

Quantum Statistics and Spacetime Topology: Quantum Surgery Formulas

Juven Wang^{1,2},  Xiao-Gang Wen³,  Shing-Tung Yau^{4,2,5} 

¹*School of Natural Sciences, Einstein Drive, Institute for Advanced Study, Princeton, NJ 08540, USA*

²*Center of Mathematical Sciences and Applications, Harvard University, Cambridge, MA 02138, USA*

³*Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*


⁴*Department of Mathematics, Harvard University, Cambridge, MA 02138, USA*


⁵*Department of Physics, Harvard University, Cambridge, MA 02138, USA*


Abstract

To formulate the universal constraints of quantum statistics data of generic long-range entangled quantum systems, we introduce the geometric-topology surgery theory on spacetime manifolds where quantum systems reside, cutting and gluing the associated quantum amplitudes, specifically in 2+1 and 3+1 spacetime dimensions. First, we introduce the fusion data for worldline and worldsheet operators capable of creating anyonic excitations of particles and strings, well-defined in gapped states of matter with intrinsic topological orders. Second, we introduce the braiding statistics data of particles and strings, such as the geometric Berry matrices for particle-string Aharonov-Bohm, 3-string, 4-string, or multi-string adiabatic loop braiding process, encoded by submanifold linkings, in the closed spacetime 3-manifolds and 4-manifolds. Third, we derive new “quantum surgery” formulas and constraints, analogous to Verlinde formula associating fusion and braiding statistics data via spacetime surgery, essential for defining the theory of topological orders, 3d and 4d TQFTs and potentially correlated to bootstrap boundary physics such as gapless modes, extended defects, 2d and 3d conformal field theories or quantum anomalies.

This article is meant to be an extended and further detailed elaboration of our previous work [1] and Chapter 6 of Ref. [2]. Our theory applies to general quantum theories and quantum mechanical systems, also applicable to, but not necessarily requiring the quantum field theory description.

 e-mail: juven@ias.edu

 e-mail: xgwen@mit.edu

 e-mail: yau@math.harvard.edu

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1 Introduction

Geometry and topology have long perceived to be intricately related to our understanding of the physics world. Our physical Universe is known to be quantum in nature. For a system with many-body quantum degrees of freedom, we have a well-motivated purpose of determining the governing laws of quantum statistics. Quantum statistics, to some extent, behave as hidden long-range “forces” or “interactions” between the quantum quasi-excitations. One of the goals of our present work is formulating the *quantum version* of constraints from geometric topology and surgery properties of spacetimes (as manifolds, either smooth differentiable, or triangulable on a lattice) in order to develop theoretical equations governing the quantum statistics of general quantum theories.

The fractional quantum Hall effect was discovered decades ago [3]. A quantum theory of the wavefunction of fractional quantum Hall effect is formulated subsequently [4]. The intrinsic relation between the topological quantum field theories (TQFT) and the topology of manifolds was found years after [5, 6]. These breakthroughs partially motivated the study of topological order [7] as a new state of matter in quantum many-body systems and in condensed matter systems [8]. Topological orders are defined as the gapped states of matter with physical properties depending on global topology (such as the ground state degeneracy (GSD)), robust against any local perturbation and any symmetry-breaking perturbation. Accordingly, topological orders cannot be characterized by the old paradigm of symmetry-breaking phases of matter via the Ginzburg-Landau theory [9, 10]. The systematic studies of 2+1 dimensional spacetime¹ (2+1D) topological orders enhance our understanding of the real-world plethora phases including quantum Hall states and spin liquids [11]. In this work, we explore the constraints between the 2+1D and 3+1D topological orders and the geometric-topology properties of 3- and 4-manifolds. We only focus on 2+1D / 3+1D topological orders with GSD insensitive to the system size and with a finite number of types of topological excitations creatable from 1D line and 2D surface operators. Specifically, the open ends of 1D line operators give rise to quasi-excitations of anyons. The open ends of 2D surface operators give rise to quasi-excitations of anyonic strings. Conversely, the worldlines of anyons become 1D line operators (see Fig.1), while the worldsheets of anyonic strings become 2D surface operators.

In this work, we mainly apply the tools of quantum mechanics in physics and surgery theory in mathematics [12, 13]. Our main results are: (1) We provide the *fusion data* for worldline and worldsheet operators creating excitations of particles (i.e. anyons [14]) and strings (i.e. anyonic strings) in topological orders. (2) We provide the *braiding statistics data* of particles and strings encoded by submanifold linking, in the 3- and 4-dimensional closed spacetime manifolds. (3) By “cutting and gluing” (or “cutting and sewing” in synonym) quantum amplitudes, we derive constraints between the fusion and braiding statistics data analogous to Verlinde formula [6, 15, 16] for 2+1 and 3+1D

¹For abbreviation, we write $n + 1D$ for an $n+1$ -dimensional spacetime. We write md for an m -dimensional spacetime. We write nD simply for the n -dimensional manifold or nD space.

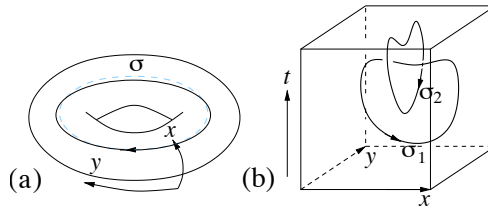


Figure 1: (a) A topologically-ordered ground state on a spatial 2-torus T_{xy}^2 is labeled by a quasiparticle σ . (b) The quantum amplitude of two linked spacetime trajectories of anyons σ_1 and σ_2 in 2+1D is proportional to a complex number $S_{\sigma_1\sigma_2}$, which is related to the modular $SL(2, \mathbb{Z})$ data to be introduced later.

topological orders.

2 Quantum Statistics: Fusion and Braiding Statistics Data

Imagine a renormalization-group-fixed-point topologically ordered quantum system on a spacetime manifold \mathcal{M} . The manifold can be viewed as a long-wavelength continuous limit of certain lattice regularization of the system.² We aim to compute the quantum amplitude from “gluing” one ket-state $|R\rangle$ with another bra-state $\langle L|$, such as $\langle L|R\rangle$. A quantum amplitude also defines a path integral or a partition function Z with the linking of worldlines/worldsheets on a d -manifold \mathcal{M}^d , read as

$$\langle L|R\rangle = Z(\mathcal{M}^d; \text{Link}[\text{worldline}, \text{worldsheet}, \dots]). \quad (2.1)$$

For example, the $|R\rangle$ state can represent a ground state of 2-torus T_{xy}^2 if we put the system on a solid torus $D_{xt}^2 \times S_y^1$ ³ (see Fig. 1(a) as the product space of 2-dimensional disk D^2 and 1-dimensional circle S^1 , where the footnote subindices label the coordinates). Note that its boundary is $\partial(D^2 \times S^1) = T^2$, and we can view the time t evolving along the radial direction. We label the trivial vacuum sector without any operator insertions as $|0_{D_{xt}^2 \times S_y^1}\rangle$, which is trivial respect to the measurement of any contractible line operator along S_x^1 . A worldline operator creates a pair of anyon and anti-anyon at its end points, if it forms a closed loop then it can be viewed as creating then annihilating a pair of anyons in a closed trajectory. Our worldline or worldsheet operator corresponds to the Wilson, ’t Hooft, or other extended operator of the gauge theory, although throughout our work, we consider a more generic quantum description without limiting to gauge theory or quantum field theory (QFT).

Inserting a line operator $W_\sigma^{S_y^1}$ in the interior of $D_{xt}^2 \times S_y^1$ gives a new state

$$W_\sigma^{S_y^1} |0_{D_{xt}^2 \times S_y^1}\rangle \equiv |\sigma_{D_{xt}^2 \times S_y^1}\rangle. \quad (2.2)$$

Here σ denotes the anyon type⁴ along the oriented line, see Fig. 1.

²Mathematically, it is proven that all smooth thus differentiable manifolds are always triangulable. This can be proven via Morse theory, which implies that we only need to show the triangulation of piecewise-linear (PL) handle-attachments. However, the converse statement may not be true in general. Thus here we focus on going from the smooth thus differentiable manifolds (the continuum) to the triangulable manifolds (the discrete).

³This is consistent with the fact that TQFT assigns a complex number to a closed manifold without boundary, while assigns a state-vector in the Hilbert space to an open manifold with a boundary.

⁴If there is a gauge theory description, then the quasi-excitation type of particle σ and string μ would be labeled by the representation for gauge charge and by the conjugacy class of the gauge group for gauge flux.

Insert all possible line operators of all σ can completely span the ground state sectors for 2+1D topological order. The gluing of $\langle 0_{D^2 \times S^1} | 0_{D^2 \times S^1} \rangle$ computes the path integral $Z(S^2 \times S^1)$. If we view the S^1 as a compact time, this counts the ground state degeneracy (GSD, i.e., the number of ground states, or equivalently the dimensions of ground-state Hilbert space $\dim(\mathcal{H})$) on a 2D spatial sphere S^2 without quasiparticle insertions. Follow the relation based on (2.1) and the above, we see that for a generic spatial manifold M_{space} , we have a relation:

$$\text{GSD}_{M_{\text{space}}} = \dim(\mathcal{H}) = |Z(M_{\text{space}} \times S^1)|. \quad (2.3)$$

For a partition function without any insertion (i.e., anyonic excitations associated extended operators), the following computes a 1-dimensional Hilbert space on a spatial S^2 in 2+1D:

$$\langle 0_{D^2 \times S^1} | 0_{D^2 \times S^1} \rangle = Z(S^2 \times S^1) = 1. \quad (2.4)$$

Similar relations hold for other dimensions, e.g. 3+1D topological orders on a S^3 without quasi-excitation yields

$$\langle 0_{D^3 \times S^1} | 0_{D^3 \times S^1} \rangle = Z(S^3 \times S^1) = 1. \quad (2.5)$$

2.1 Elementary Geometric Topology, Notations and Surgery Formulas

Below we list down some useful and elementary surgery formulas, or other math formulas, in order to prepare for the later derivation of more involved but physically more interesting surgery process.

Some of the above results are well-known [12, 13], while others are less familiar, and perhaps also novel to the literature.

1. For example, we consider the connected sum of two d -dimensional manifolds M_1 and M_2 , denoted as

$$M_1 \# M_2 \quad (2.6)$$

which becomes a new manifold formed by deleting a ball D^d inside each manifold and gluing together the resulting boundary spheres S^{d-1} .

2. We write the partition function on spacetime that is formed by disconnected manifolds M and N , which is denoted as

$$M \sqcup N, \quad (2.7)$$

obeying:

$$Z(M \sqcup N) = Z(M)Z(N). \quad (2.8)$$

Manifolds:

3-manifolds without boundaries:

$$S^3, S^2 \times S^1, (S^1)^3 = T^3, \text{ etc.}$$

3-manifolds with boundaries:

$$D^3, D^2 \times S^1, \text{ etc.}$$

4-manifolds without boundaries:

$$S^4, S^3 \times S^1, S^2 \times S^2, S^2 \times (S^1)^2 = S^2 \times T^2, (S^1)^4 = T^4, S^3 \times S^1 \# S^2 \times S^2, S^3 \times S^1 \# S^2 \times S^2 \# S^2 \times S^2.$$

4-manifolds with boundaries:

$$D^4, D^3 \times S^1, D^2 \times S^2, D^2 \times (S^1)^2 = D^2 \times T^2 \equiv C^4, S^4 \setminus D^2 \times T^2.$$

Certain 2-manifolds as the boundaries of 3-manifolds:

$$S^2, (S^1)^2 = T^2, \text{ etc.}$$

Certain 3-manifolds as the boundaries of 4-manifolds:

$$S^3, S^2 \times S^1, (S^1)^3 = T^3, \text{ etc.}$$

Surgery:

Cutting and gluing 4-manifolds:

$$S^4 = (D^3 \times S^1) \cup_{S^2 \times S^1} (S^2 \times D^2) = (D^2 \times T^2) \cup_{T^3} (S^4 \setminus D^2 \times T^2) = D^4 \cup D^4.$$

$$S^3 \times S^1 = (D^3 \times S^1) \cup_{S^2 \times S^1} (D^3 \times S^1) = (D^2 \times S^1 \times S^1) \cup_{T^3} (S^1 \times D^2 \times S^1) = (D^2 \times T^2) \cup_{T^3; S^{xyz}} (D^2 \times T^2).$$

$$S^2 \times S^2 = (D^2 \times S^2) \cup_{S^2 \times S^1} (D^2 \times S^2) = (S^4 \setminus D^2 \times T^2) \cup_{T^3; S^{xyz}} (S^4 \setminus D^2 \times T^2).$$

$$S^2 \times S^1 \times S^1 = (D^2 \times T^2) \cup_{T^3} (D^2 \times T^2).$$

$$S^3 \times S^1 \# S^2 \times S^2 = (S^4 \setminus D^2 \times T^2) \cup_{T^3; S^{xyz}} (D^2 \times T^2).$$

$$S^3 \times S^1 \# S^2 \times S^2 \# S^2 \times S^2 = (S^4 \setminus D^2 \times T^2) \cup_{T^3} (S^4 \setminus D^2 \times T^2).$$

Cutting and gluing 3-manifolds:

$$S^3 = (D^2 \times S^1) \cup_{T^2}; (S^1 \times D^2) = (D^2 \times S^1) \cup_{T^2; S^{xy}} (D^2 \times S^1) = D^3 \cup D^3.$$

$$S^2 \times S^1 = (D^2 \times S^1) \cup_{T^2} (D^2 \times S^1) = (D^2 \times S^1) \cup_{S^1 \times S^1; (T_{xy})^n} (D^2 \times S^1).$$

Mapping Class Group (MCG):

$$\text{MCG}(T^d) = \text{SL}(d, \mathbb{Z}). \quad \text{MCG}(S^2 \times S^1) = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Table 1: Manifolds, surgery formula and mapping class group (MCG) that are considered in the limited case of our study. The S^n is an n -dimensional sphere, the D^n is an n -dimensional disk, and the T^n is an n -dimensional torus. The connected sum of two d -dimensional manifolds $M_1 \# M_2$ follows the definition in Eq. (2.6). The gluing of two manifolds $M_1 \cup_{B; \varphi} M_2$ along their boundary B via the map φ follows Eq. (2.11). The complement space notation $M_1 \setminus M_2$ follows Eq. (2.12). The mapping class group $\text{MCG}(T^d) = \text{SL}(d, \mathbb{Z})$ is a special linear group with the matrix representation of integer \mathbb{Z} entries. The mapping class group $\text{MCG}(S^2 \times S^1) = \mathbb{Z}_2 \times \mathbb{Z}_2$ have two generators: A generator for the first \mathbb{Z}_2 is given by a homeomorphism that is a reflection of each of S^2 and S^1 separately, so it preserves the orientation of $S^2 \times S^1$. A generator for the second \mathbb{Z}_2 has the form $f(x, y) = (g_y(x), y)$ where g_y is the rotation of S^2 along a fixed axis by an angle that varies from 0 to 2π as y goes once around S^1 . If we do not restrict our attention to homeomorphisms that preserve orientation, then this “extended mapping class group (extended-MCG)” has twice elements whenever the manifold has an orientation-reversing homeomorphism, as in the examples we considered above. Thus for S^1, S^2 , and S^3 , the extended-MCG has \mathbb{Z}_2 , while we have the extended-MCG($S^2 \times S^1$) = $(\mathbb{Z}_2)^3$.

- For a closed manifold M glued by two pieces of d -dimensional manifolds M_U and M_D , so that we denote the gluing as

$$M = M_U \cup_B M_D \tag{2.9}$$

where the M_U and M_D share a common boundary $(d-1)$ -dimensional manifold

$$B = \partial M_U = \overline{\partial M_D}. \quad (2.10)$$

Note that ∂M_D and $\overline{\partial M_D}$ are differed by a sign of the orientation. If M_U and M_D are oriented, then the $M = M_U \cup_B M_D$ can inherit an induced natural orientation, obeying the chosen orientation of M_U and M_D . This requires an identification of $\partial M_U \simeq \overline{\partial M_D}$ by reversing the inherited orientation, as an homeomorphism.⁵

4. Moreover, we can have an extra mapping φ allowed by diffeomorphism when gluing two manifolds. The notation for gluing the boundaries via the φ is written as

$$M_U \cup_{B;\varphi} M_D. \quad (2.11)$$

It requires that the boundary to be the same, $\partial M_U = \overline{\partial M_D} = B$. In particular, we will focus on a φ of mapping class group (MCG) of the boundary B in our work. Thus, we can apply any element of $\varphi \in \text{MCG}(B)$. For $\varphi = 1$ as a trivial identity map, we can simply denote it as $\mathcal{M}_1 \cup_B \mathcal{M}_2 = \mathcal{M}_1 \cup_{B,I} \mathcal{M}_2$.

5. We denote the complement space of d -dimensional \mathcal{M}_2 out of \mathcal{M}_1 as

$$\mathcal{M}_1 \setminus \mathcal{M}_2. \quad (2.12)$$

This means we cut out \mathcal{M}_2 out of \mathcal{M}_1 . For example, to understand the connected sum $M_1 \# M_2$, we can cut a ball D^d (D for the disk D^d , which is the same a d -dimensional ball) out of the M_1 and M_2 . Each of $M_1 \setminus D^d$ and $M_2 \setminus D^d$ has a boundary of a sphere S^{d-1} . We glue the two manifolds M_1 and M_2 by a cylinder $S^{d-1} \times I^1$ where the $I^1 \equiv I$ is a 1 dimensional interval.

Throughout our article, we consistently use σ to represent the quasi-excitations of anyonic particle label (such as charge, electric charge, magnetic monopole, or representation of the gauge group) whose spacetime trajectory become 1-dimensional worldlines $W_\sigma^{S^1}$ of 1-circle. We use μ to represent the quasi-excitations of anyonic string (or loop, or gauge flux) label whose spacetime trajectory become 2-dimensional worldsheets, e.g. $V_\mu^{S^2}$ and $V_\mu^{T^2}$, for the surface of 2-sphere and 2-torus.

We can also derive some helpful homology group formulas, via Alexander duality and other relations:⁶

$$H_1(D^3 \times S^1, \mathbb{Z}) = H_2(S^4 \setminus D^3 \times S^1, \mathbb{Z}) = H_2(D^2 \times S^2, \mathbb{Z}) = \mathbb{Z}, \quad (2.16)$$

⁵Since we only focus on orientable manifolds in this work, there is no subtle problem. In this work, we will not be particularly interested in the details of orientation, so let us limit to the case that we do not emphasize $\overline{\partial M_D}$ or ∂M_D . In future work, we will consider the generalization to the case for non-orientable manifolds.

⁶To understand the following equations, suppose that X is a d -dimensional manifold with boundary (∂X) , there is a long exact sequence in homology:

$$\dots H_k(\partial X) \rightarrow H_k(X) \rightarrow H_k(X, \partial X) \rightarrow H_{k-1}(\partial X) \dots$$

The first and the left most arrow is an inclusion. The middle arrow is modding out by the image of the inclusion. The last arrow is the connecting homomorphism. Note that an element in $H_k(X, \partial X)$ is represented by a singular chain whose boundary is in ∂X . The right most arrow assigns to such a chain its boundary.

There is a dual exact sequence for cohomology:

$$\dots H^{m-1}(\partial X) \rightarrow H^m(X, \partial X) \rightarrow H^m(X) \rightarrow H^m(\partial X) \dots$$

Note that $H^m(X, \partial X)$ is represented by cocycles that are zero on all chains in (∂X) . The right most arrow here becomes the connecting homomorphism. The other two arrows are simply an algebraic inclusion (the middle one) or a pull-back.

$$H_2(D^3 \times S^1, \mathbb{Z}) = H_1(S^4 \setminus D^3 \times S^1, \mathbb{Z}) = H_1(D^2 \times S^2, \mathbb{Z}) = 0. \quad (2.17)$$

$$H_1(D^2 \times T^2, \mathbb{Z}) = H_2(S^4 \setminus D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}^2, \quad (2.18)$$

$$H_2(D^2 \times T^2, \mathbb{Z}) = H_1(S^4 \setminus D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}. \quad (2.19)$$

Follow the above definitions, we obtain a set of useful formulas, and we summarize in Table 1.⁷

2.2 Data in 2+1D

2.2.1 Quantum Fusion Data in 2+1D

In 2+1D, we consider the worldline operators creating particles. We define the *fusion data* via fusing worldline operators:

$$W_{\sigma_1}^{S_y^1} W_{\sigma_2}^{S_y^1} = \mathcal{F}_{\sigma_1 \sigma_2}^\sigma W_\sigma^{S_y^1}, \quad \text{and } G_\sigma^\alpha \equiv \langle \alpha | \sigma_{D_{xt}^2 \times S_y^1} \rangle. \quad (2.20)$$

Here G_σ^α is read from the projection to a complete basis $\langle \alpha |$. Indeed the $W_\sigma^{S_y^1}$ generates all the canonical bases from $|0_{D_{xt}^2 \times S_y^1}\rangle$. Thus the canonical projection can be

$$\langle \alpha | = \langle 0_{D_{xt}^2 \times S_y^1} | (W_{\alpha}^{S_y^1})^\dagger = \langle 0_{D_{xt}^2 \times S_y^1} | (W_{\bar{\alpha}}^{S_y^1}) = \langle \alpha_{D_{xt}^2 \times S_y^1} |, \quad (2.21)$$

then we have

$$G_\sigma^\alpha = \langle \alpha | W_\sigma^{S_y^1} | 0_{D_{xt}^2 \times S_y^1} \rangle = \langle 0_{D_{xt}^2 \times S_y^1} | (W_{\bar{\alpha}}^{S_y^1}) W_\sigma^{S_y^1} | 0_{D_{xt}^2 \times S_y^1} \rangle = Z(S^2 \times S^1; \bar{\alpha}, \sigma) = \delta_{\alpha\sigma}, \quad (2.22)$$

where a pair of particle-antiparticle σ and $\bar{\sigma}$ can fuse to the vacuum. We derive

$$\begin{aligned} \mathcal{F}_{\sigma_1 \sigma_2}^\alpha &= \langle \alpha | W_{\sigma_1}^{S_y^1} W_{\sigma_2}^{S_y^1} | 0_{D_{xt}^2 \times S_y^1} \rangle \\ &= \langle 0_{D_{xt}^2 \times S_y^1} | (W_{\bar{\alpha}}^{S_y^1}) W_{\sigma_1}^{S_y^1} W_{\sigma_2}^{S_y^1} | 0_{D_{xt}^2 \times S_y^1} \rangle \\ &= Z(S^2 \times S^1; \bar{\alpha}, \sigma_1, \sigma_2) \equiv \mathcal{N}_{\sigma_1 \sigma_2}^\alpha, \end{aligned} \quad (2.23)$$

where this path integral counts the dimension of the Hilbert space (namely the GSD or the number of channels σ_1 and σ_2 can fuse to α) on the spatial S^2 . This shows the fusion data $\mathcal{F}_{\sigma_1 \sigma_2}^\alpha$ is equivalent to the fusion rule $\mathcal{N}_{\sigma_1 \sigma_2}^\alpha$, symmetric under exchanging σ_1 and σ_2 .⁸

Duality imposes that

$$H^{d-k-1}(\partial X) = H_k(\partial X), \quad (2.13)$$

$$H^{d-k}(X, \partial X) = H_k(X), \quad (2.14)$$

$$H^{d-k}(X) = H_k(X, \partial X). \quad (2.15)$$

If we use this result and Alexander duality, we can deduce the equations: Eq. (2.16), Eq. (2.17), Eq. (2.18), and Eq. (2.19).

⁷We thank conversations with Clifford Taubes and correspondences with Robert Gompf clarifying some of these formulas. For other details, please see also our upcoming work [17].

⁸For readers who require some more background knowledge about the fusion algebra of anyons, Ref. [18–22] provide for an introduction and a review of 2+1D case. Our present work will also study the 3+1D case.

2.2.2 Quantum Braiding Data in 2+1D

More generally we can glue the T_{xy}^2 -boundary of $D_{xt}^2 \times S_y^1$ via its mapping class group (MCG), namely $\text{MCG}(T^2) = \text{SL}(2, \mathbb{Z})$ of the special linear group, (see Table 1) generated by

$$\hat{\mathcal{S}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\mathcal{T}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (2.24)$$

The $\hat{\mathcal{S}}$ identifies $(x, y) \rightarrow (-y, x)$, while $\hat{\mathcal{T}}$ identifies $(x, y) \rightarrow (x + y, y)$ of T_{xy}^2 .⁹ Based on Eq.(2.1), we write down the quantum amplitudes of the two $\text{SL}(2, \mathbb{Z})$ generators $\hat{\mathcal{S}}$ and $\hat{\mathcal{T}}$ projecting to degenerate ground states. We denote gluing two open-manifolds \mathcal{M}_1 and \mathcal{M}_2 along their boundaries \mathcal{B} under the MCG-transformation $\hat{\mathcal{U}}$ to a new manifold as $\mathcal{M}_1 \cup_{\mathcal{B}; \hat{\mathcal{U}}} \mathcal{M}_2$.¹⁰ Below we introduce three sets of quantum braiding statistics data in 2+1D,

1. Modular $\text{SL}(2, \mathbb{Z})$ modular data \mathcal{S} : It is amusing to visualize the gluing

$$D^2 \times S^1 \cup_{T^2; \hat{\mathcal{S}}} D^2 \times S^1 = S^3 \quad (2.25)$$

shows that the $\mathcal{S}_{\bar{\sigma}_1 \sigma_2}$ represents the Hopf link of two S^1 worldlines σ_1 and σ_2 (e.g. Fig.1(b)) in S^3 with the given orientation (in the canonical basis $\mathcal{S}_{\bar{\sigma}_1 \sigma_2} = \langle \sigma_1 | \hat{\mathcal{S}} | \sigma_2 \rangle$):

$$\mathcal{S}_{\bar{\sigma}_1 \sigma_2} \equiv \langle \sigma_1_{D_{xt}^2 \times S_y^1} | \hat{\mathcal{S}} | \sigma_2_{D_{xt}^2 \times S_y^1} \rangle = Z \left(\begin{array}{c} \text{Hopf link of } S^1 \text{ worldlines} \\ \text{in } S^3 \end{array} \right). \quad (2.26)$$

We note that this data \mathcal{S} has also been introduced in the work of Witten on TQFT via surgery theory [6]. However, here we actually consider more generic quantum mechanical system (a quantum theory, or a quantum many body theory) that does not necessarily requires a QFT description. We *only* require the gluing of quantum amplitudes written in the quantum mechanical bra and ket bases living in a Hilbert space associated to the quantum theory on spacetime (sub)manifolds.

2. Modular $\text{SL}(2, \mathbb{Z})$ modular data \mathcal{T} : Use the gluing

$$D^2 \times S^1 \cup_{T^2; \hat{\mathcal{T}}} D^2 \times S^1 = S^2 \times S^1, \quad (2.27)$$

we can derive a well known result written in the canonical bases,

$$\mathcal{T}_{\sigma_1 \sigma_2} \equiv \langle \sigma_1_{D_{xt}^2 \times S_y^1} | \hat{\mathcal{T}} | \sigma_2_{D_{xt}^2 \times S_y^1} \rangle = \delta_{\sigma_1 \sigma_2} e^{i\theta_{\sigma_2}}. \quad (2.28)$$

Its spacetime configuration is that two unlinked closed worldlines σ_1 and σ_2 , with the worldline σ_2 twisted by 2π . The amplitude of a twisted worldline is given by the amplitude of untwisted worldline multiplied by $e^{i\theta_{\sigma_2}}$, where $\theta_{\sigma}/2\pi$ is the spin of the σ excitation.

In summary, the above $\text{SL}(2, \mathbb{Z})$ modular data implies that $\mathcal{S}_{\bar{\sigma}_1 \sigma_2}$ measures the *mutual braiding statistics* of σ_1 -and- σ_2 , while $\mathcal{T}_{\sigma\sigma}$ measures the *spin* and *self-statistics* of σ .

⁹We may also denote the $\hat{\mathcal{S}}$ as $\hat{\mathcal{S}}^{xy}$ and the $\hat{\mathcal{T}}$ as $\hat{\mathcal{T}}^{xy}$.

¹⁰We may simplify the gluing notation $\mathcal{M}_1 \cup_{\mathcal{B}; \hat{\mathcal{U}}} \mathcal{M}_2$ to $\mathcal{M}_1 \cup_{\mathcal{B}} \mathcal{M}_2$ if the mapping class group's generator \mathcal{U} is trivial or does not affect the glued manifold. We may simplify the gluing notation further to $\mathcal{M}_1 \cup \mathcal{M}_2$ if the boundary $\mathcal{B} = \partial\mathcal{M}_1 = \partial\mathcal{M}_2$ is obvious or stated in the text earlier.

3. We can introduce additional data, the Borromean rings (BR) linking between three S^1 circles in S^3 , written as a path integral Z data with insertions. We denote this path integral Z data as

$$Z \left(\begin{array}{c} \text{1} \\ \text{---} \\ \text{S}^1 \\ \text{---} \\ \text{S}^1 \text{---} \text{S}^1 \\ \text{---} \\ \text{2} \quad \quad \quad \text{3} \end{array} \right) \equiv Z[S^3; \text{BR}[\sigma_1, \sigma_2, \sigma_3]]. \quad (2.29)$$

Although we do not know a bra-ket expression for this amplitude, we can reduce this configuration to an easier one $Z[T_{xyt}^3; \sigma'_{1x}, \sigma'_{2y}, \sigma'_{3t}]$, a path integral of 3-torus T^3 with three orthogonal line operators each inserting along a non-contractible S^1 direction. The later is a simpler expression because we can uniquely define the three line insertions exactly along the homology group generators of T^3 , namely $H_1(T^3, \mathbb{Z}) = \mathbb{Z}^3$. The two path integrals are related by three consecutive modular $\text{SL}(2, \mathbb{Z})$'s \mathcal{S} surgeries (or \mathcal{S}^{xy} surgeries) done along the T^2 -boundary of $D^2 \times S^1$ tubular neighborhood around three S^1 rings. The three-step surgeries we describe earlier sequently send the initial 3-sphere configuration with Borromean rings insertion

$$S^3 \xrightarrow{\text{1st surgery}} S^2 \times S^1 \xrightarrow{\text{2nd surgery}} S^2 \times S^1 \# S^2 \times S^1 \xrightarrow{\text{3rd surgery}} T^3 \quad (2.30)$$

to a 3-torus configuration. Here we use the notation $\mathcal{M}_1 \# \mathcal{M}_2$ means the connected sum of manifolds \mathcal{M}_1 and \mathcal{M}_2 . Namely,

$$Z[T_{xyt}^3; \sigma'_{1x}, \sigma'_{2y}, \sigma'_{3t}] = \sum_{\sigma_1, \sigma_2, \sigma_3} \mathcal{S}_{\sigma'_{1x}\sigma_1} \mathcal{S}_{\sigma'_{2y}\sigma_2} \mathcal{S}_{\sigma'_{3z}\sigma_3} Z[S^3; \text{BR}[\sigma_1, \sigma_2, \sigma_3]]. \quad (2.31)$$

2.3 Data in 3+1D

In 3+1D, there are intrinsic meanings of braidings of string-like excitations. We need to consider both the worldline and the worldsheet operators which create particles and strings. In addition to the S^1 -worldline operator $W_\sigma^{S^1}$, we introduce S^2 - and T^2 -worldsheet operators as $V_\mu^{S^2}$ and $V_{\mu'}^{T^2}$ which create closed-strings (or loops) at their spatial cross sections. We consider the vacuum sector ground state on open 4-manifolds:

$$|0_{D^3 \times S^1}\rangle, |0_{D^2 \times S^2}\rangle, |0_{D^2 \times T^2}\rangle \text{ and } |0_{S^4 \setminus D^2 \times T^2}\rangle, \quad (2.32)$$

while their boundaries are

$$\partial(D^3 \times S^1) = \partial(D^2 \times S^2) = S^2 \times S^1 \text{ and } \partial(D^2 \times T^2) = \partial(S^4 \setminus D^2 \times T^2) = T^3. \quad (2.33)$$

Here $\mathcal{M}_1 \setminus \mathcal{M}_2$ means the complement space of \mathcal{M}_2 out of \mathcal{M}_1 .

2.3.1 Quantum Fusion Data in 3+1D

Similar to 2+1D, we define the *fusion data* \mathcal{F}^M by fusing operators:

$$W_{\sigma_1}^{S^1} W_{\sigma_2}^{S^1} = (\mathcal{F}^{S^1})_{\sigma_1 \sigma_2}^\sigma W_\sigma^{S^1}, \quad (2.34)$$

$$V_{\mu_1}^{S^2} V_{\mu_2}^{S^2} = (\mathcal{F}^{S^2})_{\mu_1 \mu_2}^{\mu_3} V_{\mu_3}^{S^2}, \quad (2.35)$$

$$V_{\mu_1}^{T^2} V_{\mu_2}^{T^2} = (\mathcal{F}^{T^2})_{\mu_1 \mu_2}^{\mu_3} V_{\mu_3}^{T^2}. \quad (2.36)$$

Notice that we introduce additional upper indices in the fusion algebra \mathcal{F}^M to specify the topology of M for the fused operators.¹¹

We require normalizing worldline/sheet operators for a proper basis, so that the \mathcal{F}^M is also properly normalized in order for $Z(Y^{d-1} \times S^1; \dots)$ as the GSD on a spatial closed manifold Y^{d-1} always be a positive integer. In principle, we can derive the fusion rule of excitations in any closed spacetime 4-manifold. For instance, the fusion rule for fusing three particles on a spatial S^3 is

$$Z(S^3 \times S^1; \bar{\alpha}, \sigma_1, \sigma_2) = \langle 0_{D^3 \times S^1} | W_{\bar{\alpha}}^{S^1} W_{\sigma_1}^{S^1} W_{\sigma_2}^{S^1} | 0_{D^3 \times S^1} \rangle = (\mathcal{F}^{S^1})_{\sigma_1 \sigma_2}^{\bar{\alpha}}. \quad (2.37)$$

Many more examples of fusion rules can be derived from computing

$$Z(\mathcal{M}^4; \sigma, \mu, \dots) \quad (2.38)$$

by using \mathcal{F}^M and Eq.(2.1), here the worldline and worldsheet are submanifolds *parallel not linked* with each other.

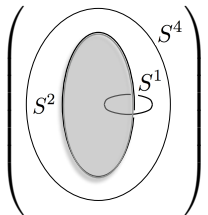
Throughout our work, we consistently use σ to represent the quasi-particle label (such as charge, electric charge, magnetic monopole, or representation of the gauge group) for worldlines $W_{\sigma}^{S^1}$, and we use μ to represent the quasi-string label for worldsheets, e.g. $V_{\mu}^{S^2}$ and $V_{\mu}^{T^2}$.

Overall we choose the operators $W_{\sigma}^{S^1}$ and $V_{\mu}^{M^2}$ carefully, so that they generate linear-independent states when acting on the $|0_{M'}\rangle$ state.

2.3.2 Quantum Braiding Data in 3+1D

If the worldline and worldsheet are linked as Eq.(2.1), then the path integral encodes the *braiding data*. Below we discuss the important braiding processes in 3+1D. We consider the following four braiding process in 4 dimensional spacetime, thus four sets of quantum braiding data in 3+1D.

1. First, the Aharonov-Bohm particle-loop braiding can be represented as a S^1 -worldline of particle and a S^2 -worldsheet of loop linked in S^4 spacetime,

$$\mathbb{L}_{\mu\sigma}^{(S^2, S^1)} \equiv \langle 0_{D^2 \times S^2} | V_{\mu}^{S^2 \dagger} W_{\sigma}^{S^1} | 0_{D^3 \times S^1} \rangle = Z \left(\left(\begin{array}{c} S^4 \\ \left(\begin{array}{c} S^2 \\ S^1 \end{array} \right) \end{array} \right) \right), \quad (2.39)$$


if we design the worldline and worldsheet along the generators of the first and the second homology group

$$H_1(D^3 \times S^1, \mathbb{Z}) = H_2(D^2 \times S^2, \mathbb{Z}) = \mathbb{Z}$$

¹¹To be further more specific, we can also specify the whole open-manifold topology $M \times V$ corresponding to the ground state sector $|0_{M \times V}\rangle$, namely we can rewrite the data in new notations to introduce the refined data: $(F^{S^1}) \rightarrow (F^{D^3 \times S^1})$, $(F^{T^2}) \rightarrow (F^{D^2 \times T^2})$ and $(F^{S^2}) \rightarrow (F^{D^2 \times S^2})$. However, because (F^M) are the fusion data in the local neighborhood around the worldline/worldsheet operators with topology M , physically it does not encode the information from the remained product space $M \times V$. Namely, at least for the most common and known theory that we can regularize on the lattice, we understand that $(F^M) = (F^{M \times V_1}) = (F^{M \times V_2}) = \dots$ for any topology V_1, V_2, \dots . So here we make a physical assumption that $(F^{S^1}) = (F^{D^3 \times S^1})$, $(F^{T^2}) = (F^{D^2 \times T^2})$ and $(F^{S^2}) = (F^{D^2 \times S^2})$.

respectively, via Alexander duality. We also use the fact

$$S^2 \times D^2 \cup_{S^2 \times S^1} D^3 \times S^1 = S^4, \quad (2.40)$$

thus

$$\langle 0_{D^2 \times S^2} | 0_{D^3 \times S^1} \rangle = Z(S^4). \quad (2.41)$$

2. Second, we can also consider particle-loop (Aharonov-Bohm) braiding as a S^1 -worldline of particle and a T^2 -worldsheet (below T^2 drawn as a S^2 with a handle) of loop linked in S^4 ,

$$\langle 0_{D^2 \times T^2} | V_\mu^{T^2 \dagger} W_\sigma^{S^1} | 0_{S^4 \setminus D^2 \times T^2} \rangle = Z \left(\begin{array}{c} S^4 \\ \text{---} \\ T^2 \text{---} \\ \text{---} \\ S^1 \end{array} \right), \quad (2.42)$$

if we design the worldline and worldsheet along the generators of the first and the second homology group

$$H_1(S^4 \setminus D^2 \times T^2, \mathbb{Z}) = H_2(D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}$$

respectively, via Alexander duality. Compare Eqs.(2.39) and (2.42), the loop excitation of S^2 -worldsheet is shrinkable, while the loop of T^2 -worldsheet needs not to be shrinkable. If there is a gauge theory description, then the loop is shrinkable implies the loop is a pure flux excitation without any net charge.

3. Third, we can represent a three-loop braiding process [23–26] as three T^2 -worldsheets *triple-linking* [27] in the spacetime S^4 (as the first figure in Eq.(2.43)). We find that

$$\begin{aligned} L_{\mu_3, \mu_2, \mu_1}^{\text{Tri}} &\equiv \langle 0_{S^4 \setminus D_{wx}^2 \times T_{yz}^2} | V_{\mu_3}^{T_{zx}^2 \dagger} V_{\mu_2}^{T_{xy}^2 \dagger} V_{\mu_1}^{T_{yz}^2} | 0_{D_{wx}^2 \times T_{yz}^2} \rangle \\ &= Z \left(\begin{array}{c} S^4 \\ \text{---} \\ 1 \text{---} \\ \text{---} \\ 3 \text{---} \\ \text{---} \\ 2 \end{array} \right) = Z \left(\begin{array}{c} S^4 \\ \text{---} \\ T^2 \text{---} \\ \text{---} \\ T^2 \text{---} \\ \text{---} \\ T^2 \end{array} \right), \end{aligned} \quad (2.43)$$

where we design the worldsheets $V_{\mu_1}^{T_{yz}^2}$ along the generator of homology group $H_2(D_{wx}^2 \times T_{yz}^2, \mathbb{Z}) = \mathbb{Z}$ while we design $V_{\mu_2}^{T_{xy}^2 \dagger}$ and $V_{\mu_3}^{T_{zx}^2 \dagger}$ along the two generators of $H_2(S^4 \setminus D_{wx}^2 \times T_{yz}^2, \mathbb{Z}) = H_1(D_{wx}^2 \times T_{yz}^2, \mathbb{Z}) = \mathbb{Z}^2$ respectively. Again, we obtain via Alexander duality that

$$H_2(S^4 \setminus D^2 \times T^2, \mathbb{Z}) = H_1(D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}^2.$$

We find that Eq.(2.43) is also equivalent to the spun surgery construction (or the so-called spinning surgery) of a Hopf link (denoted as μ_2 and μ_3) linked by a third T^2 -torus (denoted as μ_1) [28, 29]. Namely, we can view the above figure as a Hopf link of two loops spinning along the dotted path of a S^1 circle, which becomes a pair of T^2 -worldsheets μ_2 and μ_3 . Additionally the T^2 -worldsheet μ_1 (drawn in gray as a S^2 added a thin handle), together with μ_2 and μ_3 , the three worldsheets have a triple-linking topological invariance [27].

4. Fourth, we consider the four-loop braiding process, where three loops dancing in the Borromean ring trajectory while linked by a fourth loop [30], can characterize certain 3+1D non-Abelian topological orders [26]. We find it is also the spun surgery construction of Borromean rings of three loops linked by a fourth torus in the spacetime picture, and its path integral $Z[S^4; \text{Link}[\text{Spun}[\text{BR}[\mu_4, \mu_3, \mu_2]], \mu_1]]$ can be transformed:

$$\begin{aligned}
Z[S^4; \text{Link}[\text{Spun}[\text{BR}[\mu_4, \mu_3, \mu_2]], \mu_1]] &\equiv Z \left(\begin{array}{c} \text{Diagram of } S^4 \text{ with four loops labeled 1, 2, 3, 4} \end{array} \right) \xrightarrow{\text{surgery}} Z[T^4 \# S^2 \times S^2; \mu'_4, \mu'_3, \mu'_2, \mu'_1] \\
&= \langle 0_{T^4 \# S^2 \times S^2 \setminus D^2_{wx} \times T^2_{yz}} | V_{\mu'_4}^{T^2 \dagger} V_{\mu'_3}^{T^2 \dagger} V_{\mu'_2}^{T^2 \dagger} V_{\mu'_1}^{T^2 \dagger} | 0_{D^2_{wx} \times T^2_{yz}} \rangle, \tag{2.44}
\end{aligned}$$

where the surgery contains four consecutive modular \mathcal{S} -transformations done along the T^3 -boundary of $D^2 \times T^2$ tubular neighborhood around four T^2 -worldsheets.

The four-step surgeries here sequently send the initial configuration 4-sphere S^4 into:

$$\begin{aligned}
S^4 &= (D^3 \times S^1) \cup (S^2 \times D^2) = ((S^3 \setminus D^3) \times S^1) \cup (S^2 \times D^2) \tag{2.45} \\
&\xrightarrow{\text{1st surgery}} ((S^2 \times S^1 \setminus D^3) \times S^1) \cup (S^2 \times D^2) \\
&\xrightarrow{\text{2nd surgery}} ((S^2 \times S^1 \# S^2 \times S^1 \setminus D^3) \times S^1) \cup (S^2 \times D^2) \\
&\xrightarrow{\text{3rd surgery}} ((T^3 \setminus D^3) \times S^1) \cup (S^2 \times D^2) = (T^4 \setminus D^3 \times S^1) \cup (S^2 \times D^2) \\
&\xrightarrow{\text{4th surgery}} T^4 \# S^2 \times S^2
\end{aligned}$$

to a connected sum of T^4 -torus and $S^2 \times S^2$ configuration. Here we use the notation $\mathcal{M}_1 \# \mathcal{M}_2$ means the connected sum of manifolds \mathcal{M}_1 and \mathcal{M}_2 . The outcome $Z[T^4 \# S^2 \times S^2; \mu'_4, \mu'_3, \mu'_2, \mu'_1]$ has the wonderful desired property that we can design the worldsheets along the generator of homology groups so the operator insertions are well-defined. Here $V_{\mu'_1}^{T^2 \dagger}$ is the worldsheet operator acting along the only generator of homology group $H_2(D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}$. And $V_{\mu'_2}^{T^2}, V_{\mu'_3}^{T^2}, V_{\mu'_4}^{T^2}$ are the worldsheet operators acting along three among the seven generators of $H_2(T^4 \# S^2 \times S^2 \setminus D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}^7$ with the hollowed $D^2 \times T^2$ specified in the earlier text. This shows that

$$Z[T^4 \# S^2 \times S^2; \mu'_4, \mu'_3, \mu'_2, \mu'_1] \tag{2.46}$$

is a better quantum number easier to be computed than $Z[S^4; \text{Link}[\text{Spun}[\text{BR}[\mu_4, \mu_3, \mu_2]], \mu_1]]$. The final spacetime manifold that we obtained after the four-step surgery above is $T^4 \# S^2 \times S^2$, where $\#$ stands for the connected sum.

5. Modular $\text{SL}(3, \mathbb{Z})$ modular data \mathcal{S}^{xyz} and \mathcal{T}^{xy} :

We can glue the T^3 -boundary of 4-submanifolds (e.g. $D^2 \times T^2$ and $S^4 \setminus D^2 \times T^2$) via $\text{MCG}(T^3) = \text{SL}(3, \mathbb{Z})$ generated by

$$\hat{\mathcal{S}}^{xyz} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{\mathcal{T}}^{xy} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.47}$$

In this work, we define their representations as

$$\mathcal{S}_{\mu_2, \mu_1}^{xyz} \equiv \langle 0_{D_{xw}^2 \times T_{yz}^2} | V_{\mu_2}^{T_{yz}^2 \dagger} \hat{\mathcal{S}}^{xyz} V_{\mu_1}^{T_{yz}^2} | 0_{D_{xw}^2 \times T_{yz}^2} \rangle, \quad (2.48)$$

$$\mathcal{T}_{\mu_2, \mu_1}^{xy} \equiv \langle 0_{D_{xw}^2 \times T_{yz}^2} | V_{\mu_2}^{T_{yz}^2 \dagger} \hat{\mathcal{T}}^{xy} V_{\mu_1}^{T_{yz}^2} | 0_{D_{xw}^2 \times T_{yz}^2} \rangle, \quad (2.49)$$

while \mathcal{S}^{xyz} is a spun-Hopf link in $S^3 \times S^1$, and \mathcal{T}^{xy} is related to the *topological spin* and *self-statistics* of closed strings [26].

In this work, we project $\hat{\mathcal{S}}^{xyz}$ and $\hat{\mathcal{T}}^{xy}$ into the bra-ket bases of $|\mu_{D^2 \times T^2}\rangle \equiv V_{\mu}^{T_{yz}^2} |0_{D^2 \times T^2}\rangle$. Our representation of

$$\mathcal{S}_{\mu_2, \mu_1}^{xyz} \equiv \langle \mu_{2D^2 \times T^2} | \hat{\mathcal{S}}^{xyz} | \mu_{1D^2 \times T^2} \rangle = \langle 0_{D_{xw}^2 \times T_{yz}^2} | V_{\mu_2}^{T_{yz}^2 \dagger} \hat{\mathcal{S}}^{xyz} V_{\mu_1}^{T_{yz}^2} | 0_{D_{xw}^2 \times T_{yz}^2} \rangle = Z[S^3 \times S^1; \text{Spun}[\text{Hopf}[\mu_2, \mu_1]]]$$

is effectively a path integral of a spun Hopf link in the spacetime manifold

$$D^2 \times T^2 \cup_{T^3, \hat{\mathcal{S}}^{xyz}} D^2 \times T^2 = S^3 \times S^1.$$

Our representation of

$$\mathcal{T}_{\mu_2, \mu_1}^{xy} \equiv \langle \mu_{2D^2 \times T^2} | \hat{\mathcal{T}}^{xy} | \mu_{1D^2 \times T^2} \rangle = \langle 0_{D_{xw}^2 \times T_{yz}^2} | V_{\mu_2}^{T_{yz}^2 \dagger} \hat{\mathcal{T}}^{xy} V_{\mu_1}^{T_{yz}^2} | 0_{D_{xw}^2 \times T_{yz}^2} \rangle$$

is effectively a $Z[S^2 \times S^1 \times S^1]$ -path integral in the spacetime manifold

$$D^2 \times T^2 \cup_{T^3, \hat{\mathcal{T}}^{xy}} D^2 \times T^2 = S^2 \times S^1 \times S^1.$$

In addition, the worldsheet operator $V_{\mu}^{T_{yz}^2}$ effectively contains also worldline operators, e.g. $W^{S_y^1}$ and $W^{S_z^1}$ along y and z directions. Namely we mean that $V_{\mu}^{T_{yz}^2} = W^{S_y^1} W^{S_z^1} V_{\mu}^{T_{yz}^2}$, so $W^{S_y^1}$ and $W^{S_z^1}$ are along the two generators of homology group $H_1(D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}^2$, while $V_{\mu}^{T_{yz}^2}$ is along the unique one generator of homology group $H_2(D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}$. If there is a gauge theory description, then we project our $\hat{\mathcal{S}}^{xyz}$ and $\hat{\mathcal{T}}^{xy}$ into a one-flux (conjugacy class) and two-charge (representation) basis

$$|\mu_1, \sigma_2, \sigma_3\rangle \equiv V_{\mu_1}^{T_{yz}^2} W_{\sigma_2}^{S_y^1} W_{\sigma_3}^{S_z^1} |0_{D_{xw}^2 \times T_{yz}^2}\rangle. \quad (2.50)$$

So our projection here is different from the one-charge (representation) and two-flux (conjugacy class) basis $|\sigma_1, \mu_2, \mu_3\rangle$ used in Ref. [24–26]. The $|\sigma_1, \mu_2, \mu_3\rangle$ -bases can be obtained through

$$|\sigma_1, \mu_2, \mu_3\rangle \equiv W_{\sigma_1}^{S_x^1} V_{\mu_2}^{T_{xy}^2} V_{\mu_3}^{T_{xz}^2} |0_{S_4 \setminus D_{xw}^2 \times T_{yz}^2}\rangle, \quad (2.51)$$

where $W^{S_x^1}$ is along the generator of homology group $H_1(S^4 \setminus D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}$, while $V_{\mu_2}^{T_{xy}^2}$ and $V_{\mu_3}^{T_{xz}^2}$ are along the two generators of homology group $H_2(S^4 \setminus D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}^2$ via the Alexander duality. The alternate representation of the $\text{SL}(3, \mathbb{Z})$ modular data via the bases (2.51) is presented in our upcoming work [17].

3 New Quantum Surgery Formulas: Generalized Analogs of Verlinde's

3.1 Derivations of some basics of quantum surgery formulas

Now we like to derive two powerful identities (Eq. (3.2), and Eq. (3.8)) for fixed-point path integrals or partition functions for quantum theory, for example suitable for studying topological orders.

1. If the path integral formed by disconnected manifolds M and N , denoted as $M \sqcup N$, we have $Z(M \sqcup N) = Z(M)Z(N)$. Assume that
- (1) we divide both M and N into two pieces such that $M = M_U \cup_B M_D$, $N = N_U \cup_B N_D$, and their cut topology (dashed boundary denoted as B) is equivalent $B = \overline{\partial M_D} = \partial M_U = \overline{\partial N_D} = \partial N_U$, and
 - (2) the Hilbert space on the spatial slice is 1-dimensional (namely the GSD=1),¹² then we obtain

$$Z \left(\left(\begin{array}{c} M_U \\ \text{---} B \\ M_D \end{array} \right) \left(\begin{array}{c} N_U \\ \text{---} B \\ N_D \end{array} \right) \right) = Z \left(\left(\begin{array}{c} N_U \\ \text{---} B \\ M_D \end{array} \right) \left(\begin{array}{c} M_U \\ \text{---} B \\ N_D \end{array} \right) \right) \quad (3.2)$$

$$\Rightarrow \boxed{Z(M_U \cup_B M_D)Z(N_U \cup_B N_D) = Z(N_U \cup_B M_D)Z(M_U \cup_B N_D)}.$$

2. Now we derive a generic formula for our use of surgery. We consider a closed manifold M glued by two pieces M_U and M_D so that $M = M_U \cup_B M_D$ where $B = \partial M_U = \overline{\partial M_D}$. We consider there are insertions of operators in M_U and M_D . We denote the generic insertions in M_U as α_{M_U} and the generic insertions in M_D as β_{M_D} . Here both α_{M_U} and β_{M_D} may contain both worldline and worldsheet operators. We write the path integral as $Z(M; \alpha_{M_U}, \beta_{M_D}) = \langle \alpha_{M_U} | \beta_{M_D} \rangle$, while the worldline/worldsheet may be linked or may not be linked in M . Here we introduce an extra subscript M in

$$Z(M; \alpha_{M_U}, \beta_{M_D}) = \langle \alpha_{M_U} | \beta_{M_D} \rangle_M$$

to specify the glued manifold is $M_U \cup_B M_D = M$. Now we like to do surgery by cutting out the submanifold M_D out of M and re-glue it back to M_U via its mapping class group (MCG) generator

$$\hat{K} \in \text{MCG}(B) = \text{MCG}(\partial M_U) = \text{MCG}(\overline{\partial M_D}). \quad (3.3)$$

We now give some additional assumptions.

Assumption 1: The operator insertions in M are well-separated into M_U and M_D , so that no operator insertions cross the boundary B . Namely, at the boundary cut B there are no defects of point or string excitations from the cross-section of α_{M_U} , β_{M_D} or any other operators.¹³

Assumption 2: We can generate the complete bases of degenerate ground states fully spanning the dimension of Hilbert space for the spatial section of B , by inserting distinct operators (worldline/worldsheet, etc.) into M_D . Namely, we insert a set of operators Φ in the interior of $|0_{M_D}\rangle$ to obtain a new state $\Phi|0_{M_D}\rangle \equiv |\Phi_{M_D}\rangle$, such that these states $\{|\Phi|0_{M_D}\rangle\}$ are orthonormal canonical bases, and the dimension of the vector space $\dim(\{|\Phi|0_{M_D}\rangle\})$ equals to the ground state degeneracy (GSD) of the topological order on the spatial section B .

¹²Presumably there may be defect-like excitation of particles and strings on the spatial slice cross-section B . If the dimensional of Hilbert space on the spatial slice B is 1, namely the ground state degeneracy (GSD) is 1, then we can derive the gluing identity

$$\langle M_U | M_D \rangle = \langle N_U | N_D \rangle \Rightarrow \langle M_U | N_D \rangle = \langle N_U | M_D \rangle \quad (3.1)$$

because this vector space (of the Hilbert space) is 1-dimensional and all vectors are parallel in the inner product.

¹³ The readers should notice that this assumption is stronger and more restricted than the previous Eq. (3.2). In Eq. (3.2), we can have defects of point or string excitations from the cross-section and on the boundary cut B , as long as the dimension of Hilbert space associated to B is 1-dimensional.

If both assumptions hold, then we find a relation:

$$\begin{aligned}
Z(M; \alpha_{M_U}, \beta_{M_D}) &= \langle \alpha_{M_U} | \beta_{M_D} \rangle_M \\
&= \sum_{\Phi} \langle \alpha_{M_U} | \hat{K} \Phi | 0_{M_D} \rangle \langle 0_{M_D} | (\hat{K} \Phi)^\dagger | \beta_{M_D} \rangle \\
&= \sum_{\Phi} \langle \alpha_{M_U} | \hat{K} \Phi | 0_{M_D} \rangle \langle 0_{M_D} | \Phi^\dagger \hat{K}^{-1} | \beta_{M_D} \rangle \\
&= \sum_{\Phi} \langle \alpha_{M_U} | \hat{K} | \Phi_{M_D} \rangle_{M_U \cup_{B; \hat{K}} M_D} \langle \Phi_{M_D} | \hat{K}^{-1} | \beta_{M_D} \rangle_{M_D \cup_{B; \hat{K}^{-1}} M_D} \\
&= \sum_{\Phi} Z(M_U \cup_{B; \hat{K}} M_D; \alpha_{M_U}, \Phi_{M_D}) \langle \Phi_{M_D} | \hat{K}^{-1} | \beta_{M_D} \rangle_{M_D \cup_{B; \hat{K}^{-1}} M_D} \\
&= \sum_{\Phi} K_{\Phi, \beta}^{-1} Z(M_U \cup_{B; \hat{K}} M_D; \alpha_{M_U}, \Phi_{M_D})
\end{aligned} \tag{3.4}$$

We note that in the second equality, we write the identity matrix as $\mathbb{1} = \sum_{\Phi} (\hat{K} \Phi) | 0_{M_D} \rangle \langle 0_{M_D} | (\hat{K} \Phi)^\dagger$. In the third and fourth equalities, we have \hat{K}^{-1} in the inner product $\langle \Phi_{M_D} | \hat{K}^{-1} | \beta_{M_D} \rangle$, because \hat{K} as a MCG generator acts on the spatial manifold B directly. The evolution process from the first \hat{K}^{-1} on the right and the second \hat{K} on the left can be viewed as the *adiabatic evolution* of quantum states in the case of *fixed-point* topological orders. In the fifth equality, we rewrite $\langle \alpha_{M_U} | \hat{K} | \Phi_{M_D} \rangle_{M_U \cup_{B; \hat{K}} M_D} = Z(M_U \cup_{B; \hat{K}} M_D; \alpha_{M_U}, \Phi_{M_D})$ where α_{M_U} and Φ_{M_D} may or may not be linked in the new manifold $M_U \cup_{B; \hat{K}} M_D$. In the sixth equality, we assume that both $|\beta_{M_D}\rangle$ and $|\Phi_{M_D}\rangle$ are vectors in a canonical basis, then we can define

$$\langle \Phi_{M_D} | \hat{K}^{-1} | \beta_{M_D} \rangle_{M_D \cup_{B; \hat{K}^{-1}} M_D} \equiv K_{\Phi, \beta}^{-1} \tag{3.5}$$

as a matrix element of K^{-1} , which now becomes a representation of MCG in the quasi-excitation bases of $\{|\beta_{M_D}\rangle, |\Phi_{M_D}\rangle, \dots\}$. It is important to remember that $K_{\Phi, \beta}^{-1}$ is a quantum amplitude computed in the specific spacetime manifold $M_D \cup_{B; \hat{K}^{-1}} M_D$.

To summarize, so far we derive,

$$\boxed{Z(M; \alpha_{M_U}, \beta_{M_D}) = \sum_{\Phi} K_{\Phi, \beta}^{-1} Z(M_U \cup_{B; \hat{K}} M_D; \alpha_{M_U}, \Phi_{M_D})}. \tag{3.6}$$

The detailed derivation shows

$$\begin{aligned}
&\sum_{\Phi'} K_{\beta, \Phi'} Z(M; \alpha_{M_U}, \beta_{M_D}) \\
&= \sum_{\Phi'} K_{\beta, \Phi'} \sum_{\Phi} K_{\Phi, \beta}^{-1} Z(M_U \cup_{B; \hat{K}} M_D; \alpha_{M_U}, \Phi_{M_D}) \\
&= \delta_{\Phi \Phi'} Z(M_U \cup_{B; \hat{K}} M_D; \alpha_{M_U}, \Phi_{M_D}) \\
&= Z(M_U \cup_{B; \hat{K}} M_D; \alpha_{M_U}, \Phi'_{M_D}).
\end{aligned} \tag{3.7}$$

We can also derive another formula by applying the inverse transformation,

$$\boxed{Z(M_U \cup_{B; \hat{K}} M_D; \alpha_{M_U}, \Phi'_{M_D}) = \sum_{\Phi'} K_{\beta, \Phi'} Z(M; \alpha_{M_U}, \beta_{M_D})}. \tag{3.8}$$

if it satisfies $KK^{-1} = \mathbb{1}$. Again we stress that $K_{\beta, \Phi'}$ is a quantum amplitude computed in the specific spacetime manifold $M_D \cup_{B; \hat{K}^{-1}} M_D$.

3.2 Quantum Surgery Formulas in 2+1D and for 3-manifolds

In 2+1D, we can derive the renowned Verlinde formula [6,15,16] by one specific version of our Eq. (3.2):

$$\begin{aligned}
& Z \left(\begin{array}{c} S^3 \\ \text{---} \\ S^2 \\ \text{---} \\ S^1 \end{array} \right) Z \left(\begin{array}{c} S^3 \\ \text{---} \\ S^2 \\ \text{---} \\ S^1 \end{array} \right) = Z \left(\begin{array}{c} S^3 \\ \text{---} \\ S^2 \\ \text{---} \\ S^1 \end{array} \right) Z \left(\begin{array}{c} S^3 \\ \text{---} \\ S^2 \\ \text{---} \\ S^1 \end{array} \right) \\
& \Rightarrow \boxed{\mathcal{S}_{\bar{\sigma}_1 0} \sum_{\sigma_4} \mathcal{S}_{\bar{\sigma}_1 \sigma_4} \mathcal{N}_{\sigma_2 \sigma_3}^{\sigma_4} = \mathcal{S}_{\bar{\sigma}_1 \sigma_2} \mathcal{S}_{\bar{\sigma}_1 \sigma_3}}, \tag{3.9}
\end{aligned}$$

where each spacetime manifold is S^3 , with the line operator insertions such as an unlink and Hopf links. Each S^3 is cut into two D^3 pieces, so $D^3 \cup_{S^2} D^3 = S^3$, while the boundary dashed cut is $B = S^2$. The GSD for this spatial section S^2 with a pair of particle-antiparticle must be 1, so our surgery satisfies the assumptions for Eq.(3.2). The second line is derived from rewriting path integrals in terms of our data introduced before – the fusion rule $\mathcal{N}_{\sigma_2 \sigma_3}^{\sigma_4}$ comes from fusing $\sigma_2 \sigma_3$ into σ_4 which Hopf-linked with σ_1 , while Hopf links render the $SL(2, \mathbb{Z})$ modular \mathcal{S} matrices. The label 0, in $\mathcal{S}_{\bar{\sigma}_1 0}$ and hereafter, denotes a vacuum sector without operator insertions in a submanifold.

For Eq.(3.9), the only path integral we need to compute more explicitly is this:

$$\begin{aligned}
& Z \left(\begin{array}{c} S^3 \\ \text{---} \\ S^2 \\ \text{---} \\ S^1 \end{array} \right) = \langle 0_{D_{xt}^2 \times S_y^1} | (W_{\sigma_1}^{S_y^1})^\dagger \hat{S} W_{\sigma_2}^{S_y^1} W_{\sigma_3}^{S_y^1} | 0_{D_{xt}^2 \times S_y^1} \rangle \\
& = \langle 0_{D_{xt}^2 \times S_y^1} | (W_{\sigma_1}^{S_y^1})^\dagger \hat{S} W_{\sigma_4}^{S_y^1} \mathcal{F}_{\sigma_2 \sigma_3}^{\sigma_4} | 0_{D_{xt}^2 \times S_y^1} \rangle \\
& = \sum_{\alpha \sigma_4} (G_{\sigma_1}^\alpha)^* \mathcal{S}_{\alpha \sigma_4} \mathcal{F}_{\sigma_2 \sigma_3}^{\sigma_4} = \sum_{\sigma_4} \mathcal{S}_{\bar{\sigma}_1 \sigma_4} \mathcal{N}_{\sigma_2 \sigma_3}^{\sigma_4}, \tag{3.10}
\end{aligned}$$

where the last equality we use the canonical basis. Together with the previous data, we can easily derive Eq.(3.9).

Since it is convenient to express in terms of canonical bases, below for all the derivations, we will implicitly project every quantum amplitude into *canonical bases* when we write down its matrix element.

In the canonical basis when \mathcal{S} is invertible, we can massage our formula to a familiar form, which we derive that:

$$\boxed{\mathcal{N}_{\sigma_2 \sigma_3}^a = \sum_{\bar{\sigma}_1} \frac{\mathcal{S}_{\bar{\sigma}_1 \sigma_2} \mathcal{S}_{\bar{\sigma}_1 \sigma_3} (\mathcal{S}^{-1})_{\bar{\sigma}_1 a}}{\mathcal{S}_{\bar{\sigma}_1 0}}}. \tag{3.11}$$

3.3 Quantum Surgery Formulas in 3+1D and for 4-manifolds

3.3.1 Formulas for 3+1D particle-string braiding process: Link between 1-worldline and 2-worldsheet

In 3+1D, we derive that the particle-string braiding process in terms of S^4 -spacetime path integral Eq.(2.39) has the following constraint formulas:

$$\begin{aligned}
 & Z \left(\begin{array}{c} \text{Diagram 1: } S^4 \text{ with } S^2 \text{ and } S^3 \text{ components} \\ \text{Diagram 2: } S^4 \text{ with } S^2, S^3, S^1 \text{ components} \\ \text{Diagram 3: } S^4 \text{ with } S^2, S^3, S^1 \text{ components} \\ \text{Diagram 4: } S^4 \text{ with } S^2, S^3, S^1 \text{ components} \end{array} \right) \\
 & \Rightarrow \boxed{L_{\mu_1 0}^{(S^2, S^1)} \sum_{\sigma_4} L_{\mu_1 \sigma_4}^{(S^2, S^1)} (\mathcal{F}^{S^1})_{\sigma_2 \sigma_3}^{\sigma_4} = L_{\mu_1 \sigma_2}^{(S^2, S^1)} L_{\mu_1 \sigma_3}^{(S^2, S^1)}}. \quad (3.12)
 \end{aligned}$$

$$\begin{aligned}
 & Z \left(\begin{array}{c} \text{Diagram 1: } S^4 \text{ with } S^1 \text{ and } S^3 \text{ components} \\ \text{Diagram 2: } S^4 \text{ with } S^1, S^2, S^3 \text{ components} \\ \text{Diagram 3: } S^4 \text{ with } S^1, S^2, S^3 \text{ components} \\ \text{Diagram 4: } S^4 \text{ with } S^1, S^2, S^3 \text{ components} \end{array} \right) \\
 & \Rightarrow \boxed{L_{0 \sigma_1}^{(S^2, S^1)} \sum_{\mu_4} L_{\mu_4 \sigma_1}^{(S^2, S^1)} (\mathcal{F}^{S^2})_{\mu_2 \mu_3}^{\mu_4} = L_{\mu_2 \sigma_1}^{(S^2, S^1)} L_{\mu_3 \sigma_1}^{(S^2, S^1)}}. \quad (3.13)
 \end{aligned}$$

Here the gray areas mean S^2 -spheres. All the data are well-defined and introduced earlier in Eqs.(2.34), (2.35), and (2.39). Notice that Eqs.(3.12) and (3.13) are symmetric by exchanging worldsheet/worldline indices: $\mu \leftrightarrow \sigma$, except that the fusion data is different: \mathcal{F}^{S^1} fuses worldlines, while \mathcal{F}^{S^2} fuses world-sheets.

For Eq.(3.12), the only path integral we need to compute more explicitly is this:

$$\begin{aligned}
 & Z \left(\begin{array}{c} \text{Diagram: } S^4 \text{ with } S^2 \text{ and } S^3 \text{ components} \end{array} \right) = \langle 0_{D_{\varphi_w}^2 \times S_{\theta_\phi}^2} | (V_{\mu_1}^{S_{\theta_\phi}^2})^\dagger W_{\sigma_2}^{S_\varphi^1} W_{\sigma_3}^{S_\varphi^1} | 0_{D_{\theta_{\phi_w}}^3 \times S_\varphi^1} \rangle \\
 & = \langle 0_{D_{\varphi_w}^2 \times S_{\theta_\phi}^2} | (V_{\mu_1}^{S_{\theta_\phi}^2})^\dagger W_{\sigma_4}^{S_\varphi^1} (\mathcal{F}^{S^1})_{\sigma_2 \sigma_3}^{\sigma_4} | 0_{D_{\theta_{\phi_w}}^3 \times S_\varphi^1} \rangle \\
 & = \sum_{\sigma_4} L_{\mu_1 \sigma_4}^{(S^2, S^1)} (\mathcal{F}^{S^1})_{\sigma_2 \sigma_3}^{\sigma_4}, \quad (3.14)
 \end{aligned}$$

again we use the canonical basis. Together with the previous data, we can easily derive Eq.(3.12). Similarly, we can also derive Eq.(3.13), using the almost equivalent computation following Eq.(3.12).

3.3.2 Formulas for 3+1D three-string braiding process: Triple link between three sets of 2-worldsheets

We also derive a quantum surgery constraint formula for the three-loop braiding process in 3+1D in terms of an S^4 -spacetime path integral Eq.(2.43) via the \mathcal{S}^{xyz} -surgery and its matrix representation:

$$\begin{aligned}
& Z \left(\begin{array}{c} S^4 \\ \text{---} \\ T^2 \end{array} \right) Z \left(\begin{array}{c} S^4 \\ \text{---} \\ S^3 \\ \text{---} \\ T^2 \end{array} \right) = Z \left(\begin{array}{c} S^4 \\ \text{---} \\ T^2 \end{array} \right) Z \left(\begin{array}{c} S^4 \\ \text{---} \\ S^3 \\ \text{---} \\ T^2 \end{array} \right) \\
& \Rightarrow \boxed{L_{0,0,\mu_1}^{\text{Tri}} \cdot \sum_{\Gamma,\Gamma',\Gamma_1,\Gamma'_1} (\mathcal{F}^{T^2})_{\zeta_2,\zeta_4}^{\Gamma} (\mathcal{S}^{xyz})_{\Gamma',\Gamma}^{-1} (\mathcal{F}^{T^2})_{\mu_1\Gamma'}^{\Gamma_1} \mathcal{S}_{\Gamma'_1,\Gamma_1}^{xyz} L_{0,0,\Gamma'_1}^{\text{Tri}}} \\
& = \boxed{\sum_{\zeta'_2,\eta_2,\eta'_2} (\mathcal{S}^{xyz})_{\zeta'_2,\zeta_2}^{-1} (\mathcal{F}^{T^2})_{\mu_1\zeta'_2}^{\eta_2} \mathcal{S}_{\eta'_2,\eta_2}^{xyz} L_{0,0,\eta'_2}^{\text{Tri}} \cdot \sum_{\zeta'_4,\eta_4,\eta'_4} (\mathcal{S}^{xyz})_{\zeta'_4,\zeta_4}^{-1} (\mathcal{F}^{T^2})_{\mu_1\zeta'_4}^{\eta_4} \mathcal{S}_{\eta'_4,\eta_4}^{xyz} L_{0,0,\eta'_4}^{\text{Tri}}} , \quad (3.15)
\end{aligned}$$

here the μ_1 -worldsheet in gray represents a T^2 torus, while μ_2 - μ_3 -worldsheets and μ_4 - μ_5 -worldsheets are both a pair of two T^2 tori obtained by spinning the Hopf link. All our data are well-defined in Eqs.(2.36), (2.43), and (2.48) introduced earlier. For example, the $L_{0,0,\mu_1}^{\text{Tri}}$ is defined in Eq.(2.43) with 0 as a vacuum without insertion, so $L_{0,0,\mu_1}^{\text{Tri}}$ is a path integral of a T^2 worldsheet μ_1 in S^4 . The index ζ_2 is obtained from fusing μ_2 - μ_3 -worldsheets, and ζ_4 is obtained from fusing μ_4 - μ_5 -worldsheets. Only μ_1, ζ_2, ζ_4 are the fixed indices, other indices are summed over.

Now let us derive Eq.(3.15). In the first path integral, we create a pair of loop μ_1 and anti-loop $\bar{\mu}_1$ excitations and then annihilate them, in terms of the spacetime picture, we obtain that

$$Z \left(\begin{array}{c} S^4 \\ \text{---} \\ 1 \end{array} \right) = Z \left(\begin{array}{c} S^4 \\ \text{---} \\ T^2 \end{array} \right) = L_{0,0,\mu_1}^{\text{Tri}}, \quad (3.16)$$

based on the data defined earlier. Let us explain our figure expressions further:

- The grey area drawn in terms of a tube means a 2-torus T^2 in topology (the 2-surface insertion in the left hand side of path integral Eq. (3.16)).
- A 2-torus T^2 can be also regarded as a 2-sphere S^2 adding a handle in topology (the 2-surface insertion in the right hand side of path integral Eq. (3.16)).

In the third path integral $L_{\mu_3,\mu_2,\mu_1}^{\text{Tri}}$, which is Eq. (2.43), appeared in Eq. (3.15), there are two descriptions to interpret it in terms of the braiding process in spacetime:

- Here is the first description. we create a pair of loop μ_1 and anti-loop $\bar{\mu}_1$ excitations and then there a pair of μ_2 - $\bar{\mu}_2$ and another pair of μ_3 - $\bar{\mu}_3$ are created while both pairs are thread by

μ_1 . Then the μ_1 - μ_2 - μ_3 will do the three-loop braiding process, which gives the most important Berry phase or Berry matrix information into the path integral. After then the pair of μ_2 - $\bar{\mu}_2$ is annihilated and also the pair of μ_3 - $\bar{\mu}_3$ is annihilated, while all the four loops are threaded by μ_1 during the process. Finally we annihilate the pair of μ_1 and $\bar{\mu}_1$ in the end [24].

- The second description is that we take a Hopf link of μ_2 - μ_3 linking spinning around the loop of μ_1 [28,29]. We denote the Hopf link of μ_2 - μ_3 as $\text{Hopf}[\mu_3, \mu_2]$, denote its spinning as $\text{Spun}[\text{Hopf}[\mu_3, \mu_2]]$, and denote its linking with the third T^2 -worldsheet of μ_1 as $\text{Link}[\text{Spun}[\text{Hopf}[\mu_3, \mu_2]], \mu_1]$. Thus we can define $L_{\mu_3, \mu_2, \mu_1}^{\text{Tri}} \equiv Z[S^4; \text{Link}[\text{Spun}[\text{Hopf}[\mu_3, \mu_2]], \mu_1]]$. From the second description, we immediate see that $L_{\mu_3, \mu_2, \mu_1}^{\text{Tri}}$ as $Z[S^4; \text{Link}[\text{Spun}[\text{Hopf}[\mu_3, \mu_2]], \mu_1]]$ are symmetric under exchanging $\mu_2 \leftrightarrow \mu_3$, up to an overall conjugation due to the orientation of quasi-excitations.

We can view the spacetime S^4 as a $S^4 = \mathbb{R}^4 + \{\infty\}$, the Cartesian coordinate \mathbb{R}^4 plus a point at the infinity $\{\infty\}$. Similar to the embedding of Ref. [28], we embed the T^2 -worldsheets μ_1, μ_2, μ_3 into the $(X_1, X_2, X_3, X_4) \in \mathbb{R}^4$ as follows:

$$\begin{cases} X_1(u, \vec{x}) = [r_1(u) + (r_2(u) + r_3(u) \cos x) \cos y] \cos z, \\ X_2(u, \vec{x}) = [r_1(u) + (r_2(u) + r_3(u) \cos x) \cos y] \sin z, \\ X_3(u, \vec{x}) = (r_2(u) + r_3(u) \cos x) \sin y, \\ X_4(u, \vec{x}) = r_3(u) \sin x, \end{cases} \quad (3.17)$$

here $\vec{x} \equiv (x, y, z)$. We choose the T^2 -worldsheets as follows:

The T^2 -worldsheet μ_1 is parametrized by some fixed u_1 and free coordinates of (z, x) while $y = 0$ is fixed.

The T^2 -worldsheet μ_2 is parametrized by some fixed u_2 and free coordinates of (x, y) while $z = 0$ is fixed.

The T^2 -worldsheet μ_3 is parametrized by some fixed u_3 and free coordinates of (y, z) while $x = 0$ is fixed.

We can set the parameters $u_1 > u_2 > u_3$. Meanwhile, a T^3 -surface can be defined as $\mathcal{M}^3(u, \vec{x}) \equiv (X_1(u, \vec{x}), X_2(u, \vec{x}), X_3(u, \vec{x}), X_4(u, \vec{x}))$ with a fixed u and free parameters \vec{x} . The T^3 -surface $\mathcal{M}^3(u, \vec{x}) \equiv (X_1(u, \vec{x}), X_2(u, \vec{x}), X_3(u, \vec{x}), X_4(u, \vec{x}))$ encloses a 4-dimensional volume. We define the enclosed 4-dimensional volume as the $\mathcal{M}^3(u, \vec{x}) \times I^1(s)$ where $I^1(s)$ is the 1-dimensional radius interval along r_3 , such that $I^1(s) = \{s | s = [0, r_3(u)]\}$, namely $0 \leq s \leq r_3(u)$. Here we can define $r_3(0) = 0$. The topology of the enclosed 4-dimensional volume of $\mathcal{M}^3(u, \vec{x}) \times I^1(s)$ is of course the $T^3 \times I^1 = T^2 \times (S^1 \times I^1) = T^2 \times D^2$. For a $\mathcal{M}^3(u_{\text{large}}, \vec{x}) \times I^1(s)$ prescribed by a fixed larger u_{large} and free parameters \vec{x} , the $\mathcal{M}^3(u_{\text{large}}, \vec{x}) \times I^1(s)$ must enclose the 4-volume spanned by the past history of $\mathcal{M}^3(u_{\text{small}}, \vec{x}) \times I^1(s)$, for any $u_{\text{large}} > u_{\text{small}}$. Here we set $u_1 > u_2 > u_3$. And we also set $r_1(u) > r_2(u) > r_3(u)$ for any given u .

One can check that the three T^2 -worldsheet μ_1, μ_2 and μ_3 indeed have the nontrivial *triple-linking number* [27]. We can design the triple-linking number to be:

$$\begin{aligned} \text{Tlk}(\mu_2, \mu_1, \mu_3) &= \text{Tlk}(\mu_3, \mu_1, \mu_2) = 0, \quad \text{Tlk}(\mu_1, \mu_2, \mu_3) = +1, \\ \text{Tlk}(\mu_3, \mu_2, \mu_1) &= -1, \quad \text{Tlk}(\mu_2, \mu_3, \mu_1) = +1, \quad \text{Tlk}(\mu_1, \mu_3, \mu_2) = -1. \end{aligned} \quad (3.18)$$

Below we will frequently use the surgery trick by cutting out a tubular neighborhood $D^2 \times T^2$ of the T^2 -worldsheet and re-gluing this $D^2 \times T^2$ back to its complement $S^4 \setminus D^2 \times T^2$ via the modular

\mathcal{S}^{xyz} -transformation. The \mathcal{S}^{xyz} -transformation sends

$$\begin{pmatrix} x_{\text{out}} \\ y_{\text{out}} \\ z_{\text{out}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{\text{in}} \\ y_{\text{in}} \\ z_{\text{in}} \end{pmatrix}. \quad (3.19)$$

Thus, the \mathcal{S}^{xyz} -identification is $(x_{\text{out}}, y_{\text{out}}, z_{\text{out}}) \leftrightarrow (z_{\text{in}}, x_{\text{in}}, y_{\text{in}})$. The $(\mathcal{S}^{xyz})^{-1}$ -identification is $(x_{\text{out}}, y_{\text{out}}, z_{\text{out}}) \leftrightarrow (y_{\text{in}}, z_{\text{in}}, x_{\text{in}})$. The surgery on the initial S^4 outcomes a new manifold,

$$(D^2 \times T^2) \cup_{T^3; \mathcal{S}^{xyz}} (S^4 \setminus D^2 \times T^2) = S^3 \times S^1 \# S^2 \times S^2. \quad (3.20)$$

In terms of the spacetime path integral picture, use Eqs.(3.6) and (3.8), we derive:

$$\begin{aligned} Z \left(\text{Diagram} \right) &\equiv L_{\mu_3, \mu_2, \mu_1}^{\text{Tri}} = Z[S^4; \text{Link}[\text{Spun}[\text{Hopf}[\mu_3, \mu_2]], \mu_1]] \\ &= \sum_{\mu'_3} \mathcal{S}_{\mu'_3, \mu_3}^{xyz} Z(S^3 \times S^1 \# S^2 \times S^2; \mu_1, \mu_2 \parallel \mu'_3) \end{aligned} \quad (3.21)$$

$$= \sum_{\mu'_3, \Gamma_2} \mathcal{S}_{\mu'_3, \mu_3}^{xyz} (\mathcal{F}^{T^2})_{\mu_2 \mu'_3}^{\Gamma_2} Z(S^3 \times S^1 \# S^2 \times S^2; \mu_1, \Gamma_2) \quad (3.22)$$

$$= \sum_{\mu'_3, \Gamma_2, \Gamma'_2} \mathcal{S}_{\mu'_3, \mu_3}^{xyz} (\mathcal{F}^{T^2})_{\mu_2 \mu'_3}^{\Gamma_2} (\mathcal{S}^{xyz})_{\Gamma'_2, \Gamma_2}^{-1} Z(S^4; \mu_1, \Gamma'_2) \quad (3.23)$$

$$= \sum_{\mu'_3, \Gamma_2, \Gamma'_2, \Gamma''_2} \mathcal{S}_{\mu'_3, \mu_3}^{xyz} (\mathcal{F}^{T^2})_{\mu_2 \mu'_3}^{\Gamma_2} (\mathcal{S}^{xyz})_{\Gamma'_2, \Gamma_2}^{-1} (\mathcal{S}^{xyz})_{\Gamma''_2, \Gamma'_2}^{-1} Z(S^3 \times S^1 \# S^2 \times S^2; \mu_1, \Gamma''_2) \quad (3.24)$$

$$= \sum_{\mu'_3, \Gamma_2, \Gamma'_2, \Gamma''_2, \eta_2} \mathcal{S}_{\mu'_3, \mu_3}^{xyz} (\mathcal{F}^{T^2})_{\mu_2 \mu'_3}^{\Gamma_2} (\mathcal{S}^{xyz})_{\Gamma'_2, \Gamma_2}^{-1} (\mathcal{S}^{xyz})_{\Gamma''_2, \Gamma'_2}^{-1} (\mathcal{F}^{T^2})_{\mu_1 \Gamma''_2}^{\eta_2} Z(S^3 \times S^1 \# S^2 \times S^2; \eta_2) \quad (3.25)$$

$$= \sum_{\mu'_3, \Gamma_2, \Gamma'_2, \Gamma''_2, \eta_2, \eta'_2} \mathcal{S}_{\mu'_3, \mu_3}^{xyz} (\mathcal{F}^{T^2})_{\mu_2 \mu'_3}^{\Gamma_2} (\mathcal{S}^{xyz})_{\Gamma'_2, \Gamma_2}^{-1} (\mathcal{S}^{xyz})_{\Gamma''_2, \Gamma'_2}^{-1} (\mathcal{F}^{T^2})_{\mu_1 \Gamma''_2}^{\eta_2} \mathcal{S}_{\eta'_2, \eta_2}^{xyz} L_{0,0, \eta'_2}^{\text{Tri}}. \quad (3.26)$$

As usual, the repeated indices are summed over. With the trick of \mathcal{S}^{xyz} -transformation in mind, here is the step-by-step sequence of surgeries we perform.

Step 1: We cut out the tubular neighborhood $D^2 \times T^2$ of the T^2 -worldsheet of μ_3 and re-glue this $D^2 \times T^2$ back to its complement $S^4 \setminus D^2 \times T^2$ via the modular $(\mathcal{S}^{xyz})^{-1}$ -transformation. The $D^2 \times T^2$ neighborhood of μ_3 -worldsheet can be viewed as the 4-volume $\mathcal{M}^3(u_3, \vec{x}) \times I^1(s)$, which encloses neither μ_1 -worldsheet nor μ_2 -worldsheet. The $(\mathcal{S}^{xyz})^{-1}$ -transformation sends $(y_{\text{in}}, z_{\text{in}})$ of μ_3 to $(x_{\text{out}}, y_{\text{out}})$ of μ_2 . The gluing however introduces the summing-over new coordinate μ'_3 , based on Eq.(3.6). Thus Step 1 obtains Eq.(3.21).

In Step 1, as Eq.(3.21) and thereafter, we write down $\mathcal{S}_{\mu'_3, \mu_3}^{xyz}$ matrix. Based on Eq.(3.5), we stress that the $\mathcal{S}_{\mu'_3, \mu_3}^{xyz}$ is projected to the $|0_{D^2 \times T^2}\rangle$ -states with operator-insertions for both bra and ket states.

$$\begin{aligned}
\mathcal{S}_{\mu'_3, \mu_3}^{xyz} &\equiv \langle \mu'_3 D^2 \times T^2 | \hat{\mathcal{S}}^{xyz} | \mu_3 D^2 \times T^2 \rangle_{D^2 \times T^2 \cup_{T^3; \hat{\mathcal{S}}^{xyz}} D^2 \times T^2} \\
&= \langle 0_{D_{xw}^2 \times T_{yz}^2} | V_{\mu'_3}^{T_{yz}^2 \dagger} \hat{\mathcal{S}}^{xyz} V_{\mu_3}^{T_{yz}^2} | 0_{D_{xw}^2 \times T_{yz}^2} \rangle_{S^3 \times S^1}
\end{aligned} \tag{3.27}$$

Here we use the surgery fact

$$D^2 \times T^2 \cup_{T^3; \hat{\mathcal{S}}^{xyz}} D^2 \times T^2 = S^3 \times S^1. \tag{3.28}$$

So our $\mathcal{S}_{\mu'_3, \mu_3}^{xyz}$ is defined as a quantum amplitude in $S^3 \times S^1$. Two T^2 -worldsheets μ'_3 and μ_3 now become a pair of Hopf link resides in S^3 part of $S^3 \times S^1$, while share the same S^1 circle in the S^1 part of $S^3 \times S^1$. We can view the shared S^1 circle as the spinning circle of the spun surgery construction on the Hopf link in D^3 , the spun-topology would be $D^3 \times S^1$, then we glue this $D^3 \times S^1$ contains $\text{Spun}[\text{Hopf}[\mu'_3, \mu_3]]$ to another $D^3 \times S^1$, so we have $D^3 \times S^1 \cup_{S^2 \times S^1} D^3 \times S^1 = S^3 \times S^1$ as an overall new spacetime topology. Hence we also denote

$$\mathcal{S}_{\mu'_3, \mu_3}^{xyz} = Z[S^3 \times S^1; \text{Spun}[\text{Hopf}[\mu'_3, \mu_3]]]. \tag{3.29}$$

Step 2: The earlier surgery now makes the inner μ'_3 -worldsheet *parallels* to the outer μ_2 -worldsheet, since they share the same coordinates $(x_{\text{out}}, y_{\text{out}}) = (y_{\text{in}}, z_{\text{in}})$. We denote their parallel topology as $\mu_2 \parallel \mu'_3$. So we can fuse the μ_2 -worldsheet and μ'_3 -worldsheet via the fusion algebra, namely $V_{\mu_2}^{T_{x_{\text{out}}, y_{\text{out}}}^2} V_{\mu'_3}^{T_{x_{\text{out}}, y_{\text{out}}}^2} = (\mathcal{F}^{T^2})_{\mu_2 \mu'_3}^{\Gamma_2} V_{\Gamma_2}^{T_{x_{\text{out}}, y_{\text{out}}}^2}$. Thus Step 2 obtains Eq.(3.22).

Step 3: We cut out the tubular neighborhood $D^2 \times T^2$ of the T^2 -worldsheet of Γ_2 and re-glue this $D^2 \times T^2$ back to its complement $S^4 \setminus D^2 \times T^2$ via the modular \mathcal{S}^{xyz} -transformation. The $D^2 \times T^2$ neighborhood of Γ_2 -worldsheet can be viewed as the 4-volume $\mathcal{M}^3(u_2, \vec{x}) \times I^1(s)$ in the new manifold $S^3 \times S^1 \# S^2 \times S^2$, which encloses no worldsheet inside. After the surgery, the \mathcal{S}^{xyz} -transformation sends the redefined $(x_{\text{in}}, y_{\text{in}})$ of Γ_2 back to $(y_{\text{out}}, z_{\text{out}})$ of Γ'_2 . The gluing however introduces the summing-over new coordinate Γ'_2 , based on Eq.(3.6). We also transform $S^3 \times S^1 \# S^2 \times S^2$ back to S^4 again. Thus Step 3 obtains Eq.(3.23).

Step 4: We cut out the tubular neighborhood $D^2 \times T^2$ of the T^2 -worldsheet of Γ'_2 and re-glue this $D^2 \times T^2$ back to its complement $S^4 \setminus D^2 \times T^2$ via the modular \mathcal{S}^{xyz} -transformation. The $D^2 \times T^2$ neighborhood of Γ'_2 -worldsheet viewed as the 4-volume in the manifold S^4 encloses no worldsheet inside. After the surgery, the \mathcal{S}^{xyz} -transformation sends the $(x_{\text{in}}, y_{\text{in}})$ of Γ'_2 to $(z_{\text{out}}, x_{\text{out}})$ of μ_1 . The gluing however introduces the summing-over new coordinate Γ''_2 , based on Eq.(3.6). We also transform S^4 to $S^3 \times S^1 \# S^2 \times S^2$ again. Thus Step 4 obtains Eq.(3.24).

Step 5: The earlier surgery now makes the inner Γ''_2 -worldsheet *parallels* to the outer μ_1 -worldsheet, since they share the same coordinates $(z_{\text{out}}, x_{\text{out}}) = (x_{\text{in}}, y_{\text{in}})$. We denote their parallel topology as $\mu_1 \parallel \Gamma''_2$. We now fuse the μ_1 -worldsheet and Γ''_2 -worldsheet via the fusion algebra, namely $V_{\mu_1}^{T_{z_{\text{out}}, x_{\text{out}}}^2} V_{\Gamma''_2}^{T_{z_{\text{out}}, x_{\text{out}}}^2} = (\mathcal{F}^{T^2})_{\mu_1 \Gamma''_2}^{\eta_2} V_{\eta_2}^{T_{z_{\text{out}}, x_{\text{out}}}^2}$. Thus Step 5 obtains Eq.(3.25).

Step 6: We should do the inverse transformation to get back to the S^4 manifold. Thus we cut out the tubular neighborhood $D^2 \times T^2$ of the T^2 -worldsheet of η_2 and re-glue this $D^2 \times T^2$ back to its

complement via the modular $(\mathcal{S}^{xyz})^{-1}$ -transformation. We relate the original path integral to the final one $Z(S^4; \eta'_2) = L_{0,0,\eta'_2}^{\text{Tri}}$. Thus Step 6 obtains Eq.(3.26).

Similarly, in the fourth path integral of Eq.(3.15), we derive

$$\begin{aligned}
Z \left(\text{Diagram} \right) &\equiv L_{\mu_5, \mu_4, \mu_1}^{\text{Tri}} = Z[S^4; \text{Link}[\text{Spun}[\text{Hopf}[\mu_5, \mu_4]], \mu_1]] \\
&= \sum_{\mu'_5, \Gamma_4, \Gamma'_4, \Gamma''_4, \eta_4, \eta'_4} \mathcal{S}_{\mu'_5, \mu_5}^{xyz} (\mathcal{F}^{T^2})_{\mu_4 \mu'_5}^{\Gamma_4} (\mathcal{S}^{xyz})_{\Gamma'_4, \Gamma_4}^{-1} (\mathcal{S}^{xyz})_{\Gamma''_4, \Gamma'_4}^{-1} (\mathcal{F}^{T^2})_{\mu_1 \Gamma''_4}^{\eta_4} \mathcal{S}_{\eta'_4, \eta_4}^{xyz} L_{0,0,\eta'_4}^{\text{Tri}}. \quad (3.30)
\end{aligned}$$

In the second path integral of Eq.(3.15), we have the Hopf link of $\text{Hopf}[\mu_3, \mu_2]$ and the Hopf link of $\text{Hopf}[\mu_5, \mu_4]$. In the spacetime picture, all $\mu_2, \mu_3, \mu_4, \mu_5$ are T^2 -worldsheets under the spun surgery construction. We can locate the the spun object named $\text{Spun}[\text{Hopf}[\mu_3, \mu_2], \text{Hopf}[\mu_5, \mu_4]]$ inside a $D^3 \times S^1$, while this $D^3 \times S^1$ is glued with a $S^2 \times D^2$ to a S^4 . Here the $S^2 \times D^2$ contains a T^2 -worldsheet μ_1 . We can view the T^2 -worldsheet μ_1 contains a S^2 -sphere of the $S^2 \times D^2$ but attached an extra handle. We derive:

$$\begin{aligned}
Z \left(\text{Diagram} \right) &\equiv Z[S^4; \text{Link}[\text{Spun}[\text{Hopf}[\mu_3, \mu_2], \text{Hopf}[\mu_5, \mu_4]], \mu_1]] \\
&= \sum_{\mu'_3, \Gamma_2, \Gamma'_2} \sum_{\mu'_5, \Gamma_4, \Gamma'_4} \mathcal{S}_{\mu'_3, \mu_3}^{xyz} (\mathcal{F}^{T^2})_{\mu_2 \mu'_3}^{\Gamma_2} (\mathcal{S}^{xyz})_{\Gamma'_2, \Gamma_2}^{-1} \mathcal{S}_{\mu'_5, \mu_5}^{xyz} (\mathcal{F}^{T^2})_{\mu_4 \mu'_5}^{\Gamma_4} (\mathcal{S}^{xyz})_{\Gamma'_4, \Gamma_4}^{-1} Z[S^4; \text{Spun}[\Gamma'_2, \Gamma'_4], \mu_1] \quad (3.31) \\
&= \sum_{\mu'_3, \Gamma_2, \Gamma'_2} \sum_{\mu'_5, \Gamma_4, \Gamma'_4} \sum_{\Gamma} \mathcal{S}_{\mu'_3, \mu_3}^{xyz} (\mathcal{F}^{T^2})_{\mu_2 \mu'_3}^{\Gamma_2} (\mathcal{S}^{xyz})_{\Gamma'_2, \Gamma_2}^{-1} \mathcal{S}_{\mu'_5, \mu_5}^{xyz} (\mathcal{F}^{T^2})_{\mu_4 \mu'_5}^{\Gamma_4} (\mathcal{S}^{xyz})_{\Gamma'_4, \Gamma_4}^{-1} (\mathcal{F}^{T^2})_{\Gamma'_2, \Gamma'_4}^{\Gamma} Z[S^4; \Gamma, \mu_1] \quad (3.32) \\
&= \sum_{\mu'_3, \Gamma_2, \Gamma'_2} \sum_{\mu'_5, \Gamma_4, \Gamma'_4} \sum_{\Gamma, \Gamma', \Gamma_1, \Gamma'_1} \mathcal{S}_{\mu'_3, \mu_3}^{xyz} (\mathcal{F}^{T^2})_{\mu_2 \mu'_3}^{\Gamma_2} (\mathcal{S}^{xyz})_{\Gamma'_2, \Gamma_2}^{-1} \mathcal{S}_{\mu'_5, \mu_5}^{xyz} (\mathcal{F}^{T^2})_{\mu_4 \mu'_5}^{\Gamma_4} (\mathcal{S}^{xyz})_{\Gamma'_4, \Gamma_4}^{-1} \\
&\quad \cdot (\mathcal{F}^{T^2})_{\Gamma'_2, \Gamma'_4}^{\Gamma} (\mathcal{S}^{xyz})_{\Gamma', \Gamma}^{-1} (\mathcal{F}^{T^2})_{\mu_1 \Gamma'}^{\Gamma_1} \mathcal{S}_{\Gamma'_1, \Gamma_1}^{xyz} L_{0,0,\Gamma'_1}^{\text{Tri}}. \quad (3.33)
\end{aligned}$$

Here we do the Step 1, Step 2 and Step 3 surgeries on $\text{Spun}[\text{Hopf}[\mu_3, \mu_2]]$ first, then do the same 3-step surgeries on $\text{Spun}[\text{Hopf}[\mu_5, \mu_4]]$ later, then we obtain Eq.(3.31). While in Eq.(3.31), the new T^2 -worldsheets Γ'_2 and Γ'_4 have no triple-linking with the worldsheet μ_1 . Here Γ'_2 and Γ'_4 are arranged in the $D^3 \times S^1$ part of the S^4 manifold, while μ_1 is in the $S^2 \times D^2$ part of the S^4 manifold. Indeed, Γ'_2 and Γ'_4 can be fused together in parallel to a new T^2 -worldsheet Γ via the fusion algebra $(\mathcal{F}^{T^2})_{\Gamma'_2, \Gamma'_4}^{\Gamma}$, so we obtain Eq.(3.32). Then we apply the Step 4, Step 5 and Step 6 surgeries on the T^2 -worldsheets Γ and μ_1 of $Z[S^4; \text{Spun}[\Gamma], \mu_1] = Z[S^4; \Gamma, \mu_1]$ in Eq.(3.32), we obtain the final form Eq.(3.33).

Use Eqs.(3.16),(3.26),(3.30) and (3.33), and plug them into the path integral surgery relations, after massaging the relations, we derive a new quantum surgery formula (namely Eq.(3.12) in the

earlier text):

$$\begin{aligned}
& Z \left(\begin{array}{c} S^4 \\ \text{1} \end{array} \right) Z \left(\begin{array}{c} S^4 \\ \text{1} \\ \text{2} \\ \text{3} \\ \text{4} \\ \text{5} \end{array} \right) = Z \left(\begin{array}{c} S^4 \\ \text{1} \\ \text{2} \\ \text{3} \end{array} \right) Z \left(\begin{array}{c} S^4 \\ \text{1} \\ \text{4} \\ \text{5} \end{array} \right) \\
& \Rightarrow L_{0,0,\mu_1}^{\text{Tri}} \cdot \sum_{\Gamma, \Gamma', \Gamma_1, \Gamma'_1} (\mathcal{F}^{T^2})_{\Gamma'_2, \Gamma'_4}^{\Gamma} (\mathcal{S}^{xyz})_{\Gamma', \Gamma}^{-1} (\mathcal{F}^{T^2})_{\mu_1 \Gamma', \Gamma'_1}^{\Gamma_1} \mathcal{S}_{\Gamma'_1, \Gamma_1}^{xyz} L_{0,0,\Gamma'_1}^{\text{Tri}} \\
& = \sum_{\Gamma''_2, \eta_2, \eta'_2} (\mathcal{S}^{xyz})_{\Gamma''_2, \Gamma'_2}^{-1} (\mathcal{F}^{T^2})_{\mu_1 \Gamma''_2}^{\eta_2} \mathcal{S}_{\eta_2, \eta_2}^{xyz} L_{0,0,\eta'_2}^{\text{Tri}} \cdot \sum_{\Gamma'_4, \eta_4, \eta'_4} (\mathcal{S}^{xyz})_{\Gamma'_4, \Gamma'_4}^{-1} (\mathcal{F}^{T^2})_{\mu_1 \Gamma'_4}^{\eta_4} \mathcal{S}_{\eta'_4, \eta_4}^{xyz} L_{0,0,\eta'_4}^{\text{Tri}},
\end{aligned}$$

here only $\mu_1, \Gamma'_2, \Gamma'_4$ are the fixed indices, other indices are summed over.

3.3.3 More discussions

For all path integrals of S^4 in Eqs.(3.12), (3.13) and (3.15), each S^4 is cut into two D^4 pieces, so $D^4 \cup_{S^3} D^4 = S^4$. We choose all the dashed cuts for 3+1D path integral representing $B = S^3$, while we can view the S^3 as a spatial slice, with the following excitation configurations:

- (i). A loop in Eq.(3.12) on the slice $B = S^3$. Thus, we consider a natural 1-dimensional Hilbert space (GSD=1), for a single shrinkable loop excitation configuration on the spatial S^3 .
- (ii). A pair of particle-antiparticle in Eq.(3.13) on the slice $B = S^3$. Thus, we consider a natural 1-dimensional Hilbert space (GSD=1), for this particle-antiparticle configuration on the spatial S^3 .
- (iii). A pair of loop-antiloop in Eq.(3.15) on the slice $B = S^3$. In this case, here we require a stronger criterion that all loop excitations are gapped without zero modes, then the GSD is 1 for all above spatial section S^3 .

Thus all our surgeries satisfy the assumptions for Eq.(3.2).

3.3.4 Formulas for 3+1D fusion statistics

The above Verlinde-like formulas constrain the fusion data (e.g. \mathcal{N} , \mathcal{F}^{S^1} , \mathcal{F}^{S^2} , \mathcal{F}^{T^2} , etc.) and braiding data (e.g. \mathcal{S} , \mathcal{T} , $L^{(S^2, S^1)}$, L^{Tri} , \mathcal{S}^{xyz} , etc.). Moreover, we can derive constraints between the fusion data itself. Since a T^2 -worldsheet contains two non-contractible S^1 -worldlines along its two homology group generators in $H_1(T^2, \mathbb{Z}) = \mathbb{Z}^2$, the T^2 -worldsheet operator $V_{\mu}^{T^2}$ contains the data of S^1 -worldline operator $W_{\sigma}^{S^1}$. More explicitly, we can compute the state $W_{\sigma_1}^{S^1} W_{\sigma_2}^{S^1} V_{\mu_2}^{T^2} |0_{D_{wx}^2 \times T_{yz}^2}\rangle$ by fusing two $W_{\sigma}^{S^1}$ operators and one $V_{\mu}^{T^2}$ operator in different orders, then we obtain a consistency formula:

$$\sum_{\sigma_3} (\mathcal{F}^{S^1})_{\sigma_1 \sigma_2}^{\sigma_3} (\mathcal{F}^{T^2})_{\sigma_3 \mu_2}^{\mu_3} = \sum_{\mu_1} (\mathcal{F}^{T^2})_{\sigma_2 \mu_2}^{\mu_1} (\mathcal{F}^{T^2})_{\sigma_1 \mu_1}^{\mu_3}. \quad (3.34)$$

We organize our quantum statistics data of fusion and braiding, and some explicit examples of topological orders and their topological invariances in terms of our data in the Supplemental Material.

Lastly we provide more explicit calculations of Eq. (3.34), the constraint between the fusion data itself. First, we recall that

$$\begin{aligned} W_{\sigma_1}^{S^1} W_{\sigma_2}^{S^1} &= (\mathcal{F}^{S^1})_{\sigma_1 \sigma_2}^{\sigma_3} W_{\sigma_3}^{S^1}, \\ V_{\mu_1}^{T^2} V_{\mu_2}^{T^2} &= (\mathcal{F}^{T^2})_{\mu_1 \mu_2}^{\mu_3} V_{\mu_3}^{T^2}, \\ W_{\sigma_1}^{S^1} V_{\mu_2}^{T^2} &= (\mathcal{F}^{T^2})_{\sigma_1 \mu_2}^{\mu_3} V_{\mu_3}^{T^2}. \end{aligned}$$

Of course, the fusion algebra is symmetric respect to exchanging the lower indices, $(\mathcal{F}^{T^2})_{\sigma_1 \mu_2}^{\mu_3} = (\mathcal{F}^{T^2})_{\mu_2 \sigma_1}^{\mu_3}$. We can regard the fusion algebra $(\mathcal{F}^{S^1})_{\sigma_1 \sigma_2}^{\sigma_3}$ and $(\mathcal{F}^{T^2})_{\sigma_1 \mu_2}^{\mu_3}$ with worldlines as a part of a larger algebra of the fusion algebra of worldsheets $(\mathcal{F}^{T^2})_{\mu_1 \mu_2}^{\mu_3}$. We compute the state $W_{\sigma_1}^{S^1} W_{\sigma_2}^{S^1} V_{\mu_2}^{T^2} |0_{D_{wx}^2 \times T_{yz}^2}\rangle$ by fusing two $W_{\sigma}^{S^1}$ operators and one $V_{\mu}^{T^2}$ operator in different orders.

On one hand, we can fuse two worldlines first, then fuse with the worldsheet,

$$\begin{aligned} &W_{\sigma_1}^{S^1} W_{\sigma_2}^{S^1} V_{\mu_2}^{T^2} |0_{D_{wx}^2 \times T_{yz}^2}\rangle \\ &= \sum_{\sigma_3} (\mathcal{F}^{S^1})_{\sigma_1 \sigma_2}^{\sigma_3} W_{\sigma_3}^{S^1} V_{\mu_2}^{T^2} |0_{D_{wx}^2 \times T_{yz}^2}\rangle \\ &= \sum_{\sigma_3, \mu_3} (\mathcal{F}^{S^1})_{\sigma_1 \sigma_2}^{\sigma_3} (\mathcal{F}^{T^2})_{\sigma_3 \mu_2}^{\mu_3} V_{\mu_3}^{T^2} |0_{D_{wx}^2 \times T_{yz}^2}\rangle. \end{aligned} \quad (3.35)$$

On the other hand, we can fuse a worldline with the worldsheet first, then fuse with another worldline,

$$\begin{aligned} &W_{\sigma_1}^{S^1} W_{\sigma_2}^{S^1} V_{\mu_2}^{T^2} |0_{D_{wx}^2 \times T_{yz}^2}\rangle \\ &= \sum_{\mu_1} W_{\sigma_1}^{S^1} (\mathcal{F}^{T^2})_{\sigma_2 \mu_2}^{\mu_1} V_{\mu_1}^{T^2} |0_{D_{wx}^2 \times T_{yz}^2}\rangle \\ &= \sum_{\mu_1, \mu_3} (\mathcal{F}^{T^2})_{\sigma_2 \mu_2}^{\mu_1} (\mathcal{F}^{T^2})_{\sigma_1 \mu_1}^{\mu_3} V_{\mu_3}^{T^2} |0_{D_{wx}^2 \times T_{yz}^2}\rangle \end{aligned} \quad (3.36)$$

Therefore, by comparing Eqs.(3.35) and (3.36), we derive a consistency condition for fusion algebra Eq. (3.34): $\sum_{\sigma_3} (\mathcal{F}^{S^1})_{\sigma_1 \sigma_2}^{\sigma_3} (\mathcal{F}^{T^2})_{\sigma_3 \mu_2}^{\mu_3} = \sum_{\mu_1} (\mathcal{F}^{T^2})_{\sigma_2 \mu_2}^{\mu_1} (\mathcal{F}^{T^2})_{\sigma_1 \mu_1}^{\mu_3}$.

4 Summary of Quantum Statistics Data of Fusion and Braiding

We organize the quantum statistics data of fusion and braiding introduced in the earlier text into Table 2. We propose using the set of data in Table 2 to label or characterize topological orders, although such labels may only partially characterize topological orders. We also remark that Table 2 may not contain all sufficient data to characterize and classify all topological orders. What can be the missing data in Table 2? Clearly, there is the chiral central charge $c_- = c_L - c_R$, the difference between the left and right central charges, missing for 2+1D topological orders. The c_- is essential for describing 2+1D topological orders with 1+1D boundary gapless chiral edge modes. The gapless chiral edge modes cannot be fully gapped out by adding scattering terms between different modes,

Quantum statistics data of fusion and braiding

Data for **2+1D** topological orders:

- Fusion data:

$\mathcal{N}_{\sigma_1\sigma_2}^{\sigma_3} = \mathcal{F}_{\sigma_1\sigma_2}^{\sigma_3}$ (fusion tensor) from Eq. (2.23),

- Braiding data:

$\mathcal{S}^{xy}, \mathcal{T}^{xy}$ (modular $\text{SL}(2, \mathbb{Z})$ matrices from $\text{MCG}(T^2)$) from Eq. (2.26) and Eq. (2.28),

$Z[T_{xyt}^3; \sigma'_{1x}, \sigma'_{2y}, \sigma'_{3t}]$ (or $Z[S^3; \text{BR}[\sigma_1, \sigma_2, \sigma_3]]$) from Eq. (2.29), etc.

Data for **3+1D** topological orders:

- Fusion data:

$(\mathcal{F}^{S^1})_{\sigma_1\sigma_2}^{\sigma_3}, (\mathcal{F}^{S^2})_{\mu_1\mu_2}^{\mu_3}, (\mathcal{F}^{T^2})_{\mu_1\mu_2}^{\mu_3}$. (fusion tensor) from Eq. (2.34), Eq. (2.35), and Eq. (2.36).

- Braiding data:

$\mathcal{S}^{xyz}, \mathcal{T}^{xy}$ (modular $\text{SL}(3, \mathbb{Z})$ matrices from $\text{MCG}(T^3)$) from Eq. (2.48) and Eq. (2.49),

including \mathcal{S}^{xy}

$L_{0,0,\mu}^{\text{Tri}}$ (from $L_{\mu_3,\mu_2,\mu_1}^{\text{Tri}}$ of Eq. (2.43)), $L_{\mu\sigma}^{\text{Lk}(S^2, S^1)}$ from Eq. (2.39),

$Z[T^4 \# S^2 \times S^2; \mu'_4, \mu'_3, \mu'_2, \mu'_1]$ from Eq. (2.46)

(from $Z[S^4; \text{Link}[\text{Spun}[\text{BR}[\mu_4, \mu_3, \mu_2]], \mu_1]]$ of Eq. (2.44)), etc.

Table 2: Some data for 2+1D and 3+1D topological orders encodes their quantum statistics properties, such as fusion and braiding statistics of their quasi-excitations (anyonic particles and anyonic strings). However, the data is not complete because we do not account the degrees of freedom of their boundary modes, such as the chiral central charge $c_- = c_L - c_R$ for 2+1D topological orders.

because they are protected by the net chirality. So our 2+1D data only describes *2+1D non-chiral topological orders*. Similarly, our 2+1D/3+1D data may not be able to fully classify 2+1D/3+1D topological orders whose boundary modes are *protected to be gapless*. We may need additional data to encode boundary degrees of freedom for their boundary modes.

In some case, some of our data may overlap with the information given by other data. For example, the 2+1D topological order data $(\mathcal{S}_{xy}, \mathcal{T}_{xy}, \mathcal{N}_{\sigma_1\sigma_2}^{\sigma_3})$ may contain the information of $Z[T_{xyt}^3; \sigma'_{1x}, \sigma'_{2y}, \sigma'_{3t}]$ (or $Z[S^3; \text{BR}[\sigma_1, \sigma_2, \sigma_3]]$) already, since we know that we the former set of data may fully classify 2+1D bosonic topological orders.

Although it is possible that there are extra required data beyond what we list in Table 2, we find that Table 2 is sufficient enough for a large class of topological orders, at least for those described by Dijkgraaf-Witten twisted gauge theory [31] and those gauge theories with finite Abelian gauge groups. In the next Section, we will give some explicit examples of 2+1D and 3+1D topological orders described by Dijkgraaf-Witten topological gauge theory and group cohomology [31], which can be completely characterized and classified by the data given in Table 2.

5 Examples of Topological Orders, TQFTs and Topological Invariants

In Table 3, we give some explicit examples of 2+1D and 3+1D topological orders from Dijkgraaf-Witten twisted gauge theory. We like to emphasize that our quantum-surgery Verlinde-like formulas apply to generic 2+1D and 3+1D topological orders beyond the gauge theory or field theory description. So our formulas apply to quantum phases of matter or theories beyond the Dijkgraaf-Witten twisted gauge theory description. We list down these examples only because these are famous examples with a more familiar gauge theory understanding. In terms of topological order language, Dijkgraaf-Witten theory describes the low energy physics of certain bosonic topological orders which can be regularized on a lattice Hamiltonian [24, 26, 34] with local bosonic degrees of freedom (without local fermions, but there can be emergent fermions and anyons).

We also clarify that what we mean by the correspondence between the items in the same row in Table 3:

- (i) Quantum statistic braiding data,
- (ii) Group cohomology cocycles
- (iii) Topological quantum field theory (TQFT).

What we mean is that we can distinguish the topological orders of given cocycles of (ii) with the low energy TQFT of (iii) by measuring their quantum statistic Berry phase under the prescribed braiding process in the path integral of (i). The Euclidean path integral of (i) is defined through the action \mathbf{S} of (iii) via

$$Z = \int [DB_I][DA_I] \exp[-\mathbf{S}]. \quad (5.1)$$

For example, the mutual braiding (Hopf linking) measures the \mathcal{S} matrix distinguishing different types of $\int \frac{iN_I}{2\pi} B^I \wedge dA^I + \frac{i p_{IJ}}{2\pi} A^I \wedge dA^J$ with different p_{IJ} couplings; while the Borromean ring braiding can distinguish different types of $\int \frac{iN_I}{2\pi} B^I \wedge dA^I + i c_{123} A^1 \wedge A^2 \wedge A^3$ with different c_{123} couplings. However, the table does not mean that we cannot use braiding data in one row to measure the TQFT in another row. For example, \mathcal{S} matrix can also distinguish the $\int \frac{iN_I}{2\pi} B^I \wedge dA^I + i c_{123} A^1 \wedge A^2 \wedge A^3$ -type theory. However, $Z[S^3; \text{BR}[\sigma_1, \sigma_2, \sigma_3]] = Z[T_{xyt}^3; \sigma'_{1x}, \sigma'_{2y}, \sigma'_{3t}] = 1$ is trivial for $\int \frac{iN_I}{2\pi} B^I \wedge dA^I + \frac{i p_{IJ}}{2\pi} A^I \wedge dA^J$ with any p_{IJ} . Thus Borromean ring braiding cannot measure nor distinguish the nontrivial-ness of p_{IJ} -type theories.

Recently, after the appearance of our previous work [1, 2], further progress has been made on systematically and rigorously deriving the topological invariants of TQFTs, such as:

1. The TQFT link invariants [33],
2. The GSD data and the partition function [35] without extended operator insertions, with or without topological boundary,

directly from the continuum bosonic TQFT formulation.

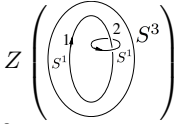
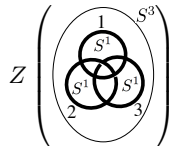
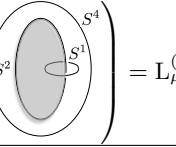
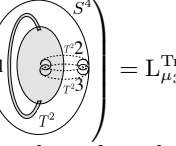
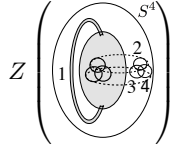
(i). Path-integral linking invariants; Quantum statistic braiding data	(ii). Group-cohomology cocycles distinguished by the braiding in (i)	(iii). TQFT actions \mathbf{S} characterized by the spacetime-braiding in (i)
2+1D		
 $Z \left(\begin{array}{c} S^3 \\ \text{1} \\ \text{2} \\ S^1 \end{array} \right)$ $= Z[S^3; \text{Hopf}[\sigma_1, \sigma_2]] = \mathcal{S}_{\sigma_1, \sigma_2}$	$\exp \left(\frac{2\pi i p_{IJ}}{N_I N_J} a_I (b_J + c_J - [b_J + c_J]) \right)$	$i \int \frac{N_I}{2\pi} B^I \wedge dA^I + \frac{p_{IJ}}{2\pi} A^I \wedge dA^J$ $A^I \rightarrow A^I + dg^I,$ $N_I B^I \rightarrow N_I B^I + d\eta^I.$
 $Z \left(\begin{array}{c} S^3 \\ \text{1} \\ \text{2} \\ \text{3} \\ S^1 \end{array} \right)$ $= Z[S^3; \text{BR}[\sigma_1, \sigma_2, \sigma_3];$ $\text{Also } Z[T_{xyt}^3; \sigma'_{1x}, \sigma'_{2y}, \sigma'_{3t}]$	$\exp \left(\frac{2\pi i p_{123}}{N_{123}} a_1 b_2 c_3 \right)$	$i \int \frac{N_I}{2\pi} B^I \wedge dA^I + c_{123} A^1 \wedge A^2 \wedge A^3$ $A^I \rightarrow A^I + dg^I,$ $N_I B^I \rightarrow N_I B^I + d\eta^I + 2\pi \tilde{c}_{IJK} A^J g^K$ $- \pi \tilde{c}_{IJK} g^J dg^K.$
3+1D		
 $Z \left(\begin{array}{c} S^4 \\ S^2 \\ S^1 \end{array} \right) = L_{\mu\sigma}^{(S^2, S^1)}$	1	$i \int \frac{N_I}{2\pi} B^I \wedge dA^I$ $A^I \rightarrow A^I + dg^I,$ $N_I B^I \rightarrow N_I B^I + d\eta^I.$
 $Z \left(\begin{array}{c} S^4 \\ \text{1} \\ \text{2} \\ \text{3} \\ T^2 \end{array} \right) = L_{\mu_3, \mu_2, \mu_1}^{\text{Tri}}$ $= Z[S^4; \text{Link}[\text{Spun}[\text{Hopf}[\mu_3, \mu_2]], \mu_1]]$	$\exp \left(\frac{2\pi i p_{IJK}}{(N_{IJ} \cdot N_K)} (a_I b_J) (c_K + d_K - [c_K + d_K]) \right)$	$i \int \frac{N_I}{2\pi} B^I \wedge dA^I + \sum_{I,J} \frac{N_I N_J p_{IJK}}{(2\pi)^2 N_{IJ}} A^I \wedge A^J \wedge dA^K$ $A^I \rightarrow A^I + dg^I,$ $N_I B^I \rightarrow N_I B^I + d\eta^I + \epsilon_{IJ} \frac{N_I N_J p_{IJK}}{2\pi N_{IJ}} dg^J \wedge A^K,$ <p style="text-align: center;">here K is fixed.</p>
 $Z \left(\begin{array}{c} S^4 \\ \text{1} \\ \text{2} \\ \text{3} \\ \text{4} \\ T^2 \end{array} \right)$ $= Z[S^4; \text{Link}[\text{Spun}[\text{BR}[\mu_4, \mu_3, \mu_2]], \mu_1]];$ $\text{Also } Z[T^4 \# S^2 \times S^2; \mu'_4, \mu'_3, \mu'_2, \mu'_1]$	$\exp \left(\frac{2\pi i p_{1234}}{N_{1234}} a_1 b_2 c_3 d_4 \right)$	$i \int \frac{N_I}{2\pi} B^I \wedge dA^I + c_{1234} A^1 \wedge A^2 \wedge A^3 \wedge A^4$ $A^I \rightarrow A^I + dg^I,$ $N_I B^I \rightarrow N_I B^I + d\eta^I - \pi \tilde{c}_{IJKL} A^J A^K g^L$ $+ \pi \tilde{c}_{IJKL} A^J g^K dg^L - \frac{\pi}{3} \tilde{c}_{IJKL} g^J dg^K dg^L.$

Table 3: Examples of topological orders and their topological invariances in terms of our data in the spacetime dimension $d + 1D$. Here some explicit examples are given as Dijkgraaf-Witten twisted gauge theory [31] with finite gauge group, such as $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3} \times \mathbb{Z}_{N_4} \times \dots$, although our quantum statistics data can be applied to more generic quantum systems without gauge or field theory description. The first column shows the path integral form which encodes the braiding process of particles and strings in the spacetime. In terms of spacetime picture, the path integral has nontrivial linkings of worldlines and worldsheets. The geometric Berry phases produced from this adiabatic braiding process of particles and strings yield the measurable quantum statistics data. This data also serves as topological invariances for topological orders. The second column shows the group-cohomology cocycle data ω as a certain partition-function solution of Dijkgraaf-Witten theory, where ω belongs to the group-cohomology group, $\omega \in \mathcal{H}^{d+1}[G, \mathbb{R}/\mathbb{Z}] = \mathcal{H}^{d+1}[G, \text{U}(1)]$. The third column shows the proposed continuous low-energy field theory action form for these theories and their gauge transformations. In 2+1D, A and B are 1-forms, while g and η are 0-forms. In 3+1D, B is a 2-form, A and η are 1-forms, while g is a 0-form. Here $I, J, K \in \{1, 2, 3, \dots\}$ belongs to the gauge subgroup indices, $N_{12\dots u} \equiv \text{gcd}(N_1, N_2, \dots, N_u)$ is defined as the greatest common divisor (gcd) of N_1, N_2, \dots, N_u . Here $p_{IJ} \in \mathbb{Z}_{N_{IJ}}, p_{123} \in \mathbb{Z}_{N_{123}}, p_{IJK} \in \mathbb{Z}_{N_{IJK}}, p_{1234} \in \mathbb{Z}_{N_{1234}}$ are integer coefficients. The $c_{IJ}, c_{123}, c_{IJK}, c_{1234}$ are quantized coefficients labeling distinct topological gauge theories, where $c_{12} = \frac{1}{(2\pi)} \frac{N_1 N_2 p_{12}}{N_{12}}, c_{123} = \frac{1}{(2\pi)^2} \frac{N_1 N_2 N_3 p_{123}}{N_{123}}, c_{1234} = \frac{1}{(2\pi)^3} \frac{N_1 N_2 N_3 N_4 p_{1234}}{N_{1234}}$. Be aware that we define both $p_{IJ\dots}$ and $c_{IJ\dots}$ as constants with *fixed-indices* I, J, \dots without summing over those indices; while we additionally define $\tilde{c}_{IJ\dots} \equiv \epsilon_{IJ\dots} c_{12\dots}$ with the $\epsilon_{IJ\dots} = \pm 1$ as an anti-symmetric Levi-Civita alternating tensor where I, J, \dots are *free indices* needed to be Einstein-summed over, but $c_{12\dots}$ is fixed. The lower and upper indices need to be summed-over, for example $\int \frac{N_I}{2\pi} B^I \wedge dA^I$ means that $\int \sum_{I=1}^s \frac{N_I}{2\pi} B^I \wedge dA^I$ where the value of s depends on the total number s of gauge subgroups $G = \prod_i^s \mathbb{Z}_{N_i}$. The quantization labelings are described and derived in [26, 32]. See the explicit verification of our proposed link invariants from continuum TQFT formulations in [33].

The relevant field theories are also discussed in Ref. [32, 36–41], here we systematically summarize and claim the field theories in Table 3 third column indeed describe the low energy TQFTs of Dijkgraaf-Witten theory. It can be checked that the continuum TQFT formulation can indeed match to Dijkgraaf-Witten theory [31] and its discrete cochain formulation [42], [26], [43, 44].

Readers can find the similarity of our Table 3 and Ref. [33]’s Table I. However, we emphasize that the derivations and the logic of our present work and Ref. [33] are somehow opposite.

I. Our present work [1, 2] starts from the following inputs:

$$\begin{aligned}
 & (1) \text{ Spacetime topology, geometric topology and surgery properties} \\
 & \xrightarrow{\text{derive}} (2) \text{ quantum surgery formulas} \\
 & \xrightarrow{\text{derive}} (3) \text{ possible link invariants and spacetime braiding process} \\
 & \xrightarrow{\text{detect}} (4) \text{ group cohomology cocycles and TQFTs,} \quad (5.2)
 \end{aligned}$$

which is going from the inputs of purely quantum mechanics and mathematical geometric topology, to obtain the left-hand-side (LHS) of Table 3, then to obtain the middle, then the right-hand-side (RHS) of Table 3.

II. Ref. [33] starts from the following inputs of:

$$\begin{aligned}
 & (4) \text{ group cohomology cocycles and continuum TQFTs} \\
 & \xrightarrow{\text{derive}} (3) \text{ link invariants and spacetime braiding process .} \quad (5.3)
 \end{aligned}$$

which is logically going from the opposite direction.

Further recently, the formulations of continuum TQFTs or discrete cochain TQFTs have also been generalized, from

1. TQFTs for the bosonic Dijkgraaf-Witten theory ([33] and References therein),
to
2. Fermionic version of TQFT gauge theory: Fermionic finite-group gauge theory and spin-TQFT, and their braiding statistics or topological link invariants [35, 45, 46]
and
3. Higher-gauge theory as TQFTs, see some selective examples in [47, 48], [37], [33], [49–53].

6 Conclusion

6.1 Comparison to Previous Works

It is known that the quantum statistics of particles in 2+1D begets anyons, beyond the familiar statistics of bosons and fermions, while Verlinde formula [15] plays a pivotal role to dictate the consistent

anyon statistics. In this work, we derive a set of quantum surgery formulas analogous to Verlinde's constraining the fusion and braiding quantum statistics of excitations of anyonic particle and anyonic string in 3+1D. We derive a set of fairly general quantum surgery formulas, which are also constraints of fusion and braiding data of topological orders. We work out the explicit derivations of Eq. (3.9) in 2+1D, Eqs. (3.12) and (3.13) in 3+1D, and then later we also derive Eq. (3.15) in 3+1D step by step. We also derive the fusion constraint Eq.(3.34) explicitly.

A further advancement of our work, comparing to the pioneer work of Witten Ref. [6] on 2+1D Chern-Simons gauge theory, is that we apply the surgery idea to generic 2+1D and 3+1D topological orders without assuming quantum field theory (QFT) or gauge theory description. Although many lattice-regularized topological orders happen to have TQFT descriptions at low energy, we may not know which topological order derives which TQFT easily. Instead we simply use quantum amplitudes written in the bra and ket (over-)complete bases, obtained from inserting worldline/sheet operators along the cycles of non-trivial homology group generators of a spacetime submanifold, to cut and glue to the desired path integrals. Consequently our approach, without the necessity of any QFT description, can be powerful to describe more generic quantum systems.¹⁴ While our result is originally based on studying specific examples of TQFT (such as Dijkgraaf-Witten gauge theory [31]), we formulate the data without using QFT. We have incorporated the necessary generic quantum statistic data and new constraints to characterize some 3+1D topological orders (including Dijkgraaf-Witten's), we will leave the issue of their sufficiency and completeness for future work. Formally, our approach can be applied to any spacetime dimensions.

6.2 Physics and Laboratory Realization, and Future Directions

Now we comment about the more physical and laboratory realization of implementing the cut-and-glue surgery procedure we discussed earlier. The matrix obtained from quantum wavefunction-overlap between different ground states of Hilbert space of a quantum matter,¹⁵ forms a representation of the mapping class group (MCG) of the real space manifold (M_{space}) where the quantum matter resides (See [54], [25, 26], and references therein). This matrix can be written as:

$$\varphi_{(\sigma', \mu', \dots), (\sigma, \mu, \dots)} \equiv \langle \psi_{\sigma', \mu', \dots} | \hat{\varphi} | \psi_{\sigma, \mu, \dots} \rangle, \quad (6.1)$$

$$\hat{\varphi} \in \text{MCG}(M_{\text{space}}), \quad | \psi_{\sigma, \mu, \dots} \rangle \in \mathcal{H}, \quad (6.2)$$

$$\text{rank}(\varphi_{(\sigma', \mu', \dots), (\sigma, \mu, \dots)}) = \text{GSD}_{M_{\text{space}}} = \dim(\mathcal{H}) = |Z(M_{\text{space}} \times S^1)|. \quad (6.3)$$

It is worthwhile to mention that for 2+1D topological order, one can construct group elements of the mapping class group of a genus g Riemann 2-surface \mathcal{M}_g , obtained via a series of projections on selecting the quasi-excitation sectors (i.e., projection by selecting the ground state sector $| \psi_{\sigma, \mu, \dots} \rangle$), along at least above a certain number of mutually intersecting non-contractible cycles on the \mathcal{M}_g [55]. It is noted that

(i) A genus g Riemann 2-surface \mathcal{M}_g can be realized in a 2D planar geometries via multi-layers or folded constructions of the desired topological orders, with the appropriate types of gapped boundaries designed.

¹⁴Namely, in our formulation and in our present derivation of quantum surgery formulas, we do *not* require the second quantization description of a quantum theory such as a QFT, which is more subtle to be formulated rigorously and mathematically. Instead we only require the first quantization description of a quantum theory via the standard mathematically well-defined standard *quantum mechanics*.

¹⁵ Preferably, the following discussion is rigorous and well-defined, when we limit our discussion such that quantum matter is topologically ordered, the so-called topological order.

(ii) The projections on selecting the quasi-excitation sectors (named topological charge projections in [55]) can be implemented as the adiabatic unitary deformations by tuning microscopic parameters of the quantum system locally (at the lattice scale or at the ultraviolet high energy locally).

The result suggests that, in 2+1D, we could potentially implement and realize the quantum representation of modular transformations or $MCG(\mathcal{M}_g)$ of topological orders in physical systems.

Future Directions:

1. It will be interesting to study the 3+1D story analogous to what [55] has done for the “mapping class group representation realization” in 2+1D, given that our present and earlier work indicate exotic braiding, fusion and “quantum surgeries” happen in 3+1D [1, 2].

2. There is a recent interest concerning a constraint of fusion and half-braiding process¹⁶ of boundary excitations of 2+1D topological orders (or 3d TQFTs), coined boundary’s defect Verlinde formula [56]. The approach relies on a tunneling matrix data \mathcal{W}_{ia} , defined in [57], which shows how an anyon i in 3d TQFT Phase A can decompose into a direct sum of a set of anyons $\oplus_a \mathcal{W}_{ia} a$ after tunneling to another 3d TQFT Phase B. Ref. [56] suggests a defect Verlinde formula (on 2d boundary of 3d TQFT) based on the relation of data of:

- The fusion rules of the boundary excitations in 2d, and
- The half-braiding or half-linking modular data (a modified modular \mathcal{S} matrix) for 3d TQFT with 2d boundary.

It will be an illuminating future direction to derive an analogous story for 3d (2+1D) boundary of a 4d (3+1D) TQFT with an analogous boundary excitation quantum surgery formulas like ours (Eqs. (3.12), (3.13) and (3.15) in 4d). Such a boundary’s defect Verlinde formula may be helpful to constrain other physical observables of systems of gapped boundary with defects, such as the *boundary topological degeneracy* [58–60] [57]. Moreover, by “the bulk-boundary 3d-2d correspondence,” we see that

♣ The 2d boundaries/interfaces of 3d TQFT systems can be regarded as the 2d defects surfaces in 3d TQFT;

via “dimensional reductions (lowering one dimension)” thus the above discussion intimately relates to

♣ The 1d boundaries/interfaces of 2d CFT, or the 1d defect lines in 2d CFT. The later direction of 2d-1d system had generated various interests in the past [61–66] and fairly recently [67–69].

Topological defects in higher dimensions viewed as the gapped interfaces/boundaries in any dimensions of group cohomology theory is explored recently in [70]. So Ref. [70]’s study may play a helpful guiding role along this direction. Possible outcomes along this direction may lead us to understand:

♠ Relation between *Verlinde formula* of 3d TQFT/2d CFT and the *defect Verlinde formula* [56] (on 2d boundary of 3d TQFT, or 1d defect line of 2d CFT).

to

♠ Relation between our *quantum surgery formulas* [1, 2] of 4d TQFT/3d CFT and the *defect analogous of these quantum surgery formulas* (on 3d boundary of 4d TQFT; or 1d defect line or 2d defect surface of 3d CFT).

3. It will be interesting to study the analogous Verlinde formula constraints for 2+1D boundary states of 3+1D bulk systems (thus relates to the direction 2), such as highly-entangled gapless modes, 3d

¹⁶The half-braiding means that the braiding of the bulk excitations in 3d (2+1D) is moving through a half-circle 1-worldline to the constrained 2d (1+1D) boundary.

conformal field theories (CFT) and 3d anomalies.¹⁷ For example, one can start from exploring the bulk-boundary correspondence between TQFT and CFT; such as for 3d TQFT and 2d CFT ([6] and the group-cohomology version of correspondence [71]), and for 4d TQFT and 3d CFT [72, 73], of some bulk quantum systems. The set of consistent quantum surgery formulas we derive may lead to an alternative effective way to *bootstrap* [74, 75] 3+1D topological states of matter and 2+1D CFT.

Note added: The formalism and some results discussed in this work have been partially reported in the first author’s Ph.D. thesis [2] and in [1]. Readers may refer to Ref. [2] for other discussions. Other related aspects of research will also be reported in upcoming work [17], [76] and [77].

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¹⁷The ’t Hooft anomalies of global symmetries can be realized as the non-onsite-ness of the global symmetry acting on the boundary states. The gauged version of the “whole” bulk-and-anomalous-boundary system also has interesting physical consequences. The interpretation of anomalies along the group cohomology theory can be found in [36, 70, 71] and References therein.

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