)). \tag{5.5}$$

We consider the gauge transformation:[15]

$$\begin{aligned}
\tilde{w}_1(TM) &\to \tilde{w}_1(TM) + \delta\alpha,\\
B &\to B + \delta\beta,\\
\tilde{c} &\to \tilde{c} + \delta\gamma + \lambda,\\
b &\to b + \delta\zeta + \mu.
\end{aligned} \tag{5.6}$$

The gauge variation shows:

$$\begin{aligned}
\mathbf{S} \to\ & \pi \int_{M^5} \frac{1}{2}(\tilde{w}_1(TM)+\delta\alpha)(B\cup B + B\cup\delta\beta + \delta\beta\cup B + \delta\beta\cup\delta\beta + B \underset{1}{\cup} \delta B + \delta\beta \underset{1}{\cup} \delta B)\\
& + \frac{1}{2}\delta\tilde{w}_1(TM)(\tilde{c}+\lambda) + (b+\mu)\delta B.
\end{aligned} \tag{5.7}$$

[14] For more guidance on the physical interpretations of link invariants, please see [11] and its Introduction.

[15] One may consider add additional terms on the gauge transformations, such as $\tilde{w}_1(TM) \to \tilde{w}_1(TM) + \delta\alpha(t,x) + \alpha_1(t,x)$ and $B \to B + \delta\beta(t,x) + \alpha_2(t,x)$, etc. However, terms such as $\alpha_1(t,x) = \alpha_1$ and $\alpha_2(t,x) = \alpha_2$ will need to be constant, which act as the higher-form "global symmetry" transformation, instead of "gauge transformation."

Note that $\delta^2\beta = 0$. The gauge variance of the action is:

$$
\begin{aligned}
\mathbf{\Delta S} &= \pi \int_{M^5} \frac{1}{2}\tilde{w}_1(TM)(\delta\beta \cup \delta\beta + 2\delta\beta \cup B + \delta(\delta\beta \underset{1}{\cup} B)) \\
&\quad + \frac{1}{2}\delta\alpha(B \cup B + B \underset{1}{\cup} \delta B + \delta\beta \cup \delta\beta + 2\delta\beta \cup B + \delta(\delta\beta \underset{1}{\cup} B)) \\
&\quad + \frac{1}{2}\delta\tilde{w}_1(TM)\lambda + \mu\delta B \qquad (5.8)\\
&= \pi \int_{M^5} \frac{1}{2}\delta\tilde{w}_1(TM)(\beta\delta\beta) + \big(\delta\tilde{w}_1(TM)(\beta B) + \tilde{w}_1(TM)\beta\delta B\big) + \frac{1}{2}\delta\tilde{w}_1(TM)(\delta\beta \underset{1}{\cup} B) \\
&\quad - \big(\alpha B\delta B + \frac{1}{2}\alpha u_2 \delta B\big) - \alpha\delta\beta\delta B + \frac{1}{2}\delta\tilde{w}_1(TM)\lambda + \mu\delta B. \qquad (5.9)
\end{aligned}
$$

In Eq. (5.8), we have used the formula[16] and again $\delta^2\beta = 0$:

$$B \cup \delta\beta - \delta\beta \cup B + \delta\beta \underset{1}{\cup} \delta B + \delta^2\beta \underset{1}{\cup} B = \delta(\delta\beta \underset{1}{\cup} B). \tag{5.11}$$

In Eq. (5.9), we have used integration by part: for a closed 5-manifold without boundary, after integration by part we can drop the boundary term $\delta(\dots)$ where $\dots$ only has effects on a 4-manifold (the 4d boundary of an open 5-manifold). Since $\delta^2 B = \delta^2\beta = \delta^2\alpha = 0$, we drop $\delta\alpha(\delta\beta \cup \delta\beta + \delta(\delta\beta \underset{1}{\cup} B))$ which has no effect on a closed 5-manifold without boundary. Here u_2 is the second Wu class, we have also used the formula in footnote 16 as

$$
\begin{aligned}
B \cup \delta B - \delta B \cup B + \delta B \underset{1}{\cup} \delta B + B \underset{1}{\cup} \delta^2 B &= \delta(B \underset{1}{\cup} \delta B), \qquad (5.12)\\
\delta B \underset{1}{\cup} \delta B = \mathrm{Sq}^2 \delta B &= u_2 \delta B. \qquad (5.13)
\end{aligned}
$$

So the above Eq. (5.9), we use $\delta(\alpha(B \cup B + B \underset{1}{\cup} \delta B)) = \delta\alpha(B \cup B + B \underset{1}{\cup} \delta B) + \alpha(\delta B \cup B + B \cup \delta B + \delta(B \underset{1}{\cup} \delta B)) = \delta\alpha(B \cup B + B \underset{1}{\cup} \delta B) + \alpha(2B \cup \delta B + u_2 \delta B)$ and we drop the total derivative term on a closed 5-manifold. The solution of gauge invariance imposes: $\mathbf{\Delta S} = 0 \Rightarrow$

$$
\begin{aligned}
\lambda &= -\beta\delta\beta - 2\beta B - \delta\beta \underset{1}{\cup} B \mod 4, \\
\mu &= -\tilde{w}_1(TM)\beta + \alpha B + \frac{1}{2}\alpha u_2 + \alpha\delta\beta \mod 2, \qquad (5.14)
\end{aligned}
$$

where we have imposed the gauge transformation for the sake of gauge invariance:[17]

---

[16] This is based on Steenrod's work "Products of Cocycles and Extensions of Mappings [61]," which derives

$$\delta(u \underset{i}{\cup} v) = (-1)^{p+q-i} u \underset{i-1}{\cup} v + (-1)^{pq+p+q} v \underset{i-1}{\cup} u + \delta u \underset{i}{\cup} v + (-1)^p u \underset{i}{\cup} \delta v \tag{5.10}$$

where $u \in C^p$, $v \in C^q$.

[17] In general, when we study the action Eq. (5.5), we have made a convenient choice with a term $\delta\tilde{w}_1(TM) \cup \tilde{c}$ instead of $\tilde{c} \cup \delta\tilde{w}_1(TM)$. For a generic 3-cochain x, $\delta\tilde{w}_1(TM)x = x\delta\tilde{w}_1(TM)$ is not true, by Steenrod's formula in footnote 16 Eq. (5.10), $\delta\tilde{w}_1(TM)x = x\delta\tilde{w}_1(TM) + \delta x \underset{1}{\cup} \delta\tilde{w}_1(TM) - \delta(x \underset{1}{\cup} \delta\tilde{w}_1(TM))$, we can only drop the total derivative terms (i.e. the coboundary terms). In our present case, we consider $x = \frac{1}{2}\beta\delta\beta + \beta B + \frac{1}{2}\delta\beta \underset{1}{\cup} B$. So if $\delta x \underset{1}{\cup} \delta\tilde{w}_1(TM)$ is a coboundary, then we can also drop it, which results in

$$\lambda = -2x = -\beta\delta\beta - 2\beta B - \delta\beta \underset{1}{\cup} B \mod 4.$$

If $\delta x \underset{1}{\cup} \delta\tilde{w}_1(TM)$ is not a coboundary, we need the extra term

$$\delta\tilde{w}_1(TM)x = x\delta\tilde{w}_1(TM) + \delta x \underset{1}{\cup} \delta\tilde{w}_1(TM) + \text{a total derivative/coboundary term.}$$

When $\delta x \underset{1}{\cup} \delta\tilde{w}_1(TM)$ is not a coboundary, this results in a modified gauge transformation to λ. By writing the action as in Eq. (5.5), we can avoid additional complications, thus we end up with a simpler gauge transformation Eq. (5.14). The graded non-commutativity of cochain fields is much more complicated than the case for continuum differential form fields. (JW thanks Pierre Deligne for a discussion on the related issues.)

We derive the 3-submanifold operator, using $\mathcal{P}(B+\delta\beta)=\mathcal{P}(B)+\delta\beta\cup\delta\beta+2\delta\beta\cup B+\delta(\delta\beta\underset{1}{\cup}B)$,

$$
\begin{aligned}
X &= \exp(\frac{\mathrm{i}\pi}{2}k(\int_{\Sigma^3}\tilde{c}+\int_{V^4}\mathcal{P}(B))) \\
&= \exp(\frac{\mathrm{i}\pi}{2}k(\int_{M^5}\delta^{\perp}(\Sigma^3)\tilde{c}+\delta^{\perp}(V^4)\mathcal{P}(B))) \qquad (5.15)
\end{aligned}
$$

is gauge invariant when we set $\delta B=0$ on the 4-submanifold Seifert volume V^4. Where k is a $\mathbb{Z}_4$ integer mod 4.

We derive the 2-submanifold (2-surface) operator

$$
\begin{aligned}
U &= \exp(\mathrm{i}\pi\ell(\int_{\Sigma^2}b-\int_{V^3}\tilde{w}_1(TM)B-\frac{1}{2}\int_{V^3}\tilde{w}_1(TM)u_2)) \\
&= \exp(\mathrm{i}\pi\ell(\int_{M^5}b\delta^{\perp}(\Sigma^2)-(\tilde{w}_1(TM)B+\frac{1}{2}\tilde{w}_1(TM)u_2)\delta^{\perp}(V^3))) \\
&= \exp(\mathrm{i}\pi\ell(\int_{M^5}b\delta^{\perp}(\Sigma^2)-(\tilde{w}_1(TM)B+\frac{1}{2}\tilde{w}_1(TM)(w_2(TM)+w_1(TM)^2))\delta^{\perp}(V^3))) \qquad (5.16)
\end{aligned}
$$

which is gauge invariant when $\delta B=\delta\tilde{w}_1(TM)=0$ on the 3-submanifold Seifert volume V^3. Where ℓ is a $\mathbb{Z}_2$ integer mod 2. Note Wu class $u_2=w_2(TM)+w_1(TM)^2$ is a cocycle thus $\delta u_2=0$ everywhere in the 5-manifold.

We insert $X, U_{(\mathbf{i})}$ and $U_{(\mathbf{ii})}$ into the path integral $\mathbf{Z}$, and write the correlation function either in the *continuum* field theory formulation, or in the *discrete* cochain field theory formulation, interchangeably as

$$
\langle XU_{(\mathbf{i})}U_{(\mathbf{ii})}\rangle=\int[\mathcal{D}B][\mathcal{D}\tilde{c}][\mathcal{D}b]\;XU_{(\mathbf{i})}U_{(\mathbf{ii})}\;\exp(\mathrm{i}\mathbf{S}). \qquad (5.17)
$$

$$
\langle XU_{(\mathbf{i})}U_{(\mathbf{ii})}\rangle=\sum_{\substack{B,b\in C^2(M^5,\mathbb{Z}_2)\\ \tilde{c}\in C^3(M^5,\mathbb{Z}_4)}}XU_{(\mathbf{i})}U_{(\mathbf{ii})}\;\exp(\mathrm{i}\pi\int_{M^5}\frac{1}{2}\delta\tilde{w}_1(TM)\cup\tilde{c}+b\cup\delta B+\frac{1}{2}\tilde{w}_1(TM)\cup\mathcal{P}(B)).
$$

Step 1, we integrate out $\tilde{c}$ in $\int[\mathcal{D}\tilde{c}]$, we get

$$
\begin{aligned}
\delta\tilde{w}_1(TM) &= k\delta^{\perp}(\Sigma^3_X), \\
\tilde{w}_1(TM) &= k\delta^{\perp}(V^4_X), \qquad (5.18)
\end{aligned}
$$

while we also have $\delta^2\tilde{w}_1(TM)=\delta(k\delta^{\perp}(\Sigma^3_W))=0$. So with the above configuration constraint, we get the double-counting mod 2 cancellation in the exponent of $\exp(\frac{\mathrm{i}\pi}{2}k(\int_{M^5}\delta^{\perp}(V^4_X)\mathcal{P}(B)))\;\exp(\mathrm{i}\pi\int_{M^5}\frac{1}{2}\tilde{w}_1(TM)\,\mathcal{P}(B))=1$. This boils down to

$$
\langle XU_{(\mathbf{i})}U_{(\mathbf{ii})}\rangle = \int[\mathcal{D}B][\mathcal{D}b]\;U_{(\mathbf{i})}U_{(\mathbf{ii})}\exp(\mathrm{i}\pi\int_{M^5}b\cup\delta B)|_{\tilde{w}_1(TM)=k\delta^{\perp}(V^4_X)}. \qquad (5.19)
$$

Step 2, we integrate out b in $\int[\mathcal{D}b]$, we get the constraint

$$
\begin{aligned}
\delta B &= \ell_{(\mathbf{i})}\delta^{\perp}(\Sigma^2_{U_{(\mathbf{i})}})+\ell_{(\mathbf{ii})}\delta^{\perp}(\Sigma^2_{U_{(\mathbf{ii})}}), \\
B &= \ell_{(\mathbf{i})}\delta^{\perp}(V^3_{U_{(\mathbf{i})}})+\ell_{(\mathbf{ii})}\delta^{\perp}(V^3_{U_{(\mathbf{ii})}}). \qquad (5.20)
\end{aligned}
$$

Step 3, finally we integrate out B in $\int[\mathcal{D}B]$, from Eq. (5.19):

$$
\begin{aligned}
&\boxed{\langle X U_{(\mathbf{i})} U_{(\mathbf{ii})}\rangle}\\
&= \int[\mathcal{D}B]\mathrm{e}^{(-\mathrm{i}\pi(\int_{M^5}(\tilde{w}_1(TM)B+\frac{1}{2}\tilde{w}_1(TM)(w_2(TM)+w_1(TM)^2))(\ell_{(\mathbf{i})}\delta^\perp(V^3_{U_{(\mathbf{i})}})+\ell_{(\mathbf{ii})}\delta^\perp(V^3_{U_{(\mathbf{ii})}}))))}\Big|_{\substack{\tilde{w}_1(TM)=k\delta^\perp(V_X^4),\\ B=\ell_{(\mathbf{i})}\delta^\perp(V^3_{U_{(\mathbf{i})}})\\ +\ell_{(\mathbf{ii})}\delta^\perp(V^3_{U_{(\mathbf{ii})}}).}}\\
&= \int[\mathcal{D}B]\mathrm{e}^{(-\mathrm{i}\pi(\int_{M^5}(\tilde{w}_1(TM)B+\frac{1}{2}\tilde{w}_1(TM)(w_2(TM)+w_1(TM)^2))B))}\Big|_{\substack{\tilde{w}_1(TM)=k\delta^\perp(V_X^4),\\ B=\ell_{(\mathbf{i})}\delta^\perp(V^3_{U_{(\mathbf{i})}})+\ell_{(\mathbf{ii})}\delta^\perp(V^3_{U_{(\mathbf{ii})}}).}}\\
&= \int[\mathcal{D}B]\mathrm{e}^{(-\mathrm{i}\pi(\int_{M^5}(\tilde{w}_1(TM)BB+\mathrm{Sq}^2\big(\frac{1}{2}\tilde{w}_1(TM)B\big))))}\Big|_{\substack{\tilde{w}_1(TM)=k\delta^\perp(V_X^4),\\ B=\ell_{(\mathbf{i})}\delta^\perp(V^3_{U_{(\mathbf{i})}})+\ell_{(\mathbf{ii})}\delta^\perp(V^3_{U_{(\mathbf{ii})}}).}} \quad (5.21)\\
&= \int[\mathcal{D}B]\mathrm{e}^{(-\mathrm{i}\pi(\int_{M^5}(\tilde{w}_1(TM)BB+\frac{1}{2}\tilde{w}_1(TM)BB+\frac{1}{2}(\frac{1}{2}\delta\tilde{w}_1(TM))(\frac{1}{2}\delta B))))}\Big|_{\substack{\tilde{w}_1(TM)=k\delta^\perp(V_X^4),\ \delta\tilde{w}_1(TM)=k\delta^\perp(\Sigma^3_X);\\ B=\ell_{(\mathbf{i})}\delta^\perp(V^3_{U_{(\mathbf{i})}})+\ell_{(\mathbf{ii})}\delta^\perp(V^3_{U_{(\mathbf{ii})}}).\\ \delta B=\ell_{(\mathbf{i})}\delta^\perp(\Sigma^2_{U_{(\mathbf{i})}})+\ell_{(\mathbf{ii})}\delta^\perp(\Sigma^2_{U_{(\mathbf{ii})}}).}} \quad (5.22)\\
&= \mathrm{e}^{(-\mathrm{i}\pi\Big(k\ell_{(\mathbf{i})}\ell_{(\mathbf{ii})}\cdot\#(V_X^4\cap V^3_{U_1}\cap V^3_{U_2})+\frac{1}{8}\delta^\perp(\Sigma^3_X)\big(\delta^\perp(\Sigma^2_{U_{(\mathbf{i})}})+\delta^\perp(\Sigma^2_{U_{(\mathbf{ii})}})\big)\Big))}\cdot(\text{Self-intersecting }\#\text{ terms}) \quad (5.23)\\
&\cong \boxed{\mathrm{e}^{(-\mathrm{i}\pi\Big(k\ell_{(\mathbf{i})}\ell_{(\mathbf{ii})}\cdot\mathrm{Tlk}^{(5)}(\Sigma^3_X,\Sigma^2_{U_{(\mathbf{i})}},\Sigma^2_{U_{(\mathbf{ii})}})\Big))}}. \quad (5.24)
\end{aligned}
$$

In Eq. (5.21), we use the fact by Wu formula on a 5-manifold that $\tilde{w}_1(TM)(w_2(TM)+w_1(TM)^2))B = \tilde{w}_1(TM)u_2B = \mathrm{Sq}^2\big(\tilde{w}_1(TM)B\big)$.
In Eq. (5.22), to derive the link invariant of $\frac{1}{2}\tilde{w}_1(TM)\mathcal{P}(B)$, we use[18]

$$
\begin{aligned}
\frac{1}{2}\tilde{w}_1(TM)u_2B &= \mathrm{Sq}^2(\frac{1}{2}\tilde{w}_1(TM)B) = \frac{1}{2}\tilde{w}_1(TM)BB + \mathrm{Sq}^1(\frac{1}{2}\tilde{w}_1(TM))\mathrm{Sq}^1B\\
&= \frac{1}{2}\tilde{w}_1(TM)BB + \frac{1}{2}(\frac{1}{2}\delta\tilde{w}_1(TM))(\frac{1}{2}\delta B).
\end{aligned}
$$

We plug in all the constraints into the path integral Eq. (5.22) to obtain Eq. (5.23).[19] We propose a set-up to remove or renormalize the (Self-intersecting # terms) appeared in Eq. (5.24), described in the footnote 19. Also the second exponent in Eq. (5.23) shows that $\int_{M^5}\delta^\perp(\Sigma^3_W)\big(\delta^\perp(\Sigma^2_{U_{(\mathbf{i})}})+\delta^\perp(\Sigma^2_{U_{(\mathbf{ii})}})\big)=$

[18] We use the Steenrod product formula: $\mathrm{Sq}^2(uv)=(\mathrm{Sq}^2u)v+(\mathrm{Sq}^1u)(\mathrm{Sq}^1v)+u\mathrm{Sq}^2(v)$ when $u,v\in\mathrm{H}^*(M,\mathbb{Z}_2)$.

[19] Here are some more explanations to derive Eq. (5.23).

- For $\int[\mathcal{D}B]\mathrm{e}^{-\mathrm{i}\pi(\int_{M^5}(\tilde{w}_1(TM)BB))}\Big|_{\substack{\tilde{w}_1(TM)=k\delta^\perp(V_X^4),\\ B=\ell_{(\mathbf{i})}\delta^\perp(V^3_{U_{(\mathbf{i})}})+\ell_{(\mathbf{ii})}\delta^\perp(V^3_{U_{(\mathbf{ii})}}).}}$, we get a mutual-quadratic crossing term $V^3_{U_{(\mathbf{i})}}\cap V^3_{U_{(\mathbf{ii})}}$ with a multiple 2π exponent in $\mathrm{e}^{\mathrm{i}2\pi\#(\mathrm{V}^4_{\mathrm{X}}\cap\mathrm{V}^3_{\mathrm{U}_{(\mathbf{i})}}\cap\mathrm{V}^3_{\mathrm{U}_{(\mathbf{ii})}})}$ which does not contribute to the expectation value. There are also two self-quadratic terms $V^3_{U_{(\mathbf{n})}}\cap V^3_{U_{(\mathbf{n})}}$ for $(\mathbf{n})=(\mathbf{i})$ or $(\mathbf{ii})$. These self-quadratic terms contribute, in principle, infinite many intersecting numbers in $\#(V_X^4\cap V^3_{U_{(\mathbf{n})}}\cap V^3_{U_{(\mathbf{n})}})$ for $(\mathbf{n})=(\mathbf{i})$ or $(\mathbf{ii})$. Since a multiple 2π exponent have zero contribution to the expectation value, therefore either we can design an even but infinite number of points on each of $\#(V_X^4\cap V^3_{U_{(\mathbf{n})}}\cap V^3_{U_{(\mathbf{n})}})$ for $(\mathbf{n})=(\mathbf{i})$ or $(\mathbf{ii})$, or we can absorb them into the (Self-intersecting # terms) in Eq. (5.23). In either cases, this term does not have any net contribution in the end at Eq. (5.24).
- For $\int[\mathcal{D}B]\mathrm{e}^{-\mathrm{i}\pi(\int_{M^5}(\frac{1}{2}\tilde{w}_1(TM)BB))}\Big|_{\substack{\tilde{w}_1(TM)=k\delta^\perp(V_X^4),\\ B=\ell_{(\mathbf{i})}\delta^\perp(V^3_{U_{(\mathbf{i})}})+\ell_{(\mathbf{ii})}\delta^\perp(V^3_{U_{(\mathbf{ii})}}).}}$, we get a mutual-quadratic crossing term $V^3_{U_{(\mathbf{i})}}\cap V^3_{U_{(\mathbf{ii})}}$ with a multiple π exponent in $\mathrm{e}^{\mathrm{i}\pi\#(\mathrm{V}^4_{\mathrm{X}}\cap\mathrm{V}^3_{\mathrm{U}_{(\mathbf{i})}}\cap\mathrm{V}^3_{\mathrm{U}_{(\mathbf{ii})}})}$, which *does* contribute to the expectation value when this intersecting number # is odd, in a 1 mod 2 effect. There are also two self-quadratic terms $V^3_{U_{(\mathbf{n})}}\cap V^3_{U_{(\mathbf{n})}}$ for $(\mathbf{n})=(\mathbf{i})$ or $(\mathbf{ii})$. Again either we can design an quadruple/four-multiplet but infinite number of points for each of $\#(V_X^4\cap V^3_{U_{(\mathbf{n})}}\cap V^3_{U_{(\mathbf{n})}})$, or we can absorb them into the (Self-intersecting # terms) in Eq. (5.23).
- For $\int[\mathcal{D}B]\mathrm{e}^{(-\mathrm{i}\pi(\int_{M^5}(\frac{1}{8}\delta\tilde{w}_1(TM)\delta B)))}\Big|_{\substack{\delta\tilde{w}_1(TM)=k\delta^\perp(\Sigma^3_X);\\ \delta B=\ell_{(\mathbf{i})}\delta^\perp(\Sigma^2_{U_{(\mathbf{i})}})+\ell_{(\mathbf{ii})}\delta^\perp(\Sigma^2_{U_{(\mathbf{ii})}})}} = \mathrm{e}^{-\mathrm{i}\pi\int_{M^5}(\frac{1}{8}(k\delta^\perp(\Sigma^3_X))\big(\ell_{(\mathbf{i})}\delta^\perp(\Sigma^2_{U_{(\mathbf{i})}})+\ell_{(\mathbf{ii})}\delta^\perp(\Sigma^2_{U_{(\mathbf{ii})}})\big))}$, we find the exponent depends on the intersecting number $\#(\Sigma^3_X\cap\Sigma^2_{U_{(\mathbf{n})}})$ for $(\mathbf{n})=(\mathbf{i})$ or $(\mathbf{ii})$, between 3-surface and 2-surface in a 5 manifold — although generically this number $\#(\Sigma^3_X\cap\Sigma^2_{U_{(\mathbf{n})}})$ is finite but can be nonzero, we design by default that there is no intersection between any of our insertions of 3-surface and 2-surface into the path integral. Thus we set $\#(\Sigma^3_X\cap\Sigma^2_{U_{(\mathbf{n})}})=0$ by default.

$\#(\Sigma^3_X \cap \Sigma^2_{U_{(\mathbf{i})}}) + \#(\Sigma^3_X \cap \Sigma^2_{U_{(\mathbf{ii})}})$, which counts the number of intersections between our insertions of 3-surface and 2-surface. However, we choose *by default* that our insertions of 3-surface and 2-surface have no intersections (to avoid unnecessary singularities) into the path integral. Namely, we set $\#(\Sigma^3_W \cap \Sigma^2_{U_{(\mathbf{n})}}) = 0$ for $(\mathbf{n}) = (\mathbf{i})$ or $(\mathbf{ii})$, and $\#(\Sigma^2_{U_{(\mathbf{i})}} \cap \Sigma^2_{U_{(\mathbf{ii})}}) = 0$ by default. Overall, under the default assumption and the footnote 19 clarification, we obtain a final relation between Eq. (5.23) and our final effective answer Eq. (5.24). We use the congruence symbol ($\cong$) to express that other unwanted terms can be removed by default design.

We derive the link invariant for the 5d TQFT $\mathbf{Z}_{\text{SET}_{(K_1=0,K_2=0)}}[M^5]$ in Eq. (5.24):

$$\boxed{\#(V^4_X \cap V^3_{U_{(\mathbf{i})}} \cap V^3_{U_{(\mathbf{ii})}}) \equiv \text{Tlk}^{(5)}_{w_1 BB}(\Sigma^3_X, \Sigma^2_{U_{(\mathbf{i})}}, \Sigma^2_{U_{(\mathbf{ii})}})}. \tag{5.25}$$

The path integral with appropriate extended operators insertions become Eq. (5.24) which provides the above link invariant.

5.2 $w_1(TM)^3 B = w_1(TM)^2 \text{Sq}^1 B$

5.2.1 Version I: $w_1(TM)^3 B$ and a Quartic Link Invariant $\mathbf{Qlk}^{(5)}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^3_{X_{(\mathrm{iii})}}, \Sigma^2_U)$

As a test example, now we consider a 5d TQFT obtained from summing over 2-form field B of $w_1(TM)^3 B$ (gauging 1-form $\mathbb{Z}_2$ of this 5d SPTs), below we convert the cochain TQFT to differential form continuum TQFT. Whose partition function and action (see footnote 10) are:

$$\begin{aligned}
\mathbf{Z} &= \int [\mathcal{D}B][\mathcal{D}b][\mathcal{D}c] \exp(\mathrm{i}\mathbf{S}). && (5.26)\\
\mathbf{S} &= \pi \int_{M^5} c\,\mathrm{d}w_1(TM) + b\,\mathrm{d}B + w_1(TM)^3 B. && (5.27)
\end{aligned}$$

This 5d TQFT is distinct from any of four classes of $\mathbf{Z}_{\text{SET}_{(K_1,K_2)}}$, but it still serves as a useful toy model.

Gauge transformations are (see footnote 15):

$$\begin{aligned}
w_1(TM) &\to w_1(TM) + \mathrm{d}\alpha,\\
B &\to B + \mathrm{d}\beta,\\
c &\to c + \mathrm{d}\gamma + \lambda,\\
b &\to b + \mathrm{d}\zeta + \mu.
\end{aligned} \tag{5.28}$$

The gauge variation shows:

$$\begin{aligned}
\mathbf{S} \to\ & \mathbf{S} + \pi \int_{M^5} \mathrm{d}\gamma\,\mathrm{d}w_1(TM) + \lambda\,\mathrm{d}w_1(TM) + \mathrm{d}\zeta\,\mathrm{d}B + \mu\,\mathrm{d}B\\
& + (\mathrm{d}\alpha\,\mathrm{d}\alpha\, w_1(TM) + w_1(TM)^2 \mathrm{d}\alpha + \mathrm{d}\alpha\,\mathrm{d}\alpha\,\mathrm{d}\alpha) B\\
& + (w_1(TM)^3 + \mathrm{d}\alpha\,\mathrm{d}\alpha\, w_1(TM) + w_1(TM)^2 \mathrm{d}\alpha + \mathrm{d}\alpha\,\mathrm{d}\alpha\,\mathrm{d}\alpha)\,\mathrm{d}\beta && (5.29)\\
=\ & \mathbf{S} + \pi \int_{M^5} \lambda\,\mathrm{d}w_1(TM) + \mu\,\mathrm{d}B + (\alpha\,\mathrm{d}\alpha B\,\mathrm{d}w_1(TM) - \alpha\,\mathrm{d}\alpha\, w_1(TM)\,\mathrm{d}B)\\
& - \alpha w_1(TM)^2 \mathrm{d}B - \alpha\,\mathrm{d}\alpha\,\mathrm{d}\alpha\,\mathrm{d}B + w_1(TM)^2 \beta\,\mathrm{d}w_1(TM) + \alpha\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}w_1(TM) && (5.30)
\end{aligned}$$

where we have used integration by part: for a closed 5-manifold without boundary, after integration by part then we can drop the boundary term $\mathrm{d}(\dots)$ where $\dots$ only has effects on a 4-manifold (the 4d boundary of an open 5-manifold) and we drop the total derivative terms which have no effect on a closed 5-manifold without boundary. The gauge variance of the action is: $\mathbf{\Delta S} = 0 \Rightarrow$

$$\begin{aligned}\lambda &= -\alpha \mathrm{d}\alpha B - w_1(TM)^2\beta - \alpha \mathrm{d}\alpha \mathrm{d}\beta,\\ \mu &= \alpha \mathrm{d}\alpha w_1(TM) + \alpha w_1(TM)^2 + \alpha \mathrm{d}\alpha \mathrm{d}\alpha.\end{aligned} \tag{5.31}$$

We derive the 3-submanifold operator:

$$\begin{aligned}X &= \exp(\mathrm{i}\pi k(\int_{\Sigma^3} c + \int_{V^4} w_1(TM)^2 B))\\ &= \exp(\mathrm{i}\pi k(\int_{M^5} (\delta^\perp(\Sigma^3) c + \delta^\perp(V^4) w_1(TM)^2 B)))\end{aligned} \tag{5.32}$$

and 2-surface operator:

$$\begin{aligned}U &= \exp(\mathrm{i}\pi \ell(\int_{\Sigma^2} b - \int_{V^3} w_1(TM)^3))\\ &= \exp(\mathrm{i}\pi \ell(\int_{M^5} (\delta^\perp(\Sigma^2) b - \delta^\perp(V^3) w_1(TM)^3)))\end{aligned} \tag{5.33}$$

are gauge invariant when $\mathrm{d}w_1(TM) = \mathrm{d}B = 0$ on the 2-surface and 3-submanifolds. Where k, ℓ are $\mathbb{Z}_2$ integers mod 2.

Insert $X_{(\mathbf{i})}, X_{(\mathbf{ii})}, X_{(\mathbf{iii})}, U$ into the path integral $\mathbf{Z}$, so we can write the *continuum* field theory formulation as

$$\langle X_{(\mathbf{i})} X_{(\mathbf{ii})} X_{(\mathbf{iii})} U\rangle = \int [\mathcal{D}B][\mathcal{D}c][\mathcal{D}b]\ X_{(\mathbf{i})} X_{(\mathbf{ii})} X_{(\mathbf{iii})} U\ \exp(\mathrm{i}\mathbf{S}). \tag{5.34}$$

$$\langle X_{(\mathbf{i})} X_{(\mathbf{ii})} X_{(\mathbf{iii})} U\rangle = \int [\mathcal{D}B][\mathcal{D}c][\mathcal{D}b] X_{(\mathbf{i})} X_{(\mathbf{ii})} X_{(\mathbf{iii})} U\ \exp(\mathrm{i}\pi \int_{M^5} c\mathrm{d}w_1(TM) + b\mathrm{d}B + w_1(TM)^3 B).$$

Step 1, we integrate out c in $\int[\mathcal{D}c]$, we get

$$\begin{aligned}\mathrm{d}w_1(TM) &= k_{(\mathbf{i})}\delta^\perp(\Sigma^3_{X_{(\mathrm{i})}}) + k_{(\mathbf{ii})}\delta^\perp(\Sigma^3_{X_{(\mathrm{ii})}}) + k_{(\mathbf{iii})}\delta^\perp(\Sigma^3_{X_{(\mathrm{iii})}}),\\ w_1(TM) &= k_{(\mathbf{i})}\delta^\perp(V^4_{X_{(\mathrm{i})}}) + k_{(\mathbf{ii})}\delta^\perp(V^4_{X_{(\mathrm{ii})}}) + k_{(\mathbf{iii})}\delta^\perp(V^4_{X_{(\mathrm{iii})}}).\end{aligned} \tag{5.35}$$

So with the above configuration constraint, we get the double-counting mod 2 cancellation in the exponent of $\exp(\mathrm{i}\pi(\int_{M^5} w_1(TM)^2 B(k_{(\mathbf{i})}\delta^\perp(V^4_{X_{(\mathrm{i})}}) + k_{(\mathbf{ii})}\delta^\perp(V^4_{X_{(\mathrm{ii})}}) + k_{(\mathbf{iii})}\delta^\perp(V^4_{X_{(\mathrm{iii})}})))) \exp(\mathrm{i}\pi \int_{M^5} w_1(TM)^3 B) = 1$. This boils down to

$$\langle X_{(\mathbf{i})} X_{(\mathbf{ii})} X_{(\mathbf{iii})} U\rangle = \int [\mathcal{D}B][\mathcal{D}b]\, U \exp(\mathrm{i}\pi \int_{M^5} b\mathrm{d}B)|_{w_1(TM) = k_{(\mathrm{i})}\delta^\perp(V^4_{X_{(\mathrm{i})}}) + k_{(\mathrm{ii})}\delta^\perp(V^4_{X_{(\mathrm{ii})}}) + k_{(\mathrm{iii})}\delta^\perp(V^4_{X_{(\mathrm{iii})}})}. \tag{5.36}$$

Step 2, we integrate out b in $\int[\mathcal{D}b]$, we get the constraint

$$\begin{aligned}\mathrm{d}B &= \ell\delta^\perp(\Sigma^2_U),\\ B &= \ell\delta^\perp(V^3_U).\end{aligned} \tag{5.37}$$

Step 3, finally we integrate out B in $\int[\mathcal{D}B]$, from Eq. (5.36):

$$\boxed{\langle X_{(\mathrm{i})} X_{(\mathrm{ii})} X_{(\mathrm{iii})} U\rangle}$$
$$= \int[\mathcal{D}B] \mathrm{e}^{-\mathrm{i}\pi(\int_{M^5} w_1(TM)^3 \ell \delta^\perp(V_U^3))}\Big|_{\substack{w_1(TM)=k_{(\mathrm{i})}\delta^\perp(V^4_{X_{(\mathrm{i})}})+k_{(\mathrm{ii})}\delta^\perp(V^4_{X_{(\mathrm{ii})}})+k_{(\mathrm{iii})}\delta^\perp(V^4_{X_{(\mathrm{iii})}}),\\ B=\ell\delta^\perp(V_U^3).}}$$
$$= \int[\mathcal{D}B] \mathrm{e}^{(-\mathrm{i}\pi(\int_{M^5} w_1(TM)^3 B))}\Big|_{\substack{w_1(TM)=k_{(\mathrm{i})}\delta^\perp(V^4_{X_{(\mathrm{i})}})+k_{(\mathrm{ii})}\delta^\perp(V^4_{X_{(\mathrm{ii})}})+k_{(\mathrm{iii})}\delta^\perp(V^4_{X_{(\mathrm{iii})}}),\\ B=\ell\delta^\perp(V_U^3).}} \tag{5.38}$$
$$= \mathrm{e}^{(-\mathrm{i}\pi\Big(k_{(\mathrm{i})}k_{(\mathrm{ii})}k_{(\mathrm{iii})}\ell\Big(\#(V^4_{X_{(\mathrm{i})}}\cap V^4_{X_{(\mathrm{ii})}}\cap V^4_{X_{(\mathrm{iii})}}\cap V_U^3)+\#(V^4_{X_{(\mathrm{ii})}}\cap V^4_{X_{(\mathrm{iii})}}\cap V^4_{X_{(\mathrm{i})}}\cap V_U^3)+\#(V^4_{X_{(\mathrm{iii})}}\cap V^4_{X_{(\mathrm{i})}}\cap V^4_{X_{(\mathrm{ii})}}\cap V_U^3)}$$
$$^{\#(V^4_{X_{(\mathrm{i})}}\cap V^4_{X_{(\mathrm{iii})}}\cap V^4_{X_{(\mathrm{ii})}}\cap V_U^3)+\#(V^4_{X_{(\mathrm{iii})}}\cap V^4_{X_{(\mathrm{ii})}}\cap V^4_{X_{(\mathrm{i})}}\cap V_U^3)+\#(V^4_{X_{(\mathrm{ii})}}\cap V^4_{X_{(\mathrm{i})}}\cap V^4_{X_{(\mathrm{iii})}}\cap V_U^3)\Big)\Big))}$$
$$\cdot(\cdots)\cdot(\text{Self-intersecting }\#\text{ terms}) \tag{5.39}$$
$$\cong \mathrm{e}^{(-\mathrm{i}\pi\Big(k_{(\mathrm{i})}k_{(\mathrm{ii})}k_{(\mathrm{iii})}\ell\cdot 6\#(V^4_{X_{(\mathrm{i})}}\cap V^4_{X_{(\mathrm{ii})}}\cap V^4_{X_{(\mathrm{iii})}}\cap V_U^3)\Big))}\cdot(\cdots) \tag{5.40}$$
$$\cong \boxed{\mathrm{e}^{(-\mathrm{i}\pi\Big(k_{(\mathrm{i})}k_{(\mathrm{ii})}k_{(\mathrm{iii})}\ell\cdot 6\mathrm{Qlk}^{(5)}(\Sigma^3_{X_{(\mathrm{i})}},\Sigma^3_{X_{(\mathrm{ii})}},\Sigma^3_{X_{(\mathrm{iii})}},\Sigma^2_U)\Big))}\cdot(\cdots)}. \tag{5.41}$$

We propose a set-up to remove or renormalize the (Self-intersecting # terms) appeared in Eq. (5.39), following the same strategy as footnote 19.

For $\mathbf{S}=\pi\int_{M^5} c\mathrm{d}w_1(TM)+b\mathrm{d}B+w_1(TM)^3B$, we derive the link invariant for the 5d TQFT $\mathbf{Z}_{\mathrm{SET}}[M^5]$ in Eq. (5.39) and Eq. (5.40):

$$\boxed{\#(V^4_{X_{(\mathrm{i})}}\cap V^4_{X_{(\mathrm{ii})}}\cap V^4_{X_{(\mathrm{iii})}}\cap V_U^3)\equiv \mathrm{Qlk}^{(5)}(\Sigma^3_{X_{(\mathrm{i})}},\Sigma^3_{X_{(\mathrm{ii})}},\Sigma^3_{X_{(\mathrm{iii})}},\Sigma^2_U)}. \tag{5.42}$$

The path integral with appropriate extended operators insertions become Eq. (5.40) which provides the above link invariant. Note however the factorial $3!=6$ causes the complex $\mathrm{e}^{\mathrm{i}\pi}$ phase becoming $\mathrm{e}^{\mathrm{i}6\pi}$ thus undetectable. It may be possible to take into account (see footnote 17) from the subtle graded non-commutativity of cochain field effect. Thus one may need to go beyond the continuum differential form TQFT formulation by using the cochain TQFT formulation in order to see the subleading effect.

5.2.2 Version II: $w_1(TM)^2\mathrm{Sq}^1B$ and a Triple Link Invariant $\mathrm{Tlk}^{(5)}_{w_1w_1\mathrm{d}B}(\Sigma^3_{X_{(\mathrm{i})}},\Sigma^3_{X_{(\mathrm{ii})}},\Sigma^2_U)$

As another test example, we consider a 5d TQFT obtained from summing over 2-form field B of $w_1(TM)^2\mathrm{Sq}^1B$ (gauging 1-form $\mathbb{Z}_2$ of this 5d SPTs), below we convert the cochain TQFT to differential form continuum TQFT.[20] Whose partition function and action (see footnote 10) are: Below we convert the cochain TQFT to a differential form continuum TQFT. Its partition function and action are (see footnote 10):

$$\mathbf{Z} = \int[\mathcal{D}B][\mathcal{D}b][\mathcal{D}c]\exp(\mathrm{i}\mathbf{S}), \tag{5.43}$$
$$\mathbf{S} = \pi\int_{M^5} c\mathrm{d}w_1(TM)+b\mathrm{d}B+w_1(TM)^2\mathrm{Sq}^1B, \tag{5.44}$$
$$\mathbf{S} = \pi\int_{M^5} c\mathrm{d}w_1(TM)+b\mathrm{d}B+w_1(TM)^2\frac{1}{2}\mathrm{d}B. \tag{5.45}$$

[20] Even though $w_1(TM)^2\mathrm{Sq}^1B$ is a rewriting of $w_1(TM)^3B$, it turns out that we still gain new insights about an additional link invariant.

Gauge transformations are:

$$
\begin{aligned}
w_1(TM) &\rightarrow w_1(TM)+\mathrm{d}\alpha,\\
B &\rightarrow B+\mathrm{d}\beta,\\
c &\rightarrow c+\mathrm{d}\gamma+\lambda,\\
b &\rightarrow b+\mathrm{d}\zeta+\mu.
\end{aligned} \tag{5.46}
$$

The gauge variation shows:

$$
\begin{aligned}
\mathbf{S} \rightarrow\; & \mathbf{S}+\pi\int_{M^5} \mathrm{d}\gamma\mathrm{d}w_1(TM)+\lambda\mathrm{d}w_1(TM)+\mathrm{d}\zeta\mathrm{d}B+\mu\mathrm{d}B\\
& +(w_1(TM)\mathrm{d}\alpha+\mathrm{d}\alpha w_1(TM)+\mathrm{d}\alpha\mathrm{d}\alpha)\frac{1}{2}\mathrm{d}B\\
& +(w_1(TM)^2+w_1(TM)\mathrm{d}\alpha+\mathrm{d}\alpha w_1(TM)+\mathrm{d}\alpha\mathrm{d}\alpha)\frac{1}{2}\mathrm{d}^2\beta \qquad (5.47)\\
=\; & \mathbf{S}+\pi\int_{M^5} \lambda\mathrm{d}w_1(TM)+\mu\mathrm{d}B+\frac{1}{2}(w_1(TM)\mathrm{d}\alpha+\mathrm{d}\alpha w_1(TM)+\mathrm{d}\alpha\mathrm{d}\alpha)\mathrm{d}B \qquad (5.48)
\end{aligned}
$$

where we have used integration by part: for a closed 5-manifold without boundary, after integration by part then we can drop the boundary term $\mathrm{d}(\dots)$ where $\dots$ only has effects on a 4-manifold (the 4d boundary of an open 5-manifold) and we drop the total derivative terms which have no effect on a closed 5-manifold without boundary. The gauge variance of the action is: $\mathbf{\Delta S}=0 \Rightarrow$

$$
\begin{aligned}
\lambda &= 0,\\
\mu &= -\frac{1}{2}(w_1(TM)\mathrm{d}\alpha+\mathrm{d}\alpha w_1(TM)+\mathrm{d}\alpha\mathrm{d}\alpha).
\end{aligned} \tag{5.49}
$$

We derive that the 3-submanifold operator:

$$
\begin{aligned}
X &= \exp(\mathrm{i}\pi k(\int_{\Sigma^3} c))\\
&= \exp(\mathrm{i}\pi k(\int_{M^5}(\delta^{\perp}(\Sigma^3)c)))
\end{aligned} \tag{5.50}
$$

and 2-surface operator:

$$
\begin{aligned}
U &= \exp(\mathrm{i}\pi\ell(\int_{\Sigma^2}(b+\frac{1}{2}w_1(TM)^2)))\\
&= \exp(\mathrm{i}\pi\ell(\int_{M^5}(\delta^{\perp}(\Sigma^2)(b+\frac{1}{2}w_1(TM)^2)))
\end{aligned} \tag{5.51}
$$

are gauge invariant. Where k,ℓ are $\mathbb{Z}_2$ integers mod 2.

Insert $X_{(\mathbf{i})}, X_{(\mathbf{ii})}, U$ into the path integral $\mathbf{Z}$, so we can write the *continuum* field theory formulation as

$$
\langle X_{(\mathbf{i})}X_{(\mathbf{ii})}U\rangle=\int[\mathcal{D}B][\mathcal{D}c][\mathcal{D}b]\; X_{(\mathbf{i})}X_{(\mathbf{ii})}U\; \exp(\mathrm{i}\mathbf{S}). \tag{5.52}
$$

$$
\langle X_{(\mathbf{i})}X_{(\mathbf{ii})}U\rangle=\int[\mathcal{D}B][\mathcal{D}c][\mathcal{D}b]X_{(\mathbf{i})}X_{(\mathbf{ii})}U\; \exp(\mathrm{i}\pi\int_{M^5} c\mathrm{d}w_1(TM)+b\mathrm{d}B+w_1(TM)^2\frac{1}{2}\mathrm{d}B).
$$

Step 1, we integrate out c in $\int[\mathcal{D}c]$, we get

$$
\begin{aligned}
\mathrm{d}w_1(TM) &= k_{(\mathbf{i})}\delta^{\perp}(\Sigma^3_{X_{(\mathbf{i})}})+k_{(\mathbf{ii})}\delta^{\perp}(\Sigma^3_{X_{(\mathbf{ii})}}),\\
w_1(TM) &= k_{(\mathbf{i})}\delta^{\perp}(V^4_{X_{(\mathbf{i})}})+k_{(\mathbf{ii})}\delta^{\perp}(V^4_{X_{(\mathbf{ii})}}).
\end{aligned} \tag{5.53}
$$

So

$$\langle X_{(\mathbf{i})} X_{(\mathbf{ii})} U \rangle = \int [\mathcal{D}B][\mathcal{D}b]\, U \exp(\mathrm{i}\pi \int_{M^5} b\mathrm{d}B + w_1(TM)^2 \frac{1}{2}\mathrm{d}B)|_{w_1(TM)=k_{(\mathbf{i})}\delta^\perp(V^4_{X_{(\mathbf{i})}})+k_{(\mathbf{ii})}\delta^\perp(V^4_{X_{(\mathbf{ii})}})}. \tag{5.54}$$

Step 2, we integrate out b in $\int[\mathcal{D}b]$, we get the constraint

$$\begin{aligned} \mathrm{d}B &= \ell\delta^\perp(\Sigma^2_U), \\ B &= \ell\delta^\perp(V^3_U). \end{aligned} \tag{5.55}$$

Step 3, finally we integrate out B in $\int[\mathcal{D}B]$, from Eq. (5.54):

$$\begin{aligned} &\boxed{\langle X_{(\mathbf{i})} X_{(\mathbf{ii})} U \rangle} \\ &= \int [\mathcal{D}B] \mathrm{e}^{-\mathrm{i}\pi(\int_{M^5} \frac{1}{2} w_1(TM)^2 \ell\delta^\perp(\Sigma^2_U) + w_1(TM)^2 \frac{1}{2}\mathrm{d}B)}\Big|_{\substack{w_1(TM)\,=\,k_{(\mathbf{i})}\delta^\perp(V^4_{X_{(\mathbf{i})}})\,+\,k_{(\mathbf{ii})}\delta^\perp(V^4_{X_{(\mathbf{ii})}}), \\ B\,=\,\ell\delta^\perp(V^3_U)}} \\ &= \int [\mathcal{D}B] \mathrm{e}^{(-\mathrm{i}\pi(\int_{M^5} \frac{1}{2} w_1(TM)^2 \mathrm{d}B + w_1(TM)^2 \frac{1}{2}\mathrm{d}B))}\Big|_{\substack{w_1(TM)\,=\,k_{(\mathbf{i})}\delta^\perp(V^4_{X_{(\mathbf{i})}})\,+\,k_{(\mathbf{ii})}\delta^\perp(V^4_{X_{(\mathbf{ii})}}), \\ B\,=\,\ell\delta^\perp(V^3_U).}} \end{aligned} \tag{5.56}$$

$$\begin{aligned} &= \mathrm{e}^{(-\mathrm{i}\pi\Big(k_{(\mathbf{i})}k_{(\mathbf{ii})}\ell\big(\#(V^4_{X_{(\mathbf{i})}} \cap V^4_{X_{(\mathbf{ii})}} \cap \Sigma^2_U) + \#(V^4_{X_{(\mathbf{ii})}} \cap V^4_{X_{(\mathbf{i})}} \cap \Sigma^2_U)\big)\Big))} \\ &\quad \cdot(\cdots)\cdot(\text{Self-intersecting } \# \text{ terms}) \end{aligned} \tag{5.57}$$

$$\cong \boxed{\mathrm{e}^{(-\mathrm{i}\pi\Big(k_{(\mathbf{i})}k_{(\mathbf{ii})}\ell\cdot(\mathrm{Tlk}^{(5)}_{w_1 w_1 \mathrm{d}B}(\Sigma^3_{X_{(\mathbf{i})}}, \Sigma^3_{X_{(\mathbf{ii})}}, \Sigma^2_U) + \mathrm{Tlk}^{(5)}_{w_1 w_1 \mathrm{d}B}(\Sigma^3_{X_{(\mathbf{ii})}}, \Sigma^3_{X_{(\mathbf{i})}}, \Sigma^2_U))\Big))} \cdot (\cdots)}. \tag{5.58}$$

We propose a set-up to remove or renormalize the (Self-intersecting # terms) appeared in Eq. (5.57), following the same strategy as footnote 19.

For $\mathbf{S} = \pi \int_{M^5} c\mathrm{d}w_1(TM) + b\mathrm{d}B + w_1(TM)^2 \frac{1}{2}\mathrm{d}B$, we derive the link invariant for the 5d TQFT $\mathbf{Z}_{\text{SET}}[M^5]$ in Eq. (5.57) and Eq. (5.58):

$$\boxed{\#(V^4_{X_{(\mathbf{i})}} \cap V^4_{X_{(\mathbf{ii})}} \cap \Sigma^2_U) \equiv \mathrm{Tlk}^{(5)}_{w_1 w_1 \mathrm{d}B}(\Sigma^3_{X_{(\mathbf{i})}}, \Sigma^3_{X_{(\mathbf{ii})}}, \Sigma^2_U)}. \tag{5.59}$$

The path integral with appropriate extended operators insertions become Eq. (5.58) which provides the above link invariant. Note however the two terms on the exponent of Eq. (5.58) are the same, which causes the complex $\mathrm{e}^{\mathrm{i}\pi}$ phase becoming $\mathrm{e}^{\mathrm{i}2\pi}$ thus undetectable. It may be possible to take into account (see footnote 17) from the subtle graded non-commutativity of cochain field effect. Thus one may need to go beyond the continuum differential form TQFT formulation by using the cochain TQFT formulation in order to see the subleading effect.

5.3 $w_3(TM)B = w_2(TM)\mathrm{Sq}^1 B$ and a Quadratic Link Invariant $\mathbf{Lk}^{(5)}_{w_2 \mathrm{d}B}(\Sigma^2_{U'}, \Sigma^2_U)$

As another interesting test example, now we consider a 5d TQFT obtained from summing over 2-form field B of $w_3(TM)B = w_2(TM)\mathrm{Sq}^1 B$ (gauging 1-form $\mathbb{Z}_2$ of this 5d SPTs), below we convert the cochain TQFT to differential form continuum TQFT. Whose partition function and action (see footnote 10) are:

$$\mathbf{Z} = \int [\mathcal{D}B][\mathcal{D}b][\mathcal{D}h] \exp(\mathrm{i}\mathbf{S}), \tag{5.60}$$

$$\mathbf{S} = \pi \int_{M^5} h\mathrm{d}w_2(TM) + b\mathrm{d}B + w_2(TM)\mathrm{Sq}^1 B, \tag{5.61}$$

$$\mathbf{S} = \pi \int_{M^5} h\mathrm{d}w_2(TM) + b\mathrm{d}B + w_2(TM)\frac{1}{2}\mathrm{d}B. \tag{5.62}$$

Gauge transformations are:

$$\begin{aligned} w_2(TM) &\to w_2(TM) + \mathrm{d}\alpha, \\ B &\to B + \mathrm{d}\beta, \\ h &\to h + \mathrm{d}\gamma + \lambda, \\ b &\to b + \mathrm{d}\zeta + \mu. \end{aligned} \tag{5.63}$$

The gauge variation shows:

$$\begin{aligned} \mathbf{S} &\to \mathbf{S} + \pi \int_{M^5} \mathrm{d}\gamma \mathrm{d}w_2(TM) + \lambda \mathrm{d}w_2(TM) + \mathrm{d}\zeta \mathrm{d}B + \mu \mathrm{d}B \\ &\quad + \mathrm{d}\alpha \frac{1}{2} \mathrm{d}B + w_2(TM) \frac{1}{2} \mathrm{d}^2\beta + \mathrm{d}\alpha \frac{1}{2} \mathrm{d}^2\beta \qquad (5.64) \\ &= \mathbf{S} + \pi \int_{M^5} \lambda \mathrm{d}w_2(TM) + \mu \mathrm{d}B + (\frac{1}{2}\mathrm{d}\alpha)\mathrm{d}B + (-\frac{1}{2}\mathrm{d}\beta)\mathrm{d}w_2(TM). \qquad (5.65) \end{aligned}$$

The gauge variance of the action is: $\Delta \mathbf{S} = 0 \Rightarrow$

$$\begin{aligned} \lambda &= (\frac{1}{2}\mathrm{d}\beta), \\ \mu &= -(\frac{1}{2}\mathrm{d}\alpha). \end{aligned} \tag{5.66}$$

We derive 2-surface operator:

$$\begin{aligned} U' &= \exp(\mathrm{i}\pi k(\int_{\Sigma^2} h - \int_{V^3} \frac{1}{2}\mathrm{d}B)) \\ &= \exp(\mathrm{i}\pi k(\int_{M^5} (\delta^\perp(\Sigma^2) h - \delta^\perp(V^3)\frac{1}{2}\mathrm{d}B))) \qquad (5.67) \\ &= \exp(\mathrm{i}\pi k(\int_{M^5} (\delta^\perp(\Sigma^2)(h - \frac{1}{2}B)))) \qquad (5.68) \end{aligned}$$

and 2-surface operator:

$$\begin{aligned} U &= \exp(\mathrm{i}\pi \ell(\int_{\Sigma^2} b + \int_{V^3} \frac{1}{2}\mathrm{d}w_2(TM))) \\ &= \exp(\mathrm{i}\pi \ell(\int_{\Sigma^2} b + \int_{\Sigma^2} \frac{1}{2}w_2(TM))) \\ &= \exp(\mathrm{i}\pi \ell(\int_{M^5} \delta^\perp(\Sigma^2)(b + \frac{1}{2}w_2(TM)))) \qquad (5.69) \end{aligned}$$

are gauge invariant. Where k, ℓ are $\mathbb{Z}_2$ integers mod 2.

Insert U', U into the path integral $\mathbf{Z}$, so we can write the *continuum* field theory formulation as

$$\langle U'U \rangle = \int [\mathcal{D}B][\mathcal{D}h][\mathcal{D}b] U'U \ \exp(\mathrm{i}\pi \int_{M^5} h \mathrm{d}w_2(TM) + b\mathrm{d}B + w_2(TM)\frac{1}{2}\mathrm{d}B).$$

Step 1, we integrate out h in $\int[\mathcal{D}h]$, we get

$$\begin{aligned} \mathrm{d}w_2(TM) &= k\delta^\perp(\Sigma^2_{U'}), \\ w_2(TM) &= k\delta^\perp(V^3_{U'}). \end{aligned} \tag{5.70}$$

We get the double-counting mod 2 cancellation in the exponent of $\exp(\mathrm{i}\,\pi(\int_{M^5} \delta^\perp(V^3_{U'})\frac{k}{2}\mathrm{d}B + w_2(TM)\frac{1}{2}\mathrm{d}B)) = 1$. This boils down to

$$\langle U'U \rangle = \int [\mathcal{D}B][\mathcal{D}b]\, U \exp(\mathrm{i}\pi \int_{M^5} b\mathrm{d}B)|_{w_2(TM) = k\delta^\perp(V^3_{U'})}. \tag{5.71}$$

Step 2, we integrate out b in $\int[\mathcal{D}b]$, we get the constraint

$$\begin{aligned} \mathrm{d}B &= \ell\delta^\perp(\Sigma_U^2), \\ B &= \ell\delta^\perp(V_U^3). \end{aligned} \tag{5.72}$$

Step 3, finally we integrate out B in $\int[\mathcal{D}B]$, from Eq. (5.71):

$$\begin{aligned} &\boxed{\langle U'U\rangle} \\ &= \int[\mathcal{D}B]\mathrm{e}^{-\mathrm{i}\pi(\int_{M^5}\frac{1}{2}w_2(TM)\ell\delta^\perp(\Sigma_U^2))}\Big|_{\substack{w_2(TM)=k\delta^\perp(V_{U'}^3),\\ B=\ell\delta^\perp(V_U^3)}} \\ &= \int[\mathcal{D}B]\mathrm{e}^{(-\mathrm{i}\pi(\int_{M^5}\frac{1}{2}w_2(TM)\mathrm{d}B))}\Big|_{\substack{w_2(TM)=k\delta^\perp(V_{U'}^3),\\ B=\ell\delta^\perp(V_U^3)}} \end{aligned} \tag{5.73}$$

$$= \mathrm{e}^{(-\mathrm{i}\pi\left(\frac{k\ell}{2}\cdot\#(V_{U'}^3\cap\Sigma_U^2)\right))} \tag{5.74}$$

$$\cong \boxed{\mathrm{e}^{(-\mathrm{i}\pi\left(\frac{k\ell}{2}\cdot\mathrm{Lk}^{(5)}(\Sigma_{U'}^2,\Sigma_U^2)\right))}}. \tag{5.75}$$

We derive the link invariant for the 5d TQFT $\mathbf{Z}_{\mathrm{SET}}[M^5]$ for $\mathbf{S}=\pi\int_{M^5}h\mathrm{d}w_2(TM)+b\mathrm{d}B+w_2(TM)\mathrm{Sq}^1B$ (Version 2) in Eq. (5.75):

$$\boxed{\#(V_{U'}^3\cap\Sigma_U^2)\equiv\mathrm{Lk}_{w_2\mathrm{d}B}^{(5)}(\Sigma_{U'}^2,\Sigma_U^2)}. \tag{5.76}$$

The path integral with appropriate extended operators insertions become Eq. (5.75) which provides the above link invariant.

5.4 $B\mathrm{Sq}^1B+(1+K_1)w_1(TM)^2\mathrm{Sq}^1B+(1+K_2)w_2(TM)\mathrm{Sq}^1B$ and More Link Invariants: $\mathbf{Tlk}_{w_1w_1\mathrm{d}B}^{(5)}(\Sigma_{X_{(\mathrm{i})}}^3,\Sigma_{X_{(\mathrm{ii})}}^3,\Sigma_U^2)$, $\mathbf{Lk}_{B\mathrm{d}B}^{(5)}(\Sigma_{U_{(\mathrm{i})}}^2,\Sigma_{U_{(\mathrm{ii})}}^2)$ and $\mathbf{Lk}_{w_2\mathrm{d}B}^{(5)}(\Sigma_{U'}^2,\Sigma_{U_{(\mathrm{ii})}}^2)$

Finally we consider the generic form including any of the four classes of $\mathbf{Z}_{\mathrm{SET}_{(K_1,K_2)}}^{5\mathrm{d}}$ in Eq. (4.5) obtained from gauging $\mathbf{Z}_{\mathrm{SPT}_{(K_1,K_2)}}^{5\mathrm{d}}$ in Eq. (2.20), with $(K_1,K_2)\in(\mathbb{Z}_2,\mathbb{Z}_2)$ labeling the four siblings. Denote $K_1':=1+K_1 \bmod 2$ and $K_2':=1+K_2 \bmod 2$. [21] The partition function and action (see footnote 10) are:

$$\mathbf{Z} = \int[\mathcal{D}B][\mathcal{D}b][\mathcal{D}h][\mathcal{D}c]\exp(\mathrm{i}\mathbf{S}). \tag{5.77}$$

$$\mathbf{S} = \pi\int_{M^5}K_1'c\mathrm{d}w_1(TM)+K_2'h\mathrm{d}w_2(TM)+b\mathrm{d}B+B\mathrm{Sq}^1B+K_1'w_1(TM)^2\mathrm{Sq}^1B+K_2'w_2(TM)\mathrm{Sq}^1B. \tag{5.78}$$

$$\mathbf{S} = \pi\int_{M^5}K_1'c\mathrm{d}w_1(TM)+K_2'h\mathrm{d}w_2(TM)+b\mathrm{d}B+B\frac{1}{2}\mathrm{d}B+K_1'w_1(TM)^2\frac{1}{2}\mathrm{d}B+K_2'w_2(TM)\frac{1}{2}\mathrm{d}B. \tag{5.79}$$

As we will see, the ordering of the path integral measures $\int[\mathcal{D}B][\mathcal{D}b][\mathcal{D}h][\mathcal{D}c]$ is based on the ordering of integrating out later. Later we will integrate out the first c, then the second h, then the third b, then the fourth B.

[21] Although in the partition function, K_i' and $K_i'+2$ are equivalent, we only consider $K_i'\in\{0,1\}$.

5.4.1 Gauge Invariance

Gauge transformations are:

$$
\begin{aligned}
w_1(TM) &\to w_1(TM) + \mathrm{d}\alpha_1,\\
w_2(TM) &\to w_2(TM) + \mathrm{d}\alpha_2,\\
B &\to B + \mathrm{d}\beta,\\
c &\to c + \mathrm{d}\gamma_1 + \lambda_1,\\
h &\to h + \mathrm{d}\gamma_2 + \lambda_2,\\
b &\to b + \mathrm{d}\zeta + \mu.
\end{aligned} \tag{5.80}
$$

The gauge variation shows:

$$
\begin{aligned}
\mathbf{S} \;\to\;& \mathbf{S} + \pi \int_{M^5} K_1' \mathrm{d}\gamma_1 \mathrm{d}w_1(TM) + K_1'\lambda_1 \mathrm{d}w_1(TM) + K_2' \mathrm{d}\gamma_2 \mathrm{d}w_2(TM) + K_2'\lambda_2 \mathrm{d}w_2(TM) + \mathrm{d}\zeta \mathrm{d}B + \mu \mathrm{d}B\\
&+\mathrm{d}\beta \frac{1}{2}\mathrm{d}B + B\frac{1}{2}\mathrm{d}^2\beta + \mathrm{d}\beta\frac{1}{2}\mathrm{d}^2\beta\\
&+K_1'(w_1(TM)\mathrm{d}\alpha_1 + \mathrm{d}\alpha_1 w_1(TM) + \mathrm{d}\alpha_1 \mathrm{d}\alpha_1)\frac{1}{2}\mathrm{d}B\\
&+K_1'(w_1(TM)^2 + w_1(TM)\mathrm{d}\alpha_1 + \mathrm{d}\alpha_1 w_1(TM) + \mathrm{d}\alpha_1 \mathrm{d}\alpha_1)\frac{1}{2}\mathrm{d}^2\beta\\
&+K_2'\mathrm{d}\alpha_2\frac{1}{2}\mathrm{d}B + K_2' w_2(TM)\frac{1}{2}\mathrm{d}^2\beta + K_2'\mathrm{d}\alpha_2\frac{1}{2}\mathrm{d}^2\beta
\end{aligned} \tag{5.81}
$$

$$
\begin{aligned}
=\;& \mathbf{S} + \pi \int_{M^5} K_1'\lambda_1 \mathrm{d}w_1(TM) + K_2'\lambda_2 \mathrm{d}w_2(TM) + \mu \mathrm{d}B + K_2'(\frac{1}{2}\mathrm{d}\alpha_2)\mathrm{d}B + K_2'(-\frac{1}{2}\mathrm{d}\beta)\mathrm{d}w_2(TM)\\
&+K_1'\frac{1}{2}(w_1(TM)\mathrm{d}\alpha_1 + \mathrm{d}\alpha_1 w_1(TM) + \mathrm{d}\alpha_1 \mathrm{d}\alpha_1)\mathrm{d}B
\end{aligned} \tag{5.82}
$$

where we have used integration by part: for a closed 5-manifold without boundary, after integration by part then we can drop the boundary term $\mathrm{d}(\dots)$ where $\dots$ only has effects on a 4-manifold (the 4d boundary of an open 5-manifold) and we drop the total derivative terms which have no effect on a closed 5-manifold without boundary.

The gauge variance of the action is: $\Delta\mathbf{S} = 0 \Rightarrow$

$$
\begin{aligned}
K_1'\lambda_1 &= 0,\\
K_2'\lambda_2 &= K_2'\frac{1}{2}\mathrm{d}\beta,\\
\mu &= -K_1'\frac{1}{2}(w_1(TM)\mathrm{d}\alpha_1 + \mathrm{d}\alpha_1 w_1(TM) + \mathrm{d}\alpha_1 \mathrm{d}\alpha_1) - K_2'\frac{1}{2}\mathrm{d}\alpha_2.
\end{aligned} \tag{5.83}
$$

5.4.2 Extended 2-Surface/3-Brane Operators and Link Invariants

We derive 3-manifold operator:

$$
\begin{aligned}
X &= \exp(\mathrm{i}\pi k K_1'(\int_{\Sigma^3} c))\\
&= \exp(\mathrm{i}\pi k(1+K_1)(\int_{M^5}(\delta^{\perp}(\Sigma^3)c))).
\end{aligned} \tag{5.84}
$$

Note that when $K_1' = 1 + K_1 = 0 \bmod 2$, the X operator vanishes.

We derive 2-surface operators:

$$
\begin{aligned}
U' &= \exp(\mathrm{i}\pi k' K_2' (\int_{\Sigma^2} h - \int_{V^3} \frac{1}{2}\mathrm{d}B)) \\
&= \exp(\mathrm{i}\pi k' K_2' (\int_{M^5} (\delta^\perp(\Sigma^2) h - \delta^\perp(V^3) \frac{1}{2}\mathrm{d}B))) \\
&= \exp(\mathrm{i}\pi k' K_2' (\int_{M^5} (\delta^\perp(\Sigma^2)(h - \frac{1}{2}B)))) \\
&= \exp(\mathrm{i}\pi k' (1+K_2) (\int_{M^5} (\delta^\perp(\Sigma^2)(h - \frac{1}{2}B)))).
\end{aligned} \tag{5.85}
$$

Note that when $K_2' = 1 + K_2 = 0 \bmod 2$, the U' operator vanishes.

$$
\begin{aligned}
U &= \exp(\mathrm{i}\pi\ell(\int_{\Sigma^2} (b + K_1' \frac{1}{2} w_1(TM)^2 + K_2' \frac{1}{2} w_2(TM)))) \\
&= \exp(\mathrm{i}\pi\ell(\int_{M^5} (\delta^\perp(\Sigma^2)(b + K_1' \frac{1}{2} w_1(TM)^2 + K_2' \frac{1}{2} w_2(TM))))) \\
&= \exp(\mathrm{i}\pi\ell(\int_{M^5} (\delta^\perp(\Sigma^2)(b + (1+K_1) \frac{1}{2} w_1(TM)^2 + (1+K_2) \frac{1}{2} w_2(TM))))).
\end{aligned} \tag{5.86}
$$

All above extended operators are gauge invariant. Where k, k', ℓ are $\mathbb{Z}_2$ integers mod 2.

Insert $X_{(\mathbf{i})}, X_{(\mathbf{ii})}, U', U_{(\mathbf{i})}, U_{(\mathbf{ii})}$ into the path integral $\mathbf{Z}$, so we can write the *continuum* field theory formulation as

$$
\langle X_{(\mathbf{i})} X_{(\mathbf{ii})} U' U_{(\mathbf{i})} U_{(\mathbf{ii})} \rangle = \int [\mathcal{D}B][\mathcal{D}b][\mathcal{D}h][\mathcal{D}c]\; X_{(\mathbf{i})} X_{(\mathbf{ii})} U' U_{(\mathbf{i})} U_{(\mathbf{ii})}\; \exp(\mathrm{i}\mathbf{S}). \tag{5.87}
$$

$$
\begin{aligned}
&\langle X_{(\mathbf{i})} X_{(\mathbf{ii})} U' U_{(\mathbf{i})} U_{(\mathbf{ii})} \rangle = \int [\mathcal{D}B][\mathcal{D}b][\mathcal{D}h][\mathcal{D}c] X_{(\mathbf{i})} X_{(\mathbf{ii})} U' U_{(\mathbf{i})} U_{(\mathbf{ii})}\; \exp(\mathrm{i}\pi \int_{M^5} K_1' c \mathrm{d}w_1(TM) \\
&+ K_2' h \mathrm{d}w_2(TM) + b \mathrm{d}B + B \frac{1}{2}\mathrm{d}B + K_1' w_1(TM)^2 \frac{1}{2}\mathrm{d}B + K_2' w_2(TM) \frac{1}{2}\mathrm{d}B).
\end{aligned}
$$

Step 1, we integrate out c in $\int[\mathcal{D}c]$, we get

$$
\begin{aligned}
K_1' \mathrm{d}w_1(TM) &= K_1' \big(k_{(\mathbf{i})} \delta^\perp(\Sigma^3_{X_{(\mathbf{i})}}) + k_{(\mathbf{ii})} \delta^\perp(\Sigma^3_{X_{(\mathbf{ii})}})\big), \\
K_1' w_1(TM) &= K_1' \big(k_{(\mathbf{i})} \delta^\perp(V^4_{X_{(\mathbf{i})}}) + k_{(\mathbf{ii})} \delta^\perp(V^4_{X_{(\mathbf{ii})}})\big).
\end{aligned} \tag{5.88}
$$

We keep K_1' on both sides, because when $K_1' = 1 \bmod 2$, we have this constraint; when $K_1' = 0 \bmod 2$, there is no such constraint. So

$$
\begin{aligned}
\langle X_{(\mathbf{i})} X_{(\mathbf{ii})} U' U_{(\mathbf{i})} U_{(\mathbf{ii})} \rangle &= \int [\mathcal{D}B][\mathcal{D}b][\mathcal{D}h]\; U' U_{(\mathbf{i})} U_{(\mathbf{ii})} \exp(\mathrm{i}\pi \int_{M^5} K_2' h \mathrm{d}w_2(TM) + b\mathrm{d}B + B\frac{1}{2}\mathrm{d}B \\
&\quad + K_1' w_1(TM)^2 \frac{1}{2}\mathrm{d}B + K_2' w_2(TM) \frac{1}{2}\mathrm{d}B)\Big|_{K_1' w_1(TM) = K_1'(k_{(\mathbf{i})}\delta^\perp(V^4_{X_{(\mathbf{i})}}) + k_{(\mathbf{ii})}\delta^\perp(V^4_{X_{(\mathbf{ii})}}))}.
\end{aligned} \tag{5.89}
$$

Step 2, we integrate out h in $\int[\mathcal{D}h]$, we get

$$
\begin{aligned}
K_2' \mathrm{d}w_2(TM) &= K_2'\; k' \delta^\perp(\Sigma^2_{U'}), \\
K_2' w_2(TM) &= K_2'\; k' \delta^\perp(V^3_{U'}).
\end{aligned} \tag{5.90}
$$

We keep K_2' on both sides, because when $K_2' = 1 \bmod 2$, we have this constraint; when $K_2' = 0 \bmod 2$, there is no such constraint. We get the double-counting mod 2 cancellation in the exponent of $\exp(\mathrm{i}\pi(\int_{M^5} K_2'\delta^\perp(V_{U'}^3)\frac{k'}{2}\mathrm{d}B + K_2' w_2(TM)\frac{1}{2}\mathrm{d}B)) = 1$. This boils down to

$$\langle X_{(\mathrm{i})} X_{(\mathrm{ii})} U' U_{(\mathrm{i})} U_{(\mathrm{ii})}\rangle = \int [\mathcal{D}B][\mathcal{D}b]\, U_{(\mathrm{i})} U_{(\mathrm{ii})} \exp(\mathrm{i}\pi \int_{M^5} b\mathrm{d}B + B\frac{1}{2}\mathrm{d}B + K_1' w_1(TM)^2\frac{1}{2}\mathrm{d}B)\Big|_{\substack{K_1' w_1(TM) = K_1'(k_{(\mathrm{i})}\delta^\perp(V^4_{X_{(\mathrm{i})}}) + k_{(\mathrm{ii})}\delta^\perp(V^4_{X_{(\mathrm{ii})}})),\\ K_2' w_2(TM) = K_2' k'\delta^\perp(V^3_{U'}).}}. \tag{5.91}$$

Step 3, we integrate out b in $\int[\mathcal{D}b]$, we get the constraint

$$\begin{aligned} \mathrm{d}B &= \ell_{(\mathrm{i})}\delta^\perp(\Sigma^2_{U_{(\mathrm{i})}}) + \ell_{(\mathrm{ii})}\delta^\perp(\Sigma^2_{U_{(\mathrm{ii})}}),\\ B &= \ell_{(\mathrm{i})}\delta^\perp(V^3_{U_{(\mathrm{i})}}) + \ell_{(\mathrm{ii})}\delta^\perp(V^3_{U_{(\mathrm{ii})}}). \end{aligned} \tag{5.92}$$

Step 4, finally we integrate out B in $\int[\mathcal{D}B]$, from Eq. (5.89):

$$\begin{aligned}
&\boxed{\langle X_{(\mathrm{i})} X_{(\mathrm{ii})} U' U_{(\mathrm{i})} U_{(\mathrm{ii})}\rangle}\\
&= \int [\mathcal{D}B] \exp(-\mathrm{i}\pi(\int_{M^5} \frac{1}{2}(K_1' w_1(TM)^2 + K_2' w_2(TM))(\ell_{(\mathrm{i})}\delta^\perp(\Sigma^2_{U_{(\mathrm{i})}}) + \ell_{(\mathrm{ii})}\delta^\perp(\Sigma^2_{U_{(\mathrm{ii})}}))\\
&+B\frac{1}{2}\mathrm{d}B + K_1' w_1(TM)^2\frac{1}{2}\mathrm{d}B))\Big|_{\substack{K_1' w_1(TM) = K_1'(k_{(\mathrm{i})}\delta^\perp(V^4_{X_{(\mathrm{i})}}) + k_{(\mathrm{ii})}\delta^\perp(V^4_{X_{(\mathrm{ii})}})),\\ K_2' w_2(TM) = K_2' k'\delta^\perp(V^3_{U'}),\\ B = \ell_{(\mathrm{i})}\delta^\perp(V^3_{U_{(\mathrm{i})}}) + \ell_{(\mathrm{ii})}\delta^\perp(V^3_{U_{(\mathrm{ii})}})}}\\
&= \int [\mathcal{D}B] \exp(-\mathrm{i}\pi(\int_{M^5} \frac{1}{2}(K_1' w_1(TM)^2 + K_2' w_2(TM))\mathrm{d}B + B\frac{1}{2}\mathrm{d}B\\
&+K_1' w_1(TM)^2\frac{1}{2}\mathrm{d}B))\Big|_{\substack{K_1' w_1(TM) = K_1'(k_{(\mathrm{i})}\delta^\perp(V^4_{X_{(\mathrm{i})}}) + k_{(\mathrm{ii})}\delta^\perp(V^4_{X_{(\mathrm{ii})}})),\\ K_2' w_2(TM) = K_2' k'\delta^\perp(V^3_{U'}),\\ B = \ell_{(\mathrm{i})}\delta^\perp(V^3_{U_{(\mathrm{i})}}) + \ell_{(\mathrm{ii})}\delta^\perp(V^3_{U_{(\mathrm{ii})}})}}
\end{aligned} \tag{5.93}$$

$$\begin{aligned}
&= \exp(-\mathrm{i}\pi\Big(K_1' k_{(\mathrm{i})} k_{(\mathrm{ii})}\cdot 2\#(V^4_{X_{(\mathrm{i})}} \cap V^4_{X_{(\mathrm{ii})}} \cap (\ell_{(\mathrm{i})}\delta^\perp(\Sigma^2_{U_{(\mathrm{i})}}) + \ell_{(\mathrm{ii})}\delta^\perp(\Sigma^2_{U_{(\mathrm{ii})}})))\\
&+K_2'(\frac{k'\ell_{(\mathrm{i})}}{2}\cdot\#(V^3_{U'}\cap\Sigma^2_{U_{(\mathrm{i})}}) + \frac{k'\ell_{(\mathrm{ii})}}{2}\cdot\#(V^3_{U'}\cap\Sigma^2_{U_{(\mathrm{ii})}})) + \frac{\ell_{(\mathrm{i})}\ell_{(\mathrm{ii})}}{2}\cdot(\#(V^3_{U_{(\mathrm{i})}}\cap\Sigma^2_{U_{(\mathrm{ii})}}) + \#(V^3_{U_{(\mathrm{ii})}}\cap\Sigma^2_{U_{(\mathrm{i})}}))\Big))\\
&\quad\cdot(\cdots)\cdot(\text{Self-intersecting \# terms})
\end{aligned} \tag{5.94}$$

$$\begin{aligned}
&\cong \exp(-\mathrm{i}\pi\Big(K_1'(k_{(\mathrm{i})} k_{(\mathrm{ii})}\ell_{(\mathrm{i})}\cdot 2\mathrm{Tlk}^{(5)}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^2_{U_{(\mathrm{i})}}) + k_{(\mathrm{i})} k_{(\mathrm{ii})}\ell_{(\mathrm{ii})}\cdot 2\mathrm{Tlk}^{(5)}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^2_{U_{(\mathrm{ii})}}))\\
&+K_2'(\frac{k'\ell_{(\mathrm{i})}}{2}\cdot\mathrm{Lk}^{(5)}(\Sigma^2_{U'}, \Sigma^2_{U_{(\mathrm{i})}}) + \frac{k'\ell_{(\mathrm{ii})}}{2}\cdot\mathrm{Lk}^{(5)}(\Sigma^2_{U'}, \Sigma^2_{U_{(\mathrm{ii})}})) + \ell_{(\mathrm{i})}\ell_{(\mathrm{ii})}\cdot\mathrm{Lk}^{(5)}(\Sigma^2_{U_{(\mathrm{i})}}, \Sigma^2_{U_{(\mathrm{ii})}})\Big))\cdot(\cdots).
\end{aligned} \tag{5.95}$$

We propose a set-up to remove or renormalize the (Self-intersecting # terms) appeared in Eq. (5.94), following the same strategy as footnote 19.

For $\mathbf{S} = \pi\int_{M^5} K_1' c\mathrm{d}w_1(TM) + K_2' h\mathrm{d}w_2(TM) + b\mathrm{d}B + B\frac{1}{2}\mathrm{d}B + K_1' w_1(TM)^2\frac{1}{2}\mathrm{d}B + K_2' w_2(TM)\frac{1}{2}\mathrm{d}B$, we derive the link invariant for the 5d TQFT $\mathbf{Z}_{\mathrm{SET}}[M^5]$ in Eq. (5.94) and Eq. (5.95):

$$\begin{aligned}
&\boxed{K_1' k_{(\mathrm{i})} k_{(\mathrm{ii})}\cdot 2\#(V^4_{X_{(\mathrm{i})}} \cap V^4_{X_{(\mathrm{ii})}} \cap (\ell_{(\mathrm{i})}\delta^\perp(\Sigma^2_{U_{(\mathrm{i})}}) + \ell_{(\mathrm{ii})}\delta^\perp(\Sigma^2_{U_{(\mathrm{ii})}})))}\\
&\boxed{+K_2'(\frac{k'\ell_{(\mathrm{i})}}{2}\cdot\#(V^3_{U'}\cap\Sigma^2_{U_{(\mathrm{i})}}) + \frac{k'\ell_{(\mathrm{ii})}}{2}\cdot\#(V^3_{U'}\cap\Sigma^2_{U_{(\mathrm{ii})}})) + \frac{\ell_{(\mathrm{i})}\ell_{(\mathrm{ii})}}{2}\cdot(\#(V^3_{U_{(\mathrm{i})}}\cap\Sigma^2_{U_{(\mathrm{ii})}}) + \#(V^3_{U_{(\mathrm{ii})}}\cap\Sigma^2_{U_{(\mathrm{i})}}))}\\
\equiv\;&\boxed{(1+K_1)(k_{(\mathrm{i})} k_{(\mathrm{ii})}\ell_{(\mathrm{i})}\cdot 2\mathrm{Tlk}^{(5)}_{w_1 w_1\mathrm{d}B}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^2_{U_{(\mathrm{i})}}) + k_{(\mathrm{i})} k_{(\mathrm{ii})}\ell_{(\mathrm{ii})}\cdot 2\mathrm{Tlk}^{(5)}_{w_1 w_1\mathrm{d}B}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^2_{U_{(\mathrm{ii})}}))}\\
&\boxed{+(1+K_2)(\frac{k'\ell_{(\mathrm{i})}}{2}\cdot\mathrm{Lk}^{(5)}_{w_2\mathrm{d}B}(\Sigma^2_{U'}, \Sigma^2_{U_{(\mathrm{i})}}) + \frac{k'\ell_{(\mathrm{ii})}}{2}\cdot\mathrm{Lk}^{(5)}_{w_2\mathrm{d}B}(\Sigma^2_{U'}, \Sigma^2_{U_{(\mathrm{ii})}})) + \ell_{(\mathrm{i})}\ell_{(\mathrm{ii})}\cdot\mathrm{Lk}^{(5)}_{B\mathrm{d}B}(\Sigma^2_{U_{(\mathrm{i})}}, \Sigma^2_{U_{(\mathrm{ii})}})}.
\end{aligned} \tag{5.96}$$

The path integral with appropriate extended operators insertions become Eq. (5.95) which provides the above link invariant.

5.4.3 $(K_1, K_2) = (0, 0)$: 1st Sibling

The $\mathbf{Z}^{5\text{d}}_{\text{SET}_{(K_1=0,K_2=0)}}$ gives rise to a 5d triple link invariant:

• $\text{Tlk}^{(5)}_{w_1 BB}$ in Sec. 5.1's Eq. (5.25). We present an exemplary link configuration later in (Sec. 6.2) that can be detected by this link invariant. On another expression, $\mathbf{Z}^{5\text{d}}_{\text{SET}_{(K_1=0,K_2=0)}}$ in Eq. (4.5) gives rise to other link invariants in Eq. (5.96) including

• $\text{Tlk}^{(5)}_{w_1 w_1 \,\mathrm{d}B}(\Sigma^3_{X_{(\text{i})}}, \Sigma^3_{X_{(\text{ii})}}, \Sigma^2_U)$, a second type of triple link in 5d (although seemly undetectable due to an exponent factor 2π in the expectation value). We present an exemplary link configuration later in (Sec. 6.3) that can be detected by this link invariant.

• $\text{Lk}^{(5)}_{w_2 \,\mathrm{d}B}(\Sigma^2_{U'}, \Sigma^2_U)$, another quadratic link of 2-surfaces in 5d. We present an exemplary link configuration later in (Sec. 6.6) that can be detected by this link invariant.

• $\text{Lk}^{(5)}_{B \,\mathrm{d}B}(\Sigma^2_{U_{(\text{i})}}, \Sigma^2_{U_{(\text{ii})}})$, a quadratic link of 2-surfaces in 5d. We present an exemplary link configuration later in (Sec. 6.5) that can be detected by this link invariant.

Physically, these link invariants may be related to each other by re-arranging the spacetime braiding process of strings/branes. It will be interesting to find a precise mathematical equality to relate these link invariants.

5.4.4 $(K_1, K_2) = (1, 0)$: 2nd Sibling

$\mathbf{Z}^{5\text{d}}_{\text{SET}_{(K_1=1,K_2=0)}}$ in Eq. (4.5) gives rise to link invariants in Eq. (5.96) including

• $\text{Lk}^{(5)}_{w_2 \,\mathrm{d}B}(\Sigma^2_{U'}, \Sigma^2_U)$, another quadratic link of 2-surfaces in 5d. We present an exemplary link configuration later in (Sec. 6.6) that can be detected by this link invariant.

• $\text{Lk}^{(5)}_{B \,\mathrm{d}B}(\Sigma^2_{U_{(\text{i})}}, \Sigma^2_{U_{(\text{ii})}})$, a quadratic link of 2-surfaces in 5d. We present an exemplary link configuration later in (Sec. 6.5) that can be detected by this link invariant.

Similarly to our comments above in Sec. 5.4.3, it will be interesting to find a precise mathematical equality to relate these link invariants.

5.4.5 $(K_1, K_2) = (0, 1)$: 3rd Sibling

$\mathbf{Z}^{5\text{d}}_{\text{SET}_{(K_1=0,K_2=1)}}$ in Eq. (4.5) gives rise to link invariants in Eq. (5.96) including

• $\text{Tlk}^{(5)}_{w_1 w_1 \,\mathrm{d}B}(\Sigma^3_{X_{(\text{i})}}, \Sigma^3_{X_{(\text{ii})}}, \Sigma^2_U)$, a second type of triple link in 5d (although seemly undetectable due to an exponent factor 2π in the expectation value). We present an exemplary link configuration later in (Sec. 6.3) that can be detected by this link invariant.

• $\text{Lk}^{(5)}_{B \,\mathrm{d}B}(\Sigma^2_{U_{(\text{i})}}, \Sigma^2_{U_{(\text{ii})}})$, a quadratic link of 2-surfaces in 5d. We present an exemplary link configuration later in (Sec. 6.5) that can be detected by this link invariant.

Similarly to our comments above in Sec. 5.4.3, it will be interesting to find a precise mathematical equality to relate these link invariants.

5.4.6 $(K_1, K_2) = (1, 1)$: 4th Sibling

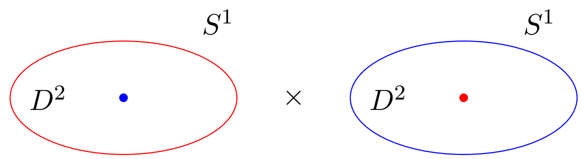
$\mathbf{Z}^{\text{5d}}_{\text{SET}_{(K_1=1,K_2=1)}}$ in Eq. (4.5) gives rise to link invariants in Eq. (5.96) including
• $\text{Lk}^{(5)}_{B\,\mathrm{d}B}(\Sigma^2_{U_{(\mathbf{i})}}, \Sigma^2_{U_{(\mathbf{ii})}})$.
Similarly to our comments above in Sec. 5.4.3, it will be interesting to find a precise mathematical equality to relate these link invariants.

6 Anyonic String/Brane Spacetime Braiding Process and Link Configurations of Extended Operators

Now we provide the exemplary spacetime braiding process of anyonic string/brane (in 5d and in other dimensions), and the link configurations of extended operators, which can be detected by the link invariants that we derived in Sec. 5.

6.1 Quadratic Link (Aharanov-Bohm) Configuration in Any Dimension

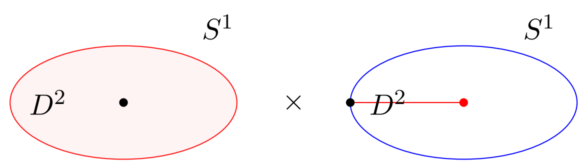
To warm up, first let us recall the quadratic link, by the Aharanov-Bohm statistics in dd due to the linking of charged particle's 1-worldline and the fractional flux's $(d-2)$d-worldsheet. In 3d spacetime, we have the following presentation



where gluing two solid tori $D^2 \times S^1$ gives rise to a 3-sphere: $(D^2_{\mathrm{L}} \times S^1_{\mathrm{R}}) \cup (S^1_{\mathrm{L}} \times D^2_{\mathrm{R}}) = S^3$. We can represent the two solid tori gluing as a blue solid tori and a red solid tori gluing: $(D^2_{\mathrm{L}} \times S^1_{\mathrm{R}}) \cup (S^1_{\mathrm{L}} \times D^2_{\mathrm{R}}) = S^3$. It is well-known that the link invariant (quadratic link) detecting this Aharanov-Bohm configuration is given by ([11] and References therein): $\mathrm{Lk}((0_{\mathrm{pt}})_{\mathrm{L}} \times S^1_{\mathrm{R}}, S^1_{\mathrm{L}} \times (0_{\mathrm{pt}})_{\mathrm{R}})$, which we also express as

$$\mathrm{Lk}((0_{\mathrm{pt}})_{\mathrm{L}} \times S^1_{\mathrm{R}}, S^1_{\mathrm{L}} \times (0_{\mathrm{pt}})_{\mathrm{R}}) \tag{6.1}$$

based on the color labeling of the inclusion of two S^1 circles in which of two solid tori. This link invariant can be computed from the intersection number,



as $\#((0_{\mathrm{pt}})_{\mathrm{L}} \times S^1_{\mathrm{R}}) \cap (D^2_{\mathrm{L}} \times (0_{\mathrm{pt}})_{\mathrm{R},-})) = 1$, which becomes

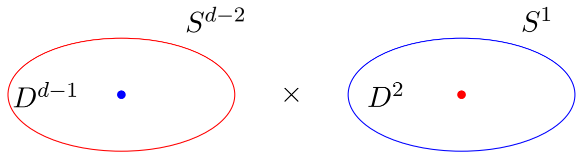
$$\#((0_{\mathrm{pt}})_{\mathrm{L}} \times S^1_{\mathrm{R}}) \cap (D^2_{\mathrm{L}} \times (0_{\mathrm{pt}-})_{\mathrm{R}})) = 1. \tag{6.2}$$

again based on the color labeling of the inclusion of S^1 and D^2 in which of two solid tori. Importantly, $(0_{\mathrm{pt}-})$ means the point (0_{pt}) now is attached with a line due to this particular way we represent the S^3 into two $D^2 \times S^1$. We see the intersection number $\#((0_{\mathrm{pt}})_{\mathrm{L}} \times S^1_{\mathrm{R}}) \cap (D^2_{\mathrm{L}} \times (0_{\mathrm{pt}-})_{\mathrm{R}})) = 1$ is right at the black dot $\bullet$.

In dd spacetime, we have the following presentation for the gluing to S^d sphere by $(D_{\mathrm{L}}^{d-1} \times S_{\mathrm{R}}^1) \cup (S_{\mathrm{L}}^{d-2} \times D_{\mathrm{R}}^2) = S^d$ or

$$(D_{\mathrm{L}}^{d-1} \times S_{\mathrm{R}}^1) \cup (S_{\mathrm{L}}^{d-2} \times D_{\mathrm{R}}^2) = S^d. \tag{6.3}$$

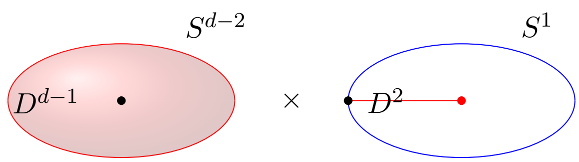
While the link configuration is



given by $\mathrm{Lk}((0_{\mathrm{pt}})_{\mathrm{L}} \times S_{\mathrm{R}}^1, S_{\mathrm{L}}^{d-2} \times (0_{\mathrm{pt}})_{\mathrm{R}})$, or

$$\mathrm{Lk}((0_{\mathrm{pt}})_{\mathrm{L}} \times S_{\mathrm{R}}^1, S_{\mathrm{L}}^{d-2} \times (0_{\mathrm{pt}})_{\mathrm{R}}) \tag{6.4}$$

with the coloring presentation explained earlier. This link invariant can be computed from the intersection number,



given by $\#((0_{\mathrm{pt}})_{\mathrm{L}} \times S_{\mathrm{R}}^1) \cap (D_{\mathrm{L}}^{d-1} \times (0_{\mathrm{pt}})_{\mathrm{R},-})) = 1$, or

$$\#((0_{\mathrm{pt}})_{\mathrm{L}} \times S_{\mathrm{R}}^1) \cap (D_{\mathrm{L}}^{d-1} \times (0_{\mathrm{pt}-})_{\mathrm{R}})) = 1. \tag{6.5}$$

with the coloring presentation explained earlier. Importantly, $(0_{\mathrm{pt}-})$ means the point (0_{pt}) now is attached with a line due to this particular way we represent the S^d. We see the intersection number $\#((0_{\mathrm{pt}})_{\mathrm{L}} \times S_{\mathrm{R}}^1) \cap (D_{\mathrm{L}}^{d-1} \times (0_{\mathrm{pt}-})_{\mathrm{R}})) = 1$. is right at the black dot $\bullet$.

6.2 The 1st Triple Link $\#(V^4_X \cap V^3_{U_{(\mathrm{i})}} \cap V^3_{U_{(\mathrm{ii})}}) \equiv \mathbf{Tlk}^{(5)}_{w_1 BB}(\Sigma^3_X, \Sigma^2_{U_{(\mathrm{i})}}, \Sigma^2_{U_{(\mathrm{ii})}})$ Configuration in 5d

We move on to discuss the triple link configuration for $\mathrm{Tlk}^{(5)}_{w_1 BB}(\Sigma^3_X, \Sigma^2_{U_{(\mathrm{i})}}, \Sigma^2_{U_{(\mathrm{ii})}})$ derived in Sec. 5.1.[22] We propose that this link invariant derived in Sec. 5.1 can detect Fig. 2.

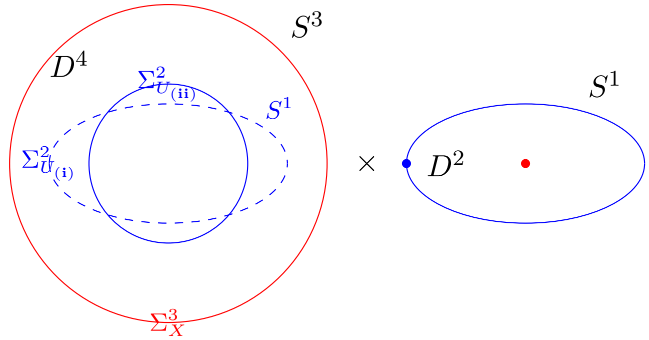


Figure 2: $S^5 = \partial D^6 = \partial(D^4 \times D^2) = S^3 \times D^2 \cup D^4 \times S^1 = S^3 \times D^2 \cup D^2 \times D^2 \times S^1$, the intersection of the two copies of $D^2 \times S^1$ in the second piece ($D^2 \times 0_{\mathrm{pt}} \times S^1$ and $0_{\mathrm{pt}} \times D^2 \times S^1$) is $0_{\mathrm{pt}} \times 0_{\mathrm{pt}} \times S^1 = 0_{\mathrm{pt}} \times S^1$, this $0_{\mathrm{pt}} \times S^1$ and $S^3 \times 0_{\mathrm{pt}}$ in the first piece are linked. In this figure, $\Sigma^3_X = S^3 \times 0_{\mathrm{pt}}$, $\Sigma^2_{U_{(\mathrm{i})}} = \partial(D^2 \times 0_{\mathrm{pt}} \times S^1)$, $\Sigma^2_{U_{(\mathrm{ii})}} = \partial(0_{\mathrm{pt}} \times D^2 \times S^1)$.

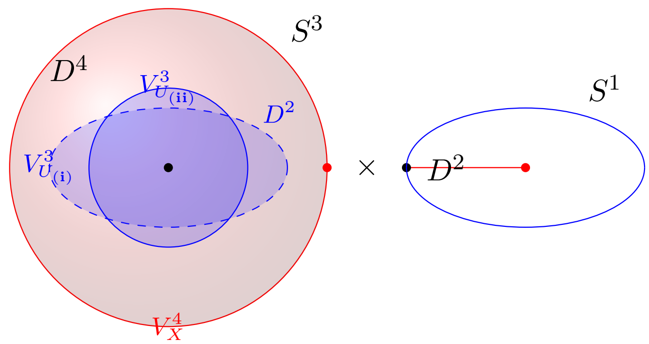


Figure 3: Following the last figure, $V^4_X = D^4 \times 0_{\mathrm{pt}}$ which bounds Σ^3_X, $V^3_{U_{(\mathrm{i})}} = D^2 \times 0_{\mathrm{pt}} \times S^1$ which bounds $\Sigma^2_{U_{(\mathrm{i})}}$, $V^3_{U_{(\mathrm{ii})}} = 0_{\mathrm{pt}} \times D^2 \times S^1$ which bounds $\Sigma^2_{U_{(\mathrm{ii})}}$. The intersection of $V^3_{U_{(\mathrm{i})}}$ and $V^3_{U_{(\mathrm{ii})}}$ is $0_{\mathrm{pt}} \times S^1$, the intersection of V^4_X and this $0_{\mathrm{pt}} \times S^1$ is a point which is the point in black in this figure.

To explain, we start by gluing into a 5-sphere via $S^5 = \partial D^6 = \partial(D^4 \times D^2) = S^3 \times D^2 \cup D^4 \times S^1 = S^3 \times D^2 \cup D^2 \times D^2 \times S^1$. We write $S^5 = (S^3_{\mathrm{L}} \times D^2_{\mathrm{R}}) \cup (D^4_{\mathrm{L}} \times S^1_{\mathrm{R}}) = (S^3_{\mathrm{L}} \times D^2_{\mathrm{R}}) \cup (D^4_{\mathrm{L}} \times S^1_{\mathrm{R}})$. We also have $S^5 = (S^3_{\mathrm{L}} \times D^2_{\mathrm{R}}) \cup (D^2_{\mathrm{L}} \times D^2_{\mathrm{L}} \times S^1_{\mathrm{R}}) = (S^3_{\mathrm{L}} \times D^2_{\mathrm{R}}) \cup (D^2_{\mathrm{L}} \times D^2_{\mathrm{L}} \times S^1_{\mathrm{R}})$.

Consider the link invariant defined by $\#(V^4_X \cap V^3_{U_{(\mathrm{i})}} \cap V^3_{U_{(\mathrm{ii})}}) \equiv \mathrm{Tlk}^{(5)}_{w_1 BB}(\Sigma^3_X, \Sigma^2_{U_{(\mathrm{i})}}, \Sigma^2_{U_{(\mathrm{ii})}})$, we see that the link configuration in Fig. 2 gives the intersection number 1 in Fig. 3. Again in the $\#(V^4_X \cap V^3_{U_{(\mathrm{i})}} \cap V^3_{U_{(\mathrm{ii})}})$ presentation in Fig. 3, $(0_{\mathrm{pt}-})$ means the point (0_{pt}) now is attached with a line due to this particular way we represent the S^d. We see the intersection number $\#(V^4_X \cap V^3_{U_{(\mathrm{i})}} \cap V^3_{U_{(\mathrm{ii})}}) = 1$ is right at the black dot $\bullet$.

[22] Effectively, $\mathrm{Tlk}^{(5)}_{w_1 BB}(\Sigma^3_X, \Sigma^2_{U_{(\mathrm{i})}}, \Sigma^2_{U_{(\mathrm{ii})}})$ can be also regarded as $\mathrm{Tlk}^{(5)}_{ABB}(\Sigma^3_X, \Sigma^2_{U_{(\mathrm{i})}}, \Sigma^2_{U_{(\mathrm{ii})}})$ where A is other $\mathbb{Z}_n$ 1-form gauge field.

6.3 The 2nd Triple Link $\#(V^4_{X_{(\mathrm{i})}} \cap V^4_{X_{(\mathrm{ii})}} \cap \Sigma^2_U) \equiv \mathbf{Tlk}^{(5)}_{w_1 w_1 \,\mathrm{d}B}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^2_U)$ Configuration in 5d

We now discuss $\mathrm{Tlk}^{(5)}_{w_1 w_1 \,\mathrm{d}B}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^2_U)$ (or $\mathrm{Tlk}^{(5)}_{AA\,\mathrm{d}B}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^2_U)$). This link invariant is derived in Sec. 5.2.2.

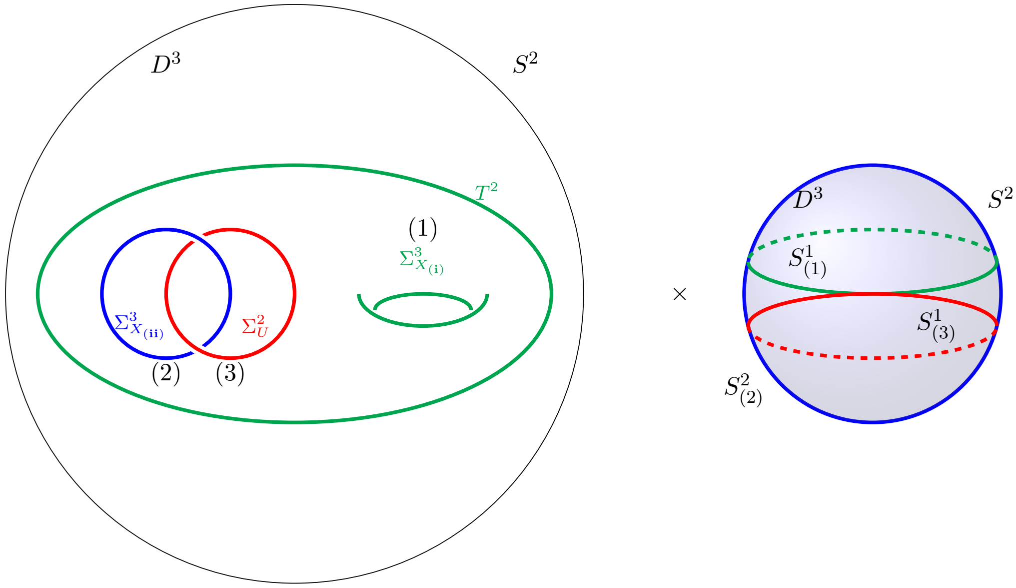


Figure 4: $S^5 = \partial D^6 = \partial(D^3 \times D^3) = S^2 \times D^3 \cup D^3 \times S^2$. Put a 2-torus (denoted by (1)) in $D^3 \times 0_{\mathrm{pt}}$, and put a Hopf link (the two circles are denoted by (2) and (3) respectively) in the solid 2-torus. Put two circles (denoted by $S^1_{(1)}$ and $S^1_{(3)}$ respectively) which intersect in only one point in $0_{\mathrm{pt}} \times S^2$ (denoted by $S^2_{(2)}$). In this figure, $\Sigma^3_{X_{(\mathrm{i})}}$ is the cartesian product of the 2-torus (1) and $S^1_{(1)}$, $\Sigma^3_{X_{(\mathrm{ii})}}$ is the cartesian product of the circle (2) and $S^2_{(2)}$, Σ^2_U is the cartesian product of the circle (3) and $S^1_{(3)}$.

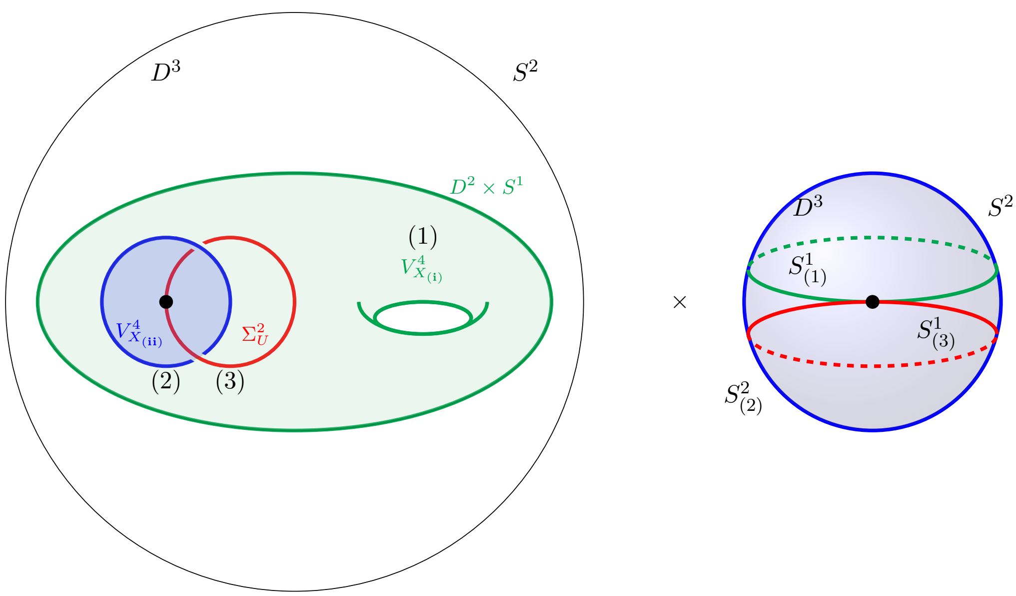


Figure 5: Following the last figure, if we fill in $\Sigma^3_{X_{(\mathrm{i})}}$ and $\Sigma^3_{X_{(\mathrm{ii})}}$, we get $V^4_{X_{(\mathrm{i})}} = D^2 \times S^1 \times S^1$ and $V^4_{X_{(\mathrm{ii})}} = D^2 \times S^2$, $V^4_{X_{(\mathrm{i})}}$, $V^4_{X_{(\mathrm{ii})}}$ and Σ^2_U will intersect in only one point which is the point in black in this figure.

Consider the link invariant defined by $\#(V^4_{X_{(\mathrm{i})}} \cap V^4_{X_{(\mathrm{ii})}} \cap \Sigma^2_U) \equiv \mathrm{Tlk}^{(5)}_{w_1 w_1 \, \mathrm{d}B}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^2_U)$, we see that the link configuration in Fig. 4 gives the intersection number 1 in Fig. 5.

6.4 Quadruple Link $\#(V^4_{X_{(\mathrm{i})}} \cap V^4_{X_{(\mathrm{ii})}} \cap V^4_{X_{(\mathrm{iii})}} \cap V^3_U) \equiv \mathbf{Qlk}^{(5)}_{w_1 w_1 w_1 B}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^3_{X_{(\mathrm{iii})}}, \Sigma^2_U)$ Configuration in 5d

We now discuss $\mathrm{Qlk}^{(5)}_{aaab}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^3_{X_{(\mathrm{iii})}})$ (or $\mathrm{Qlk}^{(5)}_{w_1 w_1 w_1 B}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^3_{X_{(\mathrm{iii})}})$).This link invariant is derived in Sec. 5.2.1.

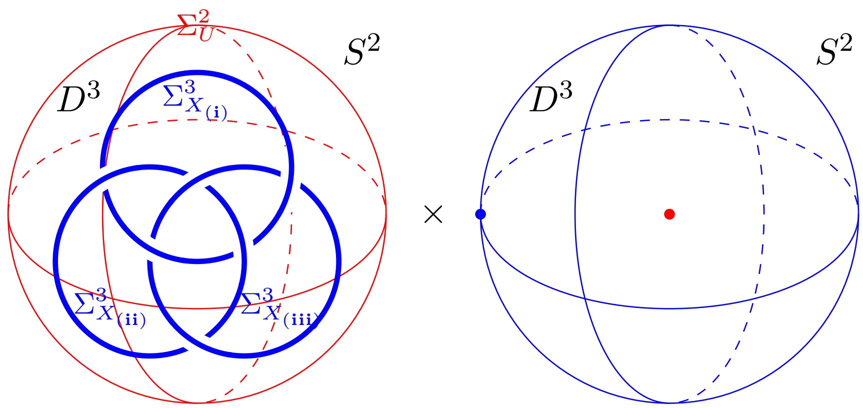


Figure 6: $S^5 = \partial D^6 = \partial(D^3 \times D^3) = S^2 \times D^3 \cup D^3 \times S^2$. Put Borromean rings in $D^3 \times 0_{\mathrm{pt}}$, If we fill in each of the three circles of the Borromean rings, then we get an intersection point, we can think of this point as 0_{pt} in D^3, then the cartesian product of each of the three circles and S^2 (denoted by $\Sigma^3_{X_{(\mathrm{i})}}$, $\Sigma^3_{X_{(\mathrm{ii})}}$ and $\Sigma^3_{X_{(\mathrm{iii})}}$ respectively) will intersect in $0_{\mathrm{pt}} \times S^2$, this $0_{\mathrm{pt}} \times S^2$ and $S^2 \times 0_{\mathrm{pt}}$ (Σ^2_U in this figure) are linked.

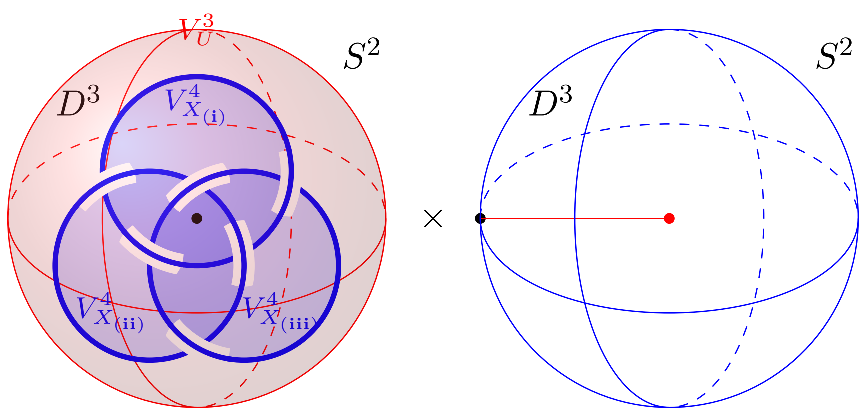


Figure 7: Following the last figure, we denote the three $D^2 \times S^2$ which bound the cartesian product of the three circles and S^2 as $V^4_{X_{(\mathrm{i})}}$, $V^4_{X_{(\mathrm{ii})}}$, $V^4_{X_{(\mathrm{iii})}}$ respectively. The intersection of $V^4_{X_{(\mathrm{i})}}$, $V^4_{X_{(\mathrm{ii})}}$ and $V^4_{X_{(\mathrm{iii})}}$ is $0_{\mathrm{pt}} \times S^2$. The intersection of $V^3_U = D^3 \times 0_{\mathrm{pt}}$ which bounds Σ^2_U and $0_{\mathrm{pt}} \times S^2$ is a point which is the point in black in this figure.

Consider the link invariant defined by $\#(V^4_{X_{(\mathrm{i})}} \cap V^4_{X_{(\mathrm{ii})}} \cap V^4_{X_{(\mathrm{iii})}} \cap V^3_U) \equiv \mathrm{Qlk}^{(5)}_{w_1 w_1 w_1 B}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^3_{X_{(\mathrm{iii})}}, \Sigma^2_U)$, we see that the link configuration in Fig. 6 gives the intersection number 1 in Fig. 7.

6.5 Quadratic Link $\#(V^3_{U_{(\mathrm{i})}} \cap \Sigma^2_{U_{(\mathrm{ii})}}) \equiv \mathbf{Lk}^{(5)}_{B\,\mathrm{d}B}(\Sigma^2_{U_{(\mathrm{i})}}, \Sigma^2_{U_{(\mathrm{ii})}})$

Now we discuss $\mathrm{Lk}^{(5)}_{B\,\mathrm{d}B}(\Sigma^2_{U_{(\mathrm{i})}}, \Sigma^2_{U_{(\mathrm{ii})}})$. This link invariant is derived in the special case $(K_1, K_2) = (1, 1)$ in Sec. 5.4.

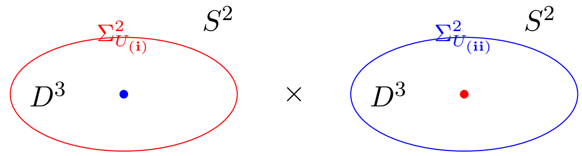


Figure 8: $S^5 = \partial D^6 = \partial(D^3 \times D^3) = S^2 \times D^3 \cup D^3 \times S^2$. The $S^2 \times 0_{\mathrm{pt}}$ in the first piece and the $0_{\mathrm{pt}} \times S^2$ in the second piece are linked. In this figure, $\Sigma^2_{U_{(\mathrm{i})}} = S^2 \times 0_{\mathrm{pt}}$, $\Sigma^2_{U_{(\mathrm{ii})}} = 0_{\mathrm{pt}} \times S^2$.

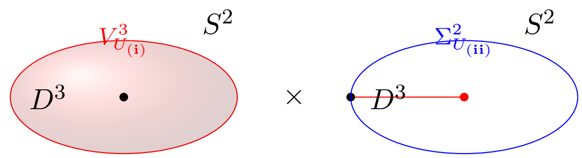


Figure 9: Following the last figure, if we fill in $S^2 \times 0_{\mathrm{pt}}$, we get $V^3_{U_{(\mathrm{i})}} = D^3 \times 0_{\mathrm{pt}}$, the intersection of $D^3 \times 0_{\mathrm{pt}}$ and $0_{\mathrm{pt}} \times S^2$ is a point which is the point in black in this figure.

Consider the link invariant defined by $\#(V^4_{X_{(\mathrm{i})}} \cap V^4_{X_{(\mathrm{ii})}} \cap V^4_{X_{(\mathrm{iii})}} \cap V^3_U) \equiv \mathrm{Qlk}^{(5)}_{w_1 w_1 w_1 B}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^3_{X_{(\mathrm{iii})}}, \Sigma^2_U)$, we see that the link configuration in Fig. 8 gives the intersection number 1 in Fig. 9.

6.6 Quadratic Link $\#(V^3_{U'} \cap \Sigma^2_U) \equiv \mathbf{Lk}^{(5)}_{w_2\,\mathrm{d}B}(\Sigma^2_U, \Sigma^2_{U'})$

Now we discuss $\mathrm{Lk}^{(5)}_{w_2\,\mathrm{d}B}(\Sigma^2_{U'}, \Sigma^2_U)$ or $\mathrm{Lk}^{(5)}_{B'\,\mathrm{d}B}(\Sigma^2_{U'}, \Sigma^2_U)$. This link invariant is derived in Sec. 5.3.

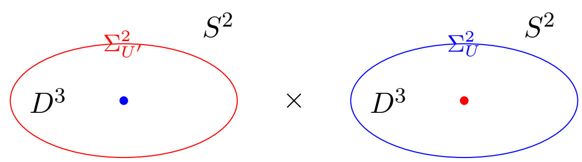


Figure 10: $S^5 = \partial D^6 = \partial(D^3 \times D^3) = S^2 \times D^3 \cup D^3 \times S^2$. The $S^2 \times 0_{\mathrm{pt}}$ in the first piece and the $0_{\mathrm{pt}} \times S^2$ in the second piece are linked. In this figure, $\Sigma^2_{U'} = S^2 \times 0_{\mathrm{pt}}$, $\Sigma^2_U = 0_{\mathrm{pt}} \times S^2$.

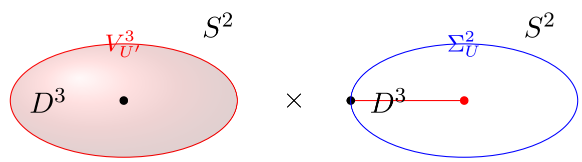


Figure 11: Following the last figure, if we fill in $S^2 \times 0_{\mathrm{pt}}$, we get $V^3_{U'} = D^3 \times 0_{\mathrm{pt}}$, the intersection of $D^3 \times 0_{\mathrm{pt}}$ and $0_{\mathrm{pt}} \times S^2$ is a point which is the point in black in this figure.

Consider the link invariant defined by $\#(V^3_{U'} \cap \Sigma^2_U) \equiv \mathrm{Lk}^{(5)}_{w_2 \mathrm{d}B}(\Sigma^2_U, \Sigma^2_{U'})$, we see that the link configuration in Fig. 10 gives the intersection number 1 in Fig. 11.

6.7 The 3rd Triple Link $\#(V^4_{X_{(\mathrm{i})}} \cap \Sigma^3_{X_{(\mathrm{ii})}} \cap V^3_U) \equiv \mathbf{Tlk}^{(5)}_{(A\mathrm{d}A)B}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^2_U)$ Configuration in 5d

Finally, we discuss a third triple link invariant $\#(V^4_{X_{(\mathrm{i})}} \cap \Sigma^3_{X_{(\mathrm{ii})}} \cap V^3_U) \equiv \mathrm{Tlk}^{(5)}_{(A\mathrm{d}A)B}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^2_U)$. We have not derived these from 4d YM-5d SET coupled systems. However, to get this, we need a topological term $(w_1(TM)\mathrm{d}w_1(TM))B$. This is possible however from $(A_I \mathrm{d}A_J)B$ type of TQFTs. We indeed can extend the dimensions to 5d from some 4d theories studied in [11] and [62].

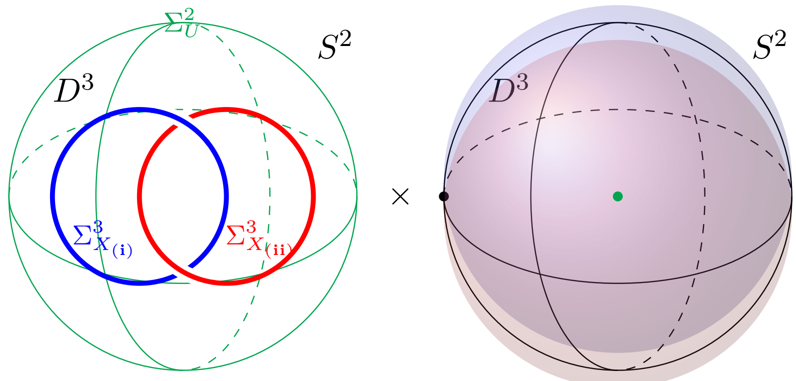


Figure 12: $S^5 = \partial D^6 = \partial(D^3 \times D^3) = S^2 \times D^3 \cup D^3 \times S^2$, put a Hopf link in $D^3 \times 0_{\mathrm{pt}}$. In this figure, $\Sigma^3_{X_{(\mathrm{i})}}$ and $\Sigma^3_{X_{(\mathrm{i})}}$ are the cartesian product of the two circles in the Hopf link and S^2 respectively, namely, they are both $S^1 \times S^2$, $\Sigma^2_U = S^2 \times 0_{\mathrm{pt}}$.

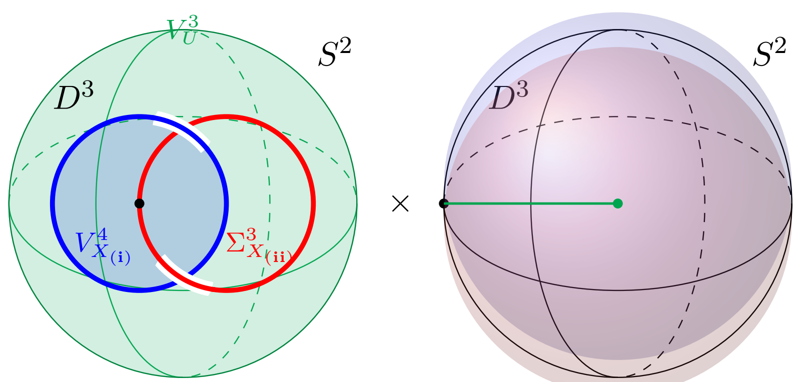


Figure 13: Following the last figure, if we fill in $\Sigma^3_{X_{(\mathrm{i})}}$, we get $V^4_{X_{(\mathrm{i})}} = D^2 \times S^2$, the intersection of $V^4_{X_{(\mathrm{i})}}$ and $\Sigma^3_{X_{(\mathrm{ii})}}$ is the cartesian product of a point (we can think of the point as 0_{pt}) and S^2. If we fill in Σ^2_U further, we get $V^3_U = D^3 \times 0_{\mathrm{pt}}$, the intersection of $D^3 \times 0_{\mathrm{pt}}$ and $0_{\mathrm{pt}} \times S^2$ is a point which is the point in black in this figure.

Consider the link invariant defined by $\#(V^4_{X_{(\mathrm{i})}} \cap \Sigma^3_{X_{(\mathrm{ii})}} \cap V^3_U) \equiv \mathrm{Tlk}^{(5)}_{(A\mathrm{d}A)B}(\Sigma^3_{X_{(\mathrm{i})}}, \Sigma^3_{X_{(\mathrm{ii})}}, \Sigma^2_U)$, we see that the link configuration in Fig. 12 gives the intersection number 1 in Fig. 13.

7 4d $\mathrm{SO}(3)_{\theta=\pi}$ Yang-Mills Gauge Theories coupled to the Boundary of 5d SETs/Long-Range Entangled TQFTs

In Sec. 2, we have shown that that the SU(2) Yang-Mills theory with $\theta = \pi$, with the gauge bundle constraint $w_2(V_{\mathrm{PSU}(2)}) = B + K_1 w_1(TM)^2 + K_2 w_2(TM)$, has four distinct t' Hooft anomalies Eq. (2.11). In this section, we further comment on gauging the 1-form $\mathbb{Z}_{2,[1]}^e$ center symmetry of the four siblings of $\mathrm{SU}(2)_{\theta=\pi}$ YM to obtain $\mathrm{SO}(3)_{\theta=\pi}$ YM theories. Since the 't Hooft anomalies involve the 1-form center symmetry and the spacetime symmetries (whose background is the Stiefel-Whitney classes $w_i(TM)$), depending on which manifold we formulate the SU(2) Yang Mills, one obtains different theories.

7.1 From SU(2) to SO(3) Gauge Theory

To illustrate, we start with gauging the 1-form symmetry [6, 63] of the time reversal symmetric and anomaly free $\mathrm{SU}(2)_{\theta=0}$ YM theories. There are still four choices of gauge bundle constraints labeled by (K_1, K_2), i.e. Eq. (3.2) except the 2-form $\mathbb{Z}_2$ gauge field is promoted to a dynamical field. Denoting $\mathbf{Z}^{4\mathrm{d}}_{\mathrm{SU(2)YM}}[B]$ as the path integral without specifying the gauge bundle constraint, the partition function with the gauge bundle constraint $w_2(E) = (B + K_1 w_1(TM)^2 + K_2 w_2(TM)) \mod 2$ is

$$\mathbf{Z}^{4\mathrm{d}}_{\mathrm{SU(2)YM}_{(K_1,K_2)}}[B] \equiv \int [D\Lambda]\ \mathbf{Z}^{4\mathrm{d}}_{\mathrm{SU(2)YM}}[B] \exp\left(\mathrm{i}\pi\Lambda \cup (w_2(E) - \Big(B + K_1 w_1(TM)^2 + K_2 w_2(TM)\Big))\right),$$

More generally, we can add Pontryagin square term $\frac{p\pi}{2}\mathcal{P}(B)$ labeled by an integer p, and define a partition function:

$$\mathbf{Z}^{4\mathrm{d}}_{\mathrm{SU(2)YM}_{(K_1,K_2)}}[B] \equiv \int [D\Lambda]\ \mathbf{Z}^{4\mathrm{d}}_{\mathrm{SU(2)YM}}[B] \exp\left(\mathrm{i}\pi(\Lambda \cup (w_2(E) - \Big(B + K_1 w_1(TM)^2 + K_2 w_2(TM)\Big)) + \frac{p}{2}\mathcal{P}(B))\right), \tag{7.1}$$

Below we like to obtain SO(3) YM by gauging 1-form $\mathbb{Z}_{2,[1]}^e$ center symmetry. The theta angle of the resulting theory is $2\pi p$. If $w_2(TM)$ is nontrivial, the resulting SO(3) theory is time reversal symmetric only when $p \in 2\mathbb{Z}$ and $p \sim p+4$. When $w_2(TM)$ is trivial, the resulting SO(3) theory is time reversal symmetric for $p \in \mathbb{Z}$ and $p \sim p+2$. In the following, we always restrict to the time reversal symmetric case. Gauging 1-form center symmetry amounts to summing over the background gauge field B (promoting B to a dynamical gauge field),

$$\mathbf{Z}^{4\mathrm{d}}_{\mathrm{SO(3)YM}_{(K_1,K_2)}} = \int [D\Lambda][DB]\ \mathbf{Z}^{4\mathrm{d}}_{\mathrm{SU(2)YM}}[B] \exp\left(\mathrm{i}\pi(\Lambda \cup (w_2(E) - \Big(B + K_1 w_1(TM)^2 + K_2 w_2(TM)\Big)) + \frac{p}{2}\mathcal{P}(B))\right), \tag{7.2}$$

By integrating out Λ enforces the relation between SO(3)-gauge bundles/connections and 2-form dynamical gauge field B. This outputs the SO(3)-gauge theory $\mathbf{Z}^{4\mathrm{d}}_{\mathrm{SO(3)YM}_{(K_1,K_2)}}$ with $\theta = 2\pi p$.

7.2 Comment on Gauging 1-form $\mathbb{Z}_{2,[1]}^e$ of SU(2) Gauge Theory with $\theta = \pi$

We proceed to discuss gauging the 1-form symmetry of SU(2) Yang Mills with $\theta = \pi$.

If one formulates the SU(2) Yang Mills on an orientable and spin manifold, i.e., $w_1 = w_2 = 0$ (hence $w_3 = 0$ as well), then one has the freedom to ignore the time reversal as a symmetry of the theory. The only symmetry of interest is the 1-form symmetry, which does not have anomaly with itself. Hence one can gauge the 1-form symmetry and the resulting theory is $\mathrm{PSU}(2) = \mathrm{SO}(3)$ Yang-Mills with $\theta = \pi$. Indeed, SO(3) Yang-Mills with $\theta = \pi$ does not respect time reversal, which maps $\theta = \pi$ to $\theta = 3\pi$ due to the identification $\theta \sim \theta + 4\pi$.

If one formulates the SU(2) Yang Mills on an orientable and non-spin manifold, one still has the freedom to ignore the time reversal as a symmetry of the theory. However, in this case, there is still nontrivial anomaly

$$\int_{M^5} K_2 \pi w_2(TM) \cup \mathrm{Sq}^1 B = \int_{M^5} K_2 \pi w_3(TM) \cup B \tag{7.3}$$

which does not vanish on an orientable manifold if $K_2 = 1$. Denoting the partition function of the $\mathrm{SU}(2)_{\theta=\pi}$ Yang-Mills coupled to B as $\mathbf{Z}_{\mathrm{SU}(2)\mathrm{YM}_{(0,K_2)}}[M^4, B]$, after promoting B to a dynamical field, the partition function of the entire 4d-5d system is

$$\sum_B \mathbf{Z}_{\mathrm{SU}(2)\mathrm{YM}_{(0,K_2)}}[M^4, B] \exp\left(\mathrm{i}\pi \int_{M^5} K_2 w_3(TM) \cup B\right). \tag{7.4}$$

If $K_2 = 0$, the 4d-5d system reduces to a intrinsic 4d system. Physically, this corresponds to the case where the gauge charge is a boson. It makes sense to gauge the 1-form symmetry which again gives raise to the SO(3) Yang-Mills theory. If $K_2 = 1$, only the entire 4d-5d system is well defined, and it does not make sense to discuss the 4d theory alone, in contrast with the case where $w_1 = w_2 = 0$. Physically, this corresponds to the case where the gauge charge is a fermion.

If one formulates the SU(2) Yang Mills on an unorientable manifold, time reversal symmetry is bulit in and too late to give it up. Promoting B to a dynamical gauge field, the partition function for the total system is

$$\sum_B \mathbf{Z}_{\mathrm{SU}(2)\mathrm{YM}_{(K_1,K_2)}}[M^4, B] \exp\left[\mathrm{i}\pi \int_{M^5} \left(B\mathrm{Sq}^1 B + \mathrm{Sq}^2\mathrm{Sq}^1 B + K_1 w_1(TM)^3 \cup B + K_2 w_3(TM) \cup B\right)\right]. \tag{7.5}$$

Since M^5 is unorientable, for all four choices of (K_1, K_2), the 5d terms do not vanish (because $\pi B\mathrm{Sq}^1 B + \pi \mathrm{Sq}^2\mathrm{Sq}^1 B$ is always non-vanishing on unorientable manifold). Hence one can only discuss the 4d-5d system rather than discussing the 4d system alone. We summarize all the above scenarios in Table 10.

$(w_1, w_2)\backslash(K_1, K_2)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	✓	✓	✓	✓
$(1,0)$	$\times$	$\times$	$\times$	$\times$
$(0,1)$	✓	✓	$\times$	$\times$
$(1,1)$	$\times$	$\times$	$\times$	$\times$

Table 10: Possibilities of gauging the $\mathrm{SU}(2)_{\theta=\pi}$ Yang-Mills theory with gauge bundle constraint (K_1, K_2) on a manifold with Stiefel-Whitney class (w_1, w_2). The ✓ means that there is a way to make sense of the resulting gauged theory as a purely 4d theory. The theories labeled by $\times$ means that it only makes sense to discuss the combined 4d-5d systems.

8 Lattice Regularization and UV completion

In this section, we formulate the background-probed-field 5d partition function of the higher SPT $\mathbf{Z}^{5\text{d}}_{\text{SPT}_{(K_1=0,K_2=0)}}[M^5;B]$ of a background 2-cochain B field on a simplicial complex spacetime. This provides a lattice regularization of the 5d SPT. We also provide lattice realization of (i) 4d higher-symmetry-extended boundary theory or (ii) 4d higher-symmetry-enriched anomalous topologically ordered boundary theory. We will generalize the approach in [52] and follow the Section IX of [64]. In condensed matter physics, this (ii) phenomenon is known as the *anomalous surface topological order* firstly noticed in [65] (typically the 2+1D boundary of 3+1D SPTs, see a review [28]).

8.1 Lattice Realization of 4d Higher-SPTs and Higher-Gauge TQFT: 4d Simplicial Complex and 3+1D Condensed Matter Realization

We warm up by considering a lattice realization of 4d Higher-SPTs given by

$$\mathbf{Z}^{4\text{d}}_{\text{SPT}}[M^4;B] = \exp\left(\mathrm{i}\frac{\pi}{2}\int_{M^4}\mathcal{P}(B)\right) = \exp\left(\mathrm{i}\frac{\pi}{2}\int_{M^4} B\cup B + B\underset{1}{\cup}\delta B\right). \tag{8.1}$$

The path integral can be regularized on a triangulated 4-manifold M^4. The building blocks of M^4 are 4-simplices. Without loss of generality, we consider an arbitrary 4-simplical which we denote as (01234) where each number labels one vertex. See Fig. 14 for a graphical representation of a 4-simplex. We denote B_{ijk} as restricting the 2-cochain B on the 2-simplex (ijk). We label the path integral amplitude on (01234) as $\omega_4(01234)$, i.e.,

$$\begin{aligned}\omega_4(01234) &= \exp\left[\mathrm{i}\pi\left(\frac{1}{2}B\cup B + \frac{1}{2}B\underset{1}{\cup}\delta B\right)_{01234}\right]\\ &= \exp\left[\mathrm{i}\frac{\pi}{2}\left(B_{012}B_{234} + B_{034}(B_{123}-B_{023}+B_{013}-B_{012}) + B_{014}(B_{234}-B_{134}+B_{124}-B_{123})\right)\right].\end{aligned} \tag{8.2}$$

It is straightforward to verify that $\omega_4(01234)$ satisfies the cocycle condition:

$$(\delta\omega_4)(012345) = \frac{\omega_4(12345)\cdot\omega_4(01345)\cdot\omega_4(01235)}{\omega_4(02345)\cdot\omega_4(01245)\cdot\omega_4(01234)} = 1. \tag{8.3}$$

8.2 Lattice Realization of 5d Higher-SPTs and Higher-Gauge SETs: 5d Simplicial Complex and 4+1D Condensed Matter Realization

The 5d partition function with $(K_1=0,K_2=0)$ is

$$\mathbf{Z}^{5\text{d}}_{\text{SPT}_{(K_1=0,K_2=0)}}[M^5] = \exp\left(\mathrm{i}\pi\int_{M^5} B\mathrm{Sq}^1B + \mathrm{Sq}^2\mathrm{Sq}^1B\right). \tag{8.4}$$

We start by triangulating the 5d closed spacetime manifold (without boundary) into 5-simplicial complex. We denote B_{ijk} as restricting the 2-cochain B on the 2-simplex (ijk). Using the identities

$$\begin{aligned}\mathrm{Sq}^1B &= B\underset{1}{\cup}B = \frac{1}{2}\delta B,\\ \mathrm{Sq}^2\mathrm{Sq}^1B &= (\mathrm{Sq}^1B)\underset{1}{\cup}(\mathrm{Sq}^1B) = \frac{1}{4}(\delta B)\cup(\delta B).\end{aligned} \tag{8.5}$$

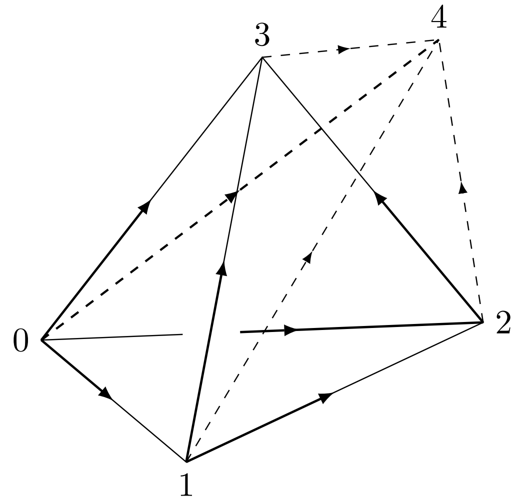


Figure 14: Graphical representation of a 4-simplex (01234).

Note that the second equality in both lines hold only when B is a cocycle, i.e., $\delta B = 0$. Since B is the classical background gauge field of $\mathbb{Z}_2$ discrete symmetry, B should obey the flat condition, hence for simplicity, one can use all the equalities in Eq. (8.5). One can express the SPT action Eq. (8.4) in terms of the sum of cup-products of B cochains over 5-simplices

$$\mathbf{Z}^{5\mathrm{d}}_{\mathrm{SPT}_{(K_1=0,K_2=0)}}[M^5] = \exp\left(\mathrm{i}\frac{\pi}{2}\sum_{M^5} B \cup \delta B + \mathrm{i}\frac{\pi}{4}\sum_{M^5} \delta B \underset{1}{\cup} \delta B\right). \tag{8.6}$$

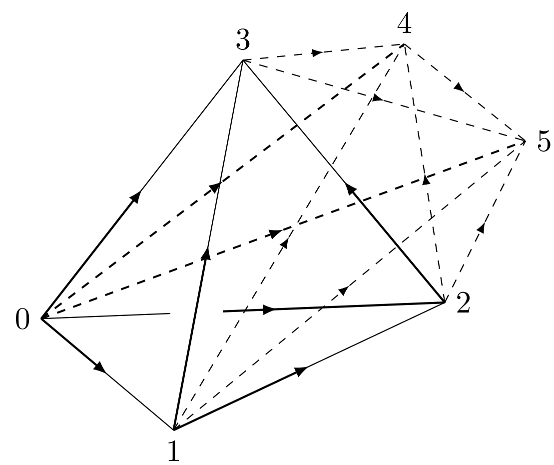


Figure 15: Graphical representation of a 5-simplex (012345).

Without loss of generality, we consider an arbitrary 5-simplex which we denote as (012345) where each number labels one vertex. See Fig. 15 for a graphical representation of a 5-simplex. We will label the path integral amplitude on the simplex (012345) as $\omega_5(012345)$, i.e.,

$$\omega_5(012345) = \exp\left[\mathrm{i}\pi\left(\frac{1}{2}B \cup \delta B + \frac{1}{4}\delta B \underset{1}{\cup} \delta B\right)_{012345}\right] \tag{8.7}$$

so that the partition function can be simplified as $\mathbf{Z}^{\text{5d}}_{\text{SPT}_{(K_1=0,K_2=0)}}[M^5] = \prod_{(ijklmn)\in M^5} \omega(ijklmn)$. Using the definition of the cup products on simplices and the identities Eq. (8.5), we have

$$\begin{aligned}
\big(\mathrm{Sq}^1 B\big)_{0123} &= \frac{1}{2}\Big(B_{123} - B_{023} + B_{013} - B_{012}\Big) \\
\big(\mathrm{Sq}^2\mathrm{Sq}^1 B\big)_{012345} &= \frac{1}{4}\Big((\delta B)_{0345}(\delta B)_{0123} + (\delta B)_{0145}(\delta B)_{1234} + (\delta B)_{0125}(\delta B)_{2345}\Big), \\
&= \frac{1}{4}\Big((-B_{045} - B_{034} + B_{035} + B_{345})\,(-B_{023} - B_{012} + B_{013} + B_{123}) \\
&\quad + (-B_{045} - B_{014} + B_{015} + B_{145})\,(-B_{134} - B_{123} + B_{124} + B_{234}) \\
&\quad + (-B_{025} - B_{012} + B_{015} + B_{125})\,(-B_{245} - B_{234} + B_{235} + B_{345}) \Big).
\end{aligned} \tag{8.8}$$

Hence the path integral amplitude on the simplex (012345) is

$$\begin{aligned}
\omega_5(012345) = \exp\Big[&\frac{\mathrm{i}\pi}{2} B_{012}(-B_{245} - B_{234} + B_{235} + B_{345}) \\
&+ \frac{\mathrm{i}\pi}{4}(-B_{045} - B_{034} + B_{035} + B_{345})\,(-B_{023} - B_{012} + B_{013} + B_{123}) \\
&+ \frac{\mathrm{i}\pi}{4}(-B_{045} - B_{014} + B_{015} + B_{145})\,(-B_{134} - B_{123} + B_{124} + B_{234}) \\
&+ \frac{\mathrm{i}\pi}{4}(-B_{025} - B_{012} + B_{015} + B_{125})\,(-B_{245} - B_{234} + B_{235} + B_{345}) \Big].
\end{aligned} \tag{8.9}$$

It is straightforward to verify that $\omega_5(012345)$ satisfies the cocycle condition:

$$(\delta\omega_5)(0123456) = \frac{\omega_5(123456)\cdot\omega_5(013456)\cdot\omega_5(012356)\cdot\omega_5(012345)}{\omega_5(023456)\cdot\omega_5(012456)\cdot\omega_5(012346)} = 1. \tag{8.10}$$

We emphasize that $\omega(012345)$ is a cocycle only when B is a cocycle, i.e., $\delta B = 0$. If B is a cochain rather than a cocycle, Eq. (8.4) is not a cocycle, hence can not be a partition function of a topological field theory. [23]

We further comment on the lattice regularization of theory with various choices of (K_1, K_2).

1. When $(K_1, K_2) = (0, 0)$, as we derived above, there is a lattice regularization of the 5d SPT partition function.
2. When $(K_1, K_2) = (1, 0)$, the path integral amplitude depends on the first Stiefel-Whitney class $w_1(TM)$. Using the method of [27], one can write down the simplicial form of $w_1(TM)^2$ using the twisted cocycle, with the coefficient in $U(1)_{\mathcal{T}}$ due to anti-unitary symmetry nature of time-reversal (in the Hamiltonian formalism of [27]). We will not write down the explicit expression for the cocycle.
3. When $(K_1, K_2) = (0, 1)$, we can use the expression

$$\mathbf{Z}^{\text{5d}}_{\text{SPT}_{(K_1=0,K_2=1)}}[M^5] = \exp\left(\mathrm{i}\pi \int_{M^5} B\mathrm{Sq}^1 B + (w_1(TM)^2)\mathrm{Sq}^1 B\right). \tag{8.11}$$

   We will see that both $B\mathrm{Sq}^1 B$ can be written on the lattice (see the next $(K_1 = 1, K_2 = 1)$), while the $(w_1(TM)^2)\mathrm{Sq}^1 B$ can be written on the lattice using the method of [27].

[23] The cocycle condition is crucial in proving the partition function to be invariant under re-triangulating the spacetime manifold M^5.

4. When $(K_1, K_2) = (1,1)$, there is also a lattice regularization of the 5d SPT partition function. To see this, we rewrite the partition function,

$$\begin{aligned} \mathbf{Z}^{\text{5d}}_{\text{SPT}_{(K_1=1,K_2=1)}}[M^5] &= \exp\left(\mathrm{i}\pi \int_{M^5} B\mathrm{Sq}^1 B + \mathrm{Sq}^2\mathrm{Sq}^1 B + (w_1(TM)^2 + w_2(TM))\mathrm{Sq}^1 B\right) \\ &= \exp\left(\mathrm{i}\pi \int_{M^5} B\mathrm{Sq}^1 B\right). \end{aligned} \tag{8.12}$$

where we have used $\mathrm{Sq}^2\mathrm{Sq}^1 B + (w_1(TM)^2 + w_2(TM))\mathrm{Sq}^1 B = 0$. Using Eq. (8.5), we obtain a cochain expression

$$\begin{aligned} \mathbf{Z}^{\text{5d}}_{\text{SPT}_{(K_1=1,K_2=1)}}[M^5] &= \prod_{(ijklmn)\in M^5} \omega_5(ijklmn), \\ \omega_5(ijklmn) &= \exp\left[\frac{i\pi}{2} B_{ijk}(-B_{kmn} - B_{klm} + B_{lmn} + B_{kln})\right]. \end{aligned} \tag{8.13}$$

Other than the probed field partition function $\mathbf{Z}^{\text{5d}}_{\text{SPT}_{(K_1,K_2)}}[M^5]$, we can also sum over B to get the the topologically ordered 5d SET $\mathbf{Z}^{\text{5d}}_{\text{SET}_{(K_1,K_2)}}[M^5]$.

Given that the 5d SPT and 5d SET path integral can be regularized on a lattice, following [27], one can write down the quantum wavefunction via the spacetime path integral. It is also possible to construct a lattice quantum Hamiltonian on the 4D space (on a constant time slice), for both SPTs and SETs, similar to the formulations of [27, 66–68]. For the topologically ordered 5d SET, we implement the method of [67, 68]:

$$\hat{\mathbf{H}} = -\sum_{\text{1-link } \ell} \hat{\mathbf{A}}_\ell - \sum_{\text{3-simplex}} \hat{\mathbf{B}}_{\text{3-simplex}} \tag{8.14}$$

where $\hat{\mathbf{A}}_\ell$ is an operator acting on the plaquettes (2-simplex) adjacent to the 1-link ℓ, and $\hat{\mathbf{B}}_{\text{3-simplex}}$ is an operator acting on the boundary of a given 3-simplex which again are plaquettes (2-simplex). The $\hat{\mathbf{A}}_\ell$ has its effect on imposing the time evolution constraint as the same as the path integral formulation: $\hat{\mathbf{A}}_\ell$ lifting the state vector to a next time slice locally around the 1-link ℓ. The $\hat{\mathbf{B}}_{\text{3-simplex}}$ imposes the zero flux condition enclosed by the 3-simplex (which is a 2-sphere S^2 in topology). We will not give the explicit expression of the quantum Hamiltonian $\hat{\mathbf{H}}$ in this paper.

8.3 Lattice Regularization of Higher-Symmetry-Extended and Higher-Symmetry-Preserving Anomalous 3+1D Topologically Ordered Gapped Boundaries

One option to saturate the anomaly inflow from the bulk 5d (4+1D) SPT is to extend the global symmetry on the 4d (3+1D) boundary. We consider the four siblings of 5d higher-SPTs labeled by (K_1, K_2), whose partition functions are

$$\mathbf{Z}^{\text{5d}}_{\text{SPT}_{(K_1,K_2)}}[M^5] = \exp\left[\mathrm{i}\pi \int_{M^5} \Big(B + (1+K_1)w_1(TM)^2 + (1+K_2)w_2(TM)\Big) \cup \mathrm{Sq}^1 B\right]. \tag{8.15}$$

Using the schematic way in [52], we find that the boundary of 5d SPT can support a 4d TQFT via symmetry extension from $\mathbb{Z}_2$ to $\mathbb{Z}_4$. Schematically, let $\omega_5^{(K_1,K_2)}$ be the 5-cocycle whose product over the 5d manifold M^5 gives the 5d SPT partition function Eq. (8.15). Let $\beta_4^{(K_1,K_2)}$ be a 4-cochain which trivializes the 5d cocycle, i.e.,

$$\omega_5^{(K_1,K_2)} = \delta\beta_4^{(K_1,K_2)}. \tag{8.16}$$

We find that the following $\beta_4^{(K_1,K_2)}$ satisfies Eq. (8.16):

$$\beta_4^{(K_1,K_2)} = \exp\left[\mathrm{i}\pi\int_{M^4}(B + (1+K_1)w_1(TM)^2 + (1+K_2)w_2(TM))\cup\gamma(C)\right]. \tag{8.17}$$

where C is a $\mathbb{Z}_4$ valued 2-cochain satisfying $B = C \mod 2$, and $\gamma : \mathbb{Z}_4 \to \mathbb{Z}_2$ is a function which maps the $\mathbb{Z}_4$ 2-cochain to a $\mathbb{Z}_2$ 2-cochain:

$$(\gamma(C))_{ijk} = \frac{(C_{ijk})^2 - C_{ijk}}{2}. \tag{8.18}$$

We comment on the lattice realization of the boundary partition function $\beta_4^{(K_1,K_2)}$:

1. When $(K_1, K_2) = (1, 1)$, the 4d partition function has a simple cocycle form, hence it shows that the partition function can be regularized on the lattice

$$\beta_4^{(1,1)} = \exp\left[\mathrm{i}\pi\int_{M^4} B\cup\gamma(C)\right] = \prod_{(ijklm)\in M^4}\exp\left[\mathrm{i}\pi B_{ijk}(\gamma(C))_{klm}\right]. \tag{8.19}$$

2. When $(K_1, K_2) = (1, 0)$, $\beta_4^{(K_1,K_2)}$ depends explicitly on $w_2(TM)$. One can use the identity $\pi w_2(TM)\cup\gamma(C) = \pi\mathcal{P}(\gamma(C)) \mod 2\pi$ to rewrite the path integral amplitude as

$$\beta_4^{(1,0)} = \exp\left[\mathrm{i}\pi\int_{M^4} B\cup\gamma(C) + \mathcal{P}(\gamma(C))\right] = \prod_{(ijklm)\in M^4}\left(\omega_4^{(1,0)}\right)_{ijklm} \tag{8.20}$$

where

$$\begin{aligned}&\left(\omega_4^{(1,0)}\right)_{01234} = \\ &\exp\Big[\mathrm{i}\pi\Big(B_{012}(\gamma(C))_{234} + (\gamma(C))_{034}((\gamma(C))_{123} - (\gamma(C))_{023} + (\gamma(C))_{013} - (\gamma(C))_{012}) \\ &\qquad + (\gamma(C))_{014}((\gamma(C))_{234} - (\gamma(C))_{134} + (\gamma(C))_{124} - (\gamma(C))_{123})\Big)\Big].\end{aligned} \tag{8.21}$$

3. When $(K_1, K_2) = (0, 1)$ and $(0, 0)$, $\beta_4^{(K_1,K_2)}$ explicitly depends on $w_1(TM)$. Following [27], it is possible to write down a cocycle expression of the path integral amplitude via the time reversal twisted cochains. We will not write it down in the present paper.

In summary, we find that for all choices of (K_1, K_2), there exist lattice realizations of the symmetry extended theory on the boundary of higher SPTs Eq. (2.19).

9 Conclusions and Discussions

1. *Summary*: In this work, we show and prove (physically from quantum field theory) that new higher 't Hooft anomalies, given by a 5d topological term Eq. (2.11) and Eq. (2.19):

$$\pi \int_{M^5} \left(B \cup \mathrm{Sq}^1 B + \mathrm{Sq}^2 \mathrm{Sq}^1 B + K_1 w_1(TM)^3 \cup B + K_2 w_3(TM) \cup B \right),$$

   of 4d time-reversal symmetric pure YM of an SU(2) gauge group with a second-Chern-class topological term at $\theta = \pi$ (i.e., $\mathrm{SU}(2)_{\theta=\pi}$ YM). We find that there are at least four siblings of $\mathrm{SU}(2)_{\theta=\pi}$ YM with bosonic UV completion, labeled by $(K_1, K_2) \in (\mathbb{Z}_2, \mathbb{Z}_2)$. Their higher 't Hooft anomalies of generalized global symmetries indicate that 4d $\mathrm{SU}(2)_{\theta=\pi}$ YM, in order to realize all global symmetries locally, necessarily couple to 5d higher symmetry-protected topological states (SPTs, as invertible TQFTs [iTQFTs], as 5d 1-form-center-symmetry-protected interacting "topological superconductors" in condensed matter).

   We explore various 4d Yang-Mills gauge theories (YM) living as boundary conditions of 5d gapped short/long-range entangled (SRE/LRE) topological states. We revisit 4d $\mathrm{SU}(2)_{\theta=\pi}$ YM-5d SRE-higher-SPTs coupled systems [5, 8] and find these "Fantastic Four Siblings" with four sets of new higher anomalies Eq. (2.19). Follow Weyl's gauge principle, by dynamically gauging the 1-form center symmetry, we transform a 5d bulk SRE SPTs into an LRE symmetry-enriched topologically ordered state (SETs); thus we obtain the 4d $\mathrm{SO}(3)_{\theta=\pi}$ YM-5d LRE-higher-SETs coupled system with dynamical higher-form gauge fields. We illustrate such 4d-5d systems schematically in Fig. 1 and Fig. 16.

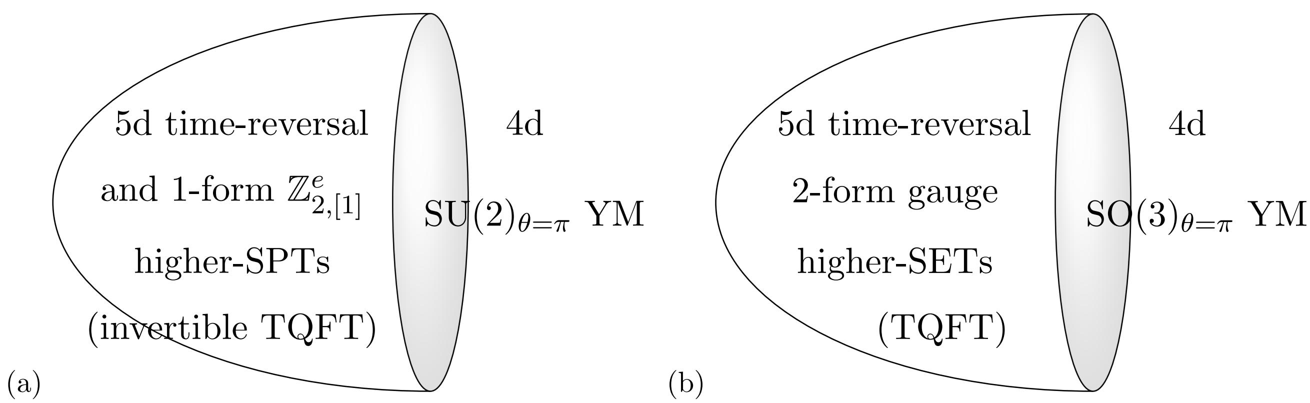


Figure 16: An alternative illustration of Fig. 1.

   The 4d SO(3) YM has a θ periodicity $\theta \sim \theta + 4\pi$ on a spin manifold, and $\theta \sim \theta + 8\pi$ on a non-spin manifold. Since time-reversal symmetry is preserved if and only if $\theta \to -\theta$ is identified, thus SO(3) YM has explicitly broken the time-reversal symmetry. In the right-hand side (b) of Fig. 1 and Fig. 16, we actually have a 5d SETs whose 4d boundary has an explicitly time-reversal symmetry breaking.

   Apply the tool introduced in [11], we derive new exotic anyonic statistics of extended objects such as 2-worldsheet of strings and 3-worldvolume of branes, which physically characterize the 5d SETs. We discover new triple and quadruple link invariants associated with the underlying 5d higher-gauge TQFT, hinting a new intrinsic relation between non-supersymmetric 4d pure YM and topological links in 5d.

2. *Appearances of mod 2 anomalies*: We note that the anomaly associated to the 5d term $\exp(\mathrm{i}\pi \int w_3(TM)B)$ has also appeared in the context of an adjoint QCD_4 theory [64,69,70]. The $\exp(\mathrm{i}\pi \int w_2(TM)w_3(TM))$

has also appeared as a new SU(2) anomaly in the SU(2) gauge theory [54]. All these anomalies and all our anomalies in Eq. (2.19) are mod 2 non-perturbative global anomalies, like the SU(2) anomalies [53, 54].

3. *Mathematical relation between 5d and 4d bordism groups*: Mathematically there seems to be an amusing relation between (1) gauging the SU(2) gauge bundle/connection under the coupling of 4d YM with 4d SPTs (4d bordism invariants of $\Omega_4^{G'}$) with G' derived from a group extension:

$$1 \to \mathrm{SU}(2) \to G' \to \mathrm{O}(d) \to 1.$$

and (2) some of the 5d bordism invariants given by $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2) = \mathbb{Z}_2^4$. It will be illuminating to explore this relation in the future.

4. *Classes of 4d SU(2)$_{\theta=\pi}$ YM*: In Ref. [36], it was noted that the Pin$^+$ and Pin$^-$ version of the above group extensions $G' = \mathrm{Pin}^+ \times_{\mathbb{Z}_2} \mathrm{SU}(2)$ and $G' = \mathrm{Pin}^- \times_{\mathbb{Z}_2} \mathrm{SU}(2)$ provide two different SPTs vacua after dynamically gauging the SU(2) symmetry give rise to two distinct 4d SU(2)$_{\theta=\pi}$ YM theories. Although Ref. [36] suggested that the Pin$^+$ and Pin$^-$ of 4d SU(2)$_{\theta=\pi}$ YM are secretly indistinguishable by correlators of local operators on orientable spacetimes nor by gapped SPT states, can be distinguished on non-orientable spacetimes or potentially by correlators of extended operators.

   In this work, we haven shown that Pin$^+$ and Pin$^-$ of 4d SU(2)$_{\theta=\pi}$ YM indeed have distinct new higher 't Hooft anomalies, given by Eq. (2.11) and Eq. (2.19), with $(K_1, K_2) = (0, 1)$ and $(K_1, K_2) = (1, 1)$ respectively. Thus we confirm that Pin$^+$ and Pin$^-$ of 4d SU(2)$_{\theta=\pi}$ YM are indeed distinct vacua.

5. *Quantum spin liquids in condensed matter*: Strong coupled gauge theories have condensed matter implications as quantum spin liquids. Time-reversal symmetric U(1) gauge theories as quantum spin liquids [28] are explored and classified based on the quantum numbers of gapped electric and magnetic excitations (Wilson and 't Hooft line operators) in Ref. [56, 57]. We will leave the interpretation of our results of non-abelian SU(2) gauge theories in the context of quantum spin liquids for a future work.

6. *Relations of link invariants and braiding statistics in various dimensions*: We have applied the tools developed in [11] to compute link invariants of 5d TQFTs. We remark that several link invariants that we find here in 5d have dimensionally reduction analogy to 4d and 3d, such that the "dimensional reduced" links in 4d and 3d are related to what had been studied in [10], [11] and References therein.

7. *Fate of IR dynamics of gauge theories, UV completion and lattice regularizations*: For the 4d-5d systems that we explore (schematically in Fig. 1 and Fig. 16), we mainly focus on their "Fantastic Four Siblings" as the UV theories. We do not yet know the IR fate of their dynamics of these strongly coupled gauge theories. However, given the potentially complete 't Hooft anomalies in Eq. (2.11) and Eq. (2.19) (at zero temperature), we can constrain the IR dynamics by UV-IR anomaly matching. The consequence of anomaly matching implies that the IR theories must be at least one of the following:

   - Time-reversal $\mathbb{Z}_2^T$ symmetry broken (spontaneously or explicitly).
   - 1-form center $\mathbb{Z}_{2,[1]}^e$ symmetry broken (spontaneously or explicitly).
   - Full symmetry-preserving anomalous TQFT. (Or a symmetry-extended anomaly-free TQFT discussed in [52], but in a more artificial setup).
   - Full symmetry-preserving gapless theory (CFT).

   In fact, in Sec. 8, we construct the 4d boundary based of the third type above as a boundary TQFT with a lattice spacetime path integral or a lattice Hamiltonian regularization; in this case, the full spacetime partition function $\mathbf{Z}[M]$ of 4d-5d system can be explicitly computed as a number (by following Sec. 9 of [52]). We will revisit the issue of dynamics in the future.

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