New Higher Anomalies, SU(N) Yang-Mills Gauge Theory and $\mathbb{CP}^{N-1}$ Sigma Model

Zheyan Wan[1] and Juven Wang[2,3,*]

[1] *School of Mathematical Sciences, USTC, Hefei 230026, China*
[2] *School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA*
[3] *Center of Mathematical Sciences and Applications, Harvard University, MA 02138, USA*

We hypothesize a new and more complete set of anomalies of certain quantum field theories (QFTs) and then give an eclectic proof. First, we propose a set of 't Hooft anomalies of 2d $\mathbb{CP}^{N-1}$-sigma models at $\theta = \pi$, with $N = 2, 3, 4$ and others, by enlisting all possible 3d cobordism invariants and selecting the matched terms. Second, we propose a set of 't Hooft higher anomalies of 4d time-reversal symmetric SU(N)-Yang-Mills (YM) gauge theory at $\theta = \pi$, via 5d cobordism invariants (higher symmetry-protected topological states) such that compactifying YM theory on a 2-torus matches the constrained 3d cobordism invariants from sigma models. Based on algebraic/geometric topology, QFT analysis, manifold generator dimensional reduction, condensed matter inputs and additional physics criteria, we derive a correspondence between 5d and 3d new invariants, thus broadly prove a more complete anomaly-matching between 4d YM and 2d $\mathbb{CP}^{N-1}$ models via a twisted 2-torus reduction, done by taking the Poincaré dual of specific cohomology class with $\mathbb{Z}_2$ coefficients. We formulate a higher-symmetry analog of "Lieb-Schultz-Mattis theorem" to constrain the low-energy dynamics.

CONTENTS

* juven@ias.edu

I. INTRODUCTION AND SUMMARY

Determining the dynamics and phase structures of strongly coupled quantum field theories (QFTs) is a challenging but important problem. For example, one of Millennium Problems is partly on showing the quantum Yang-Mills (YM) gauge theory [1] existence and mass gap: The fate of a pure YM theory with a SU(N) gauge group (i.e. we simply denote it as an SU(N)-YM), without additional matter fields, without topological term ($\theta = 0$), is confined and trivially gapped in Euclidean spacetime $\mathbb{R}^4$ [2]. A powerful tool to constrain the dynamics of QFTs is based on non-perturbative methods such as the 't Hooft anomaly-matching [3]. Although anomaly-matching may not uniquely determine the quantum dynamics, it can rule out some impossible quantum phases with mismatched anomalies, thus guiding us to focus only on favorable anomaly-matched phases for low energy phase structures of QFTs. The importance of dynamics and anomalies is not merely for a formal QFT side, but also on a more practical application to high-energy ultraviolet (UV) completion of QFTs, such as on a lattice regularization or condensed matter systems. (See, for instance [4] and references therein, a recent application of the anomalies, topological terms and dynamical constraints of SU(N)-YM gauge theories on UV-regulated condensed matter systems, obtained from dynamically gauging the SU(N)-symmetric *interacting* generalized topological superconductors/insulators [5, 6], or more generally Symmetry-Protected Topological state (SPTs) [7–9]).

In this work, we attempt to identify the potentially complete 't Hooft anomalies of 4d SU(N)-YM gauge theory and 2d $\mathbb{CP}^{\mathrm{N}-1}$-sigma model (here dd for d-dimensional spacetime) in Euclidean spacetime. Our main result is summarized in Fig. 1 and 2.

By completing 't Hooft anomalies of QFTs, we need to first identify the relevant (if not all of) the global symmetry G of QFTs. Then we couple the QFTs to classical background-symmetric gauge field of G, and try to detect the possible obstructions of such coupling [3]. Such obstructions, known as the obstruction of gauging the global symmetry, are named " 't Hooft anomalies."
In the literature, when people refer to "anomalies," it can means different things. To fix our terminology, we refer "anomalies" to one of the followings:

1. Classical global symmetry is violated at the quantum theory, such that the classical global symmetry fails to be a quantum global symmetry, e.g. the original Adler-Bell-Jackiw anomaly [10, 11].

2. Quantum global symmetry is well-defined and preserved. (Global symmetry is sensible, not only at a classical theory [if there is any classical description], but also for a quantum theory.) However, there is an obstruction to gauge the global symmetry. Specifically, we can detect a certain obstruction to even *weakly gauge* the symmetry or couple the symmetry to a *non-dynamical background probed gauge field*.[1] This is known as "'t Hooft anomalies," or sometimes regarded as "weakly gauged anomaly" in condensed matter.

3. Quantum global symmetry is well-defined and preserved. However, once we promote the global symmetry to a gauge symmetry of the dynamical gauge theory, then the gauge theory becomes ill-defined. Some people call this as a "dynamical gauge anomaly" which makes a quantum theory ill-defined.

Now "'t Hooft anomalies" (for simplicity, from now on, we may abbreviate them as "anomalies") have at least three intertwined interpretations:

Interpretation (1): In condensed matter physics, "'t Hooft anomalies" are known as the obstruction to lattice-regularize the global symmetry's quantum operator in a local on-site manner at UV due to symmetry-twists. (See [12–14] for QFT-oriented discussion and references therein.) This "non-onsite symmetry" viewpoint is generically applicable to both, *perturbative* anomalies, and *non-perturbative* anomalies:
• *perturbative* anomalies — Computable from perturbative Feynman diagram calculations.
• *non-perturbative or global* anomalies — Examples of global anomalies include the old and the new SU(2) anomalies [15, 16] (a caveat: here we mean their 't Hooft anomaly analogs if we view the SU(2) gauge field as a non-dynamical classical background, instead of dynamical field) and the global gravitational anomalies [17]. The occurrence of which types of anomalies are sensitive to the underlying UV-completion of, not only fermionic systems, but also bosonic systems [13, 18–20]. We call the anomalies of QFT whose UV-completion requires only the bosonic degrees of freedom as bosonic anomalies [18]; while those must require fermionic

[1] We will refer this kind of field simply as a background (non-dynamical gauge) field, abbreviated as "bgd.field."

degrees of freedom as fermionic anomalies.

Interpretation (2): In QFTs, the obstruction is on the impossibility of adding any counter term in its own dimension (d-d) in order to absorb a one-higher-dimensional counter term (e.g. $(d+1)$d topological term) due to background G-field [21]. This is named the "anomaly-inflow [22]." The $(d+1)$d topological term is known as the $(d+1)$d SPTs in condensed matter physics [7, 8].

Interpretation (3): In math, the dd anomalies can be systematically captured by $(d+1)$d topological invariants [15] known as cobordism invariants [23–26].

$\mathrm{N}=2$, 4d/2d anomalies and 5d/3d topological terms of 4d SU(N) YM theory and 2d $\mathbb{CP}^{\mathrm{N}-1}$-model:

5d topological invariant (4d anomaly):

$$B_2\mathrm{Sq}^1B_2+\mathrm{Sq}^2\mathrm{Sq}^1B_2+w_1(TM)^2\mathrm{Sq}^1B_2 = \frac{1}{2}\tilde{w}_1(TM)\mathcal{P}_2(B_2)+w_1(TM)^3B$$ [a]

Upper left panel:

5d manifold generator: $(\mathbb{RP}^2\times\mathbb{RP}^3, B=\alpha\cup\beta)$

↓ T^2 reduction (reduce $\alpha\beta$)

3d manifold generator: $(S^1\times\mathbb{RP}^2, w_1(E)=\gamma, w_2(V_{\mathrm{SO}(3)})=\gamma\cup\alpha)$

3d topological invariant (2d anomaly): detects $w_1(TM)w_2(V_{\mathrm{SO}(3)})$ or $w_1(E)w_1(TM)^2$

Upper right panel:

5d manifold generator: $(\mathbb{RP}^2\times\mathbb{RP}^3, B=\alpha\cup\beta+\alpha^2)$

↓ T^2 reduction (reduce $\alpha\beta$)

3d manifold generator: $(S^1\times\mathbb{RP}^2, w_1(E)=\gamma, w_2(V_{\mathrm{SO}(3)})=\gamma\cup\alpha)$

3d topological invariant (2d anomaly): detects $w_1(TM)w_2(V_{\mathrm{SO}(3)})$ or $w_1(E)w_1(TM)^2$

Middle left panel:

5d manifold generator: $(S^1\times\mathbb{RP}^2\times\mathbb{RP}^2, B=\gamma\cup\alpha_1)$

↓ T^2 reduction (reduce $\gamma\alpha_2$)

3d manifold generator: $(S^1\times\mathbb{RP}^2, w_1(E)=\gamma+\alpha, w_2(V_{\mathrm{SO}(3)})=0)$

3d topological invariant (2d anomaly): detects $w_1(E)^3$ or $w_1(E)w_1(TM)^2$

Middle right panel:

5d manifold generator: $(S^1\times\mathbb{RP}^4, B=\gamma\cup\zeta+\zeta^2)$

↓ T^2 reduction (reduce $\gamma\zeta$)

3d manifold generator: $(\mathbb{RP}^3, w_1(E)=\beta, w_2(V_{\mathrm{SO}(3)})=\beta^2)$

3d topological invariant (2d anomaly): detects $w_1(E)^3$ or $w_1(E)w_2(V_{\mathrm{SO}(3)})$

Lower left panel:

5d manifold generator: $(S^1\times\mathbb{RP}^2\times\mathbb{RP}^2, B=\gamma\cup\alpha_1)$

↓ T^2 reduction (reduce $\alpha_1\alpha_2$)

3d manifold generator: $(S^1\times S^1\times S^1, w_1(E)=\gamma_2+\gamma_3, w_2(V_{\mathrm{SO}(3)})=\gamma_1\gamma_2)$

3d topological invariant (2d anomaly): detects $w_1(E)w_2(V_{\mathrm{SO}(3)})$

[a] This formula is proved in Sec. VIII. Note that
$\mathrm{Sq}^2\mathrm{Sq}^1B_2=(w_2(TM)+w_1(TM)^2)\mathrm{Sq}^1B_2=(w_3(TM)+w_1(TM)^3)B_2$,
$B_2\mathrm{Sq}^1B_2=(\frac{1}{2}\tilde{w}_1(TM)\mathcal{P}_2(B_2)-(w_3(TM)+w_1(TM)^3)B_2)$ and $w_1(TM)^2\mathrm{Sq}^1B_2=w_1(TM)^3B_2$.

FIG. 1. The main result of our work, for the (higher) anomalies of 4d SU(N) YM and 2d $\mathbb{CP}^{\mathrm{N}-1}$ at $\mathrm{N}=2$. The 4d (higher) / 2d anomalies are uniquely specified by 5d /3d topological (cobordism) invariants and their manifold (bordism group) generators. Here $B=B_2$ is the 2-form gauge field in the YM gauge theory (at $\mathrm{N}=2$). E is a principal $\mathrm{O}(3)=\mathrm{SO}(3)\times\mathbb{Z}_2$ bundle over a 3-manifold, the corresponding principal SO(3) bundle is $V_{\mathrm{SO}(3)}$. α is the generator of $\mathrm{H}^1(\mathbb{RP}^2,\mathbb{Z}_2)$, β is the generator of $\mathrm{H}^1(\mathbb{RP}^3,\mathbb{Z}_2)$, γ is the generator of $\mathrm{H}^1(S^1,\mathbb{Z}_2)$, and ζ is the generator of $\mathrm{H}^1(\mathbb{RP}^4,\mathbb{Z}_2)$. "Reduce" means taking the Poincaré dual of certain cohomology class with $\mathbb{Z}_2$ coefficients. For $\mathrm{N}=2$, we gain an accidental $\mathbb{Z}_2^C\equiv\mathbb{Z}_2^x$ from the $\mathbb{Z}_2^x$-translation symmetry (see Sec. II C 2); this provides an explanation of the map from $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$ to $\Omega_3^{\mathrm{O}}(\mathrm{BO}(3))$: The $w_1(E)$ is reduced from $w_1(TM)$, namely $w_1(E)=w_1(TM)|_N$. The $w_2(V_{\mathrm{SO}(3)})$ is reduced from B, namely $w_2(V_{\mathrm{SO}(3)})=B|_N$. Here (N,E) is reduced from (M,B) by a 2-torus. More details are discussed in subsection VII A, the upper left panel is in 1.(a), the upper right panel is in 2.(a), the middle left panel is in 5.(a), the middle right panel is in 4.(a), the lower left panel is in 5.(b).

N = 4, 4d/2d anomalies and 5d/3d topological terms of 4d SU(N) YM theory / 2d $\mathbb{CP}^{\mathrm{N}-1}$-model:

5d topological invariant (4d anomaly):

$$\tilde{B}_2\beta_{(2,4)}B_2 + A^2\beta_{(2,4)}B_2 + AB_2w_1(TM)^2 = \frac{1}{4}\tilde{w}_1(TM)\mathcal{P}_2(B_2) + A^2\beta_{(2,4)}B_2 + AB_2w_1(TM)^2 \text{ [a]}$$

5d manifold generator:	$(S^1\times K\times T^2, A, B=\alpha'\beta'+\zeta')$	$(S^1\times K\times T^2, A, B=\alpha'\beta'+\zeta')$
	$\downarrow$ T^2 reduction, reduce ζ' mod 2	$\downarrow$ T^2 reduction, reduce $(\alpha'$ mod $2)(\beta'$ mod $2)$
3d manifold generator:	$(S^1\times K, w_1(E), w_2(E)=\alpha'\beta')$	$(S^1\times T^2, w_1(E), w_2(E)=\zeta')$
3d topological invariant: (2d anomaly)	detects $\beta_{(2,4)}w_2(E)$ $=\frac{1}{2}\tilde{w}_1(TM)w_2(E)$ [b]	detects $w_1(E)w_2(E)$
5d manifold generator:	$(S^1\times K\times \mathbb{RP}^2, A=\alpha, B=\alpha'\beta')$	$(S^1\times T^2\times \mathbb{RP}^2, A=\gamma+\gamma_2, B=\zeta')$
	$\downarrow$ T^2 reduction, reduce $(\beta'$ mod $2)\alpha$	$\downarrow$ T^2 reduction, reduce $\gamma\gamma_1$
3d manifold generator:	$(S^1\times T^2, w_1(E)=\gamma, w_2(E)=\zeta')$	$(S^1\times \mathbb{RP}^2, w_1(E)=\gamma_2, w_2(E)=0)$
3d topological invariant: (2d anomaly)	detects $w_1(E)w_2(E)$	detects $w_1(E)w_1(TM)^2$

[a] This formula is proved in Sec. VIII.
[b] This formula holds since we can prove that both LHS and RHS are bordism invariants of $\Omega_3^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2\ltimes \mathrm{PSU}(4)))$ and they coincide on the manifold generators of $\Omega_3^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2\ltimes \mathrm{PSU}(4)))$.

FIG. 2. The main result of our work, for the (higher) anomalies of 4d SU(N) YM and 2d $\mathbb{CP}^{\mathrm{N}-1}$ at N = 4. The 4d (higher) / 2d anomalies are uniquely specified by 5d /3d topological (cobordism) invariants and their manifold (bordism group) generators. Here $B=B_2$ is the 2-form gauge field in the YM gauge theory (at N = 4). K is the Klein bottle. E is a principal $\mathbb{Z}_2\ltimes\mathrm{PSU}(4)$ bundle over a 3-manifold. α' is the generator of $\mathrm{H}^1(S^1,\mathbb{Z}_4)$, β' is the generator of the $\mathbb{Z}_4$ factor of $\mathrm{H}^1(K,\mathbb{Z}_4)=\mathbb{Z}_4\times\mathbb{Z}_2$ (see Appendix C), ζ' is the generator of $\mathrm{H}^2(T^2,\mathbb{Z}_4)$. α is the generator of $\mathrm{H}^1(\mathbb{RP}^2,\mathbb{Z}_2)$, γ is the generator of $\mathrm{H}^1(S^1,\mathbb{Z}_2)$. Here $\zeta'=\alpha_1'\alpha_2'$ and α_i' mod $2=\gamma_i$. "Reduce" means taking the Poincaré dual of certain cohomology class with $\mathbb{Z}_2$ coefficients. The $w_1(E)$ is reduced from A, namely $w_1(E)=A|_N$. The $w_2(E)$ is reduced from B, namely $w_2(E)=B|_N$. Here (N,E) is reduced from (M,A,B) by a 2-torus. $\tilde{B}_2=B_2$ mod 2. More details are discussed in subsection VII B, the upper left panel is in 2.(b), the upper right panel is in 2.(a), the lower left panel is in 4.(a), the lower right panel is in 9.(b).

There is a long history of relating these two particular 4d SU(N)-YM and 2d $\mathbb{CP}^{\mathrm{N}-1}$ theories, since the work of Atiyah [27], Donaldson [28] and others, in the interplay of QFTs in physics and mathematics. Recently three key progresses shed new lights on their relations further:

(*i*) Higher symmetries and higher anomalies: The familiar 0-form global symmetry has a *charged* object of 0d measured by the *charge* operator of $(d-1)$d. The generalized q-form global symmetry, introduced by [29], demands a *charged* object of qd measured by the *charge* operator of $(d-q-1)$d (i.e. codimension-$(q+1)$). This concept turns out to be powerful to detect new anomalies, e.g. the pure SU(N)-YM at $\theta=\pi$ (See eq. (4)) has a mixed anomaly between 0-form time-reversal symmetry $\mathbb{Z}_2^T$ and 1-form center symmetry $\mathbb{Z}_{\mathrm{N},[1]}$ at an even integer N, firstly discovered in a remarkable work [30]. We review this result in Sec. IV, then we will introduce new anomalies (to our best understanding, these have not yet been identified in the previous literature) in later sections (Fig. 1 and 2.).

(*ii*) Relate (higher)-SPTs to (higher)-topological invariants: Follow the condensed matter literature, based on the earlier discussion on the *symmetry twist*, it has been recognized that the classical background-field partition function under the symmetry twist, called $\mathbf{Z}_{\text{sym.twist}}$ in $(d+1)$d can be regarded as the partition function of

$(d+1)$d SPTs $\mathbf{Z}_{\text{SPTs}}$. These descriptions are applicable to both low-energy infrared (IR) field theory, but also to the UV-regulated SPTs on a lattice, see [12, 13, and 24] and Refs. therein for a systematic set-up. Schematically, we follow the framework of [13],

$$\mathbf{Z}^{(d+1)\text{d}}_{\text{sym.twist}} = \mathbf{Z}^{(d+1)\text{d}}_{\text{SPTs}} = \mathbf{Z}^{(d+1)\text{d}}_{\text{topo.inv}} = \mathbf{Z}^{(d+1)\text{d}}_{\text{Cobordism.inv}} \longleftrightarrow d\text{d-(higher) 't Hooft anomaly.} \quad (1)$$

In general, the partition function $\mathbf{Z}_{\text{sym.twist}} = \mathbf{Z}_{\text{SPTs}}[A_1, B_2, w_i, \dots]$ is a functional containing background gauge fields of 1-form A_1, 2-form B_2 or higher forms; and can contain characteristic classes [31] such as the i-th Stiefel-Whitney class (w_i) and other geometric probes such as gravitational background fields, e.g. a gravitational Chern-Simons 3-form $\text{CS}_3(\Gamma)$ involving the Levi-Civita connection or the spin connection Γ. For convention, we use the capital letters $(A, B, ...)$ to denote *non-dynamical background gauge* fields (which, however, later they may or may not be dynamically gauged), while the little letters $(a, b, ...)$ to denote *dynamical gauge* fields.

More generally,

• For the ordinary 0-form symmetry, we can couple the charged 0d point operator to 1-form background gauge field (so the symmetry-twist occurs in the Poincaré dual codimension-1 sub-spacetime [dd] of SPTs).

• For the 1-form symmetry, we can couple the charged 1d line operator to 2-form background gauge field (so the symmetry-twist occurs in the Poincaré dual codimension-2 sub-spacetime [$(d-1)$d] of SPTs).

• For the q-form symmetry, we can couple the *charged* qd extended operator to $(q+1)$-form background gauge field. The *charged* qd extended operator can be measured by another *charge* operator of codimension-$(q+1)$ [i.e. $(d-q)$d]. So the symmetry-twist can be interpreted as the occurrence of the codimension-$(q+1)$ *charge* operator. Namely, the symmetry-twist happens at a Poincaré dual codimension-$(q+1)$ sub-spacetime [$(d-q)$d] of SPTs. We can view the measurement of a *charged* qd extended object, happening at any q-dimensional intersection between the $(q+1)$d form background gauge field and the codimension-$(q+1)$ symmetry-twist or *charge* operator of this SPT vacua.

For SPTs protected by higher symmetries (for generic q, especially for any SPTs with at least a symmetry of $q > 0$), we refer them as higher-SPTs. So our principle above is applicable to higher-SPTs [32–34]. In the following of this article, thanks to eq. (1), we can interchange the usages and interpretations of "higher SPTs $\mathbf{Z}_{\text{SPTs}}$," "higher topological terms due to symmetry-twist $\mathbf{Z}^{(d+1)\text{d}}_{\text{sym.twist}}$," "higher topological invariants $\mathbf{Z}^{(d+1)\text{d}}_{\text{topo.inv}}$," or "cobordism invariants $\mathbf{Z}^{(d+1)\text{d}}_{\text{Cobordism.inv}}$" in $(d+1)$d. They are all physically equivalent, and can uniquely determine a dd higher anomaly, when we study the anomaly of any boundary theory of the $(d+1)$d higher SPTs living on a manifold with dd boundary. Thus, we regard all of them as physically tightly-related given by eq. (1). In short, by turning on the classical background probed field (denoted as "bgd.field" in eq. (2)) coupled to dd QFT, under the symmetry transformation (i.e. symmetry twist), its partition function $\mathbf{Z}^{d\text{d}}_{\text{QFT}}$ can be *shifted*

$$\mathbf{Z}^{d\text{d}}_{\text{QFT}}\big|_{\text{bgd.field}=0} \longrightarrow \mathbf{Z}^{d\text{d}}_{\text{QFT}}\big|_{\text{bgd.field}\neq 0} \cdot \mathbf{Z}^{(d+1)\text{d}}_{\text{SPTs}}(\text{bgd.field}), \quad (2)$$

to detect the underlying $(d+1)$d topological terms/counter term/SPTs, namely the $(d+1)$d partition function $\mathbf{Z}^{(d+1)\text{d}}_{\text{SPTs}}$. To check whether the underlying $(d+1)$d SPTs really specifies a true dd 't Hooft anomaly unremovable from dd *counter term*, it means that $\mathbf{Z}^{(d+1)\text{d}}_{\text{SPTs}}(\text{bgd.field})$ cannot be absorbed by a lower-dimensional SPTs $\mathbf{Z}^{d\text{d}}_{\text{SPTs}}(\text{bgd.field})$, namely

$$\mathbf{Z}^{d\text{d}}_{\text{QFT}}\big|_{\text{bgd.field}} \cdot \mathbf{Z}^{(d+1)\text{d}}_{\text{SPTs}}(\text{bgd.field}) \neq \mathbf{Z}^{d\text{d}}_{\text{QFT}}\big|_{\text{bgd.field}} \cdot \mathbf{Z}^{d\text{d}}_{\text{SPTs}}(\text{bgd.field}). \quad (3)$$

(*iii*) Dimensional reduction: A very recent progress shows that a certain anomaly of 4d SU(N)-YM theory can be matched with another anomaly of 2d $\mathbb{CP}^{\text{N}-1}$ model under a 2-torus T^2 reduction in [35], built upon previous investigations [36, 37]. This development, together with the mathematical rigorous constraint from 4d and 2d instantons [27, 28], provides the evidence that the complete set of (higher) anomalies of 4d YM should be fully matched with 2d $\mathbb{CP}^{\text{N}-1}$ model under a T^2 reduction.[2]

In this work, we draw a wide range of knowledges, tools, comprehensions, and intuitions from:

• Condensed matter physics and lattice regularizations. Simplicial-complex regularized triangulable manifolds and smooth manifolds. This approach is related to our earlier Interpretation (1), and the progress (ii).

• QFT (continuum) methods: Path integral, higher symmetries associated to extended operators, etc. This is related to our earlier Interpretation (2), and the progress (i), (ii) and (iii).

• Mathematics: Algebraic topology methods include cobordism, cohomology and group cohomology theory. Geometric topology methods include the surgery theory, the dimensional-reduction of manifolds, and Poincaré duality, etc. This is related to our earlier Interpretation (3),

[2] The complex projective space $\mathbb{CP}^{\text{N}-1}$ is obtained from the moduli space of flat connections of SU(N) YM theory. (See [36] and Fig. 4.) This moduli space of flat connections do not have a canonical Fubini-Study metric and may have singularities. However, this subtle issue, between the $\mathbb{CP}^{\text{N}-1}$ target and the moduli space of flat connections, only affects the geometry issue, and should not affect the topological issue concerning non-perturbative global discrete anomalies that we focus on in this work.

and the progress (ii) and (iii).
Built upon previous results, we are able to derive a consistent story, which identifies, previously missing, thus, new higher anomalies in YM theory and in $\mathbb{CP}^{\mathrm{N}-1}$ model. A sublimed version of our result may count as an eclectic proof between the anomaly-matching between two theories under a 2-torus T^2 reduction from the 4d theory reduced to a 2d theory.

The outline of our article goes as follows.

In Sec. II, we comment and review on QFTs (relevant to YM theory and $\mathbb{CP}^{\mathrm{N}-1}$ model), their global symmetries, anomalies and topological invariants. This section can serve as an invitation for condensed matter colleagues, while we also review the relevant new concepts and notations to high energy/QFT theorists and mathematicians.

In Sec. III, we provide the solid results on the cobordisms, SPTs/topological terms, and manifold generators. This is relevant to our classification of all possible higher 't Hooft anomalies. Also it is relevant to our later eclectic proof on the anomalies of YM theory and $\mathbb{CP}^{\mathrm{N}-1}$ model.

In Sec. IV, we review the known anomalies in 4d YM theory and 2d $\mathbb{CP}^{\mathrm{N}-1}$ model, and explain their physical meanings, or re-derive them, in terms of mathematically precise cobordism invariants.

In Sec. V, Sec. VII and Fig. 4, we should cautiously remark that how 4d SU(N)-YM theory is related to 2d $\mathbb{CP}^{\mathrm{N}-1}$ model.

In Sec. V, in particular, we give our rules to constrain the anomalies for 4d YM theory and 2d $\mathbb{CP}^{\mathrm{N}-1}$ model, and for 5d and 3d invariants.

In Sec. VI, we present new anomalies for 2d $\mathbb{CP}^{\mathrm{N}-1}$ model.

In Sec. VII, we present mathematical formulations of dimensional reduction, from 5d to 3d of cobordism/SPTs/topological term, and from 4d to 2d of anomaly reduction.

In Sec. VIII, we present new higher anomalies for 4d SU(N) YM theory.

In Sec. IX, with the list of potentially complete 't Hooft anomalies of the above 4d SU(N)-YM and 2d $\mathbb{CP}^{\mathrm{N}-1}$-model at $\theta = \pi$, we constrain their low-energy dynamics further, based on the anomaly-matching. We discuss the higher-symmetry analog Lieb-Schultz-Mattis theorem. In particular, we check whether the 't Hooft anomalies of the above 4d SU(N)-YM and 2d $\mathbb{CP}^{\mathrm{N}-1}$-model can be saturated by a symmetric TQFT of their own dimensions, by the (higher-)symmetry-extension method generalized from the method of Ref. [14].

We conclude in Sec. X.

II. COMMENTS ON QFTS: GLOBAL SYMMETRIES AND TOPOLOGICAL INVARIANTS

A. 4d Yang-Mills Gauge Theory

Now we consider a 4d pure SU(N)-Yang-Mills gauge theory with θ-term, with a positive integer $\mathrm{N} \geq 2$, for a Euclidean partition function (such as an $\mathbb{R}^4$ spacetime) The path integral (or partition function) $\mathbf{Z}^{\text{4d}}_{\text{YM}}$ is formally written as,

$$\mathbf{Z}^{\text{4d}}_{\text{YM}} \equiv \int [\mathcal{D}a] \exp\big(-S_{\text{YM}+\theta}[a]\big) \equiv \int [\mathcal{D}a] \exp\big(-S_{\text{YM}}[a]\big) \exp\big(-S_{\theta}[a]\big) \equiv \int [\mathcal{D}a] \exp\Big(\Big(-\int_{M^4} (\frac{1}{g^2} \operatorname{Tr} F_a \wedge \star F_a) + \int_{M^4} (\frac{\mathrm{i}\theta}{8\pi^2} \operatorname{Tr} F_a \wedge F_a)\Big)\Big). \quad (4)$$

• a is the 1-form SU(N)-gauge field connection obtained from parallel transporting the principal-SU(N) bundle over the spacetime manifold M^4. The $a = a_\mu \mathrm{d}x^\mu = a^\alpha_\mu T^\alpha \mathrm{d}x^\mu$; here T^α is the generator of Lie algebra $\mathbf{g}$ for the gauge group (SU(N)), with the commutator $[T^\alpha, T^\beta] = \mathrm{i} f^{\alpha\beta\gamma} T^\gamma$, where $f^{\alpha\beta\gamma}$ is a fully anti-symmetric structure constant. Locally $\mathrm{d}x^\mu$ is a differential 1-form, the μ runs through the indices of coordinate of M^4. Then $a_\mu = a^\alpha_\mu T^\alpha$ is the Lie algebra valued gauge field, which is in the adjoint representation of the Lie algebra. (In physics, a_μ is the gluon vector field for quantum chromodynamics.) The $[\mathcal{D}a]$ is the path integral measure, for a certain configuration of the gauge field a. All allowed gauge inequivalent configurations are integrated over within the path integral measures $\int [\mathcal{D}a]$, where gauge redundancy is removed or mod out. The integration is under a weight factor $\exp\big(\mathrm{i} S_{\text{YM}+\theta}[a]\big)$.

• The $F_a = \mathrm{d}a - \mathrm{i}a \wedge a$ is the SU(N) field strength, while d is the exterior derivative and $\wedge$ is the wedge product; the $\star F_a$ is F_a's Hodge dual. The g is YM coupling constant.

• The $\operatorname{Tr}(F_a \wedge \star F_a)$ is the Yang-Mills Lagrangian [1] (a non-abelian generalization of Maxwell Lagrangian of U(1) gauge theory). The Tr denotes the trace as an invariant quadratic form of the Lie algebra of gauge group (here SU(N)). Note that $\operatorname{Tr}[F_a] = \operatorname{Tr}[\mathrm{d}a - \mathrm{i}a \wedge a] = 0$ is traceless for a SU(N) field strength. Under the variational principle, YM theory's classical equation of motion (EOM), in contrast to the linearity of U(1) Maxwell theory, is non-linear.

• The $(\frac{\theta}{8\pi^2}\mathrm{Tr}\, F_a \wedge F_a)$ term is named the θ-topological term, which does not contribute to the classical EOM.
• This path integral is physically sensible, but not precisely mathematically well-defined, because the gauge field can be freely chosen due to the gauge freedom. This problem occurs already for quantum U(1) Maxwell theory, but now becomes more troublesome due to the YM's non-abelian gauge group. One way to deal with the path integral and the quantization is the method by Faddeev-Popov [38] and De Witt [39]. However, in this work, we actually do not need to worry about of the subtlety of the gauging fixing and the details of the running coupling g for the full *quantum* theory part of this path integral. The reason is that we only aim to capture the 5d *classical* background field partition function $\mathbf{Z}^{(d+1)\mathrm{d}}_{\mathrm{sym.twist}} = \mathbf{Z}^{(d+1)\mathrm{d}}_{\mathrm{SPTs}}$ in eq. (1) that 4d YM theory must couple with in order to match the 't Hooft anomaly. Schematically, by coupling YM to background field, under the symmetry transformation, we expect that

$$\mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}\big|_{\mathrm{bgd.field}=0} \to \mathbf{Z}^{5\mathrm{d}}_{\mathrm{SPTs}}(\mathrm{bgd.field}) \cdot \mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}\big|_{\mathrm{bgd.field}\neq 0}. \tag{5}$$

For example, when a bgd.field is B,

$$\mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}(B=0) \to \mathbf{Z}^{5\mathrm{d}}_{\mathrm{SPTs}}(B\neq 0)\cdot \mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}(B\neq 0). \tag{6}$$

Our goal will be identifying the 5d topological term (5d SPTs) eq. (5) under coupling to background fields. We will focus on the Euclidean path integral of eq. (4).

B. SU(N)-YM theory: Mix higher-anomalies

Below we warm up by re-deriving the result on the mix higher-anomaly of time-reversal $\mathbb{Z}_2^T$ and 1-form center $\mathbb{Z}_\mathrm{N}$-symmetry of SU(N)-YM theory, firstly obtained in [30], from scratch. Our derivation will be as self-contained as possible, meanwhile we introduce useful notations.

1. *Global symmetry and preliminary*

For 4d SU(N)-Yang-Mills (YM) theory at $\theta = 0$ and π mod 2π, on an Euclidean $\mathbb{R}^4$ spacetime, we can identify its global symmetries: the 0-form time-reversal $\mathbb{Z}_2^T$ symmetry with time reversal $\mathcal{T}$ (see more details in Sec. II B 4), and 0-form charge conjugation $\mathbb{Z}_2^C$ with symmetry transformation $\mathcal{C}$ (see more details in Sec. II B 6). Since the parity $\mathcal{P}$ is guaranteed to be a symmetry due to $\mathcal{CPT}$ theorem (see more details in Sec. II B 6, or a version for Euclidean [40]), we can denote the full 0-form symmetry as $G_{[0]} = \mathbb{Z}_2^T \times \mathbb{Z}_2^C$. We also have the 1-form electric $G_{[1]} = \mathbb{Z}^e_{\mathrm{N},[1]}$ center symmetry [29].

So we find that the full global symmetry group "schematically" as

$$G = \mathbb{Z}^e_{\mathrm{N},[1]} \rtimes (\mathbb{Z}_2^T \times \mathbb{Z}_2^C), \tag{7}$$

which we intentionally omit the spacetime symmetry group.[3]

For N = 2, we actually have the semi-direct product "$\rtimes$" reduced to a direct product "$\times$," so we write

$$G = \mathbb{Z}^e_{2,[1]} \times \mathbb{Z}_2^T, \tag{8}$$

here we also do not have the $\mathbb{Z}_2^C$ charge conjugation global symmetry, due to that now becomes part of the SU(2) gauge group of YM theory. The non-commutative nature (the semi-direct product "$\rtimes$") of eq. (7) between 0-form and 1-form symmetries will be explained in the end of Sec. II B 6, after we first derive some preliminary knowledge below:

• The 0-form $\mathbb{Z}_2^T$ symmetry can be probed by "background symmetry-twist" if placing the system on non-orientable manifolds. The details of time-reversal symmetry transformation will be discussed in Sec. II B 4.
• The 1-form electric $\mathbb{Z}^e_{\mathrm{N},[1]}$-center symmetry (or simply 1-form $\mathbb{Z}_\mathrm{N}$-symmetry) can be coupled to 2-form background field B_2. The *charged* object of the 1-form $\mathbb{Z}^e_{\mathrm{N},[1]}$-symmetry is the gauge-invariant Wilson line

$$W_e = \mathrm{Tr}_\mathrm{R}(\mathrm{P}\exp(\mathrm{i}\oint a)). \tag{9}$$

The Wilson line W_e has the a viewed as a connection over a principal Lie group bundle (here SU(N)), which is parallel transported around the integrated closed loop resulting an element of the Lie group. P is the path ordering. The Tr is again the trace in the Lie algebra valued, over the irreducible representation R of the Lie group (here SU(N)). The spectrum of Wilson line W_e includes all representations of the given Lie group (here SU(N)). Specifying the local Lie algebra **g** is not enough, we need to also specify the gauge Lie group (here SU(N)) and other data, such as the set of extended operators and the topological terms, in order to learn the global structure and non-perturbative physics of gauge theory (See [41], and [4] for many examples).

For the SU(N) gauge theory we concern, the spectrum of purely electric Wilson line W_e includes the fundamental representation with a $\mathbb{Z}_\mathrm{N}$ class, which can be regarded as the $\mathbb{Z}_\mathrm{N}$ charge label of 1-form $\mathbb{Z}^e_{\mathrm{N},[1]}$-symmetry.

The 2-surface *charge* operator that measures the 1-form $\mathbb{Z}^e_{\mathrm{N},[1]}$-symmetry of the *charged* Wilson line is the

[3] One may wonder the role of parity $\mathcal{P}$ (details in Sec. II B 6), and a potential larger symmetry group $(\mathbb{Z}_2^T \times \mathbb{Z}_2^C \times \mathbb{Z}_2^P)$ for $G_{[0]}$. As we know that $\mathcal{CPT}$ transformation is almost a trivial and a tightly related to the spacetime symmetry group. It is at most a complex conjugation and anti-unitary operation in Minkowski signature. It is trivial in the Euclidean signature. It will become clear later we write down the full spacetime symmetry group and the internal symmetry group, when we show our cobordism calculation for the full global symmetry group including the spacetime and internal symmetries in Sec. III. More discussions on discrete symmetries in various YM gauge theories can be found in [4].

electric 2-surface operator that we denoted as U_e. The higher q-form symmetry ($q > 0$) needs to be abelian [29], thus the 1-from electric symmetry is associated to the $\mathbb{Z}_\mathrm{N}$ center subgroup part of SU(N), known as the 1-form $\mathbb{Z}^e_{\mathrm{N},[1]}$-symmetry.

If we place the Wilson line along the S^1 circle of the time or thermal circle, it is known as the Polyakov loop, which nonzero expectation value (i.e. breaking of the 0-form center dimensionally-reduced from 1-form center symmetry) serves as the *order parameter* of confinement-deconfinement transition.

Below we illuminate our understanding in details for the SU(2) YM theory (so we set N=2), which the discussion can be generalized to SU(N) YM.

1. We write the SU(2)-YM theory with a background 2-form $B \equiv B_2$ field coupling to 1-form $\mathbb{Z}^e_{\mathrm{N},[1]}$ as:

$$\begin{aligned}&\mathbf{Z}^{\mathrm{4d}}_{\mathrm{SU(2)YM}}[B] \\ &= \int [D\lambda]\, \mathbf{Z}^{\mathrm{4d}}_{\mathrm{YM}} \exp\big(\mathrm{i}\pi\lambda \cup (w_2(E) - B_2) + \mathrm{i}\frac{\pi}{2} p \mathcal{P}_2(B_2)\big),\end{aligned} \tag{10}$$

where $w_2(E)$ is the Stiefel-Whitney (SW) class of gauge bundle E, and B_2 is 2-form background field (or $\mathbb{Z}_2$-valued 2-cochain), both are non-dynamical probes. We see that integrating out λ, set $(w_2(E) - B_2) = 0 \mod 2$, thus $B_2 = w_2(E)$ is related. For $B_2 = 0$, there is no symmetry twist $w_2(E) = 0$. For $B_2 = w_2(E) \neq 0$, there is a twisted bundle or a so called symmetry twist. So we have an additional $\mathrm{i}\frac{\pi}{2} p \mathcal{P}_2(w_2(E))$ depending on $p \in \mathbb{Z}_4$. The Pontryagin square term $\mathcal{P}_2 : \mathrm{H}^2(-, \mathbb{Z}_{2^k}) \to \mathrm{H}^4(-, \mathbb{Z}_{2^{k+1}})$, here is given by

$$\mathcal{P}_2(B_2) = B_2 \cup B_2 + B_2 \underset{1}{\cup} \delta B_2 = B_2 \cup B_2 + B_2 \underset{1}{\cup} 2\mathrm{Sq}^1 B_2, \tag{11}$$

see more Sec. II B 3. With $\cup$ is a normal cup product and $\underset{1}{\cup}$ is a higher cup product. For readers who are not familiar with the mathematical details, see the introduction to mathematical background in [34]. The physical interpretation of adding $\frac{\pi}{2} p \mathcal{P}_2(B_2)$ with $p \in \mathbb{Z}_4$, is related to the fact of the YM vacua can be shifting by a higher-SPTs protected by 1-form symmetry, see Sec. II B 3.

2. The electric Wilson line W_e in the fundamental representation is dynamical and a genuine line operator. Wilson line W_e is on the boundary of a magnetic 2-surface $U_m = \exp(\mathrm{i}\pi w_2(E)) = \exp(\mathrm{i}\pi B_2)$. However, we can set $B_2 = 0$ since it is a probed field. So W_e is a genuine line operator, i.e. without the need to be at the boundary of 2-surface [29].

3. The magnetic 't Hooft line is on the boundary of an electric 2-surface $U_e = \exp(\mathrm{i}\pi\lambda)$. Since λ is dynamical, 't Hooft line is not genuine thus not in the line spectrum.

4. The electric 2-surface $U_e = \exp(\mathrm{i}\pi\lambda)$ measures 1-form e-symmetry, and it is dynamical. This can be seen from the fact that the 2-surface $w_2(E)$ is defined as a 2-surface defect (where each small 1-loop of 't Hooft line linked with this $w_2(E)$ getting a nontrivial π-phase $e^{\mathrm{i}\pi}$). The $w_2(E)$ has its boundary with Wilson loop W_e, such that $U_e U_m \sim \exp(\mathrm{i}\pi\lambda \cup w_2(E))$ specifies that when a 2-surface λ links with (i.e. wraps around) a 1-Wilson loop W_e, there is a nontrivial statistical π-phase $e^{\mathrm{i}\pi} = -1$. This type of a link of 2-surface and 1-loop in a 4d spacetime is widely known as the generalized Aharonov-Bohm type of linking, captured by a topological link invariant, see e.g. [42, 43] and references therein.

2. YM theory coupled to background fields

First we make a 2-form $\mathbb{Z}_\mathrm{N}$ field out of 2-form and 1-form U(1) fields. The 1-form global symmetry $G_{[1]}$ can be coupled to a 2-form background $\mathbb{Z}_\mathrm{N}$-gauge field B_2. In the continuum field theory, consider firstly a 2-form U(1)-gauge field B_2 and 1-form U(1)-gauge field C_1 such that

$$B_2 \text{ as a 2-form U(1) gauge field,} \tag{12}$$
$$C_1 \text{ as a 1-form U(1) gauge field,} \tag{13}$$
$$\mathrm{N}B_2 = \mathrm{d}C_1,\ B_2 \text{ as a 2-form } \mathbb{Z}_\mathrm{N} \text{ gauge field.} \tag{14}$$

that satisfactorily makes the continuum formulation of B_2 field as a 2-form $\mathbb{Z}_\mathrm{N}$-gauge field when we constrain an enclosed surface integral

$$\oint B_2 = \frac{1}{\mathrm{N}} \oint \mathrm{d}C_1 \in \frac{1}{\mathrm{N}} 2\pi\mathbb{Z}. \tag{15}$$

Now based on the relation $\mathrm{PSU(N)} = \frac{\mathrm{SU(N)}}{\mathbb{Z}_\mathrm{N}} = \frac{\mathrm{U(N)}}{\mathrm{U(1)}}$, we aim to have an SU(N) gauge theory coupled to a background 2-form $\mathbb{Z}_\mathrm{N}$ field. Here

$$\begin{aligned}&a \text{ as an SU(N) 1-form gauge field,}\\ &F_a = \mathrm{d}a - \mathrm{i}a \wedge a, \text{ as an SU(N) field strength,}\\ &\mathrm{Tr}[F_a] = \mathrm{Tr}[\mathrm{d}a - \mathrm{i}a \wedge a] = 0 \text{ traceless for SU(N).}\end{aligned} \tag{16}$$

We then promote the U(N) gauge theory with 1-form U(N) gauge field a', such that its normal subgroup U(1) is coupled to the background 1-form probed field C_1. Here we can identify the U(N) gauge field to the SU(N) and U(1) gauge fields via, up to details of gauge transformations [30],

$$\begin{aligned}&a' \text{ as an U(N) 1-form gauge field,}\\ &a' \simeq a + \mathbb{I}\frac{1}{\mathrm{N}} C_1,\\ &\mathrm{Tr}a' \simeq \mathrm{Tr}a + C_1 = C_1,\end{aligned} \tag{17}$$

$$\begin{aligned}&F_{a'} = \mathrm{d}a' - \mathrm{i}a' \wedge a', \text{ as a U(N) field strength,}\\ &\mathrm{Tr}[F_{a'}] = \mathrm{Tr}[\mathrm{d}a' - \mathrm{i}a' \wedge a'] = \mathrm{Tr}[\mathrm{d}a'] = \mathrm{d}C_1,\\ &\text{its trace is a U(1) field strength.}\end{aligned} \tag{18}$$

To associate the U(1) field strength $\mathrm{Tr}[F_{a'}] = \mathrm{Tr}[\mathrm{d}a'] = \mathrm{d}C_1$ to the background U(1) field strength, we can impose a Lagrange multiplier 2-form u,

$$\int [\mathcal{D}u] \exp\big(\mathrm{i} \int_{M^4} \frac{1}{2\pi} u \wedge (\mathrm{Tr} F_{a'} - \mathrm{d}C_1)\big) = \int [\mathcal{D}u] \exp\big(\mathrm{i} \int_{M^4} \frac{1}{2\pi} u \wedge \mathrm{d}(\mathrm{Tr} a' - C_1)\big). \quad (19)$$

We also have $\mathrm{N}B_2 = \mathrm{d}C_1$, so we can impose another Lagrange multiplier 2-form u',

$$\int [\mathcal{D}u'] \exp\big(\mathrm{i} \int_{M^4} \frac{1}{2\pi} u' \wedge (\mathrm{N}B_2 - \mathrm{d}C_1)\big) \quad (20)$$

From now we will make the YM kinetic term *implicit*s, we focus on the θ-topological term associated to the symmetry transformation. The YM kinetic term does not contribute to the anomaly (in QFT language) and is not affected under the symmetry twist (in condensed matter language [13]). Overall, with only a pair (B_2, C_1) as background fields (or sometimes simply written as (B, C)), we have,

$$\begin{aligned}
&\int [\mathcal{D}a][\mathcal{D}u][\mathcal{D}u'] \exp\big(\mathrm{i} \int_{M^4} (\frac{\theta}{8\pi^2} \mathrm{Tr}\, F_a \wedge F_a) + \mathrm{i} \int_{M^4} \frac{1}{2\pi} u \wedge \mathrm{d}(\mathrm{Tr} a' - C_1) + \mathrm{i} \int_{M^4} \frac{1}{2\pi} u' \wedge (\mathrm{N}B_2 - \mathrm{d}C_1)\big) \\
&= \int [\mathcal{D}a][\mathcal{D}u] \exp\big(\mathrm{i} \int_{M^4} (\frac{\theta}{8\pi^2} \mathrm{Tr}\, F_a \wedge F_a)\big) \exp\big(\mathrm{i} \int_{M^4} \frac{1}{2\pi} u \wedge \mathrm{d}(\mathrm{Tr} a' - C_1)\big)\big|_{\mathrm{N}B_2 = \mathrm{d}C_1} \\
&= \int [\mathcal{D}a] \exp\big(\mathrm{i} \int_{M^4} (\frac{\theta}{8\pi^2} \mathrm{Tr}\, F_a \wedge F_a)\big)\big|_{\mathrm{Tr}(F_{a'}) = \mathrm{Tr}\,\mathrm{d}a' = \mathrm{d}C_1 = \mathrm{N}B_2 = B_2(\mathrm{Tr}\,\mathbb{I})}, \quad (21)
\end{aligned}$$

here $\mathbb{I}$ is a rank-N identity matrix, thus $(\mathrm{Tr}\,\mathbb{I}) = \mathrm{N}$.

Next we rewrite the above path integral in terms of U(N) gauge field, again up to details of gauge transformations [30],

$$\begin{aligned}
&a' \simeq a + \mathbb{I}\frac{1}{\mathrm{N}} C_1, \\
&F_{a'} = \mathrm{d}a' - \mathrm{i} a' \wedge a' \\
&= (\mathrm{d}a + \mathbb{I}\frac{1}{\mathrm{N}} \mathrm{d}C_1) - \mathrm{i}(a + \mathbb{I}\frac{1}{\mathrm{N}} C_1) \wedge (a + \mathbb{I}\frac{1}{\mathrm{N}} C_1) \\
&= (\mathrm{d}a + B_2\mathbb{I}) - \mathrm{i} a \wedge a + 0 = F_a + B_2\mathbb{I} \quad (22)
\end{aligned}$$

Now, to fill in the details of gauge transformations,

$$B_2 \to B_2 + \mathrm{d}\lambda, \quad (23)$$
$$C_1 \to C_1 + \mathrm{d}\eta + \mathrm{N}\lambda, \quad (24)$$
$$a' \to a' - \lambda\mathbb{I} + \mathrm{d}\eta_a, \quad (25)$$
$$a \to a + \mathrm{d}\eta_a, \quad (26)$$

The infinitesimal and finite gauge transformations are:

$$a_\mu^\alpha \to a_\mu^\alpha + \frac{1}{g} \partial_\mu \eta_a^\alpha + f^{\alpha\beta\gamma} a_\mu^\beta \eta_a^\gamma, \quad (27)$$
$$a \to V(a + \frac{\mathrm{i}}{g}\mathrm{d})V^\dagger \equiv e^{\mathrm{i}\eta_a^\alpha T^\alpha}(a + \frac{\mathrm{i}}{g}\mathrm{d})e^{-\mathrm{i}\eta_a^\alpha T^\alpha}, \quad (28)$$

where we denote 1-form λ and 0-form η, η_a for gauge transformation parameters. Here η_a with subindex a is merely an internal label for the gauge field a's transformation η_a. Here $\alpha\beta\gamma$ are the color indices in physics, and also the indices for the adjoint representation of Lie algebra in math, which runs from $1, 2, \ldots, \mathrm{d}(G_{\text{gauge}})$ with the dimension $\mathrm{d}(G_{\text{gauge}})$ of Lie group G_{gauge} (YM gauge group), especially here $\mathrm{d}(G_{\text{gauge}}) = \mathrm{d}(\mathrm{SU(N)}) = \mathrm{N}^2 - 1$. By coupling $\mathbf{Z}_{\text{YM}}^{\text{4d}}$ to 2-form background field B, we obtain a modified partition function

$$\begin{aligned}
&\mathbf{Z}_{\text{YM}}^{\text{4d}}[B] = \int [\mathcal{D}a] \exp\big(\mathrm{i} \int_{M^4} (\frac{\theta}{8\pi^2} \mathrm{Tr}\,(F_{a'} - B_2\mathbb{I}) \wedge (F_{a'} - B_2\mathbb{I}))\big)\big|_{\mathrm{Tr}(F_{a'}) = \mathrm{Tr}\,\mathrm{d}a' = \mathrm{d}C_1 = \mathrm{N}B_2 = B_2(\mathrm{Tr}\mathbb{I})} \quad (29) \\
&= \int [\mathcal{D}a] \exp\big(\mathrm{i} \int_{M^4} (\frac{\theta}{8\pi^2} \mathrm{Tr}\,(F_{a'} \wedge F_{a'}) - \frac{2\theta\mathrm{N}}{8\pi^2} B_2 \wedge B_2 + \frac{\theta\mathrm{N}}{8\pi^2} B_2 \wedge B_2)\big) = \int [\mathcal{D}a] \exp\big(\mathrm{i} \int_{M^4} (\frac{\theta}{8\pi^2} \mathrm{Tr}\,(F_{a'} \wedge F_{a'}) - \frac{\theta\mathrm{N}}{8\pi^2} B_2 \wedge B_2)\big).
\end{aligned}$$

3. *θ periodicity and the vacua-shifting of higher SPTs*

Normally, people say θ has the 2π-periodicity,

$$\theta \simeq \theta + 2\pi. \quad (30)$$

However, this identification is imprecise. Even though the dynamics of the vacua θ and $\theta + 2\pi$ is the same, the

θ and $\theta+2\pi$ can be differed by a short-ranged entangled gapped phase of SPTs of condensed matter physics. In [30]'s language, the vacua of θ and $\theta+2\pi$ are differed by a counter term (which is the 4d higher-SPTs in condensed matter physics language). We can see the two vacua are differed by $\exp(-\mathrm{i}\int_{M^4}\frac{\Delta\theta \mathrm{N}}{8\pi^2}B_2\wedge B_2)\,|_{\Delta\theta=2\pi}$, which is

$$\exp(\mathrm{i}\int_{M^4}\frac{-\mathrm{N}}{4\pi}B_2\wedge B_2)=\exp(\mathrm{i}\int_{M^4}\frac{-\pi}{\mathrm{N}}B_2\cup B_2), \quad (31)$$

where on the right-hand-side (rhs), we switch the notation from the wedge product ($\wedge$) of differential forms to the cup product ($\cup$) of cochain field, such that $B_2\to\frac{2\pi}{N}B_2$ and $\wedge\to\cup$. More precisely, when $\mathrm{N}=2^k$ as a power of 2, the vacua is differed by

$$\exp(\mathrm{i}\int_{M^4}\frac{-\pi}{\mathrm{N}}\mathcal{P}_2(B_2)), \quad (32)$$

where a Pontryagin square term $\mathcal{P}_2:\mathrm{H}^2(-,\mathbb{Z}_{2^k})\to\mathrm{H}^4(-,\mathbb{Z}_{2^{k+1}})$ is given by eq. (11) $\mathcal{P}_2(B_2)=B_2\cup B_2+B_2\underset{1}{\cup}\delta B_2=B_2\cup B_2+B_2\underset{1}{\cup}2\mathrm{Sq}^1B_2$. This term is related to the generator of group cohomology $\mathrm{H}^4(\mathrm{B}^2\mathbb{Z}_2,\mathrm{U}(1))=\mathbb{Z}_4$ when N=2, and $\mathrm{H}^4(\mathrm{B}^2\mathbb{Z}_\mathrm{N},\mathrm{U}(1))$ for general N. This term is also related to the generator of cobordism group $\Omega^4_{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2,\mathrm{U}(1))\equiv\mathrm{Tor}\,\Omega_4^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)=\mathbb{Z}_4$ when N=2, and $\Omega^4_{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_\mathrm{N},\mathrm{U}(1))\equiv\mathrm{Tor}\,\Omega_4^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_\mathrm{N})$ for general N. For the even integer $\mathrm{N}=2^k$, we have $\Omega^4_{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_{\mathrm{N}=2^k},\mathrm{U}(1))=\mathbb{Z}_{2\mathrm{N}=2^{k+1}}$ via a $\mathbb{Z}_{2^k}$-valued 2-cochain in 2d to $\mathbb{Z}_{2^{k+1}}$ in 4d. For our concern (e.g. $\mathrm{N}=2$, 4, etc.), we have $\Omega^4_{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_\mathrm{N},\mathrm{U}(1))=\mathbb{Z}_{2\mathrm{N}}$, and the Pontryagin square is well-defined. For the odd integer N that we concern (e.g. $\mathrm{N}=3$ or say $\mathrm{N}=\mathrm{p}$ an odd prime), Pontryagin square still can be defined, but it is $\mathrm{H}^{2n}(-,\mathbb{Z}_{\mathrm{p}^k})\to\mathrm{H}^{2\mathrm{p}n}(-,\mathbb{Z}_{\mathrm{p}^{k+1}})$. So we do *not* have Pontryagin square at $\mathrm{N}=3$ in 4d. See more details on the introduction to mathematical background in [34]. Since we know that the probed-field topological term characterizes SPTs [13], which are classified by group cohomology [7, 9] or cobordism theory [24–26]; we had identified the precise SPTs (eq. (31), eq. (32)) differed between the vacua of θ and $\theta+2\pi$.

4. *Time reversal $\mathcal{T}$ transformation*

As mentioned in eq. (7), the global symmetry of YM theory (at $\theta=0$ and $\theta=\pi$) contains a time reversal symmetry $\mathcal{T}$. We denote the spacetime coordinates μ for 2-form $B\equiv B_2$ and 1-form C_1 gauge fields as $B_{2,\mu\nu}$ and $C_{1,\mu}$ respectively. Then, time reversal acts as:

$$\begin{aligned}\mathcal{T}:&a_0\to a_0,\quad a_i\to -a_i,\quad (t,x_i)\to(-t,x_i).\\ &C_{1,0}\to C_{1,0},\quad C_{1,i}\to -C_{1,i}.\\ &B_{2,0i}\to B_{2,0i},\quad B_{2,ij}\to -B_{2,ij}.\end{aligned}\quad (33)$$

Thus the path integral transforms under time reversal, schematically, becomes $\mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}[\mathcal{T}B]$. By $\mathcal{T}B$, we also mean $\mathcal{T}B\mathcal{T}^{-1}$ in the quantum operator form of B (if we canonically quantize the theory). More precisely,

$$\mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}[B]=\int[\mathcal{D}a]\exp\big(\mathrm{i}\int_{M^4}(\frac{\theta}{8\pi^2}\mathrm{Tr}\,(F_{a'}\wedge F_{a'})-\frac{\theta\mathrm{N}}{8\pi^2}B_2\wedge B_2)\big)\xrightarrow{\mathcal{T}}$$

$$\mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}[\mathcal{T}B]=\int[\mathcal{D}a]\exp\big(\mathrm{i}\int_{M^4}(\frac{-\theta}{8\pi^2}\mathrm{Tr}\,(F_{a'}\wedge F_{a'})-\frac{-\theta\mathrm{N}}{8\pi^2}B_2\wedge B_2)\big)=\mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}[B]\cdot\int[\mathcal{D}a]\exp\big(\mathrm{i}\int_{M^4}(\frac{-2\theta}{8\pi^2}\mathrm{Tr}\,(F_{a'}\wedge F_{a'})-\frac{-2\theta\mathrm{N}}{8\pi^2}B_2\wedge B_2)\big). \quad (34)$$

- When $\theta=0$, this remains the same $\mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}[\mathcal{T}B]=\mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}[B]$.
- When $\theta=\pi$, this term transforms to

$$\begin{aligned}&\mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}[B]\cdot\int[\mathcal{D}a]\exp\big(\mathrm{i}(-2\pi)\int_{M^4}(\frac{1}{8\pi^2}\mathrm{Tr}\,(F_{a'}\wedge F_{a'})-\frac{\mathrm{N}}{8\pi^2}B_2\wedge B_2)\big)\\ &=\mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}[B]\cdot\exp\big(\mathrm{i}(-2\pi)(-c_2)+\frac{(-2\pi)\mathrm{i}}{8\pi^2}\int_{M^4}(\mathrm{Tr}F_{a'}\wedge\mathrm{Tr}F_{a'}-\mathrm{N}B_2\wedge B_2)\big)|_{\mathrm{Tr}(F_{a'})=\mathrm{N}B_2}\\ &=\mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}[B]\cdot\exp\big(\mathrm{i}2\pi c_2+\frac{(-2\pi)\mathrm{i}}{8\pi^2}\int_{M^4}(\mathrm{N}(\mathrm{N}-1)B_2\wedge B_2)\big)=\mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}[B]\cdot\exp\big(\frac{-\mathrm{i}\mathrm{N}(\mathrm{N}-1)}{4\pi}\int_{M^4}(B_2\wedge B_2)\big)\end{aligned}\quad (35)$$

where we apply the 2nd Chern number c_2 identity:

$$\frac{1}{8\pi^2}\int_{M^4}(\mathrm{Tr}F_{a'}\wedge\mathrm{Tr}F_{a'}-\mathrm{Tr}\,(F_{a'}\wedge F_{a'}))=c_2\in\mathbb{Z}. \quad (36)$$

We can add a 4d SPT state (of a higher form symmetry) as a counter term. Consider again a 1-form $\mathbb{Z}_\mathrm{N}$-symmetry

($\mathbb{Z}^e_{\mathrm{N},[1]}$) protected higher-SPTs, classified by a cobordism group $\Omega^4_{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_{\mathrm{N}}, \mathrm{U}(1))$,

$$\exp(\mathrm{i} \int_{M^4} \frac{p\pi}{\mathrm{N}} \mathcal{P}_2(B_2)) \sim \exp(\mathrm{i} \frac{p\mathrm{N}}{4\pi} \int B_2 \wedge B_2), \quad (37)$$

here we again convert the 2-cochain field B_2 to 2-form field B_2 (to recall, see Sec. II B 3). For any 4-manifold, according to [29, 43],

$$\frac{\mathrm{N}p}{2} \in \mathbb{Z} \text{ (For even N, } p \in \mathbb{Z}\text{. For odd N, } p \in 2\mathbb{Z}\text{).} \quad (38)$$

$$p \simeq p + 2\mathrm{N}. \quad (39)$$

For even N, there are 2N classes of 4d higher SPTs for $p \in \mathbb{Z}$. For odd N, there are N classes of 4d higher SPTs for $p \in 2\mathbb{Z}$.

For spin 4-manifolds (when p and N are odd):

$$p \in \mathbb{Z}. \quad (40)$$

$$p \simeq p + \mathrm{N}. \quad (41)$$

In this case, there are N classes on the spin manifold. This 4d higher SPTs (counter term) under TR sym changes to: $\frac{p\mathrm{N}}{4\pi} \int B_2 \wedge B_2 \to -\frac{p\mathrm{N}}{4\pi} \int B_2 \wedge B_2$, or more precisely,

$$\int \frac{p\pi}{\mathrm{N}} \mathcal{P}_2(B_2) \to - \int \frac{p\pi}{\mathrm{N}} \mathcal{P}_2(B_2). \quad (42)$$

5. *Mix time-reversal and 1-form-symmetry anomaly*

Now we discuss the mix time-reversal $\mathcal{T}$ and 1-form $\mathbb{Z}_{\mathrm{N}}$ symmetry anomaly of [30] in details. We re-derive based on our language in [13]. The charge conjugation $\mathcal{C}$, parity $\mathcal{P}$ and time-reversal $\mathcal{T}$ form $\mathcal{CPT}$. Since $\mathcal{CPT}$ is a global symmetry for this YM theory, we can also interpret this anomaly as a mix $\mathcal{CP}$ and 1-form $\mathbb{Z}_{\mathrm{N}}$ symmetry anomaly.

So overall, $\mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}[B]$, say with a 4d higher-SPT $\frac{p\mathrm{N}}{4\pi} \int B_2 \wedge B_2$ labeled by p, is sent to

$$\mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}[B] \cdot \exp\big(\mathrm{i}(\frac{-\mathrm{N}(\mathrm{N}-1)}{4\pi} + \frac{-2p\mathrm{N}}{4\pi}) \int_{M^4} (B_2 \wedge B_2)\big) = \mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}[B] \cdot \exp\big(\frac{-\mathrm{i}\,\mathrm{N}(\mathrm{N}-1+2p)}{4\pi} \int_{M^4} (B_2 \wedge B_2)\big). \quad (43)$$

1. For even N, and $\theta = \pi$, here the 4d higher SPTs (counter term) labeled by p becomes labeled by $-(\mathrm{N}-1) - p$. To check whether there is a mixed anomaly or not, which asks for the identification of two 4d SPTs before and after time-reversal transformation. Namely $(\mathrm{N} - 1 + 2p) = 0$ (mod out the classification of 4d higher SPTs given below eq. (32)) *cannot* be satisfied for any $p \in \mathbb{Z}$ (actually $p \in \mathbb{Z}_{2^{k+1}}$ via the Pontryagin square, which sends a $\mathbb{Z}_{2^k}$-valued 2-form in 2d to $\mathbb{Z}_{2^{k+1}}$-class of 4d higher SPTs. For $\mathrm{N} = 2$, we have $p \in \mathbb{Z}_4$.)

   So this indicates that for any p (with or without 4d higher SPTs/counter term) in the YM vacua, we *detect* the mixed time-reversal $\mathcal{T}$ and 1-form $\mathbb{Z}_{\mathrm{N}}$ symmetry anomaly, which requires a 5d higher SPTs to cancel the anomaly. We will write down this 5d higher SPTs/counter term in Sec. III.

2. For even N, and $\theta = 0$, we have $\mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}[\mathcal{T}B] = \mathbf{Z}^{4\mathrm{d}}_{\mathrm{YM}}[B]$ without 4d SPTs. With 4d SPTs, the only shift is eq. (42), so to check the anomaly-free condition, we need $p = -p$, or $2p = 0$, mod out the classification of 4d higher SPTs given below eq. (32). This anomaly-free condition can be satisfied for $p = 0$. For $\mathrm{N} = 2$, we can also have $2p = 0$ mod 4, which is true for $p = 0, 2$, even with the $p = 2$-class of 4d SPTs. In that case, there is no mixed higher anomaly of $\mathcal{T}$ and $\mathbb{Z}^e_{\mathrm{N},[1]}$ symmetry,

3. For odd N, and $\theta = \pi$, the $(N - 1 + 2p) = 0$ (mod out the classification of 4d higher SPTs given below eq. (32)) can be satisfied for some $p = \frac{1-N}{2} \in \mathbb{Z}$, but p needs to be even $p \in 2\mathbb{Z}$ on a non-spin manifold. If $p = \frac{1-N}{2} \in 2\mathbb{Z}$, the 4d SPTs can be defined on a non-spin manifold. If $p = \frac{1-N}{2} \in \mathbb{Z}$, the 4d SPTs can only be defined on a spin manifold. So, for an odd N, there can be *no* mixed anomaly at $\theta = \pi$, a 4d higher SPTs/counter term of $p = \frac{1-N}{2}$ preserves the $\mathcal{T}$-symmetry and 1-form $\mathbb{Z}_{\mathrm{N}}$-symmetry (such that two symmetries can be regulated locally onsite [12–14]).

6. *Charge conjugation $\mathcal{C}$, parity $\mathcal{P}$, reflection $\mathcal{R}$, $\mathcal{CT}$, $\mathcal{CP}$ transformations, and $\mathbb{Z}_2^{CT} \times (\mathbb{Z}^e_{\mathrm{N},[1]} \rtimes \mathbb{Z}_2^C)$ and $\mathbb{Z}_2^{CT} \times \mathbb{Z}^e_{2,[1]}$ -symmetry, and their higher mixed anomalies*

Follow Sec. II B 4 and discrete $\mathcal{T}$ transformation in eq. (33), we list down additional discrete transformations including charge conjugation $\mathcal{C}$, parity $\mathcal{P}$, $\mathcal{CT}$, $\mathcal{CP}$:

$$\mathcal{C} : a_\mu \to -a^*_\mu, \quad (t, x_i) \to (t, x_i). \quad (44)$$

$$a^j_\mu(t, x) \to a^j_\mu(t, x), \quad T^j \to -T^{j*}. \quad (45)$$

$$\mathcal{P} : a_0 \to a_0, \quad a_i \to -a_i, \quad (t, x_i) \to (t, -x_i). \quad (46)$$

$$\mathcal{CT} : -a_0 \to a_0, \quad a_i \to a_i, \quad (t, x_i) \to (-t, x_i). \quad (47)$$

$$\mathcal{CP} : a_0 \to -a_0, \quad a_i \to a_i, \quad (t, x_i) \to (t, -x_i). \quad (48)$$

$$\mathcal{CPT} : a_\mu \to -a_\mu, \quad (t, x_i) \to (-t, -x_i). \quad (49)$$

The $*$ means the complex conjugation. In Euclidean spacetime, we can regard the former $\mathcal{T}$ in eq. (33) (or $\mathcal{CT}$ eq. (47)) as a reflection $\mathcal{R}$ transformation [40], which we choose to flip any of the Euclidean coordinate. See further discussions of the crucial role of discrete symmetries in YM gauge theories in [4].

We can ask whether there is any higher mixed anomalies between the above discrete symmetries and the 1-form center symmetry. We can easily check that whether the $\theta \mathrm{Tr}\,(F_{a'} \wedge F_{a'})$ term flips under any of the discrete symmetries. Among the $\mathcal{C}, \mathcal{P}$, and $\mathcal{T}$, only the $\mathcal{C}$ *does not* flip the θ term and $\mathcal{C}$ is a good global symmetry for all θ values. So the answer is that each of the

$$\mathcal{T},\, \mathcal{P},\, \mathcal{CT} \text{ and } \mathcal{CP}, \tag{50}$$

have itself mixed anomalies with the 1-form center symmetry. Only, $\mathcal{C}$, $\mathcal{PT}$ and $\mathcal{CPT}$ do not have mixed anomalies with the 1-form center symmetry.

Now, we come back to explain the non-commutative nature (the semi-direct product "$\rtimes$") of eq. (7) between 0-form and 1-form symmetries

$$\mathbb{Z}^e_{\mathrm{N},[1]} \rtimes (\mathbb{Z}_2^T \times \mathbb{Z}_2^C).$$

Obviously $(\mathbb{Z}_2^T \times \mathbb{Z}_2^C)$ is due to that $\mathcal{C}$ and $\mathcal{T}$ commute, and the combined diagonal group $\mathrm{diag}(\mathbb{Z}_2^T \times \mathbb{Z}_2^C) = \mathbb{Z}_2^{CT}$ has the group generator $\mathcal{CT}$.

We note that to physically understand some of the following statements, it may be helpful to view the symmetry transformation in the Minkowski/Lorentz signature instead of the Euclidean signature.[4]

- The non-commutative nature $\mathbb{Z}^e_{\mathrm{N},[1]} \rtimes \mathbb{Z}_2^T$ is due to that the $\mathbb{Z}_2^T$ keeps 1-Wilson loop $W_e = \mathrm{Tr}_{\mathrm{R}}(\mathrm{P}\exp(\mathrm{i} \oint a)) \to \mathrm{Tr}_{\mathrm{R}}(\mathrm{P}\exp((-\mathrm{i})(-\oint a))) = W_e$ invariant, while $\mathbb{Z}_2^T$ flips the 2-surface $U_e \to U_e^\dagger = U_e^{-1}$ due to the orientation of U_e and its boundary 't Hooft line is flipped. Thus, the 1-form $\mathbb{Z}^e_{\mathrm{N},[1]}$-symmetry charge of W_e, measured by the topological number of linking between W_e and U_e, now flips from $n \in \mathbb{Z}_{\mathrm{N}}$ to $-n = \mathrm{N} - n \in \mathbb{Z}_{\mathrm{N}}$. Since the charge operator of $\mathbb{Z}^e_{\mathrm{N},[1]}$ symmetry, U_e, is flipped thus does not commute under the $\mathbb{Z}_2^T$ symmetry, this effectively defines the semi-direct product in a dihedral group like structure of $\mathbb{Z}^e_{\mathrm{N},[1]} \rtimes \mathbb{Z}_2^T$.
- The commutative nature $\mathbb{Z}^e_{\mathrm{N},[1]} \times \mathbb{Z}_2^{CT}$ is due to that the $\mathbb{Z}_2^{CT}$ flips 1-Wilson loop $W_e = \mathrm{Tr}_{\mathrm{R}}(\mathrm{P}\exp(\mathrm{i} \oint a)) \to W_e^\dagger = W_e^{-1}$, while $\mathbb{Z}_2^{CT}$ keeps the 2-surface $U_e \to U_e$ invariant. We can see that the $\mathbb{Z}_2^{CT}$ and $\mathbb{Z}_2^T$ flips the 1-loop and 2-surface oppositely. Thus, the 1-form $\mathbb{Z}^e_{\mathrm{N},[1]}$-symmetry charge of W_e, measured by the topological number of linking between W_e and U_e, again flips from $n \in \mathbb{Z}_{\mathrm{N}}$ to $-n = \mathrm{N} - n \in \mathbb{Z}_{\mathrm{N}}$. But the charge operator of $\mathbb{Z}^e_{\mathrm{N},[1]}$ symmetry, U_e, is invariant thus does commute under the $\mathbb{Z}_2^{CT}$ symmetry, this effectively defines the direct product in a group structure of $\mathbb{Z}^e_{\mathrm{N},[1]} \times \mathbb{Z}_2^{CT}$.
- The non-commutative nature $\mathbb{Z}^e_{\mathrm{N},[1]} \rtimes \mathbb{Z}_2^C$ is due to that the $\mathbb{Z}_2^C$ in eq. (44) flips $W_e = \mathrm{Tr}_{\mathrm{R}}(\mathrm{P}\exp(\mathrm{i} \oint a)) \to \mathrm{Tr}_{\mathrm{R}}(\mathrm{P}\exp(\mathrm{i}(-\oint a^*)) = \mathrm{Tr}_{\mathrm{R}}(\mathrm{P}\exp(\mathrm{i}(\oint a))^*) = \mathrm{Tr}_{\mathrm{R}}(\mathrm{P}\exp(\mathrm{i}(\oint a))^\dagger) = W_e^\dagger = W_e^{-1}$, while $\mathbb{Z}_2^C$ also flips the 2-surface $U_e \to U_e^\dagger = U_e^{-1}$ for the same reason. Thus, the 1-form $\mathbb{Z}^e_{\mathrm{N},[1]}$-symmetry charge of W_e, measured by the topological number of linking between W_e and U_e, is invariant under $\mathbb{Z}_2^C$. But the charge operator of $\mathbb{Z}^e_{\mathrm{N},[1]}$ symmetry, U_e, is flipped thus does not commute under the $\mathbb{Z}_2^C$ symmetry, this effectively defines the semi-direct product in a dihedral group like structure of $\mathbb{Z}^e_{\mathrm{N},[1]} \rtimes \mathbb{Z}_2^C$. Note that the potentially related dihedral group structure of Yang-Mills theory under a dimensional reduction to $\mathbb{R}^3 \times S^1$ is recently explored in [30, 44].

When $\mathrm{N} = 2$, it is obvious that we simply have the direct product $\mathbb{Z}^e_{2,[1]} \times (\mathbb{Z}_2^T \times \mathbb{Z}_2^C)$ as eq. (8).

We can rewrite eq. (7)'s 0-form and 1-form symmetries

$$\boxed{\mathbb{Z}_2^{CT} \times (\mathbb{Z}^e_{\mathrm{N},[1]} \rtimes \mathbb{Z}_2^C)}. \tag{51}$$

We can rewrite eq. (8) as

$$\boxed{\mathbb{Z}_2^{CT} \times \mathbb{Z}^e_{2,[1]}}. \tag{52}$$

It is related to the fact that for SU(2) YM theory, the charge conjugation $\mathbb{Z}_2^C$ is inside the gauge group, because there is no outer automorphism of SU(2) but only an inner automorphism ($\mathbb{Z}_2$) of SU(2). For N=2, the charge conjugation matrix $\mathcal{C}_{\mathrm{SU}(2)} = e^{\mathrm{i}\frac{\pi}{2}\sigma^2} \in \mathrm{SU}(2)$ is a matrix that provides an isomorphism map between fundamental representations of $SU(2)$ and its complex conjugate. We have $\mathcal{C}_{SU(2)}\sigma^j\mathcal{C}_{SU(2)}^{-1} = -\sigma^{j*}$. Let $U_{\mathrm{SU}(2)}$ be the unitary SU(2) transformation on the SU(2)-fundamentals, so $\mathcal{C}_{SU(2)}U_{\mathrm{SU}(2)}\mathcal{C}_{SU(2)}^{-1} = \exp(-\mathrm{i}\frac{\theta}{2}\sigma^{j*}) = U^*_{\mathrm{SU}(2)}$, which is a $\mathbb{Z}_2$ inner automorphism of SU(2).

We propose that the structure of eq. (7), eq. (8), eq. (51) and eq. (52) can be regarded as an analogous 2-group. It can be helpful to further organize this 2-group like data into the context of [45].

C. 2d $\mathbb{CP}^{\mathrm{N}-1}$-sigma model

Here we consider 2d $\mathbb{CP}^{\mathrm{N}-1}$-model [46], which is a 2d sigma model with a target space $\mathbb{CP}^{\mathrm{N}-1}$. The $\mathbb{CP}^{\mathrm{N}-1}$ model is a 2d toy model which mimics some similar behaviors of 4d YM theory: dynamically-generated energy gap and asymptotically-free, etc. We will focus on 2d $\mathbb{CP}^{\mathrm{N}-1}$-model at $\theta = \pi$.

1. *Related Models*

The path integral of 2d $\mathbb{CP}^{\mathrm{N}-1}$-model is

$$\mathbf{Z}^{\rm 2d}_{\mathbb{CP}^{\rm N-1}} \equiv \int [\mathcal{D}z][\mathcal{D}\bar z][\mathcal{D}a'] \exp\big(-S_{\mathbb{CP}^{\rm N-1}+\theta}[z,\bar z,a']\big) \equiv \int [\mathcal{D}z_j][\mathcal{D}\bar z_j][\mathcal{D}a'] \exp\big(-S_{\mathbb{CP}^{\rm N-1}}[z_j,\bar z_j,a']\big)\exp\big(-S_\theta[a']\big)$$
$$\equiv \int [\mathcal{D}z][\mathcal{D}\bar z][\mathcal{D}a'] \delta(|z|^2-r^2)\exp\Big(\Big(-\int_{M^2} d^2x(\frac{1}{g'^2}|D_{a',\mu}z|^2)+\int_{M^2}(\frac{\mathrm{i}\theta}{2\pi}F_{a'})\Big)\Big). \quad (53)$$

The $z_j \in \mathbb{C}$ is a complex-valued field variable, with an index $j = 1,\dots,\mathrm{N}$. (In math, the $z_j \sim c z_j$, identified by any complex number $c \in \mathbb{C}^\times$ excluding the origin, is known as the homogeneous coordinates of the target space $\mathbb{CP}^{\rm N-1}$.) The delta function imposes a constraint: $|z|^2 \equiv \sum_{j=1}^{\rm N}|z_j|^2 = r^2$, here $r \in \mathbb{R}$ specifies the size of $\mathbb{CP}^{\rm N-1}$. The delta function $\delta(|z|^2-r^2)$ may be also replaced by a potential term, such as the $\frac{\lambda}{4}(|z|^2-r^2)^2$ potential, at large λ coupling energetically constraining $|z|^2 = r^2$. Here $|D_{a',\mu}z|^2 \equiv (D_{a',\mu}z)^\dagger(D^\mu_{a'}z)$. Here $F_{a'} = \mathrm{d}a'$ is the U(1) field strength of a'.

For 2d $\mathbb{CP}^1$ (2d $\mathbb{CP}^{\rm N-1}$ at N = 2), we can rewrite the model as the O(3) nonlinear sigma model (NLSM). The O(3) NLSM is parametrized by an O(3) = SO(3) $\times$ $\mathbb{Z}_2$ Néel vector $\vec n = (n_1,n_2,n_3)$, which is related to $\vec n = \frac{1}{r^2} z_i^\dagger \vec\sigma_{ij} z_j$ with $|\vec n|^2 = 1$ and Pauli matrix $\vec\sigma = (\sigma^1,\sigma^2,\sigma^3)$. It is called Néel vector because the 2d $\mathbb{CP}^1$ or O(3) NLSM describes the Heisenberg anti-ferromagnet phase of quantum spin system [47, 48]. To convert eq. (53) to eq. (56), notice that we do not introduce the kinetic Maxwell term $|F_{a'}|^2$ for the U(1) photon a', thus a' is an auxiliary field, that can be integrated out and eq. (53) is constrained by the EOM: $a'_\mu = -\frac{\mathrm{i}}{r^2}\sum_{j=1}^2 \bar z_j\partial_\mu z_j = \frac{\mathrm{i}}{2r^2}\sum_{j=1}^2(z_j\partial_\mu\bar z_j - \bar z_j\partial_\mu z_j)$, and we can derive:

$$|D_{a',\mu}z|^2 = \sum_{j=1}^2 |D_{a',\mu}z_j|^2 = ((\frac{r}{2})^2\partial_\mu\vec n\cdot\partial^\mu\vec n), \quad (54)$$
$$\frac{\mathrm{i}\theta}{2\pi}\epsilon^{\mu\nu}\partial_\mu a'_\nu = (\frac{\mathrm{i}\theta}{8\pi}\epsilon^{\mu\nu}\vec n\cdot(\partial_\mu\vec n\times\partial_\nu\vec n)). \quad (55)$$

Then we rewrite $\mathbf{Z}^{\rm 2d}_{\mathbb{CP}^1}$ as $\mathbf{Z}^{\rm 2d}_{\rm O(3)}$ of the O(3) NLSM path integral:

$$\mathbf{Z}^{\rm 2d}_{\rm O(3)} \equiv \int [\mathcal{D}\vec n]\delta(|\vec n|^2-1)\exp\big(-S_{\mathrm{O}(3)+\theta}[\vec n]\big)$$
$$\equiv \int [\mathcal{D}\vec n]\delta(|\vec n|^2-1)\exp\Big(\Big(-\int_{M^2} d^2x(\frac{1}{g''^2}\partial_\mu\vec n\cdot\partial^\mu\vec n)+\int_{M^2}(\frac{\mathrm{i}\theta}{8\pi}\epsilon^{\mu\nu}\vec n\cdot(\partial_\mu\vec n\times\partial_\nu\vec n))\Big)\Big). \quad (56)$$

Note that $(\frac{\mathrm{i}\theta}{8\pi}\epsilon^{\mu\nu}\vec n\cdot(\partial_\mu\vec n\times\partial_\nu\vec n)) = (\frac{\mathrm{i}\theta}{4\pi}\vec n\cdot(\partial_\tau\vec n\times\partial_x\vec n))$. The O(3) NLSM coupling g'' in $((\frac{1}{g''})^2\partial_\mu\vec n\cdot\partial^\mu\vec n) = ((\frac{r}{2g'})^2\partial_\mu\vec n\cdot\partial^\mu\vec n)$ is related to the $\mathbb{CP}^1$ model via $g'' = (2g'/r)$, which is inverse proportional to the radius size of the 2-sphere $\mathbb{CP}^1 = S^2$.

In fact, the UV high energy theory of $\mathbf{Z}^{\rm 2d}_{\mathbb{CP}^1} = \mathbf{Z}^{\rm 2d}_{\rm O(3)}$ is known to be, in Renormalization Group (RG), flowing to the same IR conformal field theory CFT from another UV model from SU(2)$_1$-WZW model (Wess-Zumino-Witten model [49–51]). The SU(N)$_k$-WZW model is

$$\mathbf{Z}^{\rm WZW}_{\mathrm{SU(N)}_k} = \int[\mathcal{D}U][\mathcal{D}U^\dagger]\exp(-\frac{k}{8\pi}\int_{M^2}\mathrm{d}^2x\mathrm{Tr}\big(\partial_\mu U^\dagger\partial^\mu U\big)+\frac{\mathrm{i}k}{12\pi}\int_{M'^3}\mathrm{Tr}\big((U^\dagger\mathrm{d}U)^3\big)), \quad (57)$$

with $M^2 = \partial(M'^3)$. At N = 2, the UV theory of $\mathbf{Z}^{\rm 2d}_{\mathbb{CP}^1} = \mathbf{Z}^{\rm 2d}_{\rm O(3)}$ flows to this 2d CFT called the SU(2)$_1$-WZW CFT at IR. The global symmetry can be preserved at IR.

For the general 2d $\mathbb{CP}^{\rm N-1}$-model, its global symmetry

[4] In the Minkowski case, we also need to regard the time-reversal symmetries ($\mathcal{T}$ and $\mathcal{CT}$) as anti-unitary symmetry, instead of the unitary symmetry (as the Euclidean case).

can also be embedded into another $\mathrm{SU(N)_1}$-WZW model at UV; although unlike $\mathrm{N}=2$ case, $\mathbb{CP}^{\mathrm{N}-1}$-models for $\mathrm{N}>2$ conventionally and generically do not flow to an IR CFT. For $\mathrm{N}>2$, there exist UV-symmetry preserving relevant deformations driving the RG flow away from an IR CFT. The global symmetry may be spontaneously broken, and the vacua can be gapped and/or degenerated. See for example [52, 53] and references therein.

2. Global symmetry: $\mathbb{Z}_2^{CT}\times \mathrm{PSU}(2)\times \mathbb{Z}_2^{C}$ *and* $\mathbb{Z}_2^{CT}\times (\mathrm{PSU(N)}\rtimes \mathbb{Z}_2^{C'})$

Let us check the global symmetry of 2d $\mathbb{CP}^{\mathrm{N}-1}$-model.

Continuous global symmetry: In eq. (53), it is easy to see the continuous global SU(N) transformation rotating between the SU(N) fundamental complex scalar multiplet z_j, $z\to Vz=V_{ij}z_j=(e^{\mathrm{i}\Theta^\alpha T^\alpha})_{ij}z_j$ which has its $\mathbb{Z}_\mathrm{N}$-center subgroup being gauged away by the U(1) gauge field a'. So we have the net continuous global symmetry

$$\mathrm{PSU(N)}=\mathrm{SU(N)}/\mathbb{Z}_\mathrm{N}=\mathrm{U(N)}/\mathrm{U(1)}, \tag{58}$$

which acts on gauge invariant object faithfully (e.g. the $\mathrm{PSU}(2)=\mathrm{SO}(3)$ symmetry can act on the gauge-invariant $\vec{n}$ vector in the 2d $\mathbb{CP}^1$-model or O(3) NLSM faithfully).

Now we explore 2d $\mathbb{CP}^{\mathrm{N}-1}$-model's discrete global symmetry as finite groups.

Discrete global symmetry for $\mathrm{N}=2$:

• $\mathbb{Z}_2^T$, there is a $\mathcal{T}$-symmetry for any θ, acting on fields and coordinates of eq. (53) and eq. (56), whose transformations become

$$\begin{aligned}\mathbb{Z}_2^T:\ & z_i\to\epsilon_{ij}\bar z_j,\quad \vec n\to-\vec n,\\ & (a'_t,a'_x)\to(a'_t,-a'_x),\quad (t,x)\to(-t,x).\end{aligned} \tag{59}$$

Here a Pauli matrix σ^2_{ij} gives $\epsilon_{ij}=\mathrm{i}\sigma^2_{ij}$.

• $\mathbb{Z}_2^x$-translation symmetry ($\equiv\mathbb{Z}_2^C$) for $\theta=0,\pi$, acts as

$$\begin{aligned}\mathbb{Z}_2^x\ (\equiv\mathbb{Z}_2^C):\ & z_i\to\epsilon_{ij}\bar z_j,\quad \vec n\to-\vec n,\\ & (a'_t,a'_x)\to-(a'_t,a'_x),\quad (t,x)\to(t,x).\end{aligned} \tag{60}$$

It is easy to understand the role of $\mathbb{Z}_2^x$-translation on the UV-lattice model of Heisenberg anti-ferromagnet (AFM) phase of quantum spin system [47, 48]. Its AFM Hamiltonian operator is

$$\hat H=\sum_{\langle i,j\rangle}|J|\hat{\vec S}_i\cdot\hat{\vec S}_j+\dots \tag{61}$$

where $\langle i,j\rangle$ is for the nearest-neighbor lattice site (i,j) AFM interaction between spin operators $\hat{\vec S}$, and $|J|>0$ is the AFM coupling. So $\mathbb{Z}_2^x$-translation flips the spin orientation, also flips the AFM's Néel vector $\vec n\to-\vec n$.

• $\mathbb{Z}_2^{C'}$-charge conjugation symmetry of $\mathbb{CP}^{\mathrm{N}-1}$-model for $\theta=0,\pi$ acts as

$$\begin{aligned}\mathbb{Z}_2^{C'}:\ & z_i\to\bar z_i,\quad (n_1,n_2,n_3)\to(n_1,-n_2,n_3),\\ & (a'_t,a'_x)\to-(a'_t,a'_x),\quad (t,x)\to(t,x).\end{aligned} \tag{62}$$

• $\mathbb{Z}_2^{C'T}$-symmetry for $\theta=0,\pi$ acts as

$$\begin{aligned}\mathbb{Z}_2^{C'T}:\ & z_i\to\epsilon_{ij}z_j,\quad (n_1,n_2,n_3)\to(-n_1,n_2,-n_3),\\ & (a'_t,a'_x)\to(-a'_t,a'_x),\quad (t,x)\to(-t,x).\end{aligned} \tag{63}$$

• $\mathbb{Z}_2^{xT}$-symmetry ($\equiv\mathbb{Z}_2^{CT}$) as another choice of time-reversal for $\theta=0,\pi$, acts as

$$\begin{aligned}\mathbb{Z}_2^{xT}\ (\equiv\mathbb{Z}_2^{CT}):\ & z_i\to z_i,\quad \vec n\to\vec n,\\ & (a'_t,a'_x)\to(-a'_t,a'_x),\quad (t,x)\to(-t,x).\end{aligned} \tag{64}$$

Next we check the commutative relation between the above continuous PSU(N) and the discrete symmetries

For $\mathrm{N}=2$, we see that $\mathbb{Z}_2^T$ commutes with PSU(2), because $\mathcal{T}Vz=\mathrm{i}\sigma^2(Vz)^*=\mathrm{i}\sigma^2V^*\bar z=V\mathrm{i}\sigma^2\bar z=V\mathcal{T}z$. Similarly, $\mathbb{Z}_2^x$ commutes with PSU(2). So, $\mathbb{Z}_2^{xT}$ commutes with PSU(2). We see that $\mathbb{Z}_2^{C'}$ does not commute with PSU(2), because $\mathcal{C}'Vz=(Vz)^*=V^*\bar z$ while $V\mathcal{C}'z=V\bar z$. Therefore, $\mathbb{Z}_2^{C'T}=\mathrm{diag}(\mathbb{Z}_2^{C'}\times\mathbb{Z}_2^T)$ also does not commute with PSU(2).

Global symmetry for $\mathrm{N}=2$:

Overall, for 2d $\mathbb{CP}^1$ model at $\theta=0,\pi$, we can combine the above to get the full 0-form global symmetries

$$\boxed{\mathbb{Z}_2^T\times\mathrm{PSU}(2)\times\mathbb{Z}_2^x=\mathbb{Z}_2^T\times\mathrm{O}(3)}, \tag{65}$$

which is the same as

$$\mathbb{Z}_2^T\times\mathrm{PSU}(2)\rtimes\mathbb{Z}_2^{C'}$$

with a semi-direct product "$\rtimes$" since PSU(2) and $\mathbb{Z}_2^{C'}$ do not commute.

It is very natural to regard $\mathbb{Z}_2^{xT}$-symmetry as the *new* $\mathbb{Z}_2^{CT}$-symmetry, because it flips the time coordinates $t\to-t$, but it does not complex conjugate the z. So we may define[5]

$$\mathbb{Z}_2^{CT}\equiv\mathbb{Z}_2^{xT}. \tag{66}$$

Similarly, we may regard the $\mathbb{Z}_2^x$-translation as a *new* charge conjugation symmetry $\mathbb{Z}_2^C\equiv\mathbb{Z}_2^x$.

[5] Above we discuss $\mathbb{Z}_2^{CT}\equiv\mathbb{Z}_2^{xT}$ and $\mathbb{Z}_2^T$ both commute with the PSU(2) (also SU(2)) for bosonic systems (bosonic QFTs). Indeed, the $\mathbb{Z}_2^{CT}$ and $\mathbb{Z}_2^T$ reminisce the discussion of [4] (e.g. Sec. II), for the case including the fermions (with the fermion parity symmetry $\mathbb{Z}_2^F$ acted by $(-1)^F$), we have the natural $\mathbb{Z}_2^{CT}$-time reversal symmetry, without taking complex conjugation on the matter fields, which gives rise to the full symmetry $\frac{\mathrm{Pin}^+\times\mathrm{SU}(2)}{\mathbb{Z}_2^F}$; while the other $\mathbb{Z}_2^T$-time reversal symmetry, involving complex conjugation on the matter fields, gives rise to $\frac{\mathrm{Pin}^-\times\mathrm{SU}(2)}{\mathbb{Z}_2^F}$.

Therefore, 0-form global symmetries eq. (65) can also be

$$\boxed{\mathbb{Z}_2^{CT} \times \mathrm{PSU}(2) \times \mathbb{Z}_2^{x} \equiv \mathbb{Z}_2^{CT} \times \mathrm{PSU}(2) \times \mathbb{Z}_2^{C}}$$
$$\equiv \boxed{\mathbb{Z}_2^{CT} \times \mathrm{O}(3)}. \tag{67}$$

Global symmetry for $\mathrm{N} > 2$:
For 2d $\mathbb{CP}^{\mathrm{N}-1}$ model eq. (53) at $\theta = 0, \pi$, $\mathrm{N} > 2$, we follow the above discussion and the footnote 5, we again can define a natural definition of $\mathbb{Z}_2^{CT}$ (without involving the complex conjugation of z fields). Then we have instead the full 0-form global symmetries:

$$\boxed{\mathbb{Z}_2^{CT} \times (\mathrm{PSU}(\mathrm{N}) \rtimes \mathbb{Z}_2^{C'})}, \tag{68}$$

where again $\mathbb{Z}_2^{C}$ acts on $z_i \to \bar{z}_i$, $a'_\mu \to -a'_\mu$ and $(t, x) \to (t, x)$ as eq. (62).

We remark that the SU(2) (or $\mathrm{N} = 2$ for $\mathbb{CP}^1$ model) is special because its order-2 automorphism is an inner automorphism. The SU(2) fundamental representation is equivalent to its conjugate. This is related to the fact that both $\mathbb{Z}_2^{CT}$ and $\mathbb{Z}_2^{T}$ can commute with the SU(2) or PSU(2), also the remark we made in the footnote 5.

For SU(N) ($\mathrm{N} > 2$ for $\mathbb{CP}^{\mathrm{N}-1}$ model) has its order-2 automorphism as an outer automorphism, which is the $\mathbb{Z}_2$ symmetry of Dynkin diagram A_{N-1} swapping fundamental with anti-fundamental representations. Although we have $\mathbb{Z}_2^{CT} \times \mathrm{PSU}(\mathrm{N})$ in eq. (68), we would have $\mathbb{Z}_2^{T} \ltimes \mathrm{PSU}(\mathrm{N})$ for $\mathrm{N} > 2$. See related and other detailed discussions in [4].

The above we have considered the "full" global symmetry (focusing on the internal symmetry) without precisely writing down their spacetime symmetry group part. In Sec. III, we like to write down the "full" global symmetry including the spacetime symmetry group.

III. COBORDISMS, TOPOLOGICAL TERMS, AND MANIFOLD GENERATORS: CLASSIFICATION OF ALL POSSIBLE HIGHER 'T HOOFT ANOMALIES

A. Mathematical preliminary and co/bordism groups

Since we have obtained the full global symmetry G (including the 0-form and higher symmetries) of 4d YM and 2d $\mathbb{CP}^{\mathrm{N}-1}$ model, we can now use the knowledge that their 't Hooft anomalies are classified by 5d and 3d cobordism invariants of the same global symmetry [26]. Namely, we can classify the 't Hooft anomalies by enlisting the complete set of all possible cobordism invariants from their corresponding 5d and 3d bordism groups, whose 5d and 3d manifold generators endorsed with the G structure.

To begin with, we should rewrite the global symmetries in previous sections (e.g. (eq. (7)/eq. (51)), (eq. (8)/eq. (52))) into the form of

$$G \equiv \left(\frac{G_{\text{spacetime}} \ltimes \mathbb{G}_{\text{internal}}}{N_{\text{shared}}}\right), \tag{69}$$

where the $G_{\text{spacetime}}$ is the spacetime symmetry, the $\mathbb{G}_{\text{internal}}$ the internal symmetry,[6] the $\ltimes$ is a semi-direct product specifying a certain "twisted" operation (e.g. due to the symmetry extension from $\mathbb{G}_{\text{internal}}$ by $G_{\text{spacetime}}$) and the N_{shared} is the shared common normal subgroup symmetry between the two numerator groups.

The theories and their 't Hooft anomalies that we concern are in dd QFTs (4d YM and 2d $\mathbb{CP}^{\mathrm{N}-1}$-model), but the topological/cobordism invariants are defined in the $D\text{d} = (d+1)$d manifolds. The manifold generators for the bordism groups are actually the closed $D\text{d} = (d+1)$d manifolds. We should clarify that although there can be 't Hooft anomalies for dd QFTs so $\mathbb{G}_{\text{internal}}$ may not be gauge-able, the SPTs/topological invariants defined in the closed $D\text{d} = (d+1)$d manifolds actually have $\mathbb{G}_{\text{internal}}$ always gauge-able in that $D\text{d} = (d+1)$d.[7] This is related to the fact that in condensed matter physics, we say that the *bulk* $D\text{d} = (d+1)$d SPTs has an onsite local internal $\mathbb{G}_{\text{internal}}$-symmetry, thus this $\mathbb{G}_{\text{internal}}$ must be gauge-able.

The *new* ingredient in our present work slightly going beyond the cobordism theory of [26] is that the $\mathbb{G}_{\text{internal}}$-symmetry may not only be an ordinary 0-form global symmetry, but also include higher global symmetries. The details of our calculation for such "*higher-symmetry-group cobordism theory*" are provided in [34].

Based on a theorem of Freed-Hopkin [26] and an extended generalization that we propose [34], there exists a 1-to-1 correspondence between "the invertible topological quantum field theories (iTQFTs) with symmetry (including higher symmetries)" and "a cobordism group." In condensed matter physics, this means that there is a 1-to-1 correspondence between "the symmetric invertible topological order with symmetry (including higher symmetries)' that can be regularized on a lattice in its own dimensions' and "a cobordism group," at least at lower dimensions.[8] More precisely, it is a 1-to-1 correspondence (isomorphism "$\cong$") between the following two

[6] Later we denote the probed background spacetime M connection over the spacetime tangent bundle TM, e.g. as $w_j(TM)$ where w_j is j-th Stiefel-Whitney (SW) class [31]. We may also denote the probed background internal-symmetry/gauge connection over the principal bundle E, e.g. as $w_j(E) = w_j(V_{\mathbb{G}_{\text{internal}}})$ where w_j is also j-th SW class. In some cases, we may alternatively denote the latter as $w'_j(E) = w'_j(V_{\mathbb{G}_{\text{internal}}})$.

[7] This idea has been pursued to study the vacua of YM theories, for example, in [4] and references therein. See more explanations in Sec. X's eq. (129)

[8] We have used a mathematical fact that all smooth and differentiable manifolds are triangulable manifolds, based on Morse theory. On the contrary, triangulable manifolds are smooth manifolds at least for dimensions up to $D = 4$ (i.e. the "if and only if" statement is true below $D \leq 4$). The concept of piecewise linear (PL) and smooth structures are equivalent in dimensions $D \leq 6$. Thus all symmetric iTQFT classified by the cobordant

mathematical well-defined objects:

$$\left\{ \begin{array}{c} \text{Deformation classes of reflection positive} \\ \text{invertible } D\text{-dimensional extended} \\ \text{topological field theories (iTQFT) with} \\ \text{symmetry group } \frac{G_{\text{spacetime}} \ltimes \mathbb{G}_{\text{internal}}}{N_{\text{shared}}} \end{array} \right\}$$

$$\cong [MT(\frac{G_{\text{spacetime}} \ltimes \mathbb{G}_{\text{internal}}}{N_{\text{shared}}}), \Sigma^{D+1} I\mathbb{Z}]_{\text{tors}}. \quad (70)$$

Let us explain the notation above: MTG is the Madsen-Tillmann spectrum [55] of the group G, Σ is the suspension, $I\mathbb{Z}$ is the Anderson dual spectrum, and tors means taking only the finite group sector (i.e. the torsion group).

Namely, we classify the deformation classes of symmetric iTQFTs and also symmetric invertible topological orders (iTOs), via this particular cobordism group

$$\Omega_G^D \equiv \Omega^D_{(\frac{G_{\text{spacetime}} \ltimes \mathbb{G}_{\text{internal}}}{N_{\text{shared}}})} \equiv \text{TP}_D(G) \equiv [MT(G), \Sigma^{n+1} I\mathbb{Z}]. \quad (71)$$

by classifying the cobordant relations of smooth, differentiable and triangulable manifolds with a stable G-structure, via associating them to the homotopy groups of Thom-Madsen-Tillmann spectra [55, 56], given by a theorem in Ref. 26. Here TP means the abbreviation of "Topological Phases" classifying the above symmetric iTQFT, where our notations follow [26] and [34]. (For an introduction of the mathematical background for physicists, the readers can consult the Appendix A of [4].)

Moreover, there are only the discrete/finite $\mathbb{Z}_n$-classes of the non-perturbative global 't Hooft anomalies for YM and $\mathbb{CP}^{\text{N}-1}$ model (so-called the *torsion* group for $\mathbb{Z}_n$-class); there is no $\mathbb{Z}$-class perturbative anomaly (so-called the free class) for our QFTs. So, we concern only the torsion group part of data in eqn. (70), this is equivalent for us to simply look at the bordism group:

$$\Omega_D^G \equiv \Omega_D^{(\frac{G_{\text{spacetime}} \ltimes \mathbb{G}_{\text{internal}}}{N_{\text{shared}}})}, \quad (72)$$

in order to classify all the 't Hooft anomalies for YM and $\mathbb{CP}^{\text{N}-1}$ model.

Therefore, below we focus on the unoriented bordism groups (and later also some oriented bordism groups, replacing the orthogonal O group to a special orthogonal SO group):

$$\Omega_D^{\text{O}}(X) = \{\text{a pair } (M, f) \text{ where } M \text{ is a closed } D\text{-manifold and } f : M \to X \text{ is a map}\}/\text{bordism}. \quad (73)$$

properties of smooth manifolds have a triangulation (thus a lattice regularization) on a simplicial complex (thus a UV competition on a lattice). This implies a correspondence between "the symmetric iTQFTs (on smooth manifolds)" and "the symmetric invertible topological orders (on triangulable manifolds)" for $D \leq 4$. See a recent application of this mathematical fact on the lattice regularization of symmetric iTQFTs and symmetric invertible topological orders in [54] for various Standard Models of particle physics.

where bordism is an equivalence relation, namely, (M, f) and (M', f') are bordant if there exists a compact $D+1$-manifold $\mathcal{M}$ and a map $h : \mathcal{M} \to X$, where X is a generic topological space, such that the boundary of $\mathcal{M}$ is the disjoint union of M and M', while we set $h|_M = f$ and $h|_{M'} = f'$.

In particular, when $X = \text{B}^2\mathbb{Z}_n$, $f : M \to \text{B}^2\mathbb{Z}_n$ is a cohomology class in $\text{H}^2(M, \mathbb{Z}_n)$. When $X = \text{B}G$, with G is a Lie group or a finite group (viewed as a Lie group with discrete topology), then $f : M \to \text{B}G$ is a principal G-bundle over M. To explain our notation, here $\text{B}G$ is a classifying space of G, and $\text{B}^2\mathbb{Z}_n$ is a higher classifying space (Eilenberg-MacLane space $K(\mathbb{Z}_n, 2)$) of $\mathbb{Z}_n$.

Our conventions in the following subsections are:

- A map is always assumed to be continuous.

- For a top degree cohomology class with coefficients $\mathbb{Z}_2$ we often suppress explicit integration over the manifold (i.e. pairing with the fundamental class $[M]$ with coefficients $\mathbb{Z}_2$), for example: $w_2(TM)w_3(TM) \equiv \int_M w_2(TM)w_3(TM)$ where M is a 5-manifold.

In the following subsections, we consider the potential cobordism invariants/topological terms (5d and 3d [higher] SPTs for 4d YM and 2d $\mathbb{CP}^{\text{N}-1}$ model), and their manifold generators for bordism groups, as the complete classification of all of their possible candidate higher 't Hooft anomalies.

First, we can convert the time reversal $\mathbb{Z}_2^{T'}$ ($\equiv \mathbb{Z}_2^T$ or $\mathbb{Z}_2^{CT}$) to the orthogonal $\text{O}(D)$-symmetry group for such an underlying UV-completion of bosonic system (all gauge-invariant operators are bosons), where the $\text{O}(D)$ is an extended symmetry group from $\text{SO}(D)$ via a short extension:

$$1 \to \text{SO}(D) \to \text{O}(D) \to \mathbb{Z}_2^{T'} \to 1. \quad (74)$$

The $\text{SO}(D)$ is the spacetime Euclidean rotational symmetry group for Dd bosonic systems.[9]

Then we can easily list their converted full symmetry group G and their relevant bordism groups, for SU(2) YM (eq. (8)/eq. (52)), SU(N) YM (eq. (7)/eq. (51)), $\mathbb{CP}^1$ model (eq. (65)/eq. (67)), and $\mathbb{CP}^{\text{N}-1}$ model (eq. (68)), into the eq. (69)'s form:

(i) $\Omega_5^{\text{O}}(\text{B}^2\mathbb{Z}_2) \equiv \Omega_5^{(\text{O}\times\text{B}\mathbb{Z}_2)}$: This is the bordism group for $\mathbb{Z}_2^{CT} \times (\mathbb{Z}_{2,[1]}^e \times \mathbb{Z}_2^C)$ in eq. (51) without $\mathbb{Z}_2^C$, which we will study in Sec. III B, here eq. (69)'s $G = \text{O}(D) \times \text{B}\mathbb{Z}_2$ or $G = \text{O}(D) \times \mathbb{Z}_{2,[1]}^e$.

[9] For the case of time reversal symmetry, where there must be an underlying UV-completion of fermionic system (some gauge-invariant operators are fermions), the more subtle time reversal extension scenario is discussed in [26] and [4].

(ii) $\Omega_5^{\rm O}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2) \equiv \Omega_5^{(\mathrm{O}\times\mathbb{Z}_2)\times \mathrm{B}\mathbb{Z}_2}$: This is the bordism group for $\mathbb{Z}_2^{CT} \times (\mathbb{Z}_{2,[1]}^e \times \mathbb{Z}_2^C)$ in eq. (51), which we will study in Sec. III C, here eq. (69)'s $G = \mathrm{O}(D) \times \mathbb{Z}_2 \times \mathrm{B}\mathbb{Z}_2$ or $G = \mathrm{O}(D) \times \mathbb{Z}_2^C \times \mathbb{Z}_{2,[1]}^e$.

(iii) $\Omega_3^{\rm O}(\mathrm{BO}(3))$: This is the bordism group for $\mathbb{Z}_2^{CT} \times \mathrm{O}(3)$ in eq. (67), which we will study in Sec. III D, here eq. (69)'s $G = \mathrm{O}(D) \times \mathrm{O}(3)$.

(iv) $\Omega_5^{\rm O}(\mathrm{B}^2\mathbb{Z}_4)$: This is the bordism group for $\mathbb{Z}_2^{CT} \times (\mathbb{Z}_{\mathrm{N},[1]}^e \rtimes \mathbb{Z}_2^C)$ in eq. (51) at $\mathrm{N} = 4$ without $\mathbb{Z}_2^C$, which we will study in Sec. III E, here eq. (69)'s $G = \mathrm{O}(D) \times \mathrm{B}\mathbb{Z}_4$ or $G = \mathrm{O}(D) \times \mathbb{Z}_{4,[1]}^e$.

(v) $\Omega_5^{\rm O}(\mathrm{B}\mathbb{Z}_2 \ltimes \mathrm{B}^2\mathbb{Z}_4) \equiv \Omega_5^{(\mathrm{O}\times(\mathbb{Z}_2\ltimes \mathrm{B}\mathbb{Z}_4))}$ and $\Omega_5^{\mathrm{O}\ltimes}(\mathrm{B}\mathbb{Z}_2 \ltimes \mathrm{B}^2\mathbb{Z}_4) \equiv \Omega_5^{(\mathrm{O}\times\mathbb{Z}_2)\ltimes \mathrm{B}\mathbb{Z}_4}$:
The first is the bordism group with a CT-time reversal, for $\mathbb{Z}_2^{CT} \times (\mathbb{Z}_{\mathrm{N},[1]}^e \rtimes \mathbb{Z}_2^C)$ in eq. (51) at $\mathrm{N} = 4$, which we will study in Sec. III F, here eq. (69)'s $G = \mathrm{O}(D) \times (\mathbb{Z}_2 \ltimes \mathrm{B}\mathbb{Z}_4)$ or $G = \mathrm{O}(D) \times (\mathbb{Z}_2^C \ltimes \mathbb{Z}_{4,[1]}^e)$.
The second is actually the re-written bordism group with a T-time reversal, for $\mathbb{Z}_{\mathrm{N},[1]}^e \rtimes (\mathbb{Z}_2^T \times \mathbb{Z}_2^C)$ at $\mathrm{N} = 4$, here eq. (69)'s $G = (\mathrm{O}(D) \times \mathbb{Z}_2) \ltimes \mathrm{B}\mathbb{Z}_4$ or $G = (\mathrm{O}(D) \times \mathbb{Z}_2^C) \ltimes \mathbb{Z}_{4,[1]}^e$. But we will *not* study this, since it is simply a more complicated re-writing of the same result of Sec. III F.

(vi) $\Omega_3^{\rm O}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)))$: This is the bordism group for $\mathbb{Z}_2^{CT} \times (\mathrm{PSU(N)} \rtimes \mathbb{Z}_2^{C'})$ in eq. (68) at $\mathrm{N} = 4$, which we will study in Sec. III G, here eq. (69)'s $G = \mathrm{O}(D) \times (\mathrm{PSU(N)} \rtimes \mathbb{Z}_2^{C'})$.

Based on the relation between bordism groups and their Dd $= (d+1)$d cobordism invariants to the dd anomalies of QFTs, below we may simply abbreviate "5d cobordism invariants for 4d YM theory's anomaly" as

"5d (Yang-Mills) terms."

We may simply abbreviate "3d cobordism invariants for 2d $\mathbb{CP}^{\mathrm{N}-1}$ model's anomaly" as

"3d ($\mathbb{CP}^{\mathrm{N}-1}$) terms."

B. $\Omega_5^{\rm O}(\mathrm{B}^2\mathbb{Z}_2)$

Follow Sec. III A, now we enlist all possible 't Hooft anomalies of 4d pure SU(2) YM at $\theta = \pi$ (but when the $\mathbb{Z}_2^C$-background field is turned off) by obtaining the 5d cobordism invariants from bordism groups of (eq. (8)/eq. (52)).

We are given a 5-manifold M and a map $f : M \to \mathrm{B}^2\mathbb{Z}_2$. Here the map $f : M \to \mathrm{B}^2\mathbb{Z}_2$ is the 2-form $B = B_2$ gauge field in the YM gauge theory eq. (10) (and eqn. (29) at $\mathrm{N} = 2$). We like to obtain the bordism invariants of $\Omega_5^{\rm O}(\mathrm{B}^2\mathbb{Z}_2)$. We find the bordism group [34][10]

$$\Omega_5^{\rm O}(\mathrm{B}^2\mathbb{Z}_2) = \mathbb{Z}_2^4, \tag{75}$$

whose cobordism invariants are generated by

$$\begin{cases} B_2 \cup \mathrm{Sq}^1 B_2, \\ \mathrm{Sq}^2\mathrm{Sq}^1 B_2, \\ w_1(TM)^2 \mathrm{Sq}^1 B_2, \\ w_2(TM) w_3(TM). \end{cases} \tag{76}$$

where TM means the spacetime tangent bundle over M, see footnote 6. Note that we derive $\mathrm{Sq}^2\mathrm{Sq}^1 B_2 = (w_2(TM)+w_1(TM)^2)\mathrm{Sq}^1 B_2 = (w_3(TM)+w_1(TM)^3)B_2$, $w_1(TM)^2\mathrm{Sq}^1 B_2 = w_1(TM)^3 B_2$ (See [34]).

Since $\mathrm{Sq}^2\mathrm{Sq}^1 B_2 = (w_2(TM) + w_1(TM)^2)\mathrm{Sq}^1 B_2$, we can rewrite the bordism invariants as $B_2 \cup \mathrm{Sq}^1 B_2$, $w_2(TM) \cup \mathrm{Sq}^1 B_2$, $w_1(TM)^2 \cup \mathrm{Sq}^1 B_2$ and $w_2(TM) w_3(TM)$.

We have a group automorphism

$$\begin{aligned} \Phi_1 : \ & \Omega_5^{\rm O}(\mathrm{B}^2\mathbb{Z}_2) \to \mathbb{Z}_2^4 \\ & (M, B_2) \mapsto (B_2 \cup \mathrm{Sq}^1 B_2, w_2(TM) \cup \mathrm{Sq}^1 B_2, \\ & w_1(TM)^2 \cup \mathrm{Sq}^1 B_2, w_2(TM) w_3(TM)). \end{aligned} \tag{77}$$

1. Let α be the generator of $\mathrm{H}^1(\mathbb{RP}^2, \mathbb{Z}_2)$, β be the generator of $\mathrm{H}^1(\mathbb{RP}^3, \mathbb{Z}_2)$.
   Since $\mathrm{Sq}^1(\alpha \cup \beta) = \alpha^2 \cup \beta + \alpha \cup \beta^2$, $w_1(T(\mathbb{RP}^2 \times \mathbb{RP}^3)) = \alpha$, $w_2(T(\mathbb{RP}^2 \times \mathbb{RP}^3)) = \alpha^2$, $w_3(T(\mathbb{RP}^2 \times \mathbb{RP}^3)) = 0$, Φ_1 maps $(\mathbb{RP}^2 \times \mathbb{RP}^3, \alpha \cup \beta)$ to $(1, 0, 0, 0)$.
2. Let γ be the generator of $\mathrm{H}^1(S^1, \mathbb{Z}_2)$, ζ be the generator of $\mathrm{H}^1(\mathbb{RP}^4, \mathbb{Z}_2)$.
   Since $\mathrm{Sq}^1(\gamma \cup \zeta) = \gamma \cup \zeta^2$, $w_1(T(S^1 \times \mathbb{RP}^4)) = \zeta$, $w_2(T(S^1 \times \mathbb{RP}^4)) = 0$, Φ_1 maps $(S^1 \times \mathbb{RP}^4, \gamma \cup \zeta)$ to $(0, 0, 1, 0)$.
3. Let W be the Wu manifold SU(3)/SO(3),
   $\mathrm{Sq}^1 w_2(T\mathrm{W}) = w_3(T\mathrm{W})$, Φ_1 maps $(\mathrm{W}, w_2(T\mathrm{W}))$ to $(1, 1, 0, 1)$, Φ_1 maps $(\mathrm{W}, 0)$ to $(0, 0, 0, 1)$.

So we conclude that a generating set of manifold generators for $\Omega_5^{\rm O}(\mathrm{B}^2\mathbb{Z}_2)$ is

$$\begin{aligned} &\{(\mathbb{RP}^2 \times \mathbb{RP}^3, \alpha \cup \beta), (\mathrm{W}, w_2(T\mathrm{W})), \\ &(S^1 \times \mathbb{RP}^4, \gamma \cup \zeta), (\mathrm{W}, 0)\}. \end{aligned} \tag{78}$$

This information will be used later to match the SU(2) YM anomalies at $\theta = \pi$.

C. $\Omega_5^{\rm O}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2) \equiv \Omega_5^{(\mathrm{O}\times\mathbb{Z}_2)\times \mathrm{B}\mathbb{Z}_2}$

Follow Sec. III A, we enlist all possible 't Hooft anomalies of 4d pure SU(4) YM at $\theta = \pi$ (when the $\mathbb{Z}_2^C$-background field can be turned on) by obtaining

[10] Interestingly, the bordism group has been studied recently in a different context in [57].

the 5d cobordism invariants from bordism groups of (eq. (8)/eq. (52)).

We are given a 5-manifold M and a 1-form field $A : M \to \mathrm{B}\mathbb{Z}_2 = \mathrm{B}\mathbb{Z}_2^C$ and a 2-form $B = B_2 : M \to \mathrm{B}^2\mathbb{Z}_2$ gauge field in the YM gauge theory eq. (10) (and eqn. (29) at N = 2). We like to obtain the bordism invariants of $\Omega_5^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$. We compute the bordism group [34]

$$\Omega_5^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2) = \mathbb{Z}_2^{12}, \tag{79}$$

whose cobordism invariants are generated by

$$\begin{cases} B_2 \cup \mathrm{Sq}^1 B_2, \\ \mathrm{Sq}^2\mathrm{Sq}^1 B_2, \\ w_1(TM)^2 \mathrm{Sq}^1 B_2, \\ w_2(TM) w_3(TM), \\ A^5,\ A^2\mathrm{Sq}^1 B_2, \\ A^3 B_2,\ A^3 w_1(TM)^2, \\ AB_2^2,\ Aw_1(TM)^4, \\ AB_2 w_1(TM)^2,\ Aw_2(TM)^2. \end{cases} \tag{80}$$

where $\mathrm{Sq}^2\mathrm{Sq}^1 B_2 = (w_2(TM) + w_1(TM)^2)\mathrm{Sq}^1 B_2 = (w_3(TM) + w_1(TM)^3)B_2$, $w_1(TM)^2\mathrm{Sq}^1 B_2 = w_1(TM)^3 B_2$ (See [34]).

$A^2\mathrm{Sq}^1 B_2 = w_1(TM)A^2 B_2$.

We also compute the oriented bordism invariants of $\Omega_5^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$, we find

$$\Omega_5^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2) = \mathbb{Z}_2^{6}, \tag{81}$$

whose cobordism invariants are generated by

$$\begin{cases} B_2 \cup \mathrm{Sq}^1 B_2 = \mathrm{Sq}^2\mathrm{Sq}^1 B_2, \\ w_2(TM) w_3(TM), \\ A^5,\ A^3 B_2, \\ AB_2^2,\ Aw_2(TM)^2. \end{cases} \tag{82}$$

The 4d Yang-Mills theory at $\theta = \pi$ have no 4d 't Hooft anomaly once the $\mathcal{CT}$ (or $\mathcal{T}$) symmetry is not preserved (as we discussed before that $\mathcal{C}$-symmetry is a good symmetry for any θ which has no anomaly directly from mixing with $\mathcal{C}$ by its own). This means that all 5d higher SPTs/cobordism invariant for 4d YM theory must vanish at $\Omega_5^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ when $\mathcal{CT}$ (or $\mathcal{T}$) is removed. So the 5d SPTs for this 4d YM are chosen among:

$$\boxed{\begin{cases} B_2 \cup \mathrm{Sq}^1 B_2 + \mathrm{Sq}^2\mathrm{Sq}^1 B_2, \\ w_1(TM)^2 \mathrm{Sq}^1 B_2, \\ A^2\mathrm{Sq}^1 B_2,\ A^3 w_1(TM)^2, \\ Aw_1(TM)^4,\ AB_2 w_1(TM)^2. \end{cases}} \tag{83}$$

Let α be the generator of $\mathrm{H}^1(\mathbb{RP}^2, \mathbb{Z}_2)$, β be the generator of $\mathrm{H}^1(\mathbb{RP}^3, \mathbb{Z}_2)$, γ be the generator of $\mathrm{H}^1(S^1, \mathbb{Z}_2)$, ζ be the generator of $\mathrm{H}^1(\mathbb{RP}^4, \mathbb{Z}_2)$.

$\mathrm{Sq}^1(\alpha \cup \beta) = \alpha^2 \cup \beta + \alpha \cup \beta^2$, $\mathrm{Sq}^2\mathrm{Sq}^1(\alpha \cup \beta) = 0$, $w_1(T(\mathbb{RP}^2 \times \mathbb{RP}^3)) = \alpha$, $\mathrm{Sq}^1(\gamma \cup \zeta) = \gamma \cup \zeta^2$, $\mathrm{Sq}^2\mathrm{Sq}^1(\gamma \cup \zeta) = \gamma \cup \zeta^4$, $w_1(T(S^1 \times \mathbb{RP}^4)) = \zeta$.

So a generating set of manifold generators for the potential candidate Yang-Mills terms[11] is

$$\begin{aligned} &\{(\mathbb{RP}^2 \times \mathbb{RP}^3, A = 0, B = \alpha \cup \beta), \\ &(S^1 \times \mathbb{RP}^4, A = 0, B = \gamma \cup \zeta), \\ &(S^1 \times \mathbb{RP}^4, A = \zeta, B = \gamma \cup \zeta), \\ &(\mathbb{RP}^2 \times \mathbb{RP}^3, A = \beta, B = 0), \\ &(S^1 \times \mathbb{RP}^4, A = \gamma, B = 0), \\ &(\mathbb{RP}^2 \times \mathbb{RP}^3, A = \beta, B = \beta^2)\}. \end{aligned} \tag{84}$$

D. $\Omega_3^{\mathrm{O}}(\mathrm{BO}(3))$

Follow Sec. III A, we enlist all possible 't Hooft anomalies of 2d $\mathbb{CP}^1$ model, or equivalently O(3) NLSM, at $\theta = \pi$, by obtaining the 3d cobordism invariants from bordism groups of (eq. (65)/eq. (67)). From physics side, we will interpret the unoriented $\mathrm{O}(D)$ spacetime symmetry with the time reversal from $\mathcal{CT}$ instead of $\mathcal{T}$.

We are given a 3-manifold M and a map $f : M \to \mathrm{BO}(3)$. Here the map $f : M \to \mathrm{BO}(3)$ is a principal O(3) bundle whose associated vector bundle is a rank 3 real vector bundle E over M.

We like to obtain the bordism invariants of $\Omega_3^{\mathrm{O}}(\mathrm{BO}(3))$. We compute the bordism group [34]

$$\Omega_3^{\mathrm{O}}(\mathrm{BO}(3)) = \mathbb{Z}_2^{4}, \tag{85}$$

whose cobordism invariants are generated by

$$\begin{cases} w_1(E)^3, \\ w_1(E)w_2(E), \\ w_3(E), \\ w_1(E)w_1(TM)^2. \end{cases} \tag{86}$$

We have a group automorphism

$$\begin{aligned} &\Phi_2 : \Omega_3^{\mathrm{O}}(\mathrm{BO}(3)) \to \mathbb{Z}_2^4 \\ &\quad (M, E) \mapsto (w_1(E)^3, w_1(E)w_2(E), \\ &\quad w_3(E), w_1(E)w_1(TM)^2). \end{aligned} \tag{87}$$

Let $l_{\mathbb{RP}^n}$ denote the tautological line bundle over $\mathbb{RP}^n$ ($\mathbb{RP}^1 = S^1$). If $x_n \in \mathrm{H}^1(\mathbb{RP}^n, \mathbb{Z}_2)$ denotes the generator, then $w(l_{\mathbb{RP}^n}) = 1 + x_n$, $w(T\mathbb{RP}^n) = (1 + x_n)^{n+1}$.

Let $\underline{n}$ denote the trivial real vector bundle of rank n, $+$ denote the direct sum.

By the Whitney sum formula, $w(E \oplus F) = w(E)w(F)$. Here $w(E) = 1 + w_1(E) + w_2(E) + \cdots$ is the total Stiefel-Whitney class of E. Then we find:

1. Since $w(3l_{\mathbb{RP}^3}) = (1 + x_3)^3 = 1 + x_3 + x_3^2 + x_3^3$, $w_1(T\mathbb{RP}^3) = 0$, Φ_2 maps $(\mathbb{RP}^3, 3l_{\mathbb{RP}^3})$ to $(1, 1, 1, 0)$.

2. Since $w(l_{\mathbb{RP}^3} + \underline{2}) = 1 + x_3$, Φ_2 maps $(\mathbb{RP}^3, l_{\mathbb{RP}^3} + \underline{2})$ to $(1, 0, 0, 0)$.

[11] We abbreviate the 5d cobordism invariants that characterize the 4d SU(N) YM theory's anomaly as "*Yang-Mills terms*."

3. Since $w(l_{S^1}+\underline{2}) = 1+x_1$, $w_1(T(S^1\times\mathbb{RP}^2)) = x_2$, Φ_2 maps $(S^1\times\mathbb{RP}^2, l_{S^1}+\underline{2})$ to $(0,0,0,1)$.

4. Since $w(l_{S^1}+l_{\mathbb{RP}^2}+\underline{1}) = (1+x_1)(1+x_2) = 1+x_1+x_2+x_1x_2$, Φ_2 maps $(S^1\times\mathbb{RP}^2, l_{S^1}+l_{\mathbb{RP}^2}+\underline{1})$ to $(1,1,0,1)$.

So a generating set of manifold generators for $\Omega_3^{\mathrm{O}}(\mathrm{BO}(3))$ is

$$\{(\mathbb{RP}^3, 3l_{\mathbb{RP}^3}), (\mathbb{RP}^3, l_{\mathbb{RP}^3}+2), (S^1\times\mathbb{RP}^2, l_{S^1}+2),\\ (S^1\times\mathbb{RP}^2, l_{S^1}+l_{\mathbb{RP}^2}+1)\}. \tag{88}$$

Note that $(S^1\times\mathbb{RP}^2, l_{S^1}+2l_{\mathbb{RP}^2})$ is also a generator. Note $w(l_{S^1}+2l_{\mathbb{RP}^2}) = (1+x_1)(1+x_2)^2 = 1+x_1+x_2^2+x_1x_2^2$, therefore Φ_2 maps $(S^1\times\mathbb{RP}^2, l_{S^1}+2l_{\mathbb{RP}^2})$ to $(0,1,1,1)$.

E. $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_4)$

Follow Sec. III A, now we enlist all possible 't Hooft anomalies of 4d pure SU(4) YM at $\theta=\pi$ (but when the $\mathbb{Z}_2^C$-background field is turned off) by obtaining the 5d cobordism invariants from bordism groups of (eq. (7)/eq. (51)).

We are given a 5-manifold M and a map $f: M\to \mathrm{B}^2\mathbb{Z}_4$. Here the map $f: M\to\mathrm{B}^2\mathbb{Z}_4$ is the 2-form $B=B_2$ gauge field in the YM gauge theory eq. (10) (and eqn. (29) at $\mathrm{N}=4$).

We compute the bordism invariants of $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_4)$, we find the bordism group [34]

$$\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_4) = \mathbb{Z}_2^4, \tag{89}$$

whose cobordism invariants are generated by

$$\begin{cases} B_2\cup\beta_{(2,4)}B_2,\\ \mathrm{Sq}^2\beta_{(2,4)}B_2,\\ w_1(TM)^2\beta_{(2,4)}B_2,\\ w_2(TM)w_3(TM). \end{cases} \tag{90}$$

where $\beta_{(2,4)}: \mathrm{H}^*(M,\mathbb{Z}_4)\to\mathrm{H}^{*+1}(M,\mathbb{Z}_2)$ is the Bockstein homomorphism associated to the extension $\mathbb{Z}_2\to\mathbb{Z}_8\to\mathbb{Z}_4$ (see Appendix A).

We have a group automorphism

$$\Phi_3: \Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_4)\to\mathbb{Z}_2^4\\ (M,B_2)\mapsto (B_2\cup\beta_{(2,4)}B_2, w_2(TM)\beta_{(2,4)}B_2,\\ w_1^2(TM)\beta_{(2,4)}B_2, w_2(TM)w_3(TM)). \tag{91}$$

Let K be the Klein bottle.

1. Let α' be the generator of $\mathrm{H}^1(S^1,\mathbb{Z}_4)$, β' be the generator of the $\mathbb{Z}_4$ factor of $\mathrm{H}^1(K,\mathbb{Z}_4) = \mathbb{Z}_4\times\mathbb{Z}_2$ (see Appendix C), γ' be the generator of $\mathrm{H}^2(S^2,\mathbb{Z}_4)$. $\beta_{(2,4)}\beta'=\sigma$ where σ is the generator of $\mathrm{H}^2(K,\mathbb{Z}_2)=\mathbb{Z}_2$ (see Appendix C).

   Since $\beta_{(2,4)}(\alpha'\cup\beta'+\gamma') = \alpha'\cup\sigma$ and $w_2(T(S^1\times K\times S^2)) = w_1(T(S^1\times K\times S^2))^2 = 0$, we find that Φ_3 maps $(S^1\times K\times S^2, \alpha'\cup\beta'+\gamma')$ to $(1,0,0,0)$.

2. Following the notation of [58], X_2 is a simply-connected 5-manifold which is orientable but non-spin. Let θ' and η' be two generators of $\mathrm{H}^2(X_2,\mathbb{Z}_4)=\mathbb{Z}_4^2$, $\beta_{(2,4)}\theta'$ is one of the two generators of $\mathrm{H}^3(X_2,\mathbb{Z}_2)=\mathbb{Z}_2^2$. Since $w_2(TX_2) = (\theta'+\eta')$ mod 2, $w_1(TX_2)=0$ and $w_3(TX_2)=0$, we find that Φ_3 maps (X_2,θ') to $(1,1,0,0)$.

3. Since $w_1(T(S^1\times K\times\mathbb{RP}^2))^2 = w_2(T(S^1\times K\times\mathbb{RP}^2)) = \alpha^2$ where α is the generator of $\mathrm{H}^1(\mathbb{RP}^2,\mathbb{Z}_2)$, we find that Φ_3 maps $(S^1\times K\times\mathbb{RP}^2, \alpha'\cup\beta')$ to $(0,1,1,0)$.

4. W is the Wu manifold, Φ_3 maps $(\mathrm{W},0)$ to $(0,0,0,1)$.

So a generating set of manifold generators for $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_4)$ is

$$\{(S^1\times K\times S^2, \alpha'\cup\beta'+\gamma'), (X_2,\theta'),\\ (S^1\times K\times\mathbb{RP}^2, \alpha'\cup\beta'), (\mathrm{W},0)\} \tag{92}$$

Note that

1. $(S^1\times K\times T^2, \alpha'\cup\beta'+\zeta')$ is also a generator where ζ' is the generator of $\mathrm{H}^2(T^2,\mathbb{Z}_4)$. Since $\beta_{(2,4)}(\alpha'\cup\beta'+\zeta') = \alpha'\cup\sigma$ and $w_2(T(S^1\times K\times T^2)) = w_1(T(S^1\times K\times T^2))^2 = 0$, we find Φ_3 maps $(S^1\times K\times T^2, \alpha'\cup\beta'+\zeta')$ to $(1,0,0,0)$.

2. $(K\times S^3/\mathbb{Z}_4, \beta'\cup\epsilon'+\phi')$ is also a generator where $S^3/\mathbb{Z}_4$ is the Lens space $L(4,1)$, ϵ' is the generator of $\mathrm{H}^1(S^3/\mathbb{Z}_4,\mathbb{Z}_4)$, ϕ' is the generator of $\mathrm{H}^2(S^3/\mathbb{Z}_4,\mathbb{Z}_4)$. Since $\beta_{(2,4)}(\beta'\cup\epsilon'+\phi') = \sigma\cup\epsilon'+\beta'\cup\phi$ where ϕ is the generator of $\mathrm{H}^2(S^3/\mathbb{Z}_4,\mathbb{Z}_2)$, and $w_2(T(K\times S^3/\mathbb{Z}_4)) = w_1(T(K\times S^3/\mathbb{Z}_4))^2 = 0$, we find that Φ_3 maps $(K\times S^3/\mathbb{Z}_4, \beta'\cup\epsilon'+\phi')$ to $(1,0,0,0)$.

The manifold generator of $B_2\cup\beta_{(2,4)}B_2$ can be chosen to be $S^1\times K\times S^2$ or $S^1\times K\times T^2$ or $K\times S^3/\mathbb{Z}_4$ or X_2.

The manifold generator of $w_1^2(TM)\beta_{(2,4)}B_2$ can be chosen to be $S^1\times K\times\mathbb{RP}^2$.

F. $\Omega_5^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2\ltimes\mathrm{B}^2\mathbb{Z}_4)\equiv\Omega_5^{(\mathrm{O}\times(\mathbb{Z}_2\ltimes\mathrm{B}\mathbb{Z}_4))}$

Follow Sec. III A, now we enlist all possible 't Hooft anomalies of 4d pure SU(4) YM at $\theta=\pi$ (when the $\mathbb{Z}_2^C$-background field can be turned on) by obtaining the 5d cobordism invariants from bordism groups of (eq. (7)/eq. (51)).

Note that again from physics side, we will interpret the unoriented $\mathrm{O}(D)$ spacetime symmetry with the time reversal from $\mathcal{CT}$ instead of $\mathcal{T}$. So we choose the former $\Omega_5^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2\ltimes\mathrm{B}^2\mathbb{Z}_4)\equiv\Omega_5^{(\mathrm{O}\times(\mathbb{Z}_2\ltimes\mathrm{B}\mathbb{Z}_4))}$ for $\mathcal{CT}$, rather than the more complicated latter $\Omega_5^{\mathrm{O}\ltimes}(\mathrm{B}\mathbb{Z}_2\ltimes\mathrm{B}^2\mathbb{Z}_4)\equiv\Omega_5^{(\mathrm{O}\times\mathbb{Z}_2)\ltimes\mathrm{B}\mathbb{Z}_4}$ for $\mathcal{T}$.

Before we dive into $\Omega_5^{\rm O}(\mathrm{B}\mathbb{Z}_2 \ltimes \mathrm{B}^2\mathbb{Z}_4) \equiv \Omega_5^{(\mathrm{O}\times(\mathbb{Z}_2\ltimes \mathrm{B}\mathbb{Z}_4))}$, we first study the simplified "untwisted" bordism group $\Omega_5^{\rm O}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_4)$.

We are given a 5-manifold M and a 1-form field $A : M \to \mathrm{B}\mathbb{Z}_2$ and a 2-form $B = B_2 : M \to \mathrm{B}^2\mathbb{Z}_4$ gauge field in the YM gauge theory eq. (10) (and eqn. (29) at $\mathrm{N} = 4$). We compute the bordism invariants of $\Omega_5^{\rm O}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_4)$, we find the bordism group [34]

$$\Omega_5^{\rm O}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_4) = \mathbb{Z}_2^{12}, \tag{93}$$

whose cobordism invariants are generated by

$$\begin{cases} B_2 \cup \beta_{(2,4)}B_2,\\ \mathrm{Sq}^2\beta_{(2,4)}B_2,\\ w_1(TM)^2\beta_{(2,4)}B_2,\\ w_2(TM)w_3(TM),\\ A^5,\ A^2\beta_{(2,4)}B_2,\\ A^3B_2,\ A^3w_1(TM)^2,\\ AB_2^2,\ Aw_1(TM)^4,\\ AB_2w_1(TM)^2,\ Aw_2(TM)^2. \end{cases} \tag{94}$$

We also compute the bordism invariants of $\Omega_5^{\rm SO}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_4)$, we find [34]

$$\Omega_5^{\rm SO}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_4) = \mathbb{Z}_2^{6}, \tag{95}$$

whose cobordism invariants are generated by

$$\begin{cases} \mathrm{Sq}^2\beta_{(2,4)}B_2,\\ w_2(TM)w_3(TM),\\ A^5,\ A^3B_2,\\ AB_2^2,\ Aw_2(TM)^2. \end{cases} \tag{96}$$

The 4d Yang-Mills theory at $\theta = \pi$ have no 4d 't Hooft anomaly once the $\mathcal{CT}$ (or $\mathcal{T}$) symmetry is not preserved (as we discussed before that $\mathcal{C}$-symmetry is a good symmetry for any θ which has no anomaly directly from mixing with $\mathcal{C}$ by its own). This means that all 5d higher SPTs/cobordism invariant for 4d YM theory must vanish at $\Omega_5^{\rm SO}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_4)$ when $\mathcal{CT}$ (or $\mathcal{T}$) is removed. So the 5d SPTs for this 4d YM are chosen among:

$$\boxed{\begin{cases} B_2 \cup \beta_{(2,4)}B_2,\\ w_1(TM)^2\beta_{(2,4)}B_2,\\ A^2\beta_{(2,4)}B_2,\ A^3w_1(TM)^2,\\ Aw_1(TM)^4,\ AB_2w_1(TM)^2. \end{cases}} \tag{97}$$

Let α' be the generator of $\mathrm{H}^1(S^1, \mathbb{Z}_4)$, β' be the generator of the $\mathbb{Z}_4$ factor of $\mathrm{H}^1(K, \mathbb{Z}_4) = \mathbb{Z}_4 \times \mathbb{Z}_2$ (see Appendix C), γ' be the generator of $\mathrm{H}^2(S^2, \mathbb{Z}_4)$. Note $\beta_{(2,4)}\beta' = \sigma$ where σ is the generator of $\mathrm{H}^2(K, \mathbb{Z}_2) = \mathbb{Z}_2$ (see Appendix C). Let α be the generator of $\mathrm{H}^1(\mathbb{RP}^2, \mathbb{Z}_2)$, β be the generator of $\mathrm{H}^1(\mathbb{RP}^3, \mathbb{Z}_2)$, γ be the generator of $\mathrm{H}^1(S^1, \mathbb{Z}_2)$.

Then a generating set of manifold generators for the Yang-Mills terms is

$$\begin{aligned} &\{(S^1 \times K \times S^2, A = 0, B = \alpha' \cup \beta' + \gamma'),\\ &(S^1 \times K \times \mathbb{RP}^2, A = 0, B = \alpha' \cup \beta'),\\ &(S^1 \times K \times \mathbb{RP}^2, A = \alpha, B = \alpha' \cup \beta'),\\ &(\mathbb{RP}^2 \times \mathbb{RP}^3, A = \beta, B = 0),\\ &(S^1 \times \mathbb{RP}^4, A = \gamma, B = 0),\\ &(S^1 \times S^2 \times \mathbb{RP}^2, A = \gamma, B = \gamma')\}. \end{aligned} \tag{98}$$

Now we discuss this group, $\Omega_5^{\rm O}(\mathrm{B}\mathbb{Z}_2 \ltimes \mathrm{B}^2\mathbb{Z}_4) \equiv \Omega_5^{(\mathrm{O}\times(\mathbb{Z}_2\ltimes \mathrm{B}\mathbb{Z}_4))}$, by Postnikov class. We have a fibration

$$\begin{array}{ccc} \mathrm{B}^2\mathbb{Z}_4 & \longrightarrow & \mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{B}\mathbb{Z}_4)\\ & & \downarrow\\ & & \mathrm{B}\mathbb{Z}_2 \end{array} \tag{99}$$

which is classified by Postnikov class in $\mathrm{H}^3(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_4) = \mathbb{Z}_2$. So $\Omega_5^{\rm O}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_4) \equiv \Omega_5^{(\mathrm{O}\times(\mathbb{Z}_2\times \mathrm{B}\mathbb{Z}_4))}$ is the trivial class with a trivial fibration, while $\Omega_5^{\rm O}(\mathrm{B}\mathbb{Z}_2 \ltimes \mathrm{B}^2\mathbb{Z}_4) \equiv \Omega_5^{(\mathrm{O}\times(\mathbb{Z}_2\ltimes \mathrm{B}\mathbb{Z}_4))}$ is the non-trivial Postnikov class with a non-trivial fibration in $\mathrm{H}^3(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_4) = \mathbb{Z}_2$.
$B \in \mathrm{H}^2(M, \mathbb{Z}_{4,A})$ which is the twisted cohomology where $A \in \mathrm{H}^1(M, \mathbb{Z}_2)$ can be viewed as a group homomorphism $\pi_1(M) \to \mathrm{Aut}(\mathbb{Z}_4) = \mathbb{Z}_2$.
We claim that among the candidates of the 5d higher SPTs/cobordism invariants for 4d SU(4) Yang-Mills theory at $\theta = \pi$, no one can vanish in $\Omega_5^{\rm O}(\mathrm{B}\mathbb{Z}_2 \ltimes \mathrm{B}^2\mathbb{Z}_4)$ (see Appendix D and [34]). Namely, we obtain that $\Omega_5^{\rm O}(\mathrm{B}\mathbb{Z}_2 \ltimes \mathrm{B}^2\mathbb{Z}_4) = \mathbb{Z}_2^{11}$, where only the A^3B_2 term is dropped, compared with $\Omega_5^{\rm O}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_4)$.

G. $\Omega_3^{\rm O}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)))$

Follow Sec. III A, we enlist all possible 't Hooft anomalies of 2d $\mathbb{CP}^{\mathrm{N}-1}$ model at $\mathrm{N} = 4$, at $\theta = \pi$, by obtaining the 3d cobordism invariants from bordism groups of (eq. (68)). From physics side, we will interpret the unoriented $\mathrm{O}(D)$ spacetime symmetry with the time reversal from $\mathcal{CT}$ instead of $\mathcal{T}$.

We are given a 3-manifold M and a map $f : M \to \mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4))$ which corresponds to a principal $\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)$ bundle E over M.

We compute the bordism invariants of $\Omega_3^{\rm O}(\mathrm{BO}(3))$, we find the bordism group [34]

$$\Omega_3^{\rm O}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4))) = \mathbb{Z}_2^{4}, \tag{100}$$

whose cobordism invariants are generated by

$$\begin{cases} w_1(E)^3,\\ w_1(E)w_1(TM)^2,\\ \beta_{(2,4)}w_2(E),\\ w_1(E)(w_2(E) \mod 2). \end{cases} \tag{101}$$

where E is a principal $\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)$ bundle over M which is a pair $(w_1(E), w_2(E)) \in \mathrm{H}^1(M, \mathbb{Z}_2) \times \mathrm{H}^2(M, \mathbb{Z}_{4,w_1(E)})$ where $\mathrm{H}^2(M, \mathbb{Z}_{4,w_1(E)})$ is the twisted cohomology, $w_1(E)$

can be viewed as a group homomorphism $\pi_1(M) \to \mathrm{Aut}(\mathbb{Z}_4) = \mathbb{Z}_2$.

In the following discussion, we use the ordinary cohomology instead of the twisted cohomology. The term $w_1(E)(w_2(E) \mod 2)$ may be modified if we use the twisted cohomology. The details of this case will be discussed in our upcoming future work.

We have a group automorphism

$$\begin{aligned}\Phi_4 : &\Omega_3^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4))) \to \mathbb{Z}_2^4\\ &(M, w_1(E), w_2(E)) \mapsto (w_1(E)^3, w_1(E)w_1(TM)^2,\\ &\beta_{(2,4)}w_2(E), w_1(E)(w_2(E) \mod 2)).\end{aligned} \tag{102}$$

1. Recall that β be the generator of $\mathrm{H}^1(\mathbb{RP}^3, \mathbb{Z}_2)$. Since $w_1(T\mathbb{RP}^3) = 0$, Φ_4 maps $(\mathbb{RP}^3, \beta, 0)$ to $(1, 0, 0, 0)$.

2. Recall that γ be the generator of $\mathrm{H}^1(S^1, \mathbb{Z}_2)$. Since $w_1(T(S^1 \times \mathbb{RP}^2)) = \alpha$ where α is the generator of $\mathrm{H}^1(\mathbb{RP}^2, \mathbb{Z}_2)$, Φ_4 maps $(S^1 \times \mathbb{RP}^2, \gamma, 0)$ to $(0, 1, 0, 0)$.

3. K is the Klein bottle. Recall that α' is the generator of $\mathrm{H}^1(S^1, \mathbb{Z}_4)$, β' is the generator of the $\mathbb{Z}_4$ factor of $\mathrm{H}^1(K, \mathbb{Z}_4) = \mathbb{Z}_4 \times \mathbb{Z}_2$ (see Appendix C). Since $\beta_{(2,4)}(\alpha' \cup \beta') = \alpha' \cup \sigma$ where σ is the generator of $\mathrm{H}^2(K, \mathbb{Z}_2)$, Φ_4 maps $(S^1 \times K, 0, \alpha' \cup \beta')$ to $(0, 0, 1, 0)$.

4. Recall that γ is the generator of $\mathrm{H}^1(S^1, \mathbb{Z}_2)$, γ' is the generator of $\mathrm{H}^2(S^2, \mathbb{Z}_4)$. Φ_4 maps $(S^1 \times S^2, \gamma, \gamma')$ to $(0, 0, 0, 1)$.

So a generating set of manifold generators for $\Omega_3^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)))$ is

$$\begin{aligned}&\{(\mathbb{RP}^3, \beta, 0), (S^1 \times \mathbb{RP}^2, \gamma, 0),\\ &(S^1 \times K, 0, \alpha' \cup \beta'), (S^1 \times S^2, \gamma, \gamma')\}\end{aligned} \tag{103}$$

Note that

1. $(S^1 \times T^2, \gamma, \zeta')$ is also a generator, where γ is the generator of $\mathrm{H}^1(S^1, \mathbb{Z}_2)$, ζ' is the generator of $\mathrm{H}^2(T^2, \mathbb{Z}_4)$. Φ_4 maps $(S^1 \times T^2, \gamma, \zeta')$ to $(0, 0, 0, 1)$

2. $(S^3/\mathbb{Z}_4, \epsilon, \phi')$ is also a generator, where $S^3/\mathbb{Z}_4$ is the Lens space $L(4, 1)$, ϵ is the generator of $\mathrm{H}^1(S^3/\mathbb{Z}_4, \mathbb{Z}_2)$, ϕ' is the generator of $\mathrm{H}^2(S^3/\mathbb{Z}_4, \mathbb{Z}_4)$. Φ_4 maps $(S^3/\mathbb{Z}_4, \epsilon, \phi')$ to $(0, 0, 0, 1)$.

IV. REVIEW AND SUMMARY OF KNOWN ANOMALIES IN COBORDISM INVARIANTS

Follow Sec. III, we have obtained the co/bordism groups relevant from the given full G-symmetry of 4d YM and 2d $\mathbb{CP}^{\mathrm{N}-1}$ models. Therefore, based on the correspondence between dd 't Hooft anomalies and Dd$=(d+1)$d topological terms/cobordism/SPTs invariants, we have obtained the classification of all possible higher 't Hooft anomalies for these 4d YM and 2d $\mathbb{CP}^{\mathrm{N}-1}$ models.

Below we first match our result to the known anomalies found in the literature, and we shall put these known anomalies into a more mathematical precise thus a more general framework, under the cobordism theory. We will write down the precise dd 't Hooft anomalies and Dd$=(d+1)$d cobordism/SPTs invariants for them. We will also clarify the physical interpretations (e.g. from condensed matter inputs) of anomalies.

A. Mix higher-anomaly of time-reversal $\mathbb{Z}_2^{CT}$ and 1-form center $\mathbb{Z}_{\mathrm{N}}$-symmetry of SU(N)-YM theory

First recall in Sec. II B 5, we re-derives the mix higher-anomaly of time-reversal $\mathbb{Z}_2^T$ and 1-form center $\mathbb{Z}_{\mathrm{N}}$-symmetry of 4d SU(N)-YM, at even N, discovered in [30]. By turning on 2-form $\mathbb{Z}_{\mathrm{N}}$-background field $B = B_2$ coupling to YM theory, the $\mathbb{Z}_2^T$-symmetry shifts the 4d YM with an additional 5d higher SPTs term eq. (II B 5). We also learned that the same mix higher-anomaly occur by replacing $\mathbb{Z}_2^T$ to eq. (50),

$$\mathbb{Z}_2^{CT},\ \mathbb{Z}_2^P,\ \text{and}\ \mathbb{Z}_2^{CP},$$

For our preference, we focus on $\mathcal{CT}$ instead of $\mathcal{T}$. This type of anomaly has the linear dependence on $\mathcal{CT}$ (thus linear also $\mathcal{T}$) and quadratic dependence on B_2. Compare with our eq. (76), we find that the precise form for 5d cobordism invariant/ 4d higher 't Hooft anomaly is:

$$\boxed{B_2 \mathrm{Sq}^1 B_2}. \tag{104}$$

We combine the Steenrod-Wu formula and the product formula of Steenrod operation to derive the equality in eq. (104). More precisely, we need to consider instead eq. (123), $B_2\mathrm{Sq}^1B_2 + \mathrm{Sq}^2\mathrm{Sq}^1B_2 = \frac{1}{2}\tilde{w}_1(TM)\mathcal{P}_2(B_2)$., see Sec. VIII A for details and derivations.

B. Mix anomaly of $\mathbb{Z}_2^C = \mathbb{Z}_2^x$- and time-reversal $\mathbb{Z}_2^{CT}$ or SO(3)-symmetry of $\mathbb{CP}^1$-model

Now we move on to 2d $\mathbb{CP}^1$ or O(3) NLSM model at $\theta = \pi$, we get the full 0-form global symmetries eq. (67),

$$\mathbb{Z}_2^{CT} \times \mathrm{PSU}(2) \times \mathbb{Z}_2^x \equiv \mathbb{Z}_2^{CT} \times \mathrm{PSU}(2) \times \mathbb{Z}_2^C = \mathbb{Z}_2^{CT} \times \mathrm{O}(3).$$

It has been known that there is a non-perturbative global discrete anomaly from the $\mathbb{Z}_2^C$ (a discrete translational $\mathbb{Z}_2^x$ symmetry) since the work of Gepner-Witten

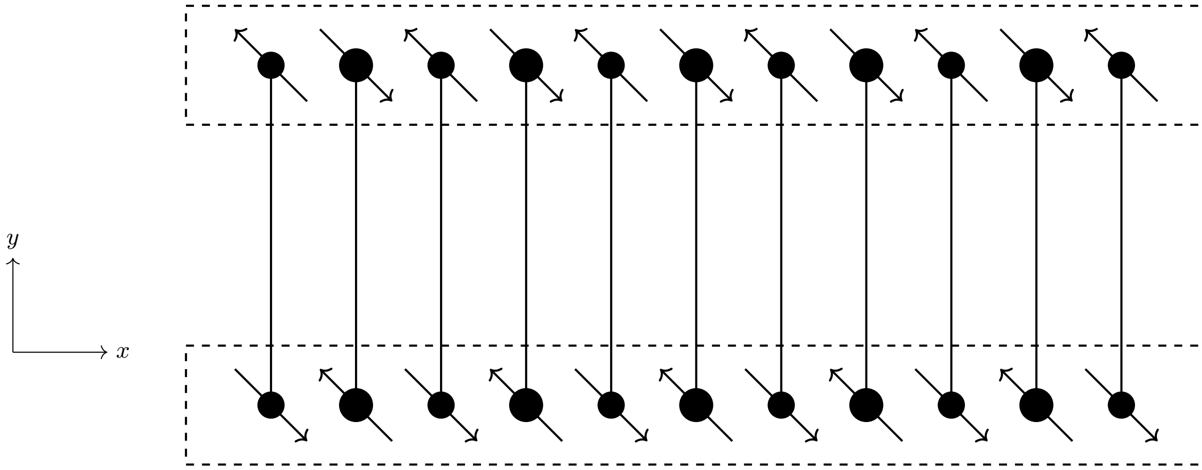


FIG. 3. An interpretation of 't Hooft anomaly $w_1(E)(w_2(V_{\mathrm{SO}(3)}) + w_1(TM)^2)$ (eq. (105)) for 2d $\mathbb{CP}^1$ or O(3) NLSM model, is obtained from the 2d dangling spin-1/2 gapless modes living on the 2d boundary a 3d layer-stacking system of the 2d spin-1 Haldane chain. Each vertical solid line represents a 2d spin-1 Haldane chain eq. (106). The 2d boundary combined from the dangling spin-1/2 gapless modes, encircled by the dashed-line rectangle, on the top and on the bottom, are effectively the 2d $\mathbb{CP}^1$ model. The $w_1(E)w_2(V_{\mathrm{SO}(3)})$ has been identified by [59]. The same trick of [59] applies to eq. (106) teaches us the more complete anomaly eq. (105). The $\mathbb{Z}_2^x \equiv \mathbb{Z}_2^C$-translational nature of Heisenberg anti-ferromagnet (AFM) is purposefully emphasized by the spin-1/2 orientation ($\uparrow$ or $\downarrow$) and the two different sizes of the black dots ($\bullet$). Notice that understand the 3d bulk nature obtained from this stacking, inform us that the 3d bulk SPTs also includes a secretly hidden $A_x \cup A_x \cup A_x \equiv A_x^3$ or the $w_1(E)^3 \equiv w_1(\mathbb{Z}_2^x)^3$-anomaly [18, 59].

[60]. More recently, this non-perturbative global discrete anomaly has been revisited by [61, 62] to understand the nature of symmetry-protected gapless critical phases.

We can compare this anomaly (associated to $\mathbb{Z}_2^x$ symmetry and to the PSU(2)-symmetry) to the 3d cobordism invariant/ 2d 't Hooft anomaly we derive in eq. (86). We find that $w_1(E)w_2(V_{\mathrm{SO}(3)})$, where $w_1(E) = w_1(V_{\mathrm{O}(3)}) = w_1(\mathbb{Z}_2^x)$, is the natural choice to describe the anomaly.

Ref. [63], detects a so-called mixed $\mathcal{CPT}$-type anomaly. We can interpret their anomaly as the mix $\mathcal{C}$ ($\mathbb{Z}_2^C = \mathbb{Z}_2^x$) with the $\mathcal{CT}$ ($\mathbb{Z}_2^{CT}$) type anomaly. We compare it to the 3d cobordism invariant/ 2d 't Hooft anomaly we derive in eq. (86), and find $w_1(E)w_1(TM)^2 = w_1(\mathbb{Z}_2^x)w_1(TM)^2$ is the natural choice to describe the anomaly.

So overall, compare with eq. (86), we can interpret the above 2d anomalies are captured by a 3d cobordism invariant for N = 2 case:

$$\boxed{w_1(E)w_2(V_{\mathrm{SO}(3)}) + w_1(E)w_1(TM)^2}. \tag{105}$$

A very natural physics derivation to understand eq. (105) is by the stacking 2d Haldane spin-1 chain picture [59], see Fig. 3. The Haldane spin-1 chain is a 2d SPTs protected by spin-1 rotation SO(3) symmetries and time-reversal (here $\mathbb{Z}_2^{CT}$); its 2d SPTs/topological term is well-known as:

$$\int_{\text{2d spin-1 chain}} w_2(V_{\mathrm{SO}(3)}) + w_1(TM)^2, \tag{106}$$

obtained from group cohomology data $\mathrm{H}^2(\mathrm{BSO}(3), \mathrm{U}(1)) = \mathbb{Z}_2$ and $\mathrm{H}^2(\mathrm{B}\mathbb{Z}_2^T, \mathrm{U}(1)) = \mathbb{Z}_2$ [7]. If the time-reversal or SO(3) symmetry is preserved, the boundary has 2-fold degenerate spin-1/2 modes on each 1d edge. The layer stacking of such spin-1/2 modes to a 2d boundary (encircled by the dashed-line rectangle in Fig. 3) can actually give rise to gapless 2d $\mathbb{CP}^1$ / O(3) NLSM / $\mathrm{SU}(2)_1$-WZW model. Part of its anomaly is captured by the $\mathbb{Z}_2^x$-translation ($w_1(E) = w_1(\mathbb{Z}_2^x)$) times the eq. (106), which renders and thus we derive eq. (105).

Ref. [64] studies the anomaly of the same system, and detects the anomaly $w_3(E)$, we can convert it to

$$\begin{aligned}
\boxed{w_3(E)} &= w_3(V_{\mathrm{O}(3)}) \\
&= w_1(V_{\mathrm{O}(3)})^3 + w_1(V_{\mathrm{O}(3)})w_2(V_{\mathrm{SO}(3)}) + w_3(V_{\mathrm{SO}(3)}) \\
&= w_1(\mathbb{Z}_2^x)^3 + w_1(\mathbb{Z}_2^x)w_2(V_{\mathrm{SO}(3)}) + w_3(V_{\mathrm{SO}(3)}) \\
&= w_1(E)^3 + w_1(E)w_2(V_{\mathrm{SO}(3)}) + w_3(V_{\mathrm{SO}(3)}) \\
&\boxed{= w_1(E)^3 + w_1(E)w_2(V_{\mathrm{SO}(3)}) + w_1(TM)w_2(E)}.
\end{aligned} \tag{107}$$

We also note that

$$\begin{aligned}
& w_1(E)w_2(E) = w_1(V_{\mathrm{O}(3)})w_2(V_{\mathrm{O}(3)}) \\
&= w_1(V_{\mathrm{O}(3)})^3 + w_1(V_{\mathrm{O}(3)})w_2(V_{\mathrm{SO}(3)}) \\
&= w_1(\mathbb{Z}_2^x)^3 + w_1(\mathbb{Z}_2^x)w_2(V_{\mathrm{SO}(3)}) \\
&= w_1(E)^3 + w_1(E)w_2(V_{\mathrm{SO}(3)}).
\end{aligned} \tag{108}$$

Similar equality and anomaly are discussed in [65] in a different topic on Chern-Simons matter theories.

To summarize, note that:
The $w_1(E)^3$ is $(1, 0, 0, 0)$ in the basis of eq. (87).
The $w_1(E)w_2(E)$ is $(0, 1, 0, 0)$ in the basis of eq. (87).
The $w_1(E)w_2(V_{\mathrm{SO}(3)})$ is $(1, 1, 0, 0)$ in the basis of eq. (87).
The $w_3(V_{\mathrm{SO}(3)}) = w_3(E) + w_1(E)w_2(E) = w_1(TM)w_2(E)$

is $(0,1,1,0)$ in the basis of eq. (87).
The $w_3(E) = w_3(V_{\mathrm{O}(3)})$ is $(0,0,1,0)$ in the basis of our eq. (87).

Therefore, Ref. [64]'s anomaly eq. (107) given by $w_3(E) = w_1(E)^3 + w_1(E)w_2(V_{\mathrm{SO}(3)}) + w_1(TM)w_2(E)$ coincides with one of the cobordism invariant as $(0,0,1,0)$ in the basis of our eq. (87). We had explained the physical meaning of $w_1(E)w_2(V_{\mathrm{SO}(3)})$ term in eq. (105). We will explain the meaning of $w_1(E)^3$ in Sec. IV C and the meaning of $w_1(TM)w_2(E)$ in Sec. IV D

C. A cubic anomaly of $\mathbb{Z}_2^C$ of $\mathbb{CP}^1$-model

Now we like to capture the physical meaning of a cubic anomaly of $\mathbb{Z}_2^C = \mathbb{Z}_2^x$-symmetry in eq. (107):

$$\boxed{w_1(E)^3 \equiv w_1(\mathbb{Z}_2^x)^3 \equiv (A_x)^3}, \tag{109}$$

which is a sensible cobordism invariant as the $(1,0,0,0)$ in the basis of eq. (87). Ref. [59] also points out this $w_1(E)^3$ or the A_x^3-anomaly, where A_x is regarded as the $\mathbb{Z}_2^x$-translational background gauge field. We know that the 2d boundary physics we look at in Fig. 3 (encircled by the dashed-line rectangle) describes the gapless CFT theory of $\mathrm{SU}(2)_1$ WZW model at $k=1$. The $\mathrm{SU}(2)_1$ WZW model at $k=1$ is equivalent to a $c=1$ compact non-chiral boson theory (the left and right chiral central charge $c_L = c_R = 1$, but the chiral central charge $c_- = c_L - c_R = 0$) at the self-dual radius [66]. Although properly we could use non-Abelian bosonization method [51], here focusing on the abelian $\mathbb{Z}_2^x$-symmetry and its anomaly, we can simply use the Abelian bosonization.

Since $\mathrm{SU}(2)_1$ WZW model at $k=1$ is equivalent to a $c=1$ compact non-chiral boson theory at the self-dual radius, we consider an action

$$S_{\mathrm{2d}} = \frac{1}{2\pi\alpha'}\int dz d\bar{z}\, (\partial_z\Phi)(\partial_{\bar{z}}\Phi) + \dots \tag{110}$$
$$S_{\mathrm{2d}} = \frac{1}{4\pi}\int dt\, dx\, \big(K_{IJ}\partial_t\phi_I\partial_x\phi_J - V_{IJ}\partial_x\phi_I\partial_x\phi_J\big) + \dots.$$

requiring a rank-2 symmetric bilinear form K-matrix,

$$K_{IJ} = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\oplus\dots; \quad V_{IJ} = \left(\begin{smallmatrix}v&0\\0&v\end{smallmatrix}\right)\oplus\dots. \tag{111}$$

The first form of the action is familiar in string theory and a $c=1$ compact non-chiral boson theory at the self-dual radius. (In string theory, we are looking at $R = \sqrt{\alpha'} = \sqrt{2}$.) The second form of the action is the familiar 2d boundary of 3d bosonic SPTs. This second description is also known as Tomonaga-Luttinger liquid theory [67–69] in condensed matter physics. It is a K-matrix multiplet generalization of the usual chiral boson theory of Floreanini and Jackiw [70]. The reason we write $\dots$ in eq. (111) is that there could be additional 3d SPTs sectors for 2d $\mathbb{CP}^1$-model (e.g. eq. (116)), more than what we focus on in this subsection. Here we trade the boson scalar Φ to ϕ_1, while ϕ_2 is the dual boson field. We can determine the bosonic anomalies [18] by looking at the anomalous symmetry transformation on the 2d theory, living on the boundary of which 3d SPTs. We use the mode expansion for a multiplet scalar boson field theory [18], with zero modes ϕ_{0I} and winding modes P_{ϕ_J}:

$$\phi_I(x) = \phi_{0I} + K^{-1}_{IJ}P_{\phi_J}\frac{2\pi}{L}x + \mathrm{i}\sum_{n\neq 0}^{n\in\mathbb{Z}}\frac{1}{n}\alpha_{I,n}e^{-\mathrm{i}nx\frac{2\pi}{L}},$$

which satisfy the commutator $[\phi_{0I}, P_{\phi_J}] = \mathrm{i}\delta_{IJ}$. The Fourier modes satisfy a generalized Kac-Moody algebra: $[\alpha_{I,n},\alpha_{J,m}] = nK^{-1}_{IJ}\delta_{n,-m}$. For a modern but self-contained pedagogical treatment on a canonical quantization of K-matrix multiplet (non-)chiral boson theory, the readers can consult Appendix B of [71].

Follow [59], based on the identification of spin observables of Hamiltonian model eq. (61) and the abelian bosonized theory, we can map the symmetry transformation to the continuum description on the boson multiplet $\phi_I(x) = (\phi_1(x),\ \phi_2(x))$. The commutation relation is $[\phi_I(x_1), K_{I'J}\partial_x\phi_J(x_2)] = 2\pi\mathrm{i}\delta_{II'}\delta(x_1-x_2)$. The continuum limit of 2d anomalous symmetry transformation is [72] [18]:

$$\mathrm{S}_{\mathrm{N}}^{(p)} = \exp[\frac{\mathrm{i}}{\mathrm{N}}(\int_0^L dx\,\partial_x\phi_2 + p\int_0^L dx\,\partial_x\phi_1)], \tag{112}$$
$$\mathrm{S}_{\mathrm{N}}^{(p)}\begin{pmatrix}\phi_1(x)\\\phi_2(x)\end{pmatrix}(\mathrm{S}_{\mathrm{N}}^{(p)})^{-1} = \begin{pmatrix}\phi_1(x)\\\phi_2(x)\end{pmatrix} + \frac{2\pi}{\mathrm{N}}\begin{pmatrix}1\\p\end{pmatrix}.$$

Here L is the compact spatial S^1 circle size of the 2d theory. For 2d $\mathbb{CP}^1$-model, we have $\mathrm{N}=2$ and $p=1$, this is indeed known as the Type I *bosonic anomaly* in [18], which also recovers one anomaly found in [59] and in [64]'s eq. (107).

D. Mix anomaly of time-reversal $\mathbb{Z}_2^T$ and 0-form flavor $\mathbb{Z}_\mathrm{N}$-center symmetry of $\mathbb{CP}^1$-model

Ref. [36, 37] point out another anomaly of $\mathbb{CP}^1$-model, which mixes between time-reversal (which we have chosen to be $\mathcal{CT}$) and the PSU(2) symmetry (which is viewed as the twisted flavor symmetry in [36, 37]). Compare with eq. (86), we can interpret the above 2d anomalies are captured by a 3d cobordism invariant for $\mathrm{N}=2$ case:

$$\boxed{w_1(TM)w_2(V_{\mathrm{SO}(3)}) = w_1(TM)w_2(E) = w_3(V_{\mathrm{SO}(3)})}. \tag{113}$$

This also coincides with the last anomaly term in eq. (107)'s $w_3(E)$. We derive the above first equality in eq. (113) based on $\mathrm{Sq}^1(w_1(E)^2) = 2w_1(E)\mathrm{Sq}^1w_1(E) = 0$ and combine Wu formula, $\mathrm{Sq}^1(w_1(E)^2) = w_1(TM)(w_1(E)^2) = 0$. Thus,

$$w_1(TM)w_2(V_{\mathrm{SO}(3)}) = w_1(TM)(w_2(E) + w_1(E)^2)$$
$$= w_1(TM)w_2(E) = w_1(TM)w_2(V_{\mathrm{O}(3)}). \tag{114}$$

The last equality in eq. (113) is due to $w_1(TM)w_2(V_{\mathrm{SO}(3)}) = \mathrm{Sq}^1 w_2(V_{\mathrm{SO}(3)}) = w_3(V_{\mathrm{SO}(3)})$.

We can combine the Steenrod-Wu formula, and Wu formula:

$$
\begin{aligned}
& w_1(E)w_2(E) + w_3(E) = \mathrm{Sq}^1(w_2(E)) = w_1(TM)w_2(E), \\
& \Rightarrow w_3(E) = (w_1(E) + w_1(TM))w_2(E) \\
& \Rightarrow w_1(TM)w_2(E) = w_3(E) + w_1(E)w_2(E),
\end{aligned} \tag{115}
$$

so we derive $w_1(TM)w_2(E)$ is $(0,1,1,0)$ in the basis of eq. (87). The physical meaning of the 2d anomaly eq. (113) will be explored later in Sec. V, Sec. VII and in Fig.4, which can be understood as the dimensional reduction of 4d anomaly of YM theory compactified on a 2-torus with twisted boundary conditions [36] [35].

In Sec. IV C, We had checked some of the 2d bosonic anomaly by dimensional reducing from 4d to 2d, can be captured by abelian bosonization method as Type I bosonic anomaly in [18]. Some of the anomalies in the above may be also related to other (Type II or Type III) bosonic discrete anomalies, when we break down the global symmetry to certain subgroups.

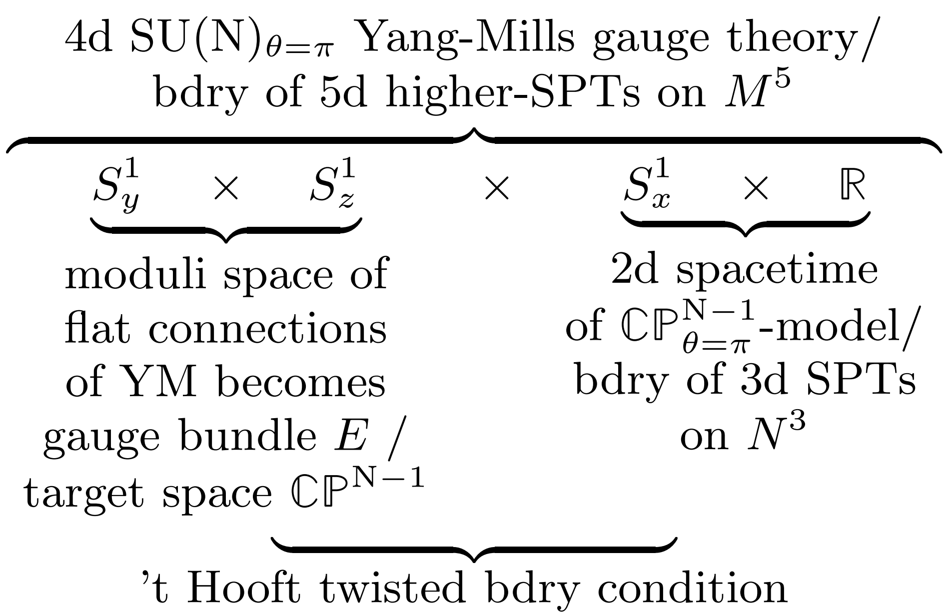


FIG. 4. Follow the setup of the twisted boundary condition induced 't Hooft boundary (bdry) condition [73] along the 2-torus $T^2_{zx} \equiv S^1_z \times S^1_x$, and the twisted compactification [36] [35], we examine that the (higher) anomaly of 4d SU(N) YM theory at $\theta = \pi$ induces the anomaly of 2d $\mathbb{CP}^{\mathrm{N}-1}$ model at $\theta = \pi$. The 4d YM on $S^1_x \times S^1_y \times S^1_z \times \mathbb{R}$ is compactified along the small size of $T^2_{yz} \equiv S^1_y \times S^1_z$, whose moduli space of flat connections becomes the target space $\mathbb{CP}^{\mathrm{N}-1}$ [74, 75], while the remained $S^1_x \times \mathbb{R}$ becomes the 2d spacetime of 2d $\mathbb{CP}^{\mathrm{N}-1}$ model. Our goal, in Sec. V, VIII and VI is to identify the underlying 't Hooft anomalies of 4d SU(N) YM and 2d $\mathbb{CP}^{\mathrm{N}-1}$ model, namely identifying their living on the boundary ($\equiv$ bdry) of 5d and 3d (higher) SPTs when all the (higher) global symmetries needed to be regularized strictly *onsite* and *local* (e.g. [12–14]). The twisted boundary condition of 4d YM for 1-form $\mathbb{Z}_{\mathrm{N}}$-center symmetry (as a higher symmetry twist of [13]) can be dimensionally reduced to the 0-form $\mathbb{Z}_{\mathrm{N}}$-flavor symmetry twisted [76] in the 2d $\mathbb{CP}^{\mathrm{N}-1}$ model. In Sec. VII, we generalize to impose twisted boundary conditions along other novel 2-submanifolds $\mathcal{U}^2$ (such as $\mathbb{RP}^2$, $\mathbb{RP}^2 \# T^2$, or $T^2 \# T^2$, etc.).

V. RULES OF THE GAME FOR ANOMALY CONSTRAINTS

With all the QFT and global symmetries information given in Sec. II, and all the possible anomalies enumerated by the cobordism theory computed in Sec. III, and all the known anomalies in the literature derived and re-written in terms of cobordism invariants organized in Sec. IV, now we are ready to set up the rules of the game to determine the full anomaly constraints for these QFTs (4d SU(N) YM theory and 2d $\mathbb{CP}^{\mathrm{N}-1}$ model at $\theta = \pi$).

Below we simply abbreviate the "5d invariant" as the 5d cobordism/(higher) SPTs invariants which captures the anomaly of 4d SU(N) YM at $\theta = \pi$ at even N, and "3d invariant" as the 3d cobordism/SPTs invariants which captures the anomaly of 2d $\mathbb{CP}^{\mathrm{N}-1}$ at $\theta = \pi$ at even N. Our convention chooses the natural time reversal symmetry transformation as $\mathcal{CT}$.

Rules:

Rule 1. For 5d invariant, for 4d SU(N) YM at $\theta = \pi$ of an even integer N must have analogous anomaly captured by 5d cobordism term of $\sim w_1(TM)(B_2)^2$ (up to some properly defined normalization and quantization).

Rule 2. The chosen 5d invariants may be non-vanished in O-bordism group, but they are vanished in SO-bordism group.

Rule 3. The 3d invariant for 2d $\mathbb{CP}^1$ model must include the 3d cobordism invariants discussed in Sec. IV, in particular, eq. (116).

Rule 4. The 3d invariant for other 2d $\mathbb{CP}^{\mathrm{N}-1}$ for even N (e.g. 2d $\mathbb{CP}^3$) model must include some of familiar terms generalizing that of 2d $\mathbb{CP}^1$ model.

Rule 5. Due to the physical meanings of $\mathcal{CT}$ and $\mathcal{T}$ (and other orientation-reversal symmetries), we must impose a swapping symmetry for 5d invariants.

Rule 6. Relating the 5d and 3d invariants: There is a dimensional reductional constraint and physical meanings between the 5d and 3d invariants, for example, by the twist-compactification on 2-torus T^2.

Rule 7. The 5d invariants for a 4d pure YM theory must involve the nontrivial 2-form B_2 field. The 5d terms that involve no B_2 dependence should be discarded.

Rule 8. For 5d invariant, for 4d SU(N) YM at $\theta = \pi$ of an even integer $\mathrm{N} > 2$ must have analogous anomaly captured by 5d cobordism term of $\sim w_1(TM)A^2B_2$ (up to some properly defined normalization and quantization).

Here are the explanations for our rules.

Rule 1 is based on Sec. II B, for 4d SU(N) YM at $\theta = \pi$ of an even integer N must have analogous anomaly captured by 5d cobordism term of $\sim w_1(TM)B_2^2$ (up to some properly defined normalization and quantization), where we choose the linear time reversal symmetry transformation from $\mathcal{CT}$ and a quadratic term of 2-form fields B_2 coupling to 1-form center symmetry.

Rule 2's physical reasoning is that the time-reversal symmetry transformation from $\mathcal{CT}$ plays an important role for the anomaly. We can see from Sec. II B 6 that only when time-reversal or orientation reversal is involved ($\mathcal{T}$, $\mathcal{P}$, $\mathcal{CT}$ and $\mathcal{CP}$), we have the mixed higher anomalies for YM theory; while for the others ($\mathcal{C}$, $\mathcal{PT}$ and $\mathcal{CPT}$), we do not gain mixed anomalies (e.g. with the 1-form center symmetry).

Rule 3 is dictated by the known physics derivations in Sec. IV and in the literature.

Rule 4 will become clear in Sec. VI.

Rule 5, the swapping symmetry for 5d invariants between $\mathcal{CT}$ and $\mathcal{T}$ (and other orientation-reversal symmetries), we will interpret the unoriented $\mathrm{O}(D)$ spacetime symmetry with the time reversal from $\mathcal{CT}$ or from $\mathcal{T}$ can be swapped. This means that we can choose the 5d topological invariant from the former $\Omega_5^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \ltimes \mathrm{B}^2\mathbb{Z}_4) \equiv \Omega_5^{(\mathrm{O}\times(\mathbb{Z}_2\ltimes\mathrm{B}\mathbb{Z}_4))}$ for $\mathcal{CT}$, rather than the more complicated latter $\Omega_5^{\mathrm{O}_\ltimes}(\mathrm{B}\mathbb{Z}_2 \ltimes \mathrm{B}^2\mathbb{Z}_4) \equiv \Omega_5^{(\mathrm{O}\times\mathbb{Z}_2)\ltimes\mathrm{B}\mathbb{Z}_4}$ for $\mathcal{T}$. We focus on the 5d terms involving $\mathcal{CT}$-symmetry.

Rule 6 about the dimensional reduction from 5d to 3d (or 4d to 2d) is explained in Fig. 4 and the main text, such as in Sec. VII. We should also find the mathematical meanings behind this constraint in Sec. VII.

Rule 7 is based on the physical input that there should be *no* obstruction to regularize a pure YM theory by imposing only ordinary 0-form symmetry alone *onsite*. The obstruction only comes from regularizing a pure YM theory with the involvement of restricting both the higher 1-form center symmetry and the ordinary 0-form symmetry to be *onsite and local*. Thus, it is necessary to turn on the 2-form background field B_2 in order to detect the 't Hooft anomaly of YM theory. Namely, the 5d cobordism invariants of the form $w1(TM)^{\mathrm{t}} \cup A^{\mathrm{a}}$ with $\mathrm{t} + \mathrm{a} = 5$ should be discarded out of the candidate list of 5d term for 4d YM anomalies.

Rule 8 is based on a QFT derivation directly from 4d SU(N) YM theory at $\theta = \pi$ of an even integer $\mathrm{N} > 2$. We find a new higher mixed anomaly between time-reversal ($\mathcal{CT}$ and $\mathcal{T}$), 0-form $\mathbb{Z}_2^C$ and 1-form center symmetry, captured by $\sim w_1(TM)A^2B_2$ (up to some properly defined normalization and quantization).

VI. NEW ANOMALIES OF 2D $\mathbb{CP}^{\mathrm{N}-1}$-MODEL

For 2d $\mathbb{CP}^1$-model at $\theta = \pi$, now we combine all the anomalies found above, including eq. (105), eq. (107), eq. (109) and eq. (113), we obtain a concise way to express the potentially complete 't Hooft anomalies of 2d $\mathbb{CP}^1$-model as:

$$\text{2d } \mathbb{CP}^1\text{-model anomaly}: \boxed{w_1(E)^3 + w_1(E)w_2(V_{\mathrm{SO}(3)}) + w_1(TM)w_2(V_{\mathrm{SO}(3)}) + w_1(E)w_1(TM)^2} \\ = w_3(E) + w_1(E)w_1(TM)^2. \tag{116}$$

Recall that, express in terms of our eq. (87), we get $w_1(E)^3$ is $(1,0,0,0)$ $w_1(E)w_2(V_{\mathrm{SO}(3)})$ is $(1,1,0,0)$ $w_1(TM)w_2(V_{\mathrm{SO}(3)})$ is $(0,1,1,0)$ $w_1(E)w_1(TM)^2$ is $(0,0,0,1)$, and $w_3(E)$ is $(0,0,1,0)$, under the basis $(w_1(E)^3, w_1(E)w_2(E), w_3(E), w_1(E)w_1(TM)^2)$ of eq. (87). To summarize, the overall anomaly of 2d $\mathbb{CP}^1$-model can be expressed as a 3d cobordism invariant/topological term eq. (116), which is $(0,0,1,1)$ under the basis $(w_1(E)^3, w_1(E)w_2(E), w_3(E), w_1(E)w_1(TM)^2)$ of eq. (87).

For 2d $\mathbb{CP}^{\mathrm{N}-1}$-model at $\theta=\pi$, at even N, Ref. [64] proposes an important quantity (called u_3 in Ref. [64]), which is an element $u_3 \in \mathrm{H}^3(\mathrm{B}(\mathrm{PSU(N)} \rtimes \mathbb{Z}_2^C), \mathbb{Z}^C)$ as an anomaly for that 2d theory. First notice that one needs to generalize the second SW class from $w_2 \in \mathrm{H}^2(\mathrm{BPSU}(2), \mathbb{Z}_2) = \mathbb{Z}_2$ to $\tilde{w}_2 \in \mathrm{H}^2(\mathrm{BPSU(N)}, \mathbb{Z}_{\mathrm{N}}) = \mathbb{Z}_{\mathrm{N}}$. Moreover, there is an additional $\mathbb{Z}_2^C$ twist modify the PSU(2)-bundle to $\mathrm{PSU(N)} \rtimes \mathbb{Z}_2^C$-bundle. Their definition u_3 is an element of $\mathrm{H}^3(\mathrm{B}(\mathrm{PSU(N)} \rtimes \mathbb{Z}_2^C), \mathbb{Z}^C) = \mathbb{Z}_{\mathrm{N}}$, where C specifies the symmetry as a charge conjugation $\mathbb{Z}_2^C$. This means that $\mathrm{d}u_3 \neq 0$, but $\mathrm{d}_A u_3 = 0$, where d_A is a twisted differential. The construction of these classes is a Bockstein operator for the extension applied to $u_2 \in \mathrm{H}^2(\mathrm{B}(\mathrm{PSU(N)} \rtimes \mathbb{Z}_2^C), \mathbb{Z}_{\mathrm{N}}^C)$. Eventually, the 3d invariant for the 2d anomaly term of Ref. [64] is $u_3 \in \mathrm{H}^3(\mathrm{B}(\mathrm{PSU(N)} \rtimes \mathbb{Z}_2^C), \mathrm{U}(1)) = \mathbb{Z}_2$.

In our setup, we consider $\tilde{w}_3(E) \equiv \tilde{w}_3(V_{\mathrm{PSU(N)} \rtimes \mathbb{Z}_2}) \in \mathrm{H}^3(\mathrm{B}(\mathrm{PSU(N)} \rtimes \mathbb{Z}_2^C), \mathbb{Z}_2) = \mathbb{Z}_2$ here E is the background gauged bundle of $\mathrm{PSU(N)} \rtimes \mathbb{Z}_2$.
For $\mathrm{N}=2$, we derive that $\tilde{w}_3(E) = w_3(E) = w_1(E)w_2(E) + w_1(TM)w_2(E)$ in eq. (107).
For $\mathrm{N}=4$, we derive that

$$\tilde{w}_3(E) = w_1(E)w_2(E) + \beta_{(2,4)}w_2(E) = w_1(E)w_2(E) + \frac{1}{2}w_1(TM)w_2(E). \tag{117}$$

Based on Rule 4 in Sec. V, we propose that 3d invariant for the anomaly of 2d $\mathbb{CP}^3$-model is:

$$\begin{aligned} \text{2d } \mathbb{CP}^3\text{-model anomaly}: &\boxed{\tilde{w}_3(E) + w_1(E)w_1(TM)^2 = w_1(E)w_2(E) + \beta_{(2,4)}w_2(E) + w_1(E)w_1(TM)^2} \\ &= w_1(E)w_2(E) + \frac{1}{2}w_1(TM)w_2(E) + w_1(E)w_1(TM)^2. \end{aligned} \tag{118}$$

This first expression is our concise way to express the potentially complete 't Hooft anomalies of 2d $\mathbb{CP}^3$-model. It also guides us to make a proposal that, based on Rule 4 in Sec. V, 3d invariant for the anomaly of 2d $\mathbb{CP}^{\mathrm{N}-1}$-model for general even N can be:

$$\text{2d } \mathbb{CP}^{\mathrm{N}-1}\text{-model anomaly}: \boxed{\tilde{w}_3(E) + w_1(E)w_1(TM)^2}. \tag{119}$$

We should mention our anomaly term contains the previous anomaly found in the literature for more generic even N. For example, our $w_1(E)w_2(E)$, with E the background gauged bundle of $\mathrm{PSU(N)} \rtimes \mathbb{Z}_2$, contains the $w_1(\mathbb{Z}_2^C)\tilde{w}_2(\mathrm{PSU(N)})$ term studied in [62, 64, 77].

VII. 5D TO 3D DIMENSIONAL REDUCTION

Now we aim to utilize the Rule 6 in Sec. V and the new anomaly of 2d $\mathbb{CP}^{\mathrm{N}-1}$-model found in Sec. VI, to deduce the new higher anomaly of 4d YM theory — which later will be organized in Sec. VIII.

From the physics side, follow [36], see Fig.4, we choose the 4d YM living on $S^1_x \times S^1_y \times S^1_z \times \mathbb{R}$, such that we the size L_y, L_z of $S^1_y \times S^1_z$ is taken to be much smaller than the size L_x of S^1_x, namely $L_y, L_z \ll L_x$. Then, below the energy gap scale

$$\Delta_E \ll {L_y}^{-1} \text{ and } {L_z}^{-1},$$

the resulting 2d theory on $S^1_z \times \mathbb{R}$ is given by a sigma model with a target space of $\mathbb{CP}^{\mathrm{N}-1}$. There are several indications that the low energy theory is a 2d $\mathbb{CP}^{\mathrm{N}-1}$-model:

- The 4d and 2d instanton matchings in [27, 28] and other mathematical works. The $\theta=\pi$-term of SU(N) YM is mapped to the $\theta=\pi$-term of 2d $\mathbb{CP}^{\mathrm{N}-1}$-model.

- The moduli space of flat connections on the 2-torus $T^2 = S^1_y \times S^1_z$ of 4d YM theory is the projective space $\mathbb{CP}^{\mathrm{N}-1}$ [74, 75] (up to the geometry details of no canonical Fubini-Study metric and singularities mentioned in [36] and footnote 2). See Fig.4.

- The 1-form $\mathbb{Z}_{\mathrm{N}}$-center symmetry of 4d YM is dimensionally reduced, in addition to 1-form symmetry itself, also to a 0-form $\mathbb{Z}_{\mathrm{N}}$-flavor of 2d $\mathbb{CP}^{\mathrm{N}-1}$ model. The twisted boundary condition of 4d YM for 1-form $\mathbb{Z}_{\mathrm{N}}$-center symmetry (as a higher symmetry twist of [13]) can be dimensionally reduced to the 0-form $\mathbb{Z}_{\mathrm{N}}$-flavor symmetry twisted [76] in the 2d $\mathbb{CP}^{\mathrm{N}-1}$ model.

- Ref. [35] derives that the physical meaning of the 2d anomaly eq. (113) is directly descended from the 4d anomaly eq. (104) of YM theory by twisted T^2

compactification.

Encouraged by the above physical and mathematical evidences, in this section, we formalize the 4d and 2d anomaly matching under twisted T^2 compactification, into a mathematical precise problem of the 5d and 3d cobordism invariants (SPTs/topological terms) matching, under the T^2 dimensional reduction.

Below we follow our notations of the bordism groups in Sec. III, and their Dd $= (d+1)$d cobordism invariants to the dd anomalies of QFTs. We may simply abbreviate,

"5d cobordism invariants for 4d YM theory's anomaly"

$\equiv$ "5d (Yang-Mills) terms."

We may simply abbreviate,

"3d cobordism invariants for 2d $\mathbb{CP}^{\mathrm{N}-1}$ model's anomaly"

$\equiv$ "3d ($\mathbb{CP}^{\mathrm{N}-1}$) terms."

If $f \in \mathrm{H}^2(M, \mathbb{Z}_2)$, since $\mathrm{H}^2(M, \mathbb{Z}_2) = \mathrm{H}_2(M, \mathbb{Z}_2)$, then f is represented by a submanifold of M. Here a homology class $\mathrm{H}_n(M, \mathbb{Z}_2)$ is represented by a submanifold N, if there is an embedding $h : N \to M$ such that $h_*([N]) = f$ where the pushforward $h_* : \mathrm{H}_n(N, \mathbb{Z}_2) \to \mathrm{H}_n(M, \mathbb{Z}_2)$ is the induced homomorphism on the homology groups, N is an n-manifold, and $[N]$ is the fundamental class of N with coefficients $\mathbb{Z}_2$.

A. From $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$ to $\Omega_3^{\mathrm{O}}(\mathrm{BO}(3))$

Now we consider the 5d cobordism invariants that characterize the 4d SU(2) YM theory's anomaly (abbreviate them as "*Yang-Mills terms*").

Below we follow eq. (73) to use the notation (M, f) to denote the pair of manifold M and the map $f : M \to X$ to a generic topological space X.

Below we define that:

- α is the generator of the singular cohomology $\mathrm{H}^1(\mathbb{RP}^2, \mathbb{Z}_2)$,
- β is the generator of $\mathrm{H}^1(\mathbb{RP}^3, \mathbb{Z}_2)$,
- γ is the generator of $\mathrm{H}^1(S^1, \mathbb{Z}_2)$,
- ζ is the generator of $\mathrm{H}^1(\mathbb{RP}^4, \mathbb{Z}_2)$.
- $\#$ is the connected sum between manifolds.

The manifold generator of $B\mathrm{Sq}^1 B + \mathrm{Sq}^2\mathrm{Sq}^1 B$ can be chosen to be $(\mathbb{RP}^2 \times \mathbb{RP}^3, \alpha \cup \beta)$, $(\mathbb{RP}^2 \times \mathbb{RP}^3, \alpha \cup \beta + \alpha^2)$, or $(S^1 \times \mathbb{RP}^4, \gamma \cup \zeta)$.

The manifold generator of $w_1(TM)^2\mathrm{Sq}^1 B$ can be chosen to be $(S^1 \times \mathbb{RP}^4, \gamma \cup \zeta)$, or $(S^1 \times \mathbb{RP}^4, \gamma \cup \zeta + \zeta^2)$, or $(S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2, \gamma \cup \alpha_1)$.

We already know that $B\mathrm{Sq}^1 B + \mathrm{Sq}^2\mathrm{Sq}^1 B$ must be a summand of the Yang-Mills term, based on Sec. V's Rule 1 and 2.

If the Yang-Mills term is $B\mathrm{Sq}^1 B + \mathrm{Sq}^2\mathrm{Sq}^1 B$, then the manifold generators of the Yang-Mills term are $(\mathbb{RP}^2 \times \mathbb{RP}^3, \alpha \cup \beta)$, $(\mathbb{RP}^2 \times \mathbb{RP}^3, \alpha \cup \beta + \alpha^2)$, or $(S^1 \times \mathbb{RP}^4, \gamma \cup \zeta)$. The corresponding cases are 1, 2, and 3 below. We cannot get the 3d topological term $w_1(E)w_2(V_{\mathrm{SO}(3)})$ under the T^2 dimensional reduction (see the discussion below) which is a contradiction to the known results.

If the Yang-Mills term is $B\mathrm{Sq}^1 B + \mathrm{Sq}^2\mathrm{Sq}^1 B + w_1(TM)^2\mathrm{Sq}^1 B$, then the manifold generators of the Yang-Mills term are $(\mathbb{RP}^2 \times \mathbb{RP}^3, \alpha \cup \beta)$, $(\mathbb{RP}^2 \times \mathbb{RP}^3, \alpha \cup \beta + \alpha^2)$, $(S^1 \times \mathbb{RP}^4, \gamma \cup \zeta + \zeta^2)$, or $(S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2, \gamma \cup \alpha_1)$. The corresponding cases are 1, 2, 4, and 5 below. We can get the full 3d topological terms under the T^2 dimensional reduction.

So we claim that the Yang-Mills term is $B\mathrm{Sq}^1 B + \mathrm{Sq}^2\mathrm{Sq}^1 B + w_1(TM)^2\mathrm{Sq}^1 B$.

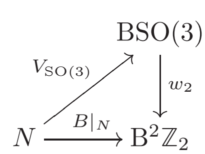
Since there is a short exact sequence

$$1 \to \mathbb{Z}_\mathrm{N} \to \mathrm{SU(N)} \to \mathrm{PSU(N)} \to 1, \qquad (120)$$

we have an induced fiber sequence

$$\mathrm{B}\mathbb{Z}_\mathrm{N} \to \mathrm{BSU(N)} \to \mathrm{BPSU(N)} \xrightarrow{w_2} \mathrm{B}^2\mathbb{Z}_\mathrm{N}. \qquad (121)$$

Following the idea in [36] and [35], the twisted boundary condition along a 2-torus T^2_{zx} is twisted by the 2-form background field B (See Fig. 4), or we can generalize the twist to $w_1(TM)^2$, where the 2-torus T^2_{zx} has a common S^1_z with the dimensional reduced 2-torus T^2_{yz} (Again see Fig. 4). Reducing a 2-torus (the effective T^2_{yz}) from the 5-manifold M, we get a 3-manifold N (obtained from taking the Poincaré dual) and we set $B|_N \in \mathrm{H}^2(N, \mathbb{Z}_2)$. Since $\pi_k\mathrm{BSU(N)} = 0$ for $k \leq 2$, by the obstruction theory, there is a principal SO(3) bundle $V_{\mathrm{SO}(3)}$ over N such that $w_2(V_{\mathrm{SO}(3)}) = B|_N$. Also since $w_1(TM)|_N \in \mathrm{H}^1(N, \mathbb{Z}_2)$, there is a principal $\mathrm{O}(3) = \mathrm{SO}(3) \times \mathbb{Z}_2$ bundle E (whose associated vector bundle is $V_{\mathrm{O}(3)}$) over N such that $w_1(E) = w_1(TM)|_N$, and $w_2(E) + w_1(E)^2 = w_2(V_{\mathrm{SO}(3)})$.



(122)

Ideally we aim to reduce a 2-torus T^2 (named as T^2_{yz} in Fig. 4), and we also aim to impose the twisted boundary condition along another T^2 (named as T^2_{zx} in Fig. 4). In this case, we abbreviate this procedure below simply as

"reduce T^2, and twist T^2."

More generally, however, we find that we sometimes need to reduce other novel 2-submanifolds $\mathcal{V}^2$ (such as $\mathbb{RP}^2$, $\mathbb{RP}^2\#T^2$, or $T^2\#T^2$, etc.) in order to do dimensional reduction successfully. In addition, we sometimes also need to impose twisted boundary conditions along other novel 2-submanifolds $\mathcal{U}^2$ (such as $\mathbb{RP}^2$, $\mathbb{RP}^2\#T^2$, or $T^2\#T^2$, etc.). In this case, we abbreviate this procedure below simply as

"reduce $\mathcal{V}^2$, and twist $\mathcal{U}^2$."

Below we list down the 5d manifold generators, and the reduced 2-submanifold, and another 2-submanifold where the twisted boundary conditions are imposed.

1. If $(M, B) = (\mathbb{RP}^2 \times \mathbb{RP}^3, \alpha \cup \beta)$, $w_1(TM)^2 = \alpha^2$:

   (a) Reduce T^2, twist $\mathbb{RP}^2$:
   Take the Poincaré dual of $\alpha\beta$, we get $N = S^1 \times \mathbb{RP}^2$, $w_1(E) = \gamma$, and $w_2(V_{\mathrm{SO}(3)}) = \gamma\alpha$. So (N, E) detects $w_1(E)w_1(TM)^2$ or $w_1(TM)w_2(V_{\mathrm{SO}(3)})$.

   However, below we elaborate other cases which do not reduce a 2-torus T^2 but other 2-manifolds.

   (i) Reduce $\mathbb{RP}^2$, twist T^2:
   Take the Poincaré dual of α^2, we get $N = \mathbb{RP}^3$, $w_1(E) = w_1(TM)|_N = 0$, and $w_2(V_{\mathrm{SO}(3)}) = 0$. So (N, E) does not detect any term.

   (ii) Reduce $\mathbb{RP}^2$, twist T^2:
   Take the Poincaré dual of β^2, we get $N = S^1 \times \mathbb{RP}^2$, $w_1(E) = \alpha$, and $w_2(V_{\mathrm{SO}(3)}) = \gamma\alpha$. So (N, E) detects $w_1(TM)w_2(V_{\mathrm{SO}(3)})$ or $w_1(E)w_2(V_{\mathrm{SO}(3)})$.

   (iii) Reduce $\mathbb{RP}^2 \# T^2$, twist T^2:
   Take the Poincaré dual of $(\alpha+\beta)\beta$, we get $N = S^1 \times \mathbb{RP}^2 \# S^1 \times \mathbb{RP}^2$ where # is the connected sum. Here we denote that α_1 and γ_1 for the first sector of $S^1 \times \mathbb{RP}^2$ of N, while α_2 and γ_2 for the second sector of $S^1 \times \mathbb{RP}^2$ of N, in the connected sum. Then we get $w_1(E) = \gamma_1 + \alpha_2$, $w_2(V_{\mathrm{SO}(3)}) = \gamma_1\alpha_1 + \gamma_2\alpha_2$. So (N, E) detects $w_1(E)w_2(V_{\mathrm{SO}(3)})$ or $w_1(E)w_1(TM)^2$.

   (iv) Reduce $\mathbb{RP}^2 \# T^2$, twist T^2:
   Take the Poincaré dual of $(\alpha + \beta)\alpha$, we get $N = \mathbb{RP}^3 \# S^1 \times \mathbb{RP}^2$, $w_1(E) = \gamma$, and $w_2(V_{\mathrm{SO}(3)}) = \gamma \cup \alpha$. So (N, E) detects $w_1(E)w_1(TM)^2$ or $w_1(TM)w_2(V_{\mathrm{SO}(3)})$.

2. If $(M, B) = (\mathbb{RP}^2 \times \mathbb{RP}^3, \alpha \cup \beta + \alpha^2)$, $w_1(TM)^2 = \alpha^2$:

   (a) Reduce T^2, twist $\mathbb{RP}^2$:
   Take the Poincaré dual of $\alpha\beta$, we get $N = S^1 \times \mathbb{RP}^2$, $w_1(E) = \gamma$, and $w_2(V_{\mathrm{SO}(3)}) = \gamma\alpha$. So (N, E) detects $w_1(E)w_1(TM)^2$ or $w_1(TM)w_2(V_{\mathrm{SO}(3)})$.

   However, below we elaborate other cases which do not reduce a 2-torus T^2 but other 2-manifolds.

   (i) Reduce $\mathbb{RP}^2$, twist $\mathbb{RP}^2 \# T^2$:
   Take the Poincaré dual of α^2, we get $N = \mathbb{RP}^3$, $w_1(E) = 0$, and $w_2(V_{\mathrm{SO}(3)}) = 0$. So (N, E) does not detect any term.

   (ii) Reduce $\mathbb{RP}^2$, twist $\mathbb{RP}^2 \# T^2$:
   Take the Poincaré dual of β^2, we get $N = S^1 \times \mathbb{RP}^2$, $w_1(E) = \alpha$, and $w_2(V_{\mathrm{SO}(3)}) = \gamma\alpha + \alpha^2$. So (N, E) detects $w_1(TM)w_2(V_{\mathrm{SO}(3)})$ or $w_1(E)w_2(V_{\mathrm{SO}(3)})$. However, this case may *not* be a reasonable choice, since β^2 has no common S^1 with both $B = \alpha(\alpha + \beta)$ and $w_1(TM)^2 = \alpha^2$.
   These are reducing $\mathbb{RP}^2$.

   (iii) Reduce $\mathbb{RP}^2 \# T^2$, twist $\mathbb{RP}^2 \# T^2$:
   Take the Poincaré dual of $(\alpha + \beta)\beta$, we get $N = S^1 \times \mathbb{RP}^2 \# S^1 \times \mathbb{RP}^2$. Here we denote that α_1 and γ_1 for the first sector of $S^1 \times \mathbb{RP}^2$ of N, while α_2 and γ_2 for the second sector of $S^1 \times \mathbb{RP}^2$ of N, in the connected sum. Then we get $w_1(E) = \gamma_1 + \alpha_2$, and $w_2(V_{\mathrm{SO}(3)}) = \gamma_1\alpha_1 + (\gamma_2 + \alpha_2)\alpha_2$. So (N, E) detects $w_1(E)w_2(V_{\mathrm{SO}(3)})$ or $w_1(E)w_1(TM)^2$.

   (iv) Reduce $\mathbb{RP}^2 \# T^2$, twist $\mathbb{RP}^2$:
   Take the Poincaré dual of $(\alpha + \beta)\alpha$, we get $N = \mathbb{RP}^3 \# S^1 \times \mathbb{RP}^2$, $w_1(E) = \gamma$, and $w_2(V_{\mathrm{SO}(3)}) = \gamma\alpha$. So (N, E) detects $w_1(E)w_1(TM)^2$.
   These are reducing $\mathbb{RP}^2 \# T^2$.

3. If $(M, B) = (S^1 \times \mathbb{RP}^4, \gamma \cup \zeta)$, $w_1(TM)^2 = \zeta^2$:

   (a) Reduce T^2, twist $\mathbb{RP}^2$:
   Take the Poincaré dual of $\gamma\zeta$, we get $N = \mathbb{RP}^3$, $w_1(E) = \beta$, and $w_2(V_{\mathrm{SO}(3)}) = 0$. So (N, E) detects $w_1(E)^3$.

   However, below we elaborate other cases which do not reduce a 2-torus T^2 but other 2-manifolds.

   (i) Reduce $\mathbb{RP}^2$, twist T^2:
   Take the Poincaré dual of ζ^2, we get $N = S^1 \times \mathbb{RP}^2$, $w_1(E) = \alpha$, and $w_2(V_{\mathrm{SO}(3)}) = \gamma\alpha$. So (N, E) detects $w_1(TM)w_2(V_{\mathrm{SO}(3)})$ or $w_1(E)w_2(V_{\mathrm{SO}(3)})$.
   This is reducing $\mathbb{RP}^2$.

   (ii) Reduce $\mathbb{RP}^2 \# T^2$, twist T^2:
   Take the Poincaré dual of $(\gamma + \zeta)\zeta$, we get $N = \mathbb{RP}^3 \# S^1 \times \mathbb{RP}^2$, $w_1(E) = \beta + \alpha$, and $w_2(V_{\mathrm{SO}(3)}) = \gamma\alpha$. So (N, E) detects $w_1(E)^3$ or $w_1(TM)w_2(V_{\mathrm{SO}(3)})$ or $w_1(E)w_2(V_{\mathrm{SO}(3)})$.
   This is reducing $\mathbb{RP}^2 \# T^2$.

4. If $(M, B) = (S^1 \times \mathbb{RP}^4, \gamma \cup \zeta + \zeta^2)$, $w_1(TM)^2 = \zeta^2$:

   (a) Reduce T^2, twist $\mathbb{RP}^2$:
   Take the Poincaré dual of $\gamma\zeta$, we get $N = \mathbb{RP}^3$, $w_1(E) = \beta$, and $w_2(V_{\mathrm{SO}(3)}) = \beta^2$. So (N, E) detects $w_1(E)^3$ or $w_1(E)w_2(V_{\mathrm{SO}(3)})$.

   However, below we elaborate other cases which do not reduce a 2-torus T^2 but other 2-manifolds.

(i) Reduce $\mathbb{RP}^2$, twist $\mathbb{RP}^2\#T^2$:

Take the Poincaré dual of ζ^2, we get $N = S^1\times\mathbb{RP}^2$, $w_1(E)=\alpha$, and $w_2(V_{\mathrm{SO}(3)})=\gamma\alpha+\alpha^2$. So (N,E) detects $w_1(TM)w_2(V_{\mathrm{SO}(3)})$ or $w_1(E)w_2(V_{\mathrm{SO}(3)})$.

This is reducing $\mathbb{RP}^2$.

(ii) Reduce $\mathbb{RP}^2\#T^2$, twist $\mathbb{RP}^2$:

Take the Poincaré dual of $(\gamma+\zeta)\zeta$, we get $N=\mathbb{RP}^3\#S^1\times\mathbb{RP}^2$, $w_1(E)=\beta+\alpha$, and $w_2(V_{\mathrm{SO}(3)})=\beta^2+\gamma\alpha+\alpha^2$. So (N,E) detects $w_1(E)^3$ or $w_1(TM)w_2(V_{\mathrm{SO}(3)})$.

This is reducing $\mathbb{RP}^2\#T^2$.

5. If $(M,B)=(S^1\times\mathbb{RP}^2\times\mathbb{RP}^2,\gamma\cup\alpha_1)$, $w_1(TM)^2=(\alpha_1+\alpha_2)^2$:

(a) Reduce T^2, twist T^2:

Take the Poincaré dual of $\gamma\alpha_2$, we get $N=S^1\times\mathbb{RP}^2$, $w_1(E)=\gamma+\alpha$, and $w_2(V_{\mathrm{SO}(3)})=0$. So (N,E) detects $w_1(E)^3$ or $w_1(E)w_1(TM)^2$.

(b) Reduce T^2, twist T^2:

Take the Poincaré dual of $\alpha_1\alpha_2$, we get $N=S^1\times S^1\times S^1$, $w_1(E)=\gamma_2+\gamma_3$, and $w_2(V_{\mathrm{SO}(3)})=\gamma_1\gamma_2$. So (N,E) detects $w_1(E)w_1(V_{\mathrm{SO}(3)})$.

There are other cases where we can reduce other topology (such as $\mathbb{RP}^2$, $T^2\#T^2$, etc) while we do not reduce T^2, but we omit our discussions on those cases.

B. From $\Omega_5^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2\ltimes\mathrm{B}^2\mathbb{Z}_4)$ to $\Omega_3^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2\ltimes\mathrm{PSU}(4)))$

Now we consider the 5d cobordism invariants that characterize the 4d SU(4) YM theory's anomaly (abbreviate them as "*Yang-Mills terms*").

Following the idea in [36] [35], the twisted boundary condition along a 2-torus T^2_{zx} is twisted by the 2-form background field B (more precisely $\tilde{B}\equiv(B \mod 2)$, see Fig. 4), or we can generalize the twist to $w_1(TM)^2$, or A^2, where the 2-torus T^2_{zx} has a common S^1_z with the dimensional reduced 2-torus T^2_{yz} (Again see Fig. 4). Reducing a 2-torus from the 5-manifold M, we get a 3-manifold N (obtained from taking the Poincaré dual) and we set $A|_N\in\mathrm{H}^1(N,\mathbb{Z}_2)$, $B|_N\in\mathrm{H}^2(N,\mathbb{Z}_4)$, since $\pi_k\mathrm{BSU(N)}=0$ for $k\leq 2$, by the obstruction theory, there is a principal $\mathbb{Z}_2\ltimes\mathrm{PSU}(4)$ bundle E over N such that $w_1(E)=A|_N$, and $w_2(E)=B|_N$.

In this subsection, all of the below, we define that
- K is the Klein bottle,
- α' is the generator of $\mathrm{H}^1(S^1,\mathbb{Z}_4)$,
- β' is the generator of the $\mathbb{Z}_4$ factor of $\mathrm{H}^1(K,\mathbb{Z}_4)=\mathbb{Z}_4\times\mathbb{Z}_2$ (see Appendix C),
- γ' is the generator of $\mathrm{H}^2(S^2,\mathbb{Z}_4)$,
- ζ' is the generator of $\mathrm{H}^2(T^2,\mathbb{Z}_4)$.
- α is the generator of $\mathrm{H}^1(\mathbb{RP}^2,\mathbb{Z}_2)$,
- β is the generator of $\mathrm{H}^1(\mathbb{RP}^3,\mathbb{Z}_2)$,
- γ is the generator of $\mathrm{H}^1(S^1,\mathbb{Z}_2)$,
- ζ is the generator of $\mathrm{H}^1(\mathbb{RP}^4,\mathbb{Z}_2)$,
- # is the connected sum between manifolds.

Note that $(\beta' \mod 2)^2=2\beta_{(2,4)}\beta'=0$.

We already know that $B_2\beta_{(2,4)}B_2$ and $A^2\beta_{(2,4)}B_2$ are summands of the Yang-Mills term based on the Rule 1 and Rule 8 in Sec. V. Since the manifold generator of $A^2\beta_{(2,4)}B_2$ is $(S^1\times K\times\mathbb{RP}^2, A=\alpha, B=\alpha'\cup\beta')$, with $w_1(TM)^2=\alpha^2$, which is also a manifold generator of $w_1(TM)^2\beta_{(2,4)}B_2$. If $w_1(TM)^2\beta_{(2,4)}B_2$ is also a summand of the Yang-Mills term, then $(S^1\times K\times\mathbb{RP}^2, A=\alpha, B=\alpha'\cup\beta')$ is no longer a manifold generator of the Yang-Mills term which is a contradiction. So $w_1(TM)^2\beta_{(2,4)}B_2$ is not a summand of the Yang-Mills term.

To get the full 3d topological terms under the T^2 dimensional reduction, we claim that the Yang-Mills term is $B_2\beta_{(2,4)}B_2+A^2\beta_{(2,4)}B_2+AB_2w_1(TM)^2$.

1. The manifold generator of $B_2\cup\beta_{(2,4)}B_2$ can be chosen to be $(S^1\times K\times S^2, A, B=\alpha'\cup\beta'+\gamma')$ and $w_1(TM)^2=0$ where A is arbitrary.

(a) Reduce T^2, twist T^2 (but the two T^2 are the same):
Take the Poincaré dual of $(\alpha' \mod 2)(\beta' \mod 2)=\gamma(\beta' \mod 2)$, we get $N=S^1\times S^2$, $w_1(E)$ is arbitrary, and $w_2(E)=\gamma'$. So (N,E) can detect $w_1(E)w_2(E)$.

However, below we elaborate other cases which do not reduce a 2-torus T^2 but other 2-manifolds.

(i) Reduce S^2, twist T^2:
Take the Poincaré dual of $\gamma' \mod 2$, we get $N=S^1\times K$, $w_1(E)$ is arbitrary, and $w_2(E)=\alpha'\beta'$. So (N,E) detects $\beta_{(2,4)}w_2(E)$. However, this case may *not* be a reasonable choice, since there is no common S^1 in the reduced S^2 and the twisted T^2.

This is reducing S^2.

2. The manifold generator of $B_2\cup\beta_{(2,4)}B_2$ can also be also chosen to be $(S^1\times K\times T^2, A, B=\alpha'\cup\beta'+\zeta')$ with $w_1(TM)^2=0$ where A is arbitrary, $\zeta'=\alpha_1'\alpha_2'$ and $\alpha_i' \mod 2=\gamma_i$.

(a) Reduce T^2, twist $T^2\#T^2$:
Take the Poincaré dual of $\gamma(\beta' \mod 2)$, we get $N=S^1\times T^2$, $w_1(E)$ is arbitrary, and $w_2(E)=\zeta'$. So (N,E) can detect $w_1(E)w_2(E)$.

(b) Reduce T^2, twist $T^2\#T^2$:
Take the Poincaré dual of $\zeta' \mod 2$, we get $N=S^1\times K$, $w_1(E)$ is arbitrary, and $w_2(E)=\alpha'\beta'$. So (N,E) detects $\beta_{(2,4)}w_2(E)$.

(c) Reduce T^2, twist $T^2\#T^2$:
Take the Poincaré dual of $\gamma\gamma_1$, we get $N = S^1 \times K$, $w_1(E)$ is arbitrary, and $w_2(E) = 0$. So (N, E) does not detect any term.

(d) Reduce T^2, twist $T^2\#T^2$:
Take the Poincaré dual of $(\beta' \mod 2)\gamma_1$, we get $N = S^1 \times S^1 \times S^1$, $w_1(E)$ is arbitrary, and $w_2(E) = \zeta'$. So (N, E) can detect $w_1(E)w_2(E)$.

3. The manifold generator of $w_1(TM)^2\beta_{(2,4)}B_2$ can be chosen to be $(S^1 \times K \times \mathbb{RP}^2, A, B = \alpha' \cup \beta')$, with $w_1(TM)^2 = \alpha^2$ where A is arbitrary but other than α.

(a) Reduce T^2, twist T^2:
Take the Poincaré dual of $(\beta' \mod 2)\alpha$, we get $N = S^1 \times S^1 \times S^1$, and $w_2(E) = \zeta'$. So (N, E) does not detect any term whatever $w_1(E)$ is since $A \neq \alpha$.

(b) Reduce T^2, twist T^2:
Take the Poincaré dual of $(\alpha' \mod 2)\alpha = \gamma\alpha$, we get $N = S^1 \times K$, and $w_2(E) = 0$. So (N, E) does not detect any term whatever $w_1(E)$ is.

However, below we elaborate other cases which do not reduce a 2-torus T^2 but other 2-manifolds.

(i) Reduce $T^2\#T^2$, twist T^2:
Take the Poincaré dual of $(\beta' \mod 2)(\gamma + \alpha)$, we get $N = S^1 \times \mathbb{RP}^2\#S^1 \times S^1 \times S^1$, and $w_2(E) = \zeta'$. So (N, E) can detect $w_1(E)w_1(TM)^2$ if $w_1(E) = A = \beta' \mod 2$.

(ii) Reduce $T^2\#T^2$, twist T^2:
Take the Poincaré dual of $\gamma(\beta' \mod 2{+}\alpha)$, we get $N = S^1 \times \mathbb{RP}^2\#S^1 \times K$, and $w_2(E) = 0$. So (N, E) can detect $w_1(E)w_1(TM)^2$ if $w_1(E) = A = \beta' \mod 2$.

(iii) Reduce $T^2\#T^2$, twist $\mathbb{RP}^2$:
Take the Poincaré dual of $\alpha(\gamma + \beta' \mod 2)$, we get $N = S^1 \times K\#S^1 \times S^1 \times S^1$, and $w_2(E) = \zeta'$. So (N, E) does not detect any term whatever $w_1(E)$ is since $A \neq \alpha$.

These are reducing $T^2\#T^2$.

4. The manifold generator of $A^2\beta_{(2,4)}B_2$ can be chosen to be $(S^1 \times K \times \mathbb{RP}^2, A = \alpha, B = \alpha' \cup \beta')$, with $w_1(TM)^2 = \alpha^2$.

(a) Reduce T^2, twist T^2:
Take the Poincaré dual of $(\beta' \mod 2)\alpha$, we get $N = S^1 \times S^1 \times S^1$, $w_1(E) = \gamma$, and $w_2(E) = \zeta'$. So (N, E) detects $w_1(E)w_2(E)$.

(b) Reduce T^2, twist T^2:
Take the Poincaré dual of $(\alpha' \mod 2)\alpha = \gamma\alpha$, we get $N = S^1 \times K$, $w_1(E) = \gamma$, and $w_2(E) = 0$. So (N, E) does not detect any term.

However, below we elaborate other cases which do not reduce a 2-torus T^2 but other 2-manifolds.

(i) Reduce $T^2\#T^2$, twist T^2:
Take the Poincaré dual of $(\beta' \mod 2)(\gamma + \alpha)$, we get $N = S^1 \times \mathbb{RP}^2\#S^1 \times S^1 \times S^1$, $w_1(E) = \alpha_1 + \gamma_2$, and $w_2(E) = \zeta'$. So (N, E) detects $w_1(E)w_2(E)$.

(ii) Reduce $T^2\#T^2$, twist T^2:
Take the Poincaré dual of $\gamma(\beta' \mod 2{+}\alpha)$, we get $N = S^1 \times \mathbb{RP}^2\#S^1 \times K$, $w_1(E) = \alpha_1 + \gamma_2$, and $w_2(E) = 0$. So (N, E) does not detect any term.

(iii) Reduce $T^2\#T^2$, twist $\mathbb{RP}^2$:
Take the Poincaré dual of $\alpha(\gamma + \beta' \mod 2)$, we get $N = S^1 \times K\#S^1 \times S^1 \times S^1$, $w_1(E) = \gamma_1 + \gamma_2$, and $w_2(E) = \zeta'$. So (N, E) detects $w_1(E)w_2(E)$.

These are reducing $T^2\#T^2$.

5. The manifold generator of $A^3w_1(TM)^2$ can be chosen to be $(\mathbb{RP}^2 \times \mathbb{RP}^3, A = \beta, B = 0)$ with $w_1(TM)^2 = \alpha^2$.

(a) Reduce T^2, twist $\mathbb{RP}^2$:
Take the Poincaré dual of $\alpha\beta$, we get $N = S^1 \times \mathbb{RP}^2$, $w_1(E) = \alpha$, and $w_2(E) = 0$. So (N, E) does not detect any term.

However, below we elaborate other cases which do not reduce a 2-torus T^2 but other 2-manifolds.

(i) Reduce $\mathbb{RP}^2\#T^2$, twist $\mathbb{RP}^2$:
Take the Poincaré dual of $\alpha(\alpha + \beta)$, we get $N = \mathbb{RP}^3\#S^1 \times \mathbb{RP}^2$, $w_1(E) = \beta + \alpha$, and $w_2(E) = 0$. So (N, E) detects $w_1(E)^3$.

(ii) Reduce $\mathbb{RP}^2\#T^2$, twist $\mathbb{RP}^2$:
Take the Poincaré dual of $\beta(\alpha + \beta)$, we get $N = S^1 \times \mathbb{RP}^2\#S^1 \times \mathbb{RP}^2$, $w_1(E) = \alpha_1 + \gamma_2$, and $w_2(E) = 0$. So (N, E) detects $w_1(E)w_1(TM)^2$.

These are reducing $\mathbb{RP}^2\#T^2$.

6. The manifold generator of $Aw_1(TM)^4$ can be chosen to be $(S^1 \times \mathbb{RP}^4, A = \gamma, B = 0)$, with $w_1(TM)^2 = \zeta^2$.

(a) Reduce T^2, twist $\mathbb{RP}^2$:
Take the Poincaré dual of $\gamma\zeta$, we get $N = \mathbb{RP}^3$, $w_1(E) = 0$, and $w_2(E) = 0$. So (N, E) does not detect any term.

However, below we elaborate other cases which do not reduce a 2-torus T^2 but other 2-manifolds.

(i) Reduce $\mathbb{RP}^2\#T^2$, twist $\mathbb{RP}^2$:
Take the Poincaré dual of $(\gamma{+}\zeta)\zeta$, we get $N = \mathbb{RP}^3\#S^1 \times \mathbb{RP}^2$, $w_1(E) = \gamma$, and $w_2(E) = 0$. So (N, E) detects $w_1(E)w_1(TM)^2$.

This is reducing $\mathbb{RP}^2 \# T^2$.

7. The manifold generator of $AB_2 w_1(TM)^2$ can be chosen to be $(S^1 \times S^2 \times \mathbb{RP}^2, A = \gamma, B = \gamma')$ with $w_1(TM)^2 = \alpha^2$.

   (a) Reduce T^2, twist $\mathbb{RP}^2$:
   Take the Poincaré dual of $\gamma\alpha$, we get $N = S^1 \times S^2$, $w_1(E) = 0$, and $w_2(E) = \gamma'$. So (N, E) does not detect any term.

   However, below we elaborate other cases which do not reduce a 2-torus T^2 but other 2-manifolds.

   (i) Reduce $\mathbb{RP}^2 \# T^2$, twist $\mathbb{RP}^2$:
   Take the Poincaré dual of $(\gamma+\alpha)\alpha$, we get $N = S^1 \times S^2 \# S^1 \times S^2$, $w_1(E) = \gamma_1$, and $w_2(E) = \gamma_1' + \gamma_2'$. So (N, E) detects $w_1(E) w_2(E)$.

   This is reducing $\mathbb{RP}^2 \# T^2$.

8. The manifold generator of $AB_2 w_1(TM)^2$ can be also chosen to be $(S^1 \times T^2 \times \mathbb{RP}^2, A = \gamma, B = \zeta')$ with $w_1(TM)^2 = \alpha^2$ where $\zeta' = \alpha_1' \alpha_2'$ and α_i' mod $2 = \gamma_i$.

   (a) Reduce T^2, twist $\mathbb{RP}^2$:
   Take the Poincaré dual of $\gamma\alpha$, we get $N = S^1 \times T^2$, $w_1(E) = 0$, and $w_2(E) = \zeta'$. So (N, E) does not detect any term.

   (b) Reduce T^2, twist T^2:
   Take the Poincaré dual of $\gamma\gamma_1$, we get $N = S^1 \times \mathbb{RP}^2$, $w_1(E) = 0$, and $w_2(E) = 0$. So (N, E) does not detect any term.

   (c) Reduce T^2, twist T^2:
   Take the Poincaré dual of $\alpha\gamma_1$, we get $N = S^1 \times S^1 \times S^1$, $w_1(E) = \gamma$, and $w_2(E) = 0$. So (N, E) does not detect any term.

   However, below we elaborate other cases which do not reduce a 2-torus T^2 but other 2-manifolds.

   (i) Reduce $\mathbb{RP}^2 \# T^2$, twist $\mathbb{RP}^2$:
   Take the Poincaré dual of $(\gamma+\alpha)\alpha$, we get $N = S^1 \times T^2 \# S^1 \times T^2$, $w_1(E) = \gamma_1$, and $w_2(E) = \zeta_1' + \zeta_2'$. So (N, E) detects $w_1(E) w_2(E)$. This is reducing $\mathbb{RP}^2 \# T^2$.

   (ii) Reduce $T^2 \# T^2$, twist T^2:
   Take the Poincaré dual of $(\gamma + \alpha)\gamma_1$, we get $N = S^1 \times T^2 \# S^1 \times \mathbb{RP}^2$, $w_1(E) = \gamma_1$, and $w_2(E) = 0$. So (N, E) does not detect any term. This is reducing $T^2 \# T^2$.

9. The manifold generator of $AB_2 w_1(TM)^2$ can also be chosen to be $(S^1 \times T^2 \times \mathbb{RP}^2, A = \gamma + \gamma_2, B = \zeta')$ with $w_1(TM)^2 = \alpha^2$ where $\zeta' = \alpha_1' \alpha_2'$ and α_i' mod $2 = \gamma_i$.

   (a) Reduce T^2, twist $\mathbb{RP}^2$:
   Take the Poincaré dual of $\gamma\alpha$, we get $N = S^1 \times T^2$, $w_1(E) = \gamma_2$, and $w_2(E) = \zeta'$. So (N, E) does not detect any term.

   (b) Reduce T^2, twist T^2:
   Take the Poincaré dual of $\gamma\gamma_1$, we get $N = S^1 \times \mathbb{RP}^2$, $w_1(E) = \gamma_2$, and $w_2(E) = 0$. So (N, E) detects $w_1(E) w_1(TM)^2$.

   (c) Reduce T^2, twist T^2:
   Take the Poincaré dual of $\alpha\gamma_1$, we get $N = S^1 \times S^1 \times S^1$, $w_1(E) = \gamma + \gamma_2$, and $w_2(E) = 0$. So (N, E) does not detect any term.

   However, below we elaborate other cases which do not reduce a 2-torus T^2 but other 2-manifolds.

   (i) Reduce $\mathbb{RP}^2 \# T^2$, twist $\mathbb{RP}^2$:
   Take the Poincaré dual of $(\gamma + \alpha)\alpha$, we get $N = S^1 \times T^2 \# S^1 \times T^2$, $w_1(E) = \gamma_{12} + \gamma_{21} + \gamma_{22}$, and $w_2(E) = \zeta_1' + \zeta_2'$. So (N, E) detects $w_1(E) w_2(E)$. This is reducing $\mathbb{RP}^2 \# T^2$.

   (ii) Reduce $T^2 \# T^2$, twist T^2:
   Take the Poincaré dual of $(\gamma + \alpha)\gamma_1$, we get $N = S^1 \times T^2 \# S^1 \times \mathbb{RP}^2$, $w_1(E) = \gamma_{11} + \gamma_{12} + \gamma_{21}$, and $w_2(E) = 0$. So (N, E) detects $w_1(E) w_1(TM)^2$. This is reducing $T^2 \# T^2$.

Next we can use the above results to deduce the new higher anomaly of 4d YM theory in the next Sec. VIII.

VIII. NEW HIGHER ANOMALIES OF 4D SU(N)-YM THEORY

We are ready to summarize and deduce the new higher anomaly of 4d YM theory written in terms of invariants given in Sec. III, and satisfying Rules in Sec. V and following the physical/mathematical 5d to 3d reduction scheme in Sec. VII.

A. SU(N)-YM at $\mathrm{N} = 2$

Let us formulate the potentially complete 't Hooft anomaly for 4d SU(N)-YM at $\mathrm{N} = 2$ at $\theta = \pi$, written in terms of a 5d cobordism invariant in Sec. III.

Base on Rule 3 and Rule 6 in Sec. V, we deduce that 4d anomaly must match 2d $\mathbb{CP}^1$-model anomaly's eq. (116) via the sum of following two terms (5d SPTs). The first term is:

$$\begin{aligned} & B_2 \mathrm{Sq}^1 B_2 + \mathrm{Sq}^2 \mathrm{Sq}^1 B_2 \\ & = \frac{1}{2} \tilde{w}_1(TM) \mathcal{P}_2(B_2). \end{aligned} \tag{123}$$

which is dictated by Rule 1 in Sec. V. (Note that $\mathrm{Sq}^2 \mathrm{Sq}^1 B_2 = (B_2 \cup_1 B_2) \cup_1 (B_2 \cup_1 B_2)$.) Here $\tilde{w}_1(TM) \in \mathrm{H}^1(M, \mathbb{Z}_{4,w_1})$ is the mod 4 reduction of the twisted first

Stiefel-Whitney class of the tangent bundle TM of a 5-manifold M which is the pullback of $\tilde{w}_1$ under the classifying map $M \to \mathrm{BO}(5)$. Here $\mathbb{Z}_{w_1}$ denotes the orientation local system, the twisted first Stiefel-Whitney class $\tilde{w}_1 \in \mathrm{H}^1(\mathrm{BO}(n), \mathbb{Z}_{w_1})$ is the pullback of the nonzero element of $\mathrm{H}^1(\mathrm{BO}(1), \mathbb{Z}_{w_1}) = \mathbb{Z}_2$ under the determinant map $\mathrm{B}\det : \mathrm{BO}(n) \to \mathrm{BO}(1)$. Since $2\tilde{w}_1 = 0$, $\tilde{w}_1(TM)\mathcal{P}_2(B_2)$ is even, so it makes sense to divide it by 2. If $w_1(TM) = 0$, then $\mathbb{Z}_{w_1} = \mathbb{Z}$ and $\mathrm{H}^1(\mathrm{BO}(1), \mathbb{Z}_{w_1}) = \mathrm{H}^1(\mathrm{BO}(1), \mathbb{Z}) = 0$, so $\tilde{w}_1 = 0$. Namely, $\frac{1}{2}\tilde{w}_1(TM)\mathcal{P}_2(B_2)$ vanishes when $w_1(TM) = 0$.

We can derive the last equality of eq. (123) by proving that both LHS and RHS are bordism invariants of $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$ and they coincide on manifold generators of $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$.

We can also prove that

$$\begin{aligned} &\beta_{(2,4)}\mathcal{P}_2(B_2) \\ &= \beta_{(2,4)}(B_2 \cup B_2 + B_2 \underset{1}{\cup} \delta B_2) \\ &= \frac{1}{4}\delta(B_2 \cup B_2 + B_2 \underset{1}{\cup} \delta B_2) \\ &= (\frac{1}{2}\delta B_2) \cup B_2 + (\frac{1}{2}\delta B_2) \underset{1}{\cup} (\frac{1}{2}\delta B_2) \\ &= B_2 \mathrm{Sq}^1 B_2 + \mathrm{Sq}^1 B_2 \underset{1}{\cup} \mathrm{Sq}^1 B_2 \\ &= B_2 \mathrm{Sq}^1 B_2 + \mathrm{Sq}^2 \mathrm{Sq}^1 B_2. \end{aligned} \tag{124}$$

The first term contains two appear together in order to satisfy Rule 2.

The other term is:

$$w_1(TM)^2 \mathrm{Sq}^1 B_2. \tag{125}$$

We also check that the sum of two terms satisfy the Rule 5 in Sec. V. Besides, Rule 7 restricts us to focus on the bordism group $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$ and discards other terms involving $\Omega_5^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$. Our final answer of 4d anomaly and 5d cobordism/SPTs invariant is combined and given in eq. (133). To our understanding, the whole expression indicates a new higher anomaly for this YM theory, new to the literature.

B. SU(N)-YM at $\mathrm{N} = 4$

Let us formulate the potentially complete 't Hooft anomaly for 4d SU(N)-YM at $\mathrm{N} = 4$ at $\theta = \pi$, written in terms of a 5d cobordism invariant in Sec. III.

Base on Rule 4 in Sec. V, we deduce the 2d $\mathbb{CP}^3$-model anomaly's eq. (118) generalizing the eq. (116). Base on Rule 3 and Rule 6, we deduce that 4d anomaly must match 2d $\mathbb{CP}^3$-model anomaly's eq. (118) via the sum of following two terms (5d SPTs). The first term is:

$$B_2 \beta_{(2,4)} B_2 = \frac{1}{4}\tilde{w}_1(TM)\mathcal{P}_2(B_2), \tag{126}$$

which is dictated by Rule 1 in Sec. V. Here $\tilde{w}_1(TM) \in \mathrm{H}^1(M, \mathbb{Z}_{8,w_1})$ is the mod 8 reduction of the twisted first Stiefel-Whitney class of the tangent bundle TM of a 5-manifold M which is the pullback of $\tilde{w}_1$ under the classifying map $M \to \mathrm{BO}(5)$. Here $\mathbb{Z}_{w_1}$ denotes the orientation local system, the twisted first Stiefel-Whitney class $\tilde{w}_1 \in \mathrm{H}^1(\mathrm{BO}(n), \mathbb{Z}_{w_1})$ is the pullback of the nonzero element of $\mathrm{H}^1(\mathrm{BO}(1), \mathbb{Z}_{w_1}) = \mathbb{Z}_2$ under the determinant map $\mathrm{B}\det : \mathrm{BO}(n) \to \mathrm{BO}(1)$. Since $2\tilde{w}_1 = 0$, $\tilde{w}_1(TM)\mathcal{P}_2(B_2)$ is divided by 4, so it makes sense to divide it by 4. If $w_1(TM) = 0$, then $\mathbb{Z}_{w_1} = \mathbb{Z}$ and $\mathrm{H}^1(\mathrm{BO}(1), \mathbb{Z}_{w_1}) = \mathrm{H}^1(\mathrm{BO}(1), \mathbb{Z}) = 0$, so $\tilde{w}_1 = 0$. Namely, $\frac{1}{4}\tilde{w}_1(TM)\mathcal{P}_2(B_2)$ vanishes when $w_1(TM) = 0$.

We can derive the last equality by proving that both LHS and RHS are bordism invariants of $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_4)$ and they coincide on manifold generators of $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_4)$.

We can also prove that

$$\begin{aligned} &\beta_{(2,8)}\mathcal{P}_2(B_2) \\ &= \beta_{(2,8)}(B_2 \cup B_2 + B_2 \underset{1}{\cup} \delta B_2) \\ &= \frac{1}{8}\delta(B_2 \cup B_2 + B_2 \underset{1}{\cup} \delta B_2) \\ &= (\frac{1}{4}\delta B_2) \cup B_2 + 2(\frac{1}{4}\delta B_2) \underset{1}{\cup} (\frac{1}{4}\delta B_2) \\ &= B_2 \beta_{(2,4)} B_2 + 2\beta_{(2,4)} B_2 \underset{1}{\cup} \beta_{(2,4)} B_2 \\ &= B_2 \beta_{(2,4)} B_2 + 2\mathrm{Sq}^2 \beta_{(2,4)} B_2 \\ &= B_2 \beta_{(2,4)} B_2. \end{aligned} \tag{127}$$

which is dictated by Rule 1 in Sec. V. (Note that $\tilde{B}_2 = B_2$ mod 2.) Other terms are:

$$A^2 \beta_{(2,4)} B_2 \text{ and } A B_2 w_1(TM)^2. \tag{128}$$

We also check that the sum of three terms satisfy the Rule 2 and Rule 5 in Sec. V. By imposing Rule 7, we can rule out thus discard many other 5d terms in the bordism group $\Omega_5^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \ltimes \mathrm{B}^2\mathbb{Z}_4)$. By imposing Rule 8, we have to select $A^2\beta_{(2,4)}B_2$ in order to match our QFT derivation of the 4d anomaly from $\sim w_1(TM)A^2B_2$ up to a normalization. In summary, our final answer of 4d anomaly and 5d cobordism/SPTs invariant is combined and given in eq. (134). To our understanding, the whole expression indicates a new higher anomaly for this YM theory, new to the literature.

IX. SYMMETRIC TQFT, SYMMETRY-EXTENSION AND HIGHER-SYMMETRY ANALOG OF LIEB-SCHULTZ-MATTIS THEOREM

Since we know the potentially complete 't Hooft anomalies of the above 4d SU(N)-YM and 2d $\mathbb{CP}^{\mathrm{N}-1}$-model at $\theta = \pi$, we wish to constrain their low-energy dynamics further, based on the anomaly-matching. This thinking can be regarded as a formulation of a higher-symmetry analog of "Lieb-Schultz-Mattis theorem [78] [79]." For example, the consequences of low-energy dynamics, under the anomaly saturation can be:

• Symmetry-breaking
- (say $\mathcal{CT}$-symmetry or other G-symmetry).
• Symmetry-preserving
- Gapless, conformal field theory (CFT),
- Intrinsic topological orders.
(Symmetry-preserving TQFT)
- Degenerate ground states.
etc.

Recently Lieb-Schultz-Mattis theorem has been applied to higher-form symmetries acting on extended objects, see [80] and references therein.

In this section, we like to ask, whether it is possible to have a fully symmetry-preserving TQFT to saturate the higher anomaly we discussed earlier, for 4d SU(N)-YM and 2d $\mathbb{CP}^{\mathrm{N}-1}$-model? We use the systematic approach developed in Ref. [14].[12]

We will trivialize the 4d and 2d 't Hooft anomaly of 4d YM and 2d-$\mathbb{CP}^{\mathrm{N}-1}$ models (we may abbreviate them as Yang-Mills and $\mathbb{CP}^{\mathrm{N}-1}$ terms) by pullback the global symmetry to the extended symmetry. If the pullback trivialization is possible, then it means that we can use the "symmetry-extension" method of [14] to construct a fully symmetry-preserving TQFT, at least as an exact solvable model.[13]

In below, when we write $[\mathrm{B}K] \to \mathrm{B}\mathbb{G} \to BG$, we mean that $[\mathrm{B}K]$ is the finite extension, while BG is the classifying space of the original full symmetry G. Moreover, the bracket in $[\mathrm{B}K]$ means that the (full-anomaly-free) K can be dynamically gauged to obtain a dynamical K gauge theory as a symmetry-G preserving TQFT, see [14].

The new ingredient and generalization here we need to go beyond the symmetry-extension method of [14] are:
(1) Higher-symmetry extension: We consider a higher group $\mathbb{G}$ or higher classifying space $\mathrm{B}\mathbb{G}$.
(2) Co/Bordism group and group cohomology of higher group $\mathbb{G}$ or higher classifying space $\mathrm{B}\mathbb{G}$.
Another companion work of ours [85] also implements this method, and explore the constraints on the low energy dynamics for adjoint quantum chromodynamics theory in 4d.

We first summarize the mathematical checks, and then we will explain their physical implications in the end of this section and in Sec. X.

A. $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$

We consider $B_2\mathrm{Sq}^1B_2 + \mathrm{Sq}^2\mathrm{Sq}^1B_2 + w_1(TM)^2\mathrm{Sq}^1B_2$ of eq. (133) for 4d SU(N)-YM at $\mathrm{N}=2$ and at $\theta=\pi$.

Since $\mathrm{Sq}^2\mathrm{Sq}^1B_2 = (w_2(TM) + w_1^2(TM))\mathrm{Sq}^1B_2$ and Sq^1B_2 can be trivialized by $\mathrm{B}^2\mathbb{Z}_4 \to \mathrm{B}^2\mathbb{Z}_2$ since when $B_2 = B_2' \mod 2$, $B_2' : M \to \mathrm{B}^2\mathbb{Z}_4$, $\mathrm{Sq}^1B_2 = 2\beta_{(2,4)}B_2' = 0$ (see Appendix A).

So $B_2\mathrm{Sq}^1B_2+\mathrm{Sq}^2\mathrm{Sq}^1B_2+w_1(TM)^2\mathrm{Sq}^1B_2$ can be trivialized via $[\mathrm{B}^2\mathbb{Z}_{2,[1]}] \to \mathrm{BO}(d) \times \mathrm{B}^2\mathbb{Z}^e_{4,[1]} \to \mathrm{BO}(d) \times \mathrm{B}^2\mathbb{Z}^e_{2,[1]}$.

B. $\Omega_3^{\mathrm{O}}(\mathrm{BO}(3))$

We consider $w_1(E)\Big(w_2(V_{\mathrm{SO}(3)}) + w_1(TM)^2\Big) + w_1(TM)w_2(V_{\mathrm{SO}(3)}) + w_1(E)^3$ of eq. (130) for 2d $\mathbb{CP}^{\mathrm{N}-1}$-model at $\mathrm{N}=2$ at $\theta=\pi$.

Since $w_2(V_{\mathrm{SO}(3)})$ can be trivialized in $\mathrm{SU}(2) = \mathrm{Spin}(3)$. Also $w_1(E)^3$ can be trivialized by $\mathbb{Z}_4^C \to \mathbb{Z}_2^C$, and $w_1(TM)^2$ can be trivialized by $\mathrm{E}(d) \to \mathrm{O}(d)$ where $\mathrm{E}(d) \subset \mathrm{O}(d) \times \mathbb{Z}_4$ is the subgroup of (A,λ) such that $\det A = \lambda^2$. It was defined in [26].

In summary, $w_1(E)(w_2(V_{\mathrm{SO}(3)}) + w_1(TM)^2) + w_1(TM)w_2(V_{\mathrm{SO}(3)}) + w_1(E)^3$ can be trivialized via $[\mathrm{B}(\mathbb{Z}_2)^3] \to \mathrm{BE}(d) \times \mathrm{BSU}(2) \times \mathrm{B}\mathbb{Z}_4^C \to \mathrm{BO}(d) \times \mathrm{BPSU}(2) \times \mathrm{B}\mathbb{Z}_2^C$.

Since $\mathrm{Sq}^2w_1(E) = (w_2(TM) + w_1(TM)^2)w_1(E) = 0$, $w_1(E)w_1(TM)^2 = w_1(E)w_2(TM)$ can also be trivialized by $\mathrm{Pin}^+(d) \to \mathrm{O}(d)$.

So $w_1(E)(w_2(V_{\mathrm{SO}(3)}) + w_1(TM)^2) + w_1(TM)w_2(V_{\mathrm{SO}(3)}) + w_1(E)^3$ can also be trivialized via $[\mathrm{B}(\mathbb{Z}_2)^3] \to \mathrm{BPin}^+(d) \times \mathrm{BSU}(2) \times \mathrm{B}\mathbb{Z}_4^C \to \mathrm{BO}(d) \times \mathrm{BPSU}(2) \times \mathrm{B}\mathbb{Z}_2^C$.

C. $\Omega_5^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \ltimes \mathrm{B}^2\mathbb{Z}_4)$

We consider $\tilde{B}_2\beta_{(2,4)}B_2 + A^2\beta_{(2,4)}B_2 + AB_2w_1(TM)^2$ of eq. (134) for 4d SU(N)-YM at $\mathrm{N}=4$ and at $\theta=\pi$.

Notice $\beta_{(2,4)}B_2$ can be trivialized by $\mathrm{B}^2\mathbb{Z}_8 \to \mathrm{B}^2\mathbb{Z}_4$, and notice that $B_2 = B_2' \mod 4$, $B_2' : M \to \mathrm{B}^2\mathbb{Z}_8$, $\beta_{(2,4)}B_2 = 2\beta_{(2,8)}B_2' = 0$ (see Appendix A). Also $w_1(TM)^2$ can be trivialized by $\mathrm{E}(d) \to \mathrm{O}(d)$

So $\tilde{B}_2\beta_{(2,4)}B_2+A^2\beta_{(2,4)}B_2+AB_2w_1(TM)^2$ can be trivialized via $[\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_{2,[1]}] \to \mathrm{BE}(d) \times \mathrm{B}\mathbb{Z}_2^C \ltimes \mathrm{B}^2\mathbb{Z}^e_{8,[1]} \to \mathrm{BO}(d) \times \mathrm{B}\mathbb{Z}_2^C \ltimes \mathrm{B}^2\mathbb{Z}^e_{4,[1]}$.

D. $\Omega_3^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)))$

We consider $w_1(E)\Big(w_2(E) + w_1(TM)^2\Big) + \frac{1}{2}\tilde{w}_1(TM)w_2(E)$ of eq. (131) for 2d $\mathbb{CP}^{\mathrm{N}-1}$-model at $\mathrm{N}=4$ at $\theta=\pi$.

Since there is a short exact sequence of groups: $1 \to \mathbb{Z}_4 \to \mathbb{Z}_2^C \ltimes \mathrm{SU}(4) \to \mathbb{Z}_2^C \ltimes \mathrm{PSU}(4) \to 1$, we have a fiber sequence: $\mathrm{B}\mathbb{Z}_4 \to \mathrm{B}(\mathbb{Z}_2^C \ltimes \mathrm{SU}(4)) \to \mathrm{B}(\mathbb{Z}_2^C \ltimes \mathrm{PSU}(4)) \xrightarrow{w_2}$

---

[12] One can also formulate a lattice realization of version given in [81]. Closely related work on this symmetry-extension method include [19, 82–84] and references therein.

[13] A caveat: One needs to beware that the dimensionality affects the dynamics and stability of long-range entanglement, the symmetry-preserving TQFT at 2d or below can be destroyed by local perturbations. See detailed explorations in [14].

$B^2\mathbb{Z}_4$, so $w_2(E)$ can be trivialized by $B(\mathbb{Z}_2^C \ltimes SU(4)) \to B(\mathbb{Z}_2^C \ltimes PSU(4))$.

Also $w_1(TM)^2$ can be trivialized by $E(d) \to O(d)$.

So $w_1(E)\Big(w_2(E)+w_1(TM)^2\Big) + \frac{1}{2}\tilde{w}_1(TM)w_2(E)$ can be trivialized via $[B(\mathbb{Z}_2 \times \mathbb{Z}_4)] \to BE(d) \times B(\mathbb{Z}_2^C \ltimes SU(4)) \to BO(d) \times B(\mathbb{Z}_2^C \ltimes PSU(4))$.

Since $Sq^2 w_1(E) = (w_2(TM) + w_1(TM)^2)w_1(E) = 0$, $w_1(E)w_1(TM)^2 = w_1(E)w_2(TM)$ can also be trivialized by $Pin^+(d) \to O(d)$.

So $w_1(E)\Big(w_2(E)+w_1(TM)^2\Big) + \frac{1}{2}\tilde{w}_1(TM)w_2(E)$ can also be trivialized via $[B(\mathbb{Z}_2 \times \mathbb{Z}_4)] \to BPin^+(d) \times B(\mathbb{Z}_2^C \ltimes SU(4)) \to BO(d) \times B(\mathbb{Z}_2^C \ltimes PSU(4))$.

In summary, again, in this section, we obtain various possible symmetry-G preserving TQFTs to saturate (higher) 't Hooft anomalies of YM theories and $\mathbb{CP}^{N-1}$-model, from the $[BK]$ extension. This means that the (full-anomaly-free) K can be dynamically gauged to obtain a dynamical K gauge theory, subject to a caveat in footnote 13, see [14].

X. CONCLUSION AND MORE COMMENTS: ANOMALIES FOR THE GENERAL N

In this work, we propose a new and more complete set of 't Hooft anomalies of certain quantum field theories (QFTs): time-reversal symmetric 4d SU(N)-Yang-Mills (YM) and 2d-$\mathbb{CP}^{N-1}$ models with a topological term $\theta = \pi$, and then give an eclectic "proof" of the existence of these full anomalies (of ordinary 0-form global symmetries or higher symmetries) to match these QFTs. Our "proof" is formed by a set of analyses and arguments, combining algebraic/geometric topology, QFT analysis, condensed matter inputs and additional physical criteria

We mainly focus on N = 2 and N = 4 cases. As known in the literature, we actually know that N = 3 case is absent from the strict 't Hooft anomaly. The absence of obvious 't Hooft anomalies also apply to the more general odd integer N case (although one needs to be careful about the global consistency or global inconsistency, see [30]). For a general even N integer, it has not been clear in the literature what are the complete 't Hoot anomalies for these QFTs.

Physically we follow the idea that coupling the global symmetry of dd QFTs to background fields, we can detect the higher dimensional ($d+1$d) SPTs/counter term as eq. (2):

$$\mathbf{Z}^{dd}_{\text{QFT}}\big|_{\text{bgd.field}=0} \longrightarrow \mathbf{Z}^{(d+1)\text{d}}_{\text{SPTs}}(\text{bgd.field}) \cdot \mathbf{Z}^{dd}_{\text{QFT}}\big|_{\text{bgd.field}\neq 0},$$

that cannot be absorbed by dd SPTs. (Here, for condensed matter oriented terminology, we follow the conventions of [13].) This underlying d+1d SPTs means that the dd QFTs have an obstruction to be regularized with all the relevant (higher) global symmetries strictly local or onsite. Thus this indicates the obstruction of gauging, which indicates the dd 't Hooft anomalies (See [12–14] for QFT-oriented discussion and references therein).

We comment that the above idea eq. (2) is *distinct* from another idea also relating to coupling QFTs to SPTs, for example used in [4]: There one couples dd QFTs to dd SPTs/topological terms,

$$\mathbf{Z}^{dd}_{\text{QFT}}(A_1, B_2, .)\big|_{\text{bgd.field}} \xrightarrow{\text{dynamical gauging + } dd \text{ SPTs}} \int [\mathcal{D}A_1][\mathcal{D}B_2]\dots \mathbf{Z}^{dd}_{\text{QFT}}(A_1, B_2, .) \cdot \mathbf{Z}^{dd}_{\text{SPTs}}(A_1, B_2, .), \tag{129}$$

with the allowed global symmetries, and then dynamically gauging some of global symmetries. A similar framework outlining the above two ideas, on coupling QFTs to SPTs and gauging, is also explored in [21].

Follow the idea of eq. (2) and the QFT and global symmetries information given in Sec. II, we classify all the possible anomalies enumerated by the cobordism theory computed in Sec. III. Then constrained by the known anomalies in the literature Sec. IV, we follow the rules for the anomaly constraint we set in Sec. V and a dimensional reduction method in Sec. VII, we deduce the new anomalies of 2d-$\mathbb{CP}^{N-1}$ models in Sec. VI and of 4d SU(N)-Yang-Mills (YM) in Sec. VIII. To summarize the dd anomalies and the $(d+1)$ cobordism/SPTs invariants of the above QFTs,

we propose that a general anomaly formula (3d cobordism/SPT invariant) for 2d $\mathbb{CP}^{N-1}_{\theta=\pi}$ model at N = 2 as:

$$\begin{aligned}
&\mathbf{Z}^{2\text{d}}_{\mathbb{CP}^1{}_{\theta=\pi}}(w_j(TM), w_j(E), \dots)\mathbf{Z}^{3\text{d}}_{\text{SPTs}}\\
&\equiv \mathbf{Z}^{2\text{d}}_{\mathbb{CP}^1{}_{\theta=\pi}}(w_j(TM), w_j(E), \dots) \exp(\text{i}\pi \int_{M^3} \big(w_1(E)^3 + w_1(E)w_2(V_{\text{SO}(3)}) + w_1(TM)w_2(V_{\text{SO}(3)}) + w_1(E)w_1(TM)^2\big)\\
&= \mathbf{Z}^{2\text{d}}_{\mathbb{CP}^1_{\theta=\pi}}(w_j(TM), w_j(E), \dots) \exp(\text{i}\pi \int_{M^3} \big(w_3(E) + w_1(E)w_1(TM)^2\big).
\end{aligned} \tag{130}$$

We propose that a general anomaly formula (3d cobordism/SPT invariant) for 2d $\mathbb{CP}^{\mathrm{N}-1}_{\theta=\pi}$ model at N = 4 as:

$$\begin{aligned}&\mathbf{Z}^{\text{2d}}_{\mathbb{CP}^3_{\theta=\pi}}(w_j(TM),\tilde{w}_j(E),\dots)\mathbf{Z}^{\text{3d}}_{\text{SPTs}}\\&\equiv \mathbf{Z}^{\text{2d}}_{\mathbb{CP}^3_{\theta=\pi}}(w_j(TM),\tilde{w}_j(E),\dots)\exp(\mathrm{i}\pi\int_{M^3}\big(w_1(E)w_2(E)+\frac{1}{2}w_1(TM)w_2(E)+w_1(E)w_1(TM)^2\big)\\&=\mathbf{Z}^{\text{2d}}_{\mathbb{CP}^3_{\theta=\pi}}(w_j(TM),\tilde{w}_j(E),\dots)\exp(\mathrm{i}\pi\int_{M^3}\big(\tilde{w}_3(E)+w_1(E)w_1(TM)^2\big).\end{aligned}\tag{131}$$

For all the above case, we propose that a general anomaly formula (3d cobordism/SPT invariant) for 2d $\mathbb{CP}^{\mathrm{N}-1}_{\theta=\pi}$ model at N is an even integer:

$$\mathbf{Z}^{\text{2d}}_{\mathbb{CP}^{\mathrm{N}-1}_{\theta=\pi}}(w_j(TM),\tilde{w}_j(E),\dots)\mathbf{Z}^{\text{3d}}_{\text{SPTs}}\equiv \mathbf{Z}^{\text{2d}}_{\mathbb{CP}^{\mathrm{N}-1}_{\theta=\pi}}(w_j(TM),\tilde{w}_j(E),\dots)\exp(\mathrm{i}\pi\int_{M^3}\big(\tilde{w}_3(E)+w_1(E)w_1(TM)^2\big).\tag{132}$$

$\tilde{w}_3(E)\in \mathrm{H}^3(\mathrm{B}(\mathrm{PSU(N)}\rtimes\mathbb{Z}_2),\mathbb{Z}_2)=\mathbb{Z}_2$ when N is even. (Our $\tilde{w}_3(E)$ is related to Ref. [64] named u_3, while our convention of u_j is normally called the Wu class instead.)

We propose that a general anomaly formula (5d cobordism/higher SPT invariant) for 4d $\mathrm{SU(N)}_{\theta=\pi}$-YM theory at N = 2 as:

$$\begin{aligned}&\mathbf{Z}^{\text{4d}}_{\text{SU(2)YM}_{\theta=\pi}}(w_j(TM),A,B_2,\dots)\mathbf{Z}^{\text{5d}}_{\text{higher-SPTs}}\\&\equiv \mathbf{Z}^{\text{4d}}_{\text{SU(2)YM}_{\theta=\pi}}(w_j(TM),A,B_2,\dots)\exp(\mathrm{i}\pi\int_{M^5}\big(B_2\mathrm{Sq}^1B_2+\mathrm{Sq}^2\mathrm{Sq}^1B_2+w_1(TM)^2\mathrm{Sq}^1B_2\big))\\&=\mathbf{Z}^{\text{4d}}_{\text{SU(2)YM}_{\theta=\pi}}(w_j(TM),A,B_2,\dots)\exp(\mathrm{i}\pi\int_{M^5}\big(\frac{1}{2}\tilde{w}_1(TM)\mathcal{P}_2(B_2)+w_1(TM)^2\mathrm{Sq}^1B_2\big)).\end{aligned}\tag{133}$$

We propose that a general anomaly formula (5d cobordism/higher SPT invariant) for 4d $\mathrm{SU(N)}_{\theta=\pi}$-YM theory at N = 4 as:

$$\begin{aligned}&\mathbf{Z}^{\text{4d}}_{\text{SU(4)YM}_{\theta=\pi}}(w_j(TM),A,B_2,\dots)\mathbf{Z}^{\text{5d}}_{\text{higher-SPTs}}\\&\equiv \mathbf{Z}^{\text{4d}}_{\text{SU(4)YM}_{\theta=\pi}}(w_j(TM),A,B_2,\dots)\exp(\mathrm{i}\pi\int_{M^5}\big(B_2\beta_{(2,4)}B_2+A^2\beta_{(2,4)}B_2+AB_2w_1(TM)^2\big).\end{aligned}\tag{134}$$

When $\mathrm{N}=2^n$ is a power of 2, with some positive integer $n>1$, we propose that a general anomaly formula (5d cobordism/higher SPT invariant) for 4d $\mathrm{SU(N)}_{\theta=\pi}$-YM theory

$$\begin{aligned}&\mathbf{Z}^{\text{4d}}_{\text{SU(N)YM}_{\theta=\pi}}(w_j(TM),A,B_2,\dots)\mathbf{Z}^{\text{5d}}_{\text{higher-SPTs}}\\&\equiv \mathbf{Z}^{\text{4d}}_{\text{SU(N)YM}_{\theta=\pi}}(w_j(TM),A,B_2,\dots)\exp(\mathrm{i}\pi\int_{M^5}\big(B_2\beta_{(2,\mathrm{N})}B_2+A^2\beta_{(2,\mathrm{N})}B_2+AB_2w_1(TM)^2\big).\end{aligned}\tag{135}$$

Note that we can derive $B_2\beta_{(2,\mathrm{N}=2^n)}B_2=\frac{1}{\mathrm{N}}\tilde{w}_1(TM)\mathcal{P}_2(B_2)$, where Pontryagin square $\mathcal{P}_2:\mathrm{H}^2(-,\mathbb{Z}_{2^n})\to\mathrm{H}^4(-,\mathbb{Z}_{2^{n+1}})$. Only when $\mathrm{N}=2=2^1$, we have the exceptional result obtained in our eq. (133), distinct from the form of our eq. (135) for $\mathrm{N}=2^n$ with $n>1$.

We notice that the above anomalies we discussed all are (mod 2) classes, captured by cobordism invariants of $\mathbb{Z}_2$ classes. These all are *non-perturbative global* anomalies.

We have commented about the higher symmetry analog of "Lieb-Schultz-Mattis theorem" in Sec. IX, for example, the consequences of low-energy dynamics due to the anomalies. (For the early-history and the recent explorations on the emergent dynamical gauge fields and anomalous higher symmetries in quantum mechanical and in condensed matter systems, see for example, [86] and [87] respectively, and references therein.) We hope to address more about the dynamics in future work.

XI. ACKNOWLEDGMENTS

The authorship is listed in the alphabetical order by convention. JW thanks the participants of Developments in Quantum Field Theory and Condensed Matter Physics (November 5-7, 2018) at Simons Center for Geometry and Physics at SUNY Stony Brook University for giving valuable feedback where this work is publicly reported. We thank Pavel Putrov and Edward Witten for helpful remarks or comments. JW especially thanks Zohar Komargodski, and also Clay Cordova, for discussions on the issue of the potentially missing anomalies of YM theories [88]. JW also thanks Kantaro Ohmori, Nathan Seiberg, Masahito Yamazaki, and Yunqin Zheng for conversations. ZW acknowledges support from NSFC grants 11431010 and 11571329. JW acknowledges the Corning Glass Works Foundation Fellowship and NSF Grant PHY-1606531. This work is also supported by NSF Grant DMS-1607871 "Analysis, Geometry and Mathematical Physics" and Center for Mathematical Sciences and Applications at Harvard University.

Appendix A: Bockstein Homomorphism

In general, given a chain complex $C_\bullet$ and a short exact sequence of abelian groups:

$$0 \to A' \to A \to A'' \to 0, \tag{A1}$$

we have a short exact sequence of cochain complexes:

$$\begin{aligned} &0 \to \mathrm{Hom}(C_\bullet, A') \to \mathrm{Hom}(C_\bullet, A) \\ &\to \mathrm{Hom}(C_\bullet, A'') \to 0. \end{aligned} \tag{A2}$$

Hence we obtain a long exact sequence of cohomology groups:

$$\begin{aligned} &\cdots \to \mathrm{H}^n(C_\bullet, A') \to \mathrm{H}^n(C_\bullet, A) \to \mathrm{H}^n(C_\bullet, A'') \\ &\xrightarrow{\partial} \mathrm{H}^{n+1}(C_\bullet, A') \to \cdots, \end{aligned} \tag{A3}$$

the connecting homomorphism ∂ is called Bockstein homomorphism.

For example, $\beta_{(n,m)} : \mathrm{H}^*(-, \mathbb{Z}_m) \to \mathrm{H}^{*+1}(-, \mathbb{Z}_n)$ is the Bockstein homomorphism associated to the extension $\mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_{nm} \to \mathbb{Z}_m$ where $\cdot m$ is the group homomorphism given by multiplication by m. In particular, $\beta_{(2,2^n)} = \frac{1}{2^n}\delta$ mod 2.

Since there is a commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}_n & \xrightarrow{\cdot m} & \mathbb{Z}_{nm} & \xrightarrow{\text{mod } m} & \mathbb{Z}_m \\ \| & & \downarrow{\cdot k} & & \downarrow{\cdot k} \\ \mathbb{Z}_n & \xrightarrow{\cdot km} & \mathbb{Z}_{knm} & \xrightarrow{\text{mod } km} & \mathbb{Z}_{km}, \end{array} \tag{A4}$$

by the naturality of connecting homomorphism, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^*(-, \mathbb{Z}_m) & \xrightarrow{\beta_{(n,m)}} & \mathrm{H}^{*+1}(-, \mathbb{Z}_n) \\ \downarrow{\cdot k} & & \| \\ \mathrm{H}^*(-, \mathbb{Z}_{km}) & \xrightarrow{\beta_{(n,km)}} & \mathrm{H}^{*+1}(-, \mathbb{Z}_n). \end{array} \tag{A5}$$

Hence we prove that

$$\beta_{(n,m)} = \beta_{(n,km)} \cdot k. \tag{A6}$$

In particular, since $\mathrm{Sq}^1 = \beta_{(2,2)}$, we have $\mathrm{Sq}^1 = \beta_{(2,4)} \cdot 2$. This formula is used in Sec. IX.

Appendix B: Poincaré Duality

An orientable manifold is R-orientable for any ring R, while a non-orientable manifold is R-orientable iff R contains a unit of order 2, which is equivalent to having $2 = 0$ in R. Thus every manifold is $\mathbb{Z}_2$-orientable.

Poincaré Duality: Let M be a closed connected n-dimensional manifold, R is a ring, if M is R-orientable, let $[M] \in \mathrm{H}_n(M, R)$ be the fundamental class for M with coefficients in R, then the map $\mathrm{PD} : \mathrm{H}^k(M, R) \to \mathrm{H}_{n-k}(M, R)$ defined by $\mathrm{PD}(\alpha) = [M] \cap \alpha$ is an isomorphism for all k.

Fact: $\mathrm{H}_k(M, R)$ can be represented by a submanifold of M when

(1) $R = \mathbb{Z}_2$;

(2) $R = \mathbb{Z}$, $k \leq 6$.

Appendix C: Cohomology of Klein bottle with coefficients $\mathbb{Z}_4$

In this Appendix, we derive the relation of $\beta_{(2,4)} x = z$, where x is the generator of the $\mathbb{Z}_4$ factor of $\mathrm{H}^1(K, \mathbb{Z}_4) = \mathbb{Z}_4 \times \mathbb{Z}_2$ and z is the generator of $\mathrm{H}^2(K, \mathbb{Z}_2) = \mathbb{Z}_2$.

One Δ-complex structure of Klein bottle is shown in Fig. 5. Let α_i denote the dual cochain of the 1-simplex a_i with coefficients $\mathbb{Z}_4$, λ_i the dual cochain of the 2-simplex u_i with coefficients $\mathbb{Z}_4$, let $\tilde{}$ denote its mod 2 reduction and let $\{\ \}$ denote the cohomology class.

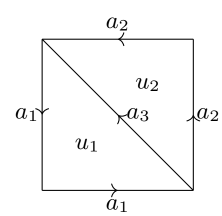


FIG. 5. One Δ-complex structure of Klein bottle

The 2-simplexes and 1-simplexes are related by the boundary differential ∂ of chains, namely $\partial u_1 = 2a_1 + a_3$, $\partial u_2 = 2a_2 - a_3$, so we deduce that the boundary differential δ of cochains have the following relation: $\delta\alpha_1 = 2\lambda_1$, $\delta\alpha_2 = 2\lambda_2$, $\delta\alpha_3 = \lambda_1 - \lambda_2$. So we deduce that the cohomology classes $\{\lambda_1\} = \{\lambda_2\}$ are the same.

Since $\delta(\alpha_1 - \alpha_2 - 2\alpha_3) = 0$, $\delta(2\alpha_1) = 0$, $\mathrm{H}^1(K, \mathbb{Z}_4) = \mathbb{Z}_4 \times \mathbb{Z}_2$. Let $x = \{\alpha_1 - \alpha_2 - 2\alpha_3\}$, $y = \{2\alpha_1\}$, then x generates $\mathbb{Z}_4$, y generates $\mathbb{Z}_2$, $x \mod 2 = \{\tilde{\alpha}_1 + \tilde{\alpha}_2\}$, $y \mod 2 = 0$.

By the definition of cup product, $\alpha_1^2(u_1) = \alpha_1(a_1) \cdot \alpha_1(a_1) = 1$, $\alpha_1^2(u_2) = \alpha_1(a_2) \cdot \alpha_1(a_2) = 0$, so $\alpha_1^2 = \lambda_1$, similarly $\alpha_2^2 = \lambda_2$.

$\{\tilde{\alpha}_1 + \tilde{\alpha}_2\}^2 = \{\tilde{\alpha}_1\}^2 + \{\tilde{\alpha}_2\}^2 = 2z = 0$ where $z = \{\tilde{\lambda}_1\} = \{\tilde{\lambda}_2\}$ is the generator of $\mathrm{H}^2(K, \mathbb{Z}_2) = \mathbb{Z}_2$, so $\beta_{(2,4)} x = z$.

Appendix D: Cohomology of $B\mathbb{Z}_2 \ltimes B^2\mathbb{Z}_4$

In order to compute $\Omega_5^{O}(B\mathbb{Z}_2 \ltimes B^2\mathbb{Z}_4)$, we need the data of $H^n(B\mathbb{Z}_2 \ltimes B^2\mathbb{Z}_4, \mathbb{Z}_2)$ for $n \leq 5$.

Let $\mathbb{G}$ be a 2-group with $B\mathbb{G} = B\mathbb{Z}_2 \ltimes B^2\mathbb{Z}_4$. By the Universal Coefficient Theorem,

$$\begin{aligned} H^n(B\mathbb{G}, \mathbb{Z}_2) = H^n(B\mathbb{G}, \mathbb{Z}) \otimes \mathbb{Z}_2 \oplus \\ \mathrm{Tor}(H^{n+1}(B\mathbb{G}, \mathbb{Z}), \mathbb{Z}_2). \end{aligned} \tag{D1}$$

So we need only compute $H^n(B\mathbb{Z}_2 \ltimes B^2\mathbb{Z}_4, \mathbb{Z})$ for $n \leq 6$.

$H^n(B^2\mathbb{Z}_4, \mathbb{Z})$ is computed in Appendix C of [89].

$$H^n(B^2\mathbb{Z}_4, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n = 1 \\ 0 & n = 2 \\ \mathbb{Z}_4 & n = 3 \\ 0 & n = 4 \\ \mathbb{Z}_8 & n = 5 \\ \mathbb{Z}_2 & n = 6 \end{cases} \tag{D2}$$

For the 2-group $\mathbb{G}$ defined by the nontrivial action ρ of $\mathbb{Z}_2$ on $\mathbb{Z}_4$ and nontrivial fibration

$$\begin{array}{ccc} B^2\mathbb{Z}_4 & \longrightarrow & B\mathbb{G} \\ & & \downarrow \\ & & B\mathbb{Z}_2 \end{array} \tag{D3}$$

classified by the nonzero Postnikov class $\pi \in H^3(B\mathbb{Z}_2, \mathbb{Z}_4)$. Here we consider the fiber sequence $B^2\mathbb{Z}_{4,[1]} \to B\mathbb{G} \to B\mathbb{Z}_2 \to B^3\mathbb{Z}_{4,[1]} \to \dots$ induced from a short exact sequence $1 \to \mathbb{Z}_{4,[1]} \to \mathbb{G} \to \mathbb{Z}_2 \to 1$. We have the Serre spectral sequence

$$H^p(B\mathbb{Z}_2, H^q(B^2\mathbb{Z}_4, \mathbb{Z})) \Rightarrow H^{p+q}(B\mathbb{G}, \mathbb{Z}), \tag{D4}$$

the E_2 page of the Serre spectral sequence is the ρ-equivariant cohomology $H^p(B\mathbb{Z}_2, H^q(B^2\mathbb{Z}_4, \mathbb{Z}))$. The shape of the relevant piece is shown in Fig. 6.

Note that p labels the columns and q labels the rows.

The bottom row is $H^p(B\mathbb{Z}_2, \mathbb{Z})$.

The universal coefficient theorem tells us that $H^3(B^2\mathbb{Z}_4, \mathbb{Z}) = H^2(B^2\mathbb{Z}_4, \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}(H_2(B^2\mathbb{Z}_4, \mathbb{Z}), \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}(\pi_2(B^2\mathbb{Z}_4), \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}_4, \mathbb{R}/\mathbb{Z}) = \hat{\mathbb{Z}}_4$, so the $q = 3$ row is $H^p(B\mathbb{Z}_2, \hat{\mathbb{Z}}_4)$, where $\mathbb{Z}_2$ acts on $\mathbb{Z}_4$ via ρ. For example, $H^0(B\mathbb{Z}_2, \hat{\mathbb{Z}}_4)$ is the subgroup of $\mathbb{Z}_2$-invariant characters in $\hat{\mathbb{Z}}_4$.

It is also known that $H^5(B^2\mathbb{Z}_4, \mathbb{Z}) = H^4(B^2\mathbb{Z}_4, \mathbb{R}/\mathbb{Z})$ is the group of quadratic functions $q : \mathbb{Z}_4 \to \mathbb{R}/\mathbb{Z}$. The group at $(p, q) = (0, 5)$ is then the subgroup of $\mathbb{Z}_2$-invariant quadratic forms.

The first possibly non-zero differential is on the E_3 page:

$$H^0(B\mathbb{Z}_2, H^5(B^2\mathbb{Z}_4, \mathbb{Z})) \to H^3(B\mathbb{Z}_2, \hat{\mathbb{Z}}_4). \tag{D5}$$

Following the appendix of [90], this map sends a $\mathbb{Z}_2$-invariant quadratic form $q : \mathbb{Z}_4 \to \mathbb{R}/\mathbb{Z}$ to $\langle \pi, - \rangle_q$, where the bracket denotes the bilinear pairing $\langle x, y \rangle_q = q(x + y) - q(x) - q(y)$.

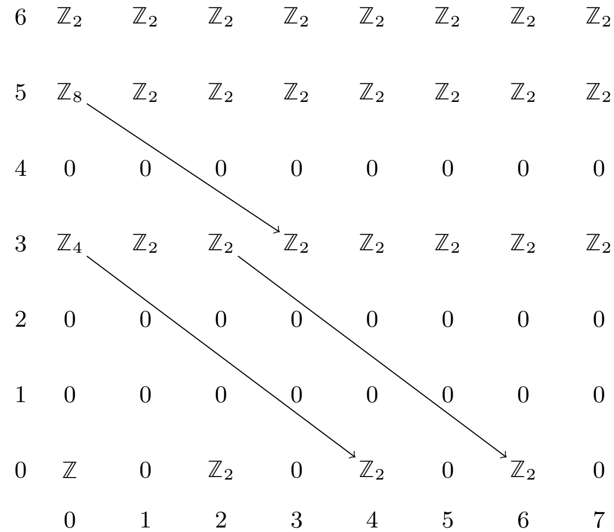


FIG. 6. Serre spectral sequence for $(B\mathbb{Z}_2, B^2\mathbb{Z}_4)$

The next possibly non-zero differentials are on the E_4 page:

$$H^j(B\mathbb{Z}_2, \hat{\mathbb{Z}}_4) \to H^{j+3}(B\mathbb{Z}_2, \mathbb{R}/\mathbb{Z}) \xrightarrow{\sim} H^{j+4}(B\mathbb{Z}_2, \mathbb{Z}) \tag{D6}$$

The first map is contraction with π.

The last relevant possibly non-zero differential is on the E_6 page:

$$H^0(B\mathbb{Z}_2, H^5(B^2\mathbb{Z}_4, \mathbb{Z})) \to H^6(B\mathbb{Z}_2, \mathbb{Z}). \tag{D7}$$

Following the appendix of [90], this differential is actually zero.

So the only possible differentials of Serre spectral sequence are d_3 from $(0, 5)$ to $(3, 3)$ and d_4 from the third row to the zeroth row.

$\langle \pi, \pi \rangle_q = 2q(\pi)$, $8q(\pi) = 0$, there are 2 among the 8 choices of $q(\pi)$ such that $q \to \langle \pi, - \rangle_q$ maps to the dual linear function of π, if we identify $\hat{\mathbb{Z}}_4$ with $\mathbb{Z}_4$, then the nonzero element in the image of $q \to \langle \pi, - \rangle_q$ is just π. So the differential $d_3^{(0,5)}$ is nontrivial.

The differential $d_4^{(0,3)} : H^0(B\mathbb{Z}_2, \hat{\mathbb{Z}}_4) \to H^3(B\mathbb{Z}_2, \mathbb{R}/\mathbb{Z})$ is defined by

$$d_4^{(2,3)}(\lambda)(v_0, \dots, v_3) = \lambda(\pi(v_0, \dots, v_3))$$

which is actually zero since $\pi(v_0, \dots, v_3) \in 2\mathbb{Z}_4$.

The differential $d_4^{(2,3)} : H^2(B\mathbb{Z}_2, \hat{\mathbb{Z}}_4) \to H^5(B\mathbb{Z}_2, \mathbb{R}/\mathbb{Z})$ is defined by

$$d_4^{(2,3)}(\chi)(v_0, \dots, v_5) = (\chi(v_0, \dots, v_2))(\pi(v_2, \dots, v_5))$$

which is also actually zero since $\pi(v_2, \dots, v_5) \in 2\mathbb{Z}_4$.

So only the A^3B_2 is vanished in $H^5(B\mathbb{Z}_2 \ltimes B^2\mathbb{Z}_4, \mathbb{Z}_2)$, hence in $\Omega_5^{O}(B\mathbb{Z}_2 \ltimes B^2\mathbb{Z}_4)$.

[1] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).
[2] A. Jaffe and E. Witten, Clay Mathematics Institute Millennium Prize Problems (2004).
[3] G. 't Hooft, C. Itzykson, A. Jaffe, H. Lehmann, P. K. Mitter, I. M. Singer, and R. Stora, NATO Sci. Ser. B **59** (1980).
[4] M. Guo, P. Putrov, and J. Wang, Annals of Physics **394**, 244 (2018), arXiv:1711.11587 [cond-mat.str-el].
[5] M. Z. Hasan and C. L. Kane, Reviews of Modern Physics **82**, 3045 (2010), arXiv:1002.3895 [cond-mat.mes-hall].
[6] X.-L. Qi and S.-C. Zhang, Reviews of Modern Physics **83**, 1057 (2011), arXiv:1008.2026 [cond-mat.mes-hall].
[7] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. **B87**, 155114 (2013), arXiv:1106.4772 [cond-mat.str-el].
[8] T. Senthil, Annual Review of Condensed Matter Physics **6**, 299 (2015), arXiv:1405.4015 [cond-mat.str-el].
[9] X.-G. Wen, Reviews of Modern Physics **89**, 041004 (2017), arXiv:1610.03911 [cond-mat.str-el].
[10] S. L. Adler, Phys. Rev. **177**, 2426 (1969), [,241(1969)].
[11] J. S. Bell and R. Jackiw, Nuovo Cim. **A60**, 47 (1969).
[12] X.-G. Wen, Phys. Rev. **D88**, 045013 (2013), arXiv:1303.1803 [hep-th].
[13] J. C. Wang, Z.-C. Gu, and X.-G. Wen, Phys. Rev. Lett. **114**, 031601 (2015), arXiv:1405.7689 [cond-mat.str-el].
[14] J. Wang, X.-G. Wen, and E. Witten, Phys. Rev. **X8**, 031048 (2018), arXiv:1705.06728 [cond-mat.str-el].
[15] E. Witten, Phys. Lett. **B117**, 324 (1982), [,230(1982)].
[16] J. Wang, X.-G. Wen, and E. Witten, (2018), arXiv:1810.00844 [hep-th].
[17] E. Witten, Commun. Math. Phys. **100**, 197 (1985), [,197(1985)].
[18] J. Wang, L. H. Santos, and X.-G. Wen, Phys. Rev. **B91**, 195134 (2015), arXiv:1403.5256 [cond-mat.str-el].
[19] A. Kapustin and R. Thorngren, (2014), arXiv:1404.3230 [hep-th].
[20] J. C. Wang, *Aspects of symmetry, topology and anomalies in quantum matter*, Ph.D. thesis, Massachusetts Institute of Technology (2015), arXiv:1602.05569 [cond-mat.str-el].
[21] A. Kapustin and N. Seiberg, JHEP **04**, 001 (2014), arXiv:1401.0740 [hep-th].
[22] C. G. Callan, Jr. and J. A. Harvey, Nucl. Phys. **B250**, 427 (1985).
[23] X.-Z. Dai and D. S. Freed, J. Math. Phys. **35**, 5155 (1994), [Erratum: J. Math. Phys.42,2343(2001)], arXiv:hep-th/9405012 [hep-th].
[24] A. Kapustin, (2014), arXiv:1403.1467 [cond-mat.str-el].
[25] A. Kapustin, R. Thorngren, A. Turzillo, and Z. Wang, JHEP **12**, 052 (2015), [JHEP12,052(2015)], arXiv:1406.7329 [cond-mat.str-el].
[26] D. S. Freed and M. J. Hopkins, ArXiv e-prints (2016), arXiv:1604.06527 [hep-th].
[27] M. F. Atiyah, Comm. Math. Phys. **93**, 437 (1984).
[28] S. K. Donaldson, Commun. Math. Phys. **93**, 453 (1984).
[29] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, JHEP **02**, 172 (2015), arXiv:1412.5148 [hep-th].
[30] D. Gaiotto, A. Kapustin, Z. Komargodski, and N. Seiberg, JHEP **05**, 091 (2017), arXiv:1703.00501 [hep-th].
[31] J. Milnor and J. Stasheff, *Characteristic Classes, by Milnor and Stasheff*, Annals of Mathematics Studies, No. 76 (Princeton University Press, 1974).
[32] R. Thorngren and C. von Keyserlingk, (2015), arXiv:1511.02929 [cond-mat.str-el].
[33] C. Delcamp and A. Tiwari, JHEP **10**, 049 (2018), arXiv:1802.10104 [cond-mat.str-el].
[34] Z. Wan and J. Wang, (2018), arXiv:1812.11967 [hep-th].
[35] M. Yamazaki, JHEP **10**, 172 (2018), arXiv:1711.04360 [hep-th].
[36] M. Yamazaki and K. Yonekura, JHEP **07**, 088 (2017), arXiv:1704.05852 [hep-th].
[37] Y. Tanizaki, T. Misumi, and N. Sakai, JHEP **12**, 056 (2017), arXiv:1710.08923 [hep-th].
[38] L. D. Faddeev and V. N. Popov, Phys. Lett. **B25**, 29 (1967), [,325(1967)].
[39] B. S. DeWitt, Phys. Rev. **162**, 1195 (1967), [,298(1967)].
[40] E. Witten, Phys. Rev. **B94**, 195150 (2016), arXiv:1605.02391 [hep-th].
[41] O. Aharony, N. Seiberg, and Y. Tachikawa, JHEP **08**, 115 (2013), arXiv:1305.0318 [hep-th].
[42] J. Wang, X.-G. Wen, and S.-T. Yau, (2016), arXiv:1602.05951 [cond-mat.str-el].
[43] P. Putrov, J. Wang, and S.-T. Yau, Annals Phys. **384**, 254 (2017), arXiv:1612.09298 [cond-mat.str-el].
[44] K. Aitken, A. Cherman, and M. Unsal, (2018), arXiv:1804.05845 [hep-th].
[45] F. Benini, C. Cordova, and P.-S. Hsin, (2018), arXiv:1803.09336 [hep-th].
[46] E. Witten, Nucl. Phys. **B149**, 285 (1979).
[47] F. D. M. Haldane, Phys. Lett. **A93**, 464 (1983).
[48] F. D. M. Haldane, Phys. Rev. Lett. **50**, 1153 (1983).
[49] J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971).
[50] E. Witten, Nucl. Phys. **B223**, 422 (1983).
[51] E. Witten, Commun. Math. Phys. **92**, 455 (1984), [,201(1983)].
[52] I. Affleck and F. D. M. Haldane, Phys. Rev. **B36**, 5291 (1987).
[53] I. Affleck, Nucl. Phys. **B305**, 582 (1988).
[54] J. Wang and X.-G. Wen, (2018), arXiv:1809.11171 [hep-th].
[55] S. Galatius, I. Madsen, U. Tillmann, and M. Weiss, Acta Math. **202**, 195 (2009), arXiv:math/0605249.
[56] R. Thom, Commentarii Mathematici Helvetici **28**, 17 (1954).
[57] A. Kapustin and R. Thorngren, JHEP **10**, 080 (2017), arXiv:1701.08264 [cond-mat.str-el].

[58] D. Barden, Annals of Mathematics , 365 (1965).
[59] M. A. Metlitski and R. Thorngren, Phys. Rev. **B98**, 085140 (2018), arXiv:1707.07686 [cond-mat.str-el].
[60] D. Gepner and E. Witten, Nucl. Phys. **B278**, 493 (1986).
[61] S. C. Furuya and M. Oshikawa, Phys. Rev. Lett. **118**, 021601 (2017), arXiv:1503.07292 [cond-mat.stat-mech].
[62] Y. Yao, C.-T. Hsieh, and M. Oshikawa, (2018), arXiv:1805.06885 [cond-mat.str-el].
[63] T. Sulejmanpasic and Y. Tanizaki, Phys. Rev. **B97**, 144201 (2018), arXiv:1802.02153 [hep-th].
[64] Z. Komargodski, A. Sharon, R. Thorngren, and X. Zhou, (2017), arXiv:1705.04786 [hep-th].
[65] C. Cordova, P.-S. Hsin, and N. Seiberg, SciPost Phys. **4**, 021 (2018), arXiv:1711.10008 [hep-th].
[66] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics (Springer-Verlag, New York, 1997).
[67] S. Tomonaga, Prog. Theor. Phys. **5**, 544 (1950).
[68] J. M. Luttinger, J. Math. Phys. **4**, 1154 (1963).
[69] F. D. M. Haldane, Journal of Physics C Solid State Physics **14**, 2585 (1981).
[70] R. Floreanini and R. Jackiw, Phys. Rev. Lett. **59**, 1873 (1987).
[71] B. Lian and J. Wang, Phys. Rev. **B97**, 165124 (2018), arXiv:1801.10149 [cond-mat.mes-hall].
[72] Y.-M. Lu and A. Vishwanath, Phys. Rev. **B86**, 125119 (2012), [Erratum: Phys. Rev.B89,no.19,199903(2014)], arXiv:1205.3156 [cond-mat.str-el].
[73] G. 't Hooft, Nucl. Phys. **B153**, 141 (1979).
[74] E. Looijenga, Inventiones mathematicae **38**, 17 (1976).
[75] R. Friedman, J. Morgan, and E. Witten, Commun. Math. Phys. **187**, 679 (1997), arXiv:hep-th/9701162 [hep-th].
[76] G. V. Dunne and M. Unsal, JHEP **11**, 170 (2012), arXiv:1210.2423 [hep-th].
[77] K. Ohmori, N. Seiberg, and S.-H. Shao, (2018), arXiv:1809.10604 [hep-th].
[78] E. H. Lieb, T. Schultz, and D. Mattis, Annals Phys. **16**, 407 (1961).
[79] M. B. Hastings, Phys. Rev. **B69**, 104431 (2004), arXiv:cond-mat/0305505 [cond-mat].
[80] R. Kobayashi, K. Shiozaki, Y. Kikuchi, and S. Ryu, Phys. Rev. **B99**, 014402 (2019), arXiv:1805.05367 [cond-mat.stat-mech].
[81] A. Prakash, J. Wang, and T.-C. Wei, Phys. Rev. **B98**, 125108 (2018), arXiv:1804.11236 [quant-ph].
[82] Y. Tachikawa, (2017), arXiv:1712.09542 [hep-th].
[83] J. Wang, K. Ohmori, P. Putrov, Y. Zheng, Z. Wan, M. Guo, H. Lin, P. Gao, and S.-T. Yau, Progress of Theoretical and Experimental Physics **2018**, 053A01 (2018), arXiv:1801.05416 [cond-mat.str-el].
[84] M. Guo, K. Ohmori, P. Putrov, Z. Wan, and J. Wang, (2018), arXiv:1812.11959 [hep-th].
[85] Z. Wan and J. Wang, (2018), arXiv:1812.11955 [hep-th].
[86] F. Wilczek and A. Zee, Phys. Rev. Lett. **52**, 2111 (1984).
[87] X.-G. Wen, (2018), arXiv:1812.02517 [cond-mat.str-el].
[88] C. Cordova, Z. Komargodski, and J. Wang, conversations (2017).
[89] A. Clement, Doctoral Thesis (2002).
[90] A. Kapustin and R. Thorngren, ArXiv e-prints (2013), arXiv:1309.4721 [hep-th].