Non-Abelian Gauge Theories, Sigma Models, Higher Anomalies, Symmetries, and Cobordisms I:

Classification of Higher-Symmetry Protected Topological States and Higher Anomalies via a generalized cobordism theory

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Abstract

We explore the higher-form generalized global symmetries and higher anomalies based on a generalized cobordism theory. Our cobordism calculation guides us to classify higher anomalies and topological terms for Yang-Mills (YM) gauge theories and sigma models in various dimensions. Some of YM gauge theories can be obtained from dynamically gauging the SU(N) time-reversal symmetric cobordism invariants (SU(N)-generalized topological superconductors/insulators [arXiv:1711.11587]). We elaborate the cases of YM theory with a compact Lie gauge group (such as SU(N) with N=2, 3 and others) particularly in a 4d spacetime. We provide the relevant homotopy and cobordism group calculations of higher classifying spaces, based on mathematical tools of algebraic topology, to support the physics stories.

This is a companion article with further detailed calculations supporting other shorter articles.

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1 Introduction and Summary

1.1 A Preliminary Introduction

The purpose of this article is a companion article with further detailed mathematical calculations in order to support other shorter articles [1].

The major motivation of our work is to generalize the calculations and the cobordism theory of Freed-Hopkins [2] — such that we consider the cobordism theory with higher classifying spaces, in order to study the higher-form generalized global symmetries and higher anomalies in physical theories, such as quantum field theories (QFTs) including Yang-Mills gauge theories [3] and sigma models.

Freed-Hopkins's work [2] is motivated by the development of cobordism theory classification [4,5] of so-called the Symmetry Protected Topological (SPT) state in condensed matter physics [6]. In a very short summary, Freed-Hopkins's work [2] applies the theory of Thom-Madsen-Tillmann spectra [7,8], to prove a theorem relating the "Topological Phases" (which later will be abbreviated as TP) or certain deformation classes of reflection positive invertible n-dimensional extended topological field theories (iTQFT) with symmetry group (or in short, symmetric iTQFT), to Madsen-Tillmann spectrum [8] of the symmetry group.

In this work, we will consider the generalization of [2] to include higher symmetries [9], for example, including both 0-form symmetry of group $G_{(0)}$ and 1-form symmetry of group $G_{(1)}$, or in certain cases, as higher symmetry group of higher n-group.[1] Other physics motivations to study higher group can be found in [11–14] and references therein.

We generalize the work of Freed-Hopkins [2]: there is a 1:1 correspondence

$$\left\{\begin{array}{c}\text{deformation classes of reflection positive}\\ \text{invertible } n\text{-dimensional extended topological}\\ \text{field theories with symmetry group } H_n\times\mathbb{G}\end{array}\right\}\cong[MT(H\times\mathbb{G}),\Sigma^{n+1}I\mathbb{Z}]_{\text{tors}}. \tag{1.1}$$

where H is the space time symmetry, $\mathbb{G}$ is the internal symmetry which is possibly a higher group, $MT(H\times\mathbb{G})$ is the Madsen-Tillmann spectrum [8] of the group $H\times\mathbb{G}$, Σ is the suspension, $I\mathbb{Z}$ is the Anderson dual spectrum, and tors means the torsion part.

Since there is an exact sequence

$$0\to\text{Ext}^1(\pi_n\mathcal{B},\mathbb{Z})\to[\mathcal{B},\Sigma^{n+1}I\mathbb{Z}]\to\text{Hom}(\pi_{n+1}\mathcal{B},\mathbb{Z})\to 0 \tag{1.2}$$

for any spectrum $\mathcal{B}$, especially for $MT(H\times\mathbb{G})$. The torsion part $[MT(H\times\mathbb{G}),\Sigma^{n+1}I\mathbb{Z}]_{\text{tors}}$ is $\text{Ext}^1((\pi_nMT(H\times\mathbb{G}))_{\text{tors}},\mathbb{Z})=\text{Hom}((\pi_nMT(H\times\mathbb{G}))_{\text{tors}},\text{U}(1))$.

[1] For the physics application of our result, please see [1]. Some of these 4d non-Abelian SU(N) Yang-Mills [3]-like gauge theories can be obtained from gauging the time-reversal symmetric SU(N)-SPT generalization of topological insulator/superconductor (TI/SC) [10]. We can understand their anomalies of 0-form symmetry of group $G_{(0)}$ and 1-form symmetry of group $G_{(1)}$, as the obstruction to regularize the global symmetries locally in its own dimensions (4d for YM theory). Instead, in order to regularize the global symmetries locally and onsite, the 4d gauge theories need to be placed on the boundary of 5d higher SPTs. The 5d higher SPTs corresponds to the nontrivial generators of cobordism groups of higher classifying spaces. We write $G_{(0)}$ or G_a to indicate some 0-form symmetry probed by 1-form a field. We write $G_{(1)}$ or G_b to indicate some 1-form symmetry probed by 2-form b field.

By the generalized Pontryagin-Thom isomorphism (6.27), $\pi_n MT(H \times \mathbb{G}) = \Omega_n^{H \times \mathbb{G}} = \Omega_n^H(\mathrm{B}\mathbb{G})$ which is the bordism group defined in definition 114.

Namely, we can classify the deformation classes of symmetric iTQFTs and also symmetric invertible topological orders (iTOs), via the particular group

$$\mathrm{TP}_n(H \times \mathbb{G}) \equiv [MT(H \times \mathbb{G}), \Sigma^{n+1} I\mathbb{Z}]. \tag{1.3}$$

Here TP means the abbreviation of "Topological Phases" classifying the above symmetric iTQFT, the torsion part of $\mathrm{TP}_n(H \times \mathbb{G})$ and $\Omega_n^H(\mathrm{B}\mathbb{G})$ are the same.

In this paper, we compute the (co)bordism groups $\Omega_d^H(\mathrm{B}\mathbb{G})$ $(\mathrm{TP}_d(H \times \mathbb{G}))$ for $H = \mathrm{O}/\mathrm{SO}/\mathrm{Spin}/\mathrm{Pin}^{\pm}$ and several $\mathbb{G}$, we also consider $\Omega_d^{\mathbb{G}}$ where $\mathrm{B}\mathbb{G}$ is the total space of the nontrivial fibration with base space BO and fiber $\mathrm{B}^2\mathbb{Z}_2$ in section 3.1.

For readers who wishes to explore other physics stories and introduction materials, we suggest to look at the introduction of [10] and that of the shorter articles [1]. In particular, we encourage to read the Section III of [1].

For readers who wishes to explore other mathematical introductory materials, we suggest to look at the [2] and Appendices of [10].

Readers may be also interested in other recent work along the cobordism theory applications to physics [15] [16] [17] [18].

1.2 The convention of notations

We explain the convention for our notations and terminology below. Most of our conventions follow [2] and [10].

- We denote O an orthogonal group, SO a special orthogonal group, Spin the spin group, and $\mathrm{Pin}^{\pm}$ the two ways of $\mathbb{Z}_2$ extension (related to the time reversal symmetry) of Spin group.
- $\mathbb{Z}_n$ is the finite group of order n.
- A map between topological spaces is always assumed to be continuous.
- For a (pointed) topological space X, Σ denotes a suspension $\Sigma X = S^1 \wedge X = (S^1 \times X)/(S^1 \vee X)$ where $\wedge$ and $\vee$ are smash product and wedge sum (one point union) of pointed topological spaces respectively. For a graded algebra A, ΣA is obtained from A by shifting its degree by 1.
- For a (pointed) topological space X, ΩX is the loop space of X:

$$\Omega X = \{\gamma : I \to X \text{ continuous} | \gamma(0) = \gamma(1)\}. \tag{1.4}$$

- A spectrum M is a collection of (pointed) topological spaces M_n together with structure maps $\Sigma M_n \to M_{n+1}$ such that the adjoints $M_n \to \Omega M_{n+1}$ of the structure maps are homeomorphisms.

- $\mathrm{H}^*(M,A)$ is the reduced cohomology with coefficients in A if M is a spectrum and the ordinary cohomology with coefficients in A if M is a topological space.
- We will abbreviate the cup product $x \cup y$ by xy.
- M_d (or simply M) is a d-dimensional (possibly non-orientable) manifold.
- TM_d (or simply TM) is the tangent bundle over M_d (or M).
- Rank r real (complex) vector bundle V is a bundle with fibers being real (complex) vector spaces of real (complex) dimension r.
- $w_i(V)$ is the i-th Stiefel-Whitney class of a real vector bundle V (which may be also complex rank r but considered as real rank $2r$).
- $p_i(V)$ is the i-th Pontryagin class of a real vector bundle V.
- $c_i(V)$ is the i-th Chern class of a complex vector bundle V. Pontryagin classes are closely related to Chern classes via complexification:

$$p_i(V) = (-1)^i c_{2i}(V \otimes_{\mathbb{R}} \mathbb{C}) \tag{1.5}$$

  where $V \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of the real vector bundle V. The relation between Pontryagin classes and Stiefel-Whitney classes is

$$p_i(V) = w_{2i}(V)^2 \mod 2. \tag{1.6}$$

- For a top degree cohomology class with coefficients $\mathbb{Z}_2$ we often suppress explicit integration over the manifold (i.e. pairing with the fundamental class $[M]$ with coefficients $\mathbb{Z}_2$), for example: $w_2(TM)w_3(TM) \equiv \int_M w_2(TM)w_3(TM)$ where M is a 5-manifold.
- If x is an element of a graded vector space, $|x|$ denotes the degree of x.
- For an odd prime p and a non-negatively and integrally graded vector space V over $\mathbb{Z}_p$, let V^{even} and V^{odd} be even and odd graded parts of V . The free algebra $F_{\mathbb{Z}_p}[V]$ generated by the graded vector space V is the tensor product of the polynomial algebra on V^{even} and the exterior algebra on V^{odd}:

$$F_{\mathbb{Z}_p}[V] = \mathbb{Z}_p[V^{\text{even}}] \otimes \Lambda_{\mathbb{Z}_p}(V^{\text{odd}}). \tag{1.7}$$

  We sometimes replace the vector space with a set of bases of it.
- $\mathcal{A}_p$ denotes the mod p Steenrod algebra where p is a prime.
- Sq^n is the n-th Steenrod square, it is an element of $\mathcal{A}_2$.
- $\mathcal{A}_2(1)$ denotes the subalgebra of $\mathcal{A}_2$ generated by Sq^1 and Sq^2.
- $\beta_{(n,m)} : \mathrm{H}^*(-,\mathbb{Z}_m) \to \mathrm{H}^{*+1}(-,\mathbb{Z}_n)$ is the Bockstein homomorphism associated to the extension $\mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_{nm} \to \mathbb{Z}_m$, when $n = m = p$ is a prime, it is an element of $\mathcal{A}_p$. If $p = 2$, then $\beta_{(2,2)} = \mathrm{Sq}^1$.
- $P_p^n : \mathrm{H}^*(-,\mathbb{Z}_p) \to \mathrm{H}^{*+2n(p-1)}(-,\mathbb{Z}_p)$ is the n-th Steenrod power, it is an element of $\mathcal{A}_p$ where p is an odd prime. For odd primes p, we only consider $p = 3$, so we abbreviate P_3^n by P^n.

- $\mathcal{P}_2$ is the Pontryagin square operation $\mathrm{H}^{2i}(M, \mathbb{Z}_{2^k}) \to \mathrm{H}^{4i}(M, \mathbb{Z}_{2^{k+1}})$. Explicitly, $\mathcal{P}_2$ is given by

$$\mathcal{P}_2(x) = x \cup x + x \underset{1}{\cup} \delta x \mod 2^{k+1} \tag{1.8}$$

and it satisfies

$$\mathcal{P}_2(x) = x \cup x \mod 2^k. \tag{1.9}$$

Here $\underset{1}{\cup}$ is the higher cup product.

- Postnikov square $\mathfrak{P}_3 : \mathrm{H}^2(-, \mathbb{Z}_{3^k}) \to \mathrm{H}^5(-, \mathbb{Z}_{3^{k+1}})$ is given by

$$\mathfrak{P}_3(u) = \beta_{(3^{k+1}, 3^k)}(u \cup u) \tag{1.10}$$

where $\beta_{(3^{k+1}, 3^k)}$ is the Bockstein homomorphism associated to $0 \to \mathbb{Z}_{3^{k+1}} \to \mathbb{Z}_{3^{2k+1}} \to \mathbb{Z}_{3^k} \to 0$.

- For a finitely generated abelian group G and a prime p, $G_p^\wedge = \lim_n G/p^n G$ is the p-completion of G.

- $\pi_d(M)$ has two meanings: one is the d-th (ordinary) homotopy group of the M if M refers to a topological space, the other one is the d-th stable homotopy group of M if M refers to a spectrum,

$$\pi_d(M) = \mathrm{colim}_{k \to \infty} \pi_{d+k} M_k. \tag{1.11}$$

The colimit above can be understood as a limiting group in the sequence $\pi_d M_0 \to \pi_{d+1} M_1 \to \pi_{d+2} M_2 \to \cdots$.

- For an abelian group G, the Eilenberg-MacLane space $K(G, n)$ is a space with homotopy groups satisfying

$$\pi_i K(G, n) = \begin{cases} G, & i = n. \\ 0, & i \neq n. \end{cases} \tag{1.12}$$

The Eilenberg-MacLane spectrum HG is the spectrum whose n-th space is $K(G, n)$.

- Let X, Y be topological spaces, $[X, Y]$ is the set of homotopy classes of maps from X to Y.

- Let G be a group, the classifying space of G, $\mathrm{B}G$ is a topological space such that

$$[X, \mathrm{B}G] = \{\text{isomorphism classes of principal } G\text{-bundles over } X\} \tag{1.13}$$

for any topological space X. In particular, if G is an abelian group, then $\mathrm{B}G$ is a group.

- There is a vector bundle associated to a principal G-bundle P_G: $P_G \times_G V = (P_G \times V)/G$ which is the quotient of $P_G \times V$ by the right G-action

$$(p, v)g = (pg, g^{-1}v) \tag{1.14}$$

where V is the vector space which G acts on. For characteristic classes of a principal G-bundle, we mean the characteristic classes of the associated vector bundle.

$\Omega_d^H(-)$	$B^2\mathbb{Z}_2$	$B^2\mathbb{Z}_3$	BPSU(2)	BPSU(3)	$B\mathbb{Z}_2 \times B^2\mathbb{Z}_2$	$B\mathbb{Z}_3 \times B^2\mathbb{Z}_3$	$BPSU(2)\times B^2\mathbb{Z}_2$	$BPSU(3)\times B^2\mathbb{Z}_3$
2 SO	$\mathbb{Z}_2$: x_2	$\mathbb{Z}_3$: x_2'	$\mathbb{Z}_2$: w_2'	$\mathbb{Z}_3$: z_2	$\mathbb{Z}_2$: x_2	$\mathbb{Z}_3$: x_2'	$\mathbb{Z}_2^2$: w_2', x_2	$\mathbb{Z}_3^2$: x_2', z_2
2 Spin	$\mathbb{Z}_2^2$: x_2, Arf	$\mathbb{Z}_2 \times \mathbb{Z}_3$: Arf, x_2'	$\mathbb{Z}_2^2$: w_2', Arf	$\mathbb{Z}_2 \times \mathbb{Z}_3$: Arf, z_2	$\mathbb{Z}_2^3$: x_2, Arf, $a\tilde{\eta}$[2]	$\mathbb{Z}_2 \times \mathbb{Z}_3$: Arf, x_2'	$\mathbb{Z}_2^3$: w_2', x_2, Arf	$\mathbb{Z}_2 \times \mathbb{Z}_3^2$: Arf, x_2', z_2
2 O	$\mathbb{Z}_2^2$: x_2, w_1^2	$\mathbb{Z}_2$: w_1^2	$\mathbb{Z}_2^2$: w_2', w_1^2	$\mathbb{Z}_2$: w_1^2	$\mathbb{Z}_2^3$: a^2, x_2, w_1^2	$\mathbb{Z}_2$: w_1^2	$\mathbb{Z}_2^3$: w_2', x_2, w_1^2	$\mathbb{Z}_2$: w_1^2
2 Pin^+	$\mathbb{Z}_2^2$: $x_2, w_1\tilde{\eta}$	$\mathbb{Z}_2$: $w_1\tilde{\eta}$	$\mathbb{Z}_2^2$: $w_2', w_1\tilde{\eta}$	$\mathbb{Z}_2$: $w_1\tilde{\eta}$	$\mathbb{Z}_2^3$: $w_1 a = a^2, x_2$, $w_1\tilde{\eta}$	$\mathbb{Z}_2$: $w_1\tilde{\eta}$	$\mathbb{Z}_2^3$: w_2', x_2, $w_1\tilde{\eta}$	$\mathbb{Z}_2$: $w_1\tilde{\eta}$
2 Pin^-	$\mathbb{Z}_2 \times \mathbb{Z}_8$: x_2, ABK	$\mathbb{Z}_8$: ABK	$\mathbb{Z}_2 \times \mathbb{Z}_8$: w_2', ABK	$\mathbb{Z}_8$: ABK	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$: $x_2, f_s(a)$[3], ABK	$\mathbb{Z}_8$: ABK	$\mathbb{Z}_2^2 \times \mathbb{Z}_8$: w_2', x_2, ABK	$\mathbb{Z}_8$: ABK

Table 1: 2d bordism groups.

$\Omega_d^H(-)$	$B^2\mathbb{Z}_2$	$B^2\mathbb{Z}_3$	BPSU(2)	BPSU(3)	$B\mathbb{Z}_2 \times B^2\mathbb{Z}_2$	$B\mathbb{Z}_3 \times B^2\mathbb{Z}_3$	$BPSU(2)\times B^2\mathbb{Z}_2$	$BPSU(3)\times B^2\mathbb{Z}_3$
3 SO	0	0	0	0	$\mathbb{Z}_2^2$: ax_2, a^3	$\mathbb{Z}_3^2$: $a'b', a'x_2'$	0	0
3 Spin	0	0	0	0	$\mathbb{Z}_2 \times \mathbb{Z}_8$: ax_2, aABK	$\mathbb{Z}_3^2$: $a'b', a'x_2'$	0	0
3 O	$\mathbb{Z}_2$: $x_3 = w_1x_2$	0	$\mathbb{Z}_2$: $w_3' = w_1w_2'$	0	$\mathbb{Z}_2^4$: $x_3 = w_1x_2, ax_2$, aw_1^2, a^3	0	$\mathbb{Z}_2^2$: $x_3 = w_1x_2, w_3' = w_1w_2'$	0
3 Pin^+	$\mathbb{Z}_2^2$: $w_1x_2 = x_3$, w_1Arf	$\mathbb{Z}_2$: w_1Arf	$\mathbb{Z}_2^2$: $w_1w_2' = w_3'$, w_1Arf	$\mathbb{Z}_2$: w_1Arf	$\mathbb{Z}_2^5$: $a^3, w_1x_2 = x_3$, $ax_2, w_1a\tilde{\eta}$, w_1Arf	$\mathbb{Z}_2$: w_1Arf	$\mathbb{Z}_2^3$: $w_1w_2' = w_3', w_1x_2 = x_3$, w_1Arf	$\mathbb{Z}_2$: w_1Arf
3 Pin^-	$\mathbb{Z}_2$: $w_1x_2 = x_3$	0	$\mathbb{Z}_2$: $w_1w_2' = w_3'$	0	$\mathbb{Z}_2^4$: a^3, w_1^2a, $x_3 = w_1x_2$, ax_2	0	$\mathbb{Z}_2^2$: $w_1w_2' = w_3'$, $w_1x_2 = x_3$	0

Table 2: 3d bordism groups.

$\Omega_d^H(-)$	$\mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}^2\mathbb{Z}_3$	$\mathrm{BPSU}(2)$	$\mathrm{BPSU}(3)$	$\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3$	$\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{BPSU}(3)\times \mathrm{B}^2\mathbb{Z}_3$
4 SO	$\mathbb{Z}\times\mathbb{Z}_4$: $\sigma, \mathcal{P}_2(x_2)$	$\mathbb{Z}\times\mathbb{Z}_3$: $\sigma, x_2'^2$	$\mathbb{Z}^2$: σ, p_1'	$\mathbb{Z}^2$: σ, c_2	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_4$: σ, $ax_3 = a^2x_2$, $\mathcal{P}_2(x_2)$	$\mathbb{Z}\times\mathbb{Z}_3^2$: σ, $a'x_3' = b'x_2'$, $x_2'^2$	$\mathbb{Z}^2\times\mathbb{Z}_2\times\mathbb{Z}_4$: σ, p_1', $w_2'x_2, \mathcal{P}_2(x_2)$	$\mathbb{Z}^2\times\mathbb{Z}_3^2$: σ, c_2, $x_2'^2, x_2'z_2$
4 Spin	$\mathbb{Z}\times\mathbb{Z}_2$: $\frac{\sigma}{16}$, $\frac{\mathcal{P}_2(x_2)}{2}$	$\mathbb{Z}\times\mathbb{Z}_3$: $\frac{\sigma}{16}, x_2'^2$	$\mathbb{Z}^2$: $\frac{\sigma}{16}, \frac{p_1'}{2}$	$\mathbb{Z}^2$: $\frac{\sigma}{16}, c_2$	$\mathbb{Z}\times\mathbb{Z}_2^2$: $\frac{\sigma}{16}$, $ax_3 = a^2x_2$, $\frac{\mathcal{P}_2(x_2)}{2}$	$\mathbb{Z}\times\mathbb{Z}_3^2$: $\frac{\sigma}{16}$, $a'x_3' = b'x_2'$, $x_2'^2$	$\mathbb{Z}^2\times\mathbb{Z}_2^2$: $\frac{\sigma}{16}, \frac{p_1'}{2}$, $w_2'x_2$, $\frac{\mathcal{P}_2(x_2)}{2}$	$\mathbb{Z}^2\times\mathbb{Z}_3^2$: $\frac{\sigma}{16}, c_2$, $x_2'^2, x_2'z_2$
4 O	$\mathbb{Z}_2^4$: x_2^2, w_1^4, $w_1^2x_2, w_2^2$	$\mathbb{Z}_2^2$: w_1^4, w_2^2	$\mathbb{Z}_2^4$: $w_2'^2, w_1^4$, $w_1^2w_2', w_2^2$	$\mathbb{Z}_2^3$: w_1^4, w_2^2, $c_2 (\text{mod } 2)$	$\mathbb{Z}_2^8$: w_1^4, w_2^2, a^4, a^2x_2, ax_3, x_2^2, $w_1^2a^2, w_1^2x_2$	$\mathbb{Z}_2^2$: w_1^4, w_2^2	$\mathbb{Z}_2^7$: w_1^4, w_2^2, $x_2^2, w_2'^2$, $x_2w_1^2, w_2'w_1^2$, $w_2'x_2$	$\mathbb{Z}_2^3$: w_1^4, w_2^2, $c_2 (\text{mod } 2)$
4 Pin^+	$\mathbb{Z}_4 \times \mathbb{Z}_{16}$: $q_s(x_2)$[4], η[5]	$\mathbb{Z}_{16}$: η	$\mathbb{Z}_4 \times \mathbb{Z}_{16}$: $q_s(w_2')$[6], η	$\mathbb{Z}_2\times\mathbb{Z}_{16}$: $c_2 (\text{mod } 2)$, η	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_8\times\mathbb{Z}_{16}$: $ax_3, w_1ax_2 = a^2x_2 + ax_3$, $q_s(x_2)$, $w_1a\mathrm{ABK}$, η	$\mathbb{Z}_{16}$: η	$\mathbb{Z}_4^2\times\mathbb{Z}_{16}\times\mathbb{Z}_2$: $q_s(w_2')$, $q_s(x_2)$, $\eta, w_2'x_2$	$\mathbb{Z}_2\times\mathbb{Z}_{16}$: $c_2 (\text{mod } 2)$, η
4 Pin^-	$\mathbb{Z}_2$: $w_1^2x_2$	0	$\mathbb{Z}_2$: $w_1^2w_2'$	$\mathbb{Z}_2$: $c_2 (\text{mod } 2)$	$\mathbb{Z}_2^3$: $w_1^2x_2, ax_3$, $w_1ax_2 = a^2x_2 + ax_3$	0	$\mathbb{Z}_2^3$: $w_1^2w_2'$, $w_1^2x_2, w_2'x_2$	$\mathbb{Z}_2$: $c_2 (\text{mod } 2)$

Table 3: $4d$ bordism groups.

1.3 Tables and Summary of Some Co/Bordism Groups

[2] $\tilde{\eta}$ is the "mod 2 index" of the 1d Dirac operator (#zero eigenvalues mod 2, no contribution from spectral asymmetry).

[3] $f_s : \mathrm{H}^1(M,\mathbb{Z}_2) \to \mathbb{Z}_4$ is a $\mathbb{Z}_4$ valued quadratic refinement (dependent on the choice of Pin^- structure $s \in \mathrm{Pin}^-(M)$) of the intersection form

$$\langle , \rangle : \mathrm{H}^1(M,\mathbb{Z}_2) \times \mathrm{H}^1(M,\mathbb{Z}_2) \to \mathbb{Z}_2$$

i.e. so that $f_s(x+y) - f_s(x) - f_s(y) = 2\langle x, y\rangle \in \mathbb{Z}_4$ (in particular $f_s(x) = \langle x, x\rangle \mod 2$)

The space of Pin^- structures is acted upon freely and transitively by $\mathrm{H}^1(M,\mathbb{Z}_2)$, and the dependence of f_s on the Pin^- structure should satisfy

$$f_{s+h}(x) - f_s(x) = 2hx, \text{ for any } h \in \mathrm{H}^1(M,\mathbb{Z}_2)$$

(note that any two quadratic functions differ by a linear function)

[4] $q_s : \mathrm{H}^2(M,\mathbb{Z}_2) \to \mathbb{Z}_4$ is a $\mathbb{Z}_4$ valued quadratic refinement (dependent on the choice of Pin^+ structure $s \in \mathrm{Pin}^+(M)$) of the intersection form

$$\langle , \rangle : \mathrm{H}^2(M,\mathbb{Z}_2) \times \mathrm{H}^2(M,\mathbb{Z}_2) \to \mathbb{Z}_2$$

i.e. so that $q_s(x+y) - q_s(x) - q_s(y) = 2\langle x, y\rangle \in \mathbb{Z}_4$ (in particular $q_s(x) = \langle x, x\rangle \mod 2$)

The space of Pin^+ structures is acted upon freely and transitively by $\mathrm{H}^1(M,\mathbb{Z}_2)$, and the dependence of q_s on the Pin^+ structure should satisfy

$$q_{s+h}(x) - q_s(x) = 2w_1(TM)hx, \text{ for any } h \in \mathrm{H}^1(M,\mathbb{Z}_2)$$

(note that any two quadratic functions differ by a linear function)

$\Omega_d^H(-)$	$B^2\mathbb{Z}_2$	$B^2\mathbb{Z}_3$	BPSU(2)	BPSU(3)	$B\mathbb{Z}_2 \times B^2\mathbb{Z}_2$	$B\mathbb{Z}_3 \times B^2\mathbb{Z}_3$	$\text{BPSU}(2)\times B^2\mathbb{Z}_2$	$\text{BPSU}(3)\times B^2\mathbb{Z}_3$
5 SO	$\mathbb{Z}_2^2$: $x_5 = x_2x_3$, w_2w_3	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^2$: w_2w_3, $w_2'w_3'$	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^6$: ax_2^2, a^5, $x_5 = x_2x_3, a^3x_2$, w_2w_3, aw_2^2	$\mathbb{Z}_2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_9$: w_2w_3, $a'b'x_2'$, $a'x_2'^2$, $\mathfrak{P}_3(b')$	$\mathbb{Z}_2^4$: $w_2'w_3', x_5 = x_2x_3$, $w_3'x_2 = w_2'x_3$, w_2w_3	$\mathbb{Z}_2 \times \mathbb{Z}_3$: w_2w_3, $z_2x_3' = -z_3x_2'$
5 Spin	0	0	0	0	$\mathbb{Z}_2$: a^3x_2	$\mathbb{Z}_3^2 \times \mathbb{Z}_9$: $a'b'x_2'$, $a'x_2'^2$, $\mathfrak{P}_3(b')$	$\mathbb{Z}_2$: $w_3'x_2 = w_2'x_3$	$\mathbb{Z}_3$: $z_2x_3' = -z_3x_2'$
5 O	$\mathbb{Z}_2^4$: x_2x_3, x_5, $w_1^2x_3$, w_2w_3	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^3$: w_2w_3, $w_1^2w_3'$, $w_2'w_3'$	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^{12}$: a^5, a^2x_3, $a^3x_2, a^3w_1^2$, ax_2^2, aw_1^4, $ax_2w_1^2, aw_2^2$, $x_2x_3, w_1^2x_3$, x_5, w_2w_3	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^8$: $w_2'w_3', x_2w_3'$, $w_1^2w_3', w_2'x_3$, $x_2x_3, w_1^2x_3$, x_5, w_2w_3	$\mathbb{Z}_2$: w_2w_3
5 Pin^+	$\mathbb{Z}_2^2$: x_2x_3, $w_1^2x_3 = x_5$	0	$\mathbb{Z}_2$: $w_1^2w_3' = w_2'w_3'$	0	$\mathbb{Z}_2^7$: w_1^4a, $a^5 = w_1^2a^3$, $w_1^2x_3 = x_5$, x_2x_3, $w_1^2ax_2 = ax_2^2 + a^2x_3$, $w_1ax_3 = a^2x_3$, a^3x_2	0	$\mathbb{Z}_2^5$: $w_1^2w_3' = w_2'w_3'$, $w_1^2x_3 = x_5$, $x_2x_3, w_3'x_2$, $w_1w_2'x_2 = w_2'x_3 + w_3'x_2$	0
5 Pin^-	$\mathbb{Z}_2$: x_2x_3	0	0	0	$\mathbb{Z}_2^5$: $w_1^2a^3, x_2x_3$, $w_1^2ax_2$, $w_1ax_3 = a^2x_3$, a^3x_2	0	$\mathbb{Z}_2^3$: $x_2x_3, w_3'x_2$, $w_1w_2'x_2 = w_2'x_3 + w_3'x_2$	0

Table 4: $5d$ bordism groups.

If $w_1(TM) = 0$, then $q_s(x)$ is independent on the Pin^+ structure $s \in \text{Pin}^+(M)$, it reduces to $\mathcal{P}_2(x)$ where $\mathcal{P}_2(x)$ is the Pontryagin square of x.

[5]Here η is the usual Atiyah-Patodi-Singer eta-invariant of the 4d Dirac operator (="#zero eigenvalues + spectral asymmetry").

[6] one can also define this $\mathbb{Z}_4$ invariant as

$$(\eta_{\text{SO}(3)} - 3\eta)/4 \in \mathbb{Z}_4 \quad (*)$$

where $\eta \in \mathbb{Z}_{16}$ is the (properly normalized) eta-invariant of the ordinary Dirac operator, and $\eta_{\text{SO}(3)} \in \mathbb{Z}_{16}$ is the eta invariant of the twisted Dirac operator acting on the $S \otimes V_3$ where S is the spinor bundle and V_3 is the bundle associated to 3-dim representation of SO(3). Note that $(*)$ is well defined because $\eta_{\text{SO}(3)} = 3\eta \mod 4$.

Note that on non-orientable manifold, if $w_2(V_3) = 0$, then since $w_1(V_3) = 0$, we also have $w_3(V_3) = 0$, hence V_3 is stably trivial, $\eta_{\text{SO}(3)} = 3\eta$.

Also note that on oriented manifold one can use Atiyah-Patodi-Singer index theorem to show that (here the

$\mathrm{TP}_d(H\times -)$	$\mathrm{B}\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_3$	$\mathrm{PSU}(2)$	$\mathrm{PSU}(3)$	$\mathbb{Z}_2\times\mathrm{B}\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathrm{B}\mathbb{Z}_3$	$\mathrm{PSU}(2)\times\mathrm{B}\mathbb{Z}_2$	$\mathrm{PSU}(3)\times\mathrm{B}\mathbb{Z}_3$
2 SO	$\mathbb{Z}_2: x_2$	$\mathbb{Z}_3: x_2'$	$\mathbb{Z}_2: w_2'$	$\mathbb{Z}_3: z_2$	$\mathbb{Z}_2: x_2$	$\mathbb{Z}_3: x_2'$	$\mathbb{Z}_2^2: w_2', x_2$	$\mathbb{Z}_3^2: x_2', z_2$
2 Spin	$\mathbb{Z}_2^2: x_2, \mathrm{Arf}$	$\mathbb{Z}_2\times\mathbb{Z}_3: \mathrm{Arf}, x_2'$	$\mathbb{Z}_2^2: w_2', \mathrm{Arf}$	$\mathbb{Z}_2\times\mathbb{Z}_3: \mathrm{Arf}, z_2$	$\mathbb{Z}_2^3: x_2, \mathrm{Arf}, a\tilde{\eta}$	$\mathbb{Z}_2\times\mathbb{Z}_3: \mathrm{Arf}, x_2'$	$\mathbb{Z}_2^3: w_2', x_2, \mathrm{Arf}$	$\mathbb{Z}_2\times\mathbb{Z}_3^2: \mathrm{Arf}, x_2', z_2$
2 O	$\mathbb{Z}_2^2: x_2, w_1^2$	$\mathbb{Z}_2: w_1^2$	$\mathbb{Z}_2^2: w_2', w_1^2$	$\mathbb{Z}_2: w_1^2$	$\mathbb{Z}_2^3: a^2, x_2, w_1^2$	$\mathbb{Z}_2: w_1^2$	$\mathbb{Z}_2^3: w_2', x_2, w_1^2$	$\mathbb{Z}_2: w_1^2$
2 Pin^+	$\mathbb{Z}_2^2: x_2, w_1\tilde{\eta}$	$\mathbb{Z}_2: w_1\tilde{\eta}$	$\mathbb{Z}_2^2: w_2', w_1\tilde{\eta}$	$\mathbb{Z}_2: w_1\tilde{\eta}$	$\mathbb{Z}_2^3: w_1a = a^2, x_2, w_1\tilde{\eta}$	$\mathbb{Z}_2: w_1\tilde{\eta}$	$\mathbb{Z}_2^3: w_2', x_2, w_1\tilde{\eta}$	$\mathbb{Z}_2: w_1\tilde{\eta}$
2 Pin^-	$\mathbb{Z}_2\times\mathbb{Z}_8: x_2, \mathrm{ABK}$	$\mathbb{Z}_8: \mathrm{ABK}$	$\mathbb{Z}_2\times\mathbb{Z}_8: w_2', \mathrm{ABK}$	$\mathbb{Z}_8: \mathrm{ABK}$	$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_8: x_2, f_s(a), \mathrm{ABK}$	$\mathbb{Z}_8: \mathrm{ABK}$	$\mathbb{Z}_2^2\times\mathbb{Z}_8: w_2', x_2, \mathrm{ABK}$	$\mathbb{Z}_8: \mathrm{ABK}$

Table 5: TP_2.

$\mathrm{TP}_d(H\times -)$	$\mathrm{B}\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_3$	$\mathrm{PSU}(2)$	$\mathrm{PSU}(3)$	$\mathbb{Z}_2\times\mathrm{B}\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathrm{B}\mathbb{Z}_3$	$\mathrm{PSU}(2)\times\mathrm{B}\mathbb{Z}_2$	$\mathrm{PSU}(3)\times\mathrm{B}\mathbb{Z}_3$
3 SO	$\mathbb{Z}: \frac{1}{3}\mathrm{CS}_3^{(TM)}$ [7]	$\mathbb{Z}: \frac{1}{3}\mathrm{CS}_3^{(TM)}$	$\mathbb{Z}^2: \frac{1}{3}\mathrm{CS}_3^{(TM)}, \mathrm{CS}_3^{(\mathrm{SO}(3))}$ [8]	$\mathbb{Z}^2: \frac{1}{3}\mathrm{CS}_3^{(TM)}, \mathrm{CS}_3^{(\mathrm{PSU}(3))}$ [9]	$\mathbb{Z}\times\mathbb{Z}_2^2: \frac{1}{3}\mathrm{CS}_3^{(TM)}, ax_2, a^3$	$\mathbb{Z}\times\mathbb{Z}_3^2: \frac{1}{3}\mathrm{CS}_3^{(TM)}, a'b', a'x_2'$	$\mathbb{Z}^2: \frac{1}{3}\mathrm{CS}_3^{(TM)}, \mathrm{CS}_3^{(\mathrm{SO}(3))}$	$\mathbb{Z}^2: \frac{1}{3}\mathrm{CS}_3^{(TM)}, \mathrm{CS}_3^{(\mathrm{PSU}(3))}$
3 Spin	$\mathbb{Z}: \frac{1}{48}\mathrm{CS}_3^{(TM)}$	$\mathbb{Z}: \frac{1}{48}\mathrm{CS}_3^{(TM)}$	$\mathbb{Z}^2: \frac{1}{48}\mathrm{CS}_3^{(TM)}, \frac{1}{2}\mathrm{CS}_3^{(\mathrm{SO}(3))}$	$\mathbb{Z}^2: \frac{1}{48}\mathrm{CS}_3^{(TM)}, \mathrm{CS}_3^{(\mathrm{PSU}(3))}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_8: \frac{1}{48}\mathrm{CS}_3^{(TM)}, ax_2, a\mathrm{ABK}$	$\mathbb{Z}\times\mathbb{Z}_3^2: \frac{1}{48}\mathrm{CS}_3^{(TM)}, a'b', a'x_2'$	$\mathbb{Z}^2: \frac{1}{48}\mathrm{CS}_3^{(TM)}, \frac{1}{2}\mathrm{CS}_3^{(\mathrm{SO}(3))}$	$\mathbb{Z}^2: \frac{1}{48}\mathrm{CS}_3^{(TM)}, \mathrm{CS}_3^{(\mathrm{PSU}(3))}$
3 O	$\mathbb{Z}_2: x_3 = w_1x_2$	0	$\mathbb{Z}_2: w_3' = w_1w_2'$	0	$\mathbb{Z}_2^4: x_3 = w_1x_2, ax_2, aw_1^2, a^3$	0	$\mathbb{Z}_2^2: x_3 = w_1x_2, w_3' = w_1w_2'$	0
3 Pin^+	$\mathbb{Z}_2^2: w_1x_2 = x_3, w_1\mathrm{Arf}$	$\mathbb{Z}_2: w_1\mathrm{Arf}$	$\mathbb{Z}_2^2: w_1w_2' = w_3', w_1\mathrm{Arf}$	$\mathbb{Z}_2: w_1\mathrm{Arf}$	$\mathbb{Z}_2^5: a^3, w_1x_2 = x_3, ax_2, w_1a\tilde{\eta}, w_1\mathrm{Arf}$	$\mathbb{Z}_2: w_1\mathrm{Arf}$	$\mathbb{Z}_2^3: w_1w_2' = w_3', w_1x_2 = x_3, w_1\mathrm{Arf}$	$\mathbb{Z}_2: w_1\mathrm{Arf}$
3 Pin^-	$\mathbb{Z}_2: w_1x_2 = x_3$	0	$\mathbb{Z}_2: w_1w_2' = w_3'$	0	$\mathbb{Z}_2^4: a^3, w_1^2a, x_3 = w_1x_2, ax_2$	0	$\mathbb{Z}_2^2: w_1w_2' = w_3', w_1x_2 = x_3$	0

Table 6: TP_3.

normalization of eta-invariants is such that η is an integer mod 16 on a general non-oriented 4-manifold)

$$\eta = -\frac{\sigma(M)}{2},$$

$$\eta_{\mathrm{SO}(3)} = -\frac{3\sigma(M)}{2} + 4p_1(\mathrm{SO}(3)).$$

$\mathrm{TP}_d(H\times -)$	$\mathrm{B}\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_3$	PSU(2)	PSU(3)	$\mathbb{Z}_2\times \mathrm{B}\mathbb{Z}_2$	$\mathbb{Z}_3\times \mathrm{B}\mathbb{Z}_3$	$\mathrm{PSU}(2)\times \mathrm{B}\mathbb{Z}_2$	$\mathrm{PSU}(3)\times \mathrm{B}\mathbb{Z}_3$
4 SO	$\mathbb{Z}_4$: $\mathcal{P}_2(x_2)$	$\mathbb{Z}_3$: $x_2'^2$	0	0	$\mathbb{Z}_2\times\mathbb{Z}_4$: $ax_3=a^2x_2$, $\mathcal{P}_2(x_2)$	$\mathbb{Z}_3^2$: $a'x_3'=b'x_2'$, $x_2'^2$	$\mathbb{Z}_2\times\mathbb{Z}_4$: $w_2'x_2, \mathcal{P}_2(x_2)$	$\mathbb{Z}_3^2$: $x_2'^2, x_2'z_2$
4 Spin	$\mathbb{Z}_2$: $\frac{\mathcal{P}_2(x_2)}{2}$	$\mathbb{Z}_3$: $x_2'^2$	0	0	$\mathbb{Z}_2^2$: $ax_3=a^2x_2$, $\frac{\mathcal{P}_2(x_2)}{2}$	$\mathbb{Z}_3^2$: $a'x_3'=b'x_2'$, $x_2'^2$	$\mathbb{Z}_2^2$: $w_2'x_2$, $\frac{\mathcal{P}_2(x_2)}{2}$	$\mathbb{Z}_3^2$: $x_2'^2, x_2'z_2$
4 O	$\mathbb{Z}_2^4$: $x_2^2, w_1^4, w_1^2x_2, w_2^2$	$\mathbb{Z}_2^2$: w_1^4, w_2^2	$\mathbb{Z}_2^4$: $w_2'^2, w_1^4, w_1^2w_2', w_2^2$	$\mathbb{Z}_2^3$: w_1^4, w_2^2, $c_2 (\bmod 2)$	$\mathbb{Z}_2^8$: $w_1^4, w_2^2, a^4, a^2x_2, ax_3, x_2^2, w_1^2a^2, w_1^2x_2$	$\mathbb{Z}_2^2$: w_1^4, w_2^2	$\mathbb{Z}_2^7$: $w_1^4, w_2^2, x_2^2, w_2'^2, x_2w_1^2, w_2'w_1^2, w_2'x_2$	$\mathbb{Z}_2^3$: w_1^4, w_2^2, $c_2 (\bmod 2)$
4 Pin^+	$\mathbb{Z}_4\times\mathbb{Z}_{16}$: $q_s(x_2)$, η	$\mathbb{Z}_{16}$: η	$\mathbb{Z}_4\times\mathbb{Z}_{16}$: $q_s(w_2')$, η	$\mathbb{Z}_2\times\mathbb{Z}_{16}$: $c_2 (\bmod 2)$, η	$\mathbb{Z}_2^2\times\mathbb{Z}_4\times\mathbb{Z}_8\times\mathbb{Z}_{16}$: ax_3, $w_1ax_2=a^2x_2+ax_3$, $q_s(x_2)$, $w_1a\mathrm{ABK}$, η	$\mathbb{Z}_{16}$: η	$\mathbb{Z}_4^2\times\mathbb{Z}_{16}\times\mathbb{Z}_2$: $q_s(w_2')$, $q_s(x_2)$, $\eta, w_2'x_2$	$\mathbb{Z}_2\times\mathbb{Z}_{16}$: $c_2 (\bmod 2)$, η
4 Pin^-	$\mathbb{Z}_2$: $w_1^2x_2$	0	$\mathbb{Z}_2$: $w_1^2w_2'$	$\mathbb{Z}_2$: $c_2 (\bmod 2)$	$\mathbb{Z}_2^3$: $w_1^2x_2, ax_3$, $w_1ax_2=a^2x_2+ax_3$	0	$\mathbb{Z}_2^3$: $w_1^2w_2'$, $w_1^2x_2, w_2'x_2$	$\mathbb{Z}_2$: $c_2 (\bmod 2)$

Table 7: TP_4.

In Section 3.1, we compute the topological terms (involving the cohomology classes of $\mathrm{B}^2\mathbb{Z}_2$) of $\Omega_5^{\mathbb{G}}$ where $\mathbb{G}$ is a 2-group with $G_a=\mathrm{O}$, $G_b=\mathbb{Z}_2$ We find that the term x_2w_3 (or x_3w_2) survives only for $\beta=0, w_1^3$ (the Postnikov class $\beta\in\mathrm{H}^3(\mathrm{BO},\mathbb{Z}_2)=\mathbb{Z}_2^3$ which is generated by w_1^3, w_1w_2, w_3). This term also appears in eq. 2.57 of [19].

2 Difference between a previous cobordism theory and this work

Difference between a previous cobordism theory [10] and this work. In all Adams charts of the computation in [10], there are no nonzero differentials, while in this paper we encounter nonzero differentials d_n due to the (p,p^n)-Bocksteins in the computation involving $\mathrm{B}^2\mathbb{Z}_{p^n}$ and $\mathrm{B}\mathbb{Z}_{p^n}$.

So

$$(\eta_{\mathrm{SO}(3)}-3\eta)/4 = p_1(\mathrm{SO}(3)) \mod 4 = \mathcal{P}_2(w_2(\mathrm{SO}(3))).$$

$q_s(w_2(\mathrm{SO}(3)))$ also reduces to $\mathcal{P}_2(w_2(\mathrm{SO}(3)))$ in the oriented case.

[7] $\mathrm{CS}_3(TM)\equiv \mathrm{CS}_3^{(TM)}$ is the Chern-Simons 3-form of the tangent bundle.

[8] $\mathrm{CS}_3(\mathrm{SO}(3))\equiv \mathrm{CS}_3^{(\mathrm{SO}(3))}$ is the Chern-Simons 3-form of the SO(3) gauge bundle.

[9] $\mathrm{CS}_3(\mathrm{PSU}(3))\equiv \mathrm{CS}_3^{(\mathrm{PSU}(3))}$ is the Chern-Simons 3-form of the PSU(3) gauge bundle.

[10] $\mathrm{CS}_5(\mathrm{PSU}(3))\equiv \mathrm{CS}_5^{(\mathrm{PSU}(3))}$ is the Chern-Simons 5-form of the PSU(3) gauge bundle.

$\mathrm{TP}_d(H\times -)$	$\mathrm{B}\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_3$	$\mathrm{PSU}(2)$	$\mathrm{PSU}(3)$	$\mathbb{Z}_2\times\mathrm{B}\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathrm{B}\mathbb{Z}_3$	$\mathrm{PSU}(2)\times\mathrm{B}\mathbb{Z}_2$	$\mathrm{PSU}(3)\times\mathrm{B}\mathbb{Z}_3$
5 SO	$\mathbb{Z}_2^2$: $x_5=x_2x_3$, w_2w_3	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^2$: w_2w_3, $w_2'w_3'$	$\mathbb{Z}\times\mathbb{Z}_2$: $\mathrm{CS}_5^{(\mathrm{PSU}(3))}$[10], w_2w_3	$\mathbb{Z}_2^6$: ax_2^2, a^5, $x_5=x_2x_3, a^3x_2$, w_2w_3, aw_2^2	$\mathbb{Z}_2\times\mathbb{Z}_3^2\times\mathbb{Z}_9$: w_2w_3, $a'b'x_2'$, $a'x_2'^2$, $\mathfrak{P}_3(b')$	$\mathbb{Z}_2^4$: $w_2'w_3', x_5=x_2x_3$, $w_3'x_2=w_2'x_3$, w_2w_3	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_3$: $\mathrm{CS}_5^{(\mathrm{PSU}(3))}$), w_2w_3, $z_2x_3'=-z_3x_2'$
5 Spin	0	0	0	$\mathbb{Z}$: $\frac{1}{2}\mathrm{CS}_5^{(\mathrm{PSU}(3))}$	$\mathbb{Z}_2$: a^3x_2	$\mathbb{Z}_3^2\times\mathbb{Z}_9$: $a'b'x_2'$, $a'x_2'^2$, $\mathfrak{P}_3(b')$	$\mathbb{Z}_2$: $w_3'x_2=w_2'x_3$	$\mathbb{Z}\times\mathbb{Z}_3$: $\frac{1}{2}\mathrm{CS}_5^{(\mathrm{PSU}(3))}$, $z_2x_3'=-z_3x_2'$
5 O	$\mathbb{Z}_2^4$: x_2x_3, $x_5=(w_2+w_1^2)x_3=(w_3+w_1^3)x_2$, $w_1^2x_3=w_1^3x_2$, w_2w_3	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^3$: w_2w_3, $w_1^2w_3'=w_1^3w_2'$, $w_2'w_3'$.	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^{12}$: a^5, a^2x_3, $a^3x_2, a^3w_1^2$, ax_2^2, aw_1^4, $ax_2w_1^2, aw_2^2$, $x_2x_3, w_1^2x_3$, x_5, w_2w_3	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^8$: $w_2'w_3', x_2w_3'$, $w_1^2w_3', w_2'x_3$, $x_2x_3, w_1^2x_3$, x_5, w_2w_3	$\mathbb{Z}_2$: w_2w_3
5 Pin^+	$\mathbb{Z}_2^2$: x_2x_3, $x_5=w_1^2x_3=w_1^3x_2$	0	$\mathbb{Z}_2$: $w_1^2w_3'=w_2'w_3'$	0	$\mathbb{Z}_2^7$: w_1^4a, $a^5=w_1^2a^3$, $w_1^2x_3=x_5$, x_2x_3, $w_1^2ax_2=ax_2^2+a^2x_3$, $w_1ax_3=a^2x_3$, a^3x_2	0	$\mathbb{Z}_2^5$: $w_1^2w_3'=w_2'w_3'$, $w_1^2x_3=x_5$, $x_2x_3, w_3'x_2$, $w_1w_2'x_2=w_2'x_3+w_3'x_2$	0
5 Pin^-	$\mathbb{Z}_2$: x_2x_3	0	0	0	$\mathbb{Z}_2^5$: $w_1^2a^3, x_2x_3$, $w_1^2ax_2$, $w_1ax_3=a^2x_3$, a^3x_2	0	$\mathbb{Z}_2^3$: $x_2x_3, w_3'x_2$, $w_1w_2'x_2=w_2'x_3+w_3'x_2$	0

Table 8: TP_5.

3 Higher Group Cobordisms and Non-trivial Fibrations

If G_a is a group, G_b is an abelian group, then $\mathrm{B}G_b$ is a group. Consider the group extension

$$0\to\mathrm{B}G_b\to\mathbb{G}\to G_a\to 0, \tag{3.1}$$

we have a fibration

$$\begin{array}{ccc} \mathrm{B}^2 G_b & \longrightarrow & \mathrm{B}\mathbb{G} \\ & & \big\downarrow \\ & & \mathrm{B}G_a \end{array} \tag{3.2}$$

which is classified by the Postnikov class $\beta \in \mathrm{H}^3(\mathrm{B}G_a, G_b)$.

3.1 $(\mathrm{B}G_a, \mathrm{B}^2 G_b) : (\mathrm{B}\mathbf{O}, \mathrm{B}^2\mathbb{Z}_2)$

If $G_a = \mathrm{O}$ and $G_b = \mathbb{Z}_2$. Then $\mathrm{H}^n(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z})$ is computed in Appendix C of [20].

$$\mathrm{H}^n(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n = 1 \\ 0 & n = 2 \\ \mathbb{Z}_2 & n = 3 \\ 0 & n = 4 \\ \mathbb{Z}_4 & n = 5 \\ \mathbb{Z}_2 & n = 6 \end{cases} \tag{3.3}$$

For the fibration

$$\begin{array}{ccc} \mathrm{B}^2 \mathbb{Z}_2 & \longrightarrow & \mathrm{B}\mathbb{G} \\ & & \big\downarrow \\ & & \mathrm{BO}, \end{array} \tag{3.4}$$

the E_2 page of the Serre spectral sequence is $\mathrm{H}^p(\mathrm{BO}, \mathrm{H}^q(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}))$. The shape of the relevant piece is shown in Figure 1.

Note that p labels the columns and q labels the rows.

The bottom row is $\mathrm{H}^p(\mathrm{BO}, \mathbb{Z})$.

The universal coefficient theorem (6.15) tells us that $\mathrm{H}^3(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}) = \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}), \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}(\pi_2(\mathrm{B}^2\mathbb{Z}_2), \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}_2, \mathbb{R}/\mathbb{Z}) = \hat{\mathbb{Z}}_2$, so the $q = 3$ row is $\mathrm{H}^p(\mathrm{BO}, \hat{\mathbb{Z}}_2)$.

It is also known that $\mathrm{H}^5(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}) = \mathrm{H}^4(\mathrm{B}^2\mathbb{Z}_2, \mathbb{R}/\mathbb{Z})$ is the group of quadratic functions $q : \mathbb{Z}_2 \to \mathbb{R}/\mathbb{Z}$ [21]. The isomorphism is discussed in detail in [22].

The first possibly non-zero differential is on the E_3 page:

$$\mathrm{H}^0(\mathrm{BO}, \mathrm{H}^5(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z})) \to \mathrm{H}^3(\mathrm{BO}, \hat{\mathbb{Z}}_2). \tag{3.5}$$

Following the appendix of [23], this map sends a quadratic form $q : \mathbb{Z}_2 \to \mathbb{R}/\mathbb{Z}$ to $\langle \beta, - \rangle_q$, where the bracket denotes the bilinear pairing $\langle x, y \rangle_q = q(x + y) - q(x) - q(y)$.

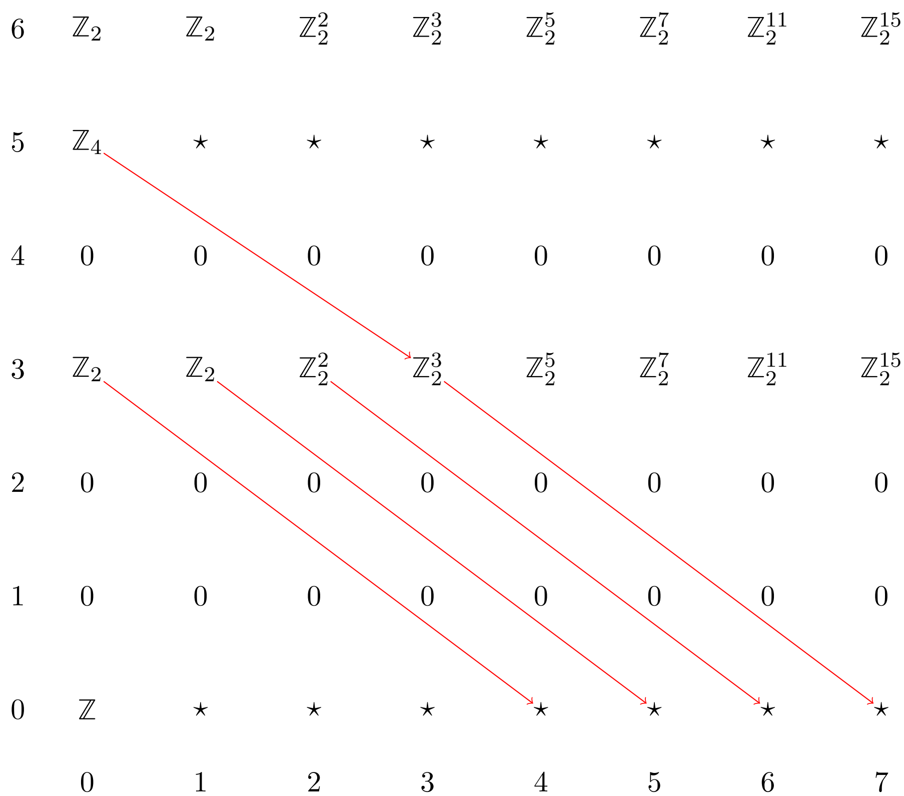


Figure 1: Serre spectral sequence for $(\mathrm{BO}, \mathrm{B}^2\mathbb{Z}_2)$

The next possibly non-zero differentials are on the E_4 page:

$$\mathrm{H}^j(\mathrm{BO}, \hat{\mathbb{Z}}_2) \to \mathrm{H}^{j+3}(\mathrm{BO}, \mathbb{R}/\mathbb{Z}) \to \mathrm{H}^{j+4}(\mathrm{BO}, \mathbb{Z}). \tag{3.6}$$

The first map is contraction with β. The second map comes from the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^n(\mathrm{BO}, \mathbb{R}) \longrightarrow \mathrm{H}^n(\mathrm{BO}, \mathbb{R}/\mathbb{Z}) \longrightarrow \mathrm{H}^{n+1}(\mathrm{BO}, \mathbb{Z}) \longrightarrow \mathrm{H}^{n+1}(\mathrm{BO}, \mathbb{R}) \longrightarrow \cdots. \tag{3.7}$$

If $\mathrm{H}^n(\mathrm{BO}, \mathbb{R}) = \mathrm{H}^{n+1}(\mathrm{BO}, \mathbb{R}) = 0$, then $\mathrm{H}^n(\mathrm{BO}, \mathbb{R}/\mathbb{Z}) = \mathrm{H}^{n+1}(\mathrm{BO}, \mathbb{Z})$. Since $\mathrm{H}^n(\mathrm{BO}, \mathbb{R}) = \mathrm{H}^n(\mathrm{BO}, \mathbb{Z}) \otimes \mathbb{R}$ and $\mathrm{H}^n(\mathrm{BO}, \mathbb{Z})$ is finite if n is not divisible by 4, $\mathrm{H}^n(\mathrm{BO}, \mathbb{R}) = 0$ if n is not divisible by 4, thus $\mathrm{H}^n(\mathrm{BO}, \mathbb{R}/\mathbb{Z}) = \mathrm{H}^{n+1}(\mathrm{BO}, \mathbb{Z})$ for $n = 1, 2 \mod 4$.

The last relevant possibly non-zero differential is on the E_6 page:

$$\mathrm{H}^0(\mathrm{BO}, \mathrm{H}^5(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z})) \to \mathrm{H}^6(\mathrm{BO}, \mathbb{Z}). \tag{3.8}$$

Following the appendix of [23], this differential is actually zero.

So the only possible differentials of Serre spectral sequence are d_3 from $(0, 5)$ to $(3, 3)$ and d_4 from the third row to the zeroth row.

By the Universal Coefficient Theorem (6.20),

$$\mathrm{H}^n(\mathrm{B}\mathbb{G}, \mathbb{Z}_2) = \mathrm{H}^n(\mathrm{B}\mathbb{G}, \mathbb{Z}) \otimes \mathbb{Z}_2 \oplus \mathrm{Tor}(\mathrm{H}^{n+1}(\mathrm{B}\mathbb{G}, \mathbb{Z}), \mathbb{Z}_2). \tag{3.9}$$

The Madsen-Tillmann spectrum $MT\mathbb{G}=\text{Thom}(\text{B}\mathbb{G};-V)$ where V is the induced virtual bundle over $\text{B}\mathbb{G}$ (of dimension 0) from $\text{B}\mathbb{G}\to\text{BO}$.

By Thom isomorphism, $\text{H}^*(MT\mathbb{G},\mathbb{Z}_2)=\text{H}^*(\text{B}\mathbb{G},\mathbb{Z}_2)U$ where U is the Thom class with $\text{Sq}^iU=\bar{w}_iU$ where $(1+\bar{w}_1+\bar{w}_2+\cdots)(1+w_1+w_2+\cdots)=1$.

We have the Adams spectral sequence

$$\text{Ext}_{\mathcal{A}_2}^{s,t}(\text{H}^*(MT\mathbb{G},\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\pi_{t-s}(MT\mathbb{G}) \tag{3.10}$$

where $\mathcal{A}_2$ is the mod 2 Steenrod algebra.

The $\mathcal{A}_2$-module structure of $\text{H}^*(MT\mathbb{G},\mathbb{Z}_2)$ below degree 5 is shown in Figure 2 where we intentionally omit terms that don't involve the cohomology classes of $\text{B}^2\mathbb{Z}_2$.

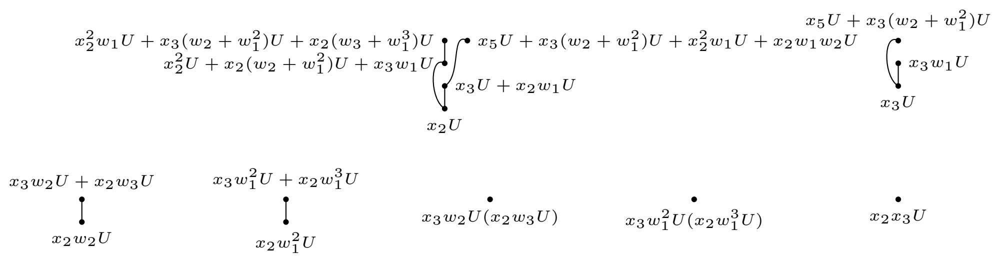


Figure 2: The $\mathcal{A}_2$-module structure of $\text{H}^*(MT\mathbb{G},\mathbb{Z}_2)$ below degree 5

Note that the position $(0,3)$ contributes to both $\text{H}^2(\text{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$ which is generated by x_2 and $\text{H}^3(\text{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$ which is generated by x_3, $\text{Sq}^1x_2=x_3$. The position $(2,3)$ corresponds to xw_1^2,xw_2, the position $(3,3)$ corresponds to xw_1^3,xw_1w_2,xw_3 for $x=x_2,x_3$.

$\langle\beta,\beta\rangle_q=-2q(\beta)$, $4q(\beta)=q(2\beta)=0$, there are 2 among the 4 choices of $q(\beta)$ such that $q\to\langle\beta,-\rangle_q$ maps to the dual linear function of β, if we identify $\hat{\mathbb{Z}}_2$ with $\mathbb{Z}_2$, then the nonzero element in the image of $q\to\langle\beta,-\rangle_q$ is just β. $\text{Im}d_3^{(0,5)}$ is spanned by $x\beta$.

The differential $d_4^{(2,3)}:\text{H}^2(\text{BO},\hat{\mathbb{Z}}_2)\to\text{H}^5(\text{BO},\mathbb{R}/\mathbb{Z})$ is defined by

$$d_4^{(2,3)}(\alpha)(v_0,\dots,v_5)=(\alpha(v_0,\dots,v_2))(\beta(v_2,\dots,v_5)).$$

The differential $d_4^{(3,3)}:\text{H}^3(\text{BO},\hat{\mathbb{Z}}_2)\to\text{H}^6(\text{BO},\mathbb{R}/\mathbb{Z})$ is defined by

$$d_4^{(3,3)}(\alpha)(v_0,\dots,v_6)=(\alpha(v_0,\dots,v_3))(\beta(v_3,\dots,v_6)).$$

Let $\beta=aw_1^3+bw_1w_2+cw_3$, if we identify $\hat{\mathbb{Z}}_2$ with $\mathbb{Z}_2$, $\text{Ker}d_4^{(3,3)}$ is spanned by $x\gamma$ where $\gamma=a'w_1^3+b'w_1w_2+c'w_3$ with $aa'+bb'+cc'=0\mod 2$.

In the following cases, we only consider the topological terms involving the cohomology classes of $\text{B}^2\mathbb{Z}_2$.

Case 1: $\beta=0$, $\text{B}\mathbb{G}=\text{BO}\times\text{B}^2\mathbb{Z}_2$, $MT\mathbb{G}=M\text{O}\wedge(\text{B}^2\mathbb{Z}_2)_+$, $\pi_dMT\mathbb{G}=\Omega_d^{\text{O}}(\text{B}^2\mathbb{Z}_2)$. The 5d topological terms are x_3w_2 (or x_2w_3), $x_3w_1^2$ (or $x_2w_1^3$) and x_2x_3. This case will be discussed later.

Case 2: $\beta = w_1^3$, $x_2w_1^3$ is killed in the E_3 page, x_2w_3 (or x_3w_2) survives in the E_∞ page, so the $5d$ topological terms are x_2w_3 (or x_3w_2) and x_2x_3.

Case 3: $\beta = w_1w_2$, x_3w_2 is killed in the E_4 page, $x_3w_1^2$ (or $x_2w_1^3$) survives in the E_∞ page, so the 5d topological terms are $x_3w_1^2$ (or $x_2w_1^3$) and x_2x_3.

Case 4: $\beta = w_3$, x_2w_3 is killed in the E_3 page, $x_3w_1^2$ (or $x_2w_1^3$) survives in the E_∞ page, so the 5d topological terms are $x_3w_1^2$ (or $x_2w_1^3$) and x_2x_3.

Case 5: $\beta = w_1^3 + w_1w_2$, $x_2w_1^3$ is identified with $x_2w_1w_2$ in the E_3 page, but $x_2w_1w_2 = \mathrm{Sq}^3x_2 = 0$, x_3w_2 is killed in the E_4 page, so the 5d topological term is x_2x_3.

Case 6: $\beta = w_1w_2 + w_3$, x_2w_3 is identified with $x_2w_1w_2$ in the E_3 page, but $x_2w_1w_2 = \mathrm{Sq}^3x_2 = 0$, $x_3w_1^2$ (or $x_2w_1^3$) survives in the E_∞ page, so the 5d topological terms are $x_3w_1^2$ (or $x_2w_1^3$) and x_2x_3.

Case 7: $\beta = w_1^3 + w_3$, x_2w_3 is identified with $x_2w_1^3$ in the E_3 page, $x_2w_1^3$ is killed in the E_4 page, so the 5d topological term is x_2x_3.

Case 8: $\beta = w_1^3 + w_1w_2 + w_3$, x_2w_3 is identified with $x_2w_1^3 + x_2w_1w_2$ in the E_3 page, but $x_2w_1w_2 = \mathrm{Sq}^3x_2 = 0$, $x_2w_1^3$ is killed in the E_4 page, so the 5d topological term is x_2x_3.

4 O/SO/Spin/Pin$^\pm$ bordism groups of classifying spaces

In this section, we compute the O/SO/Spin/Pin$^\pm$ bordism groups of the classifying space of the group $\mathbb{G} = G_a \times \mathrm{B}G_b$: $\mathrm{B}\mathbb{G} = \mathrm{B}G_a \times \mathrm{B}^2G_b$. Here $\mathrm{B}G_b$ is a group since G_b is abelian.

4.1 Introduction

For $H = $ O/SO/Spin/Pin$^\pm$ and the group $H \times \mathbb{G}$, define

$$MT(H \times \mathbb{G}) := \mathrm{Thom}(\mathrm{B}(H \times \mathbb{G}); -V) \tag{4.1}$$

where V is the induced virtual bundle over $\mathrm{B}(H \times \mathbb{G})$ by the composition $\mathrm{B}(H \times \mathbb{G}) \to \mathrm{B}H \to \mathrm{BO}$ where the first map is the projection, the second map is the natural homomorphism.

By the Pontryagin-Thom isomorphism (6.27) and the property of Thom space (6.24), $\Omega_d^H(\mathrm{B}\mathbb{G}) = \pi_d(MTH \wedge \mathrm{B}\mathbb{G}_+) = \pi_d(MT(H \times \mathbb{G}))$. Hence we can define

$$\Omega_d^{H\times\mathbb{G}} := \pi_d(MT(H \times \mathbb{G})) = \Omega_d^H(\mathrm{B}\mathbb{G}). \tag{4.2}$$

$$\mathrm{TP}_n(H \times \mathbb{G}) := [MT(H \times \mathbb{G}), \Sigma^{n+1}I\mathbb{Z}] \tag{4.3}$$

Here X_+ is the disjoint union of X and a point. $MT\mathrm{O} = M\mathrm{O}$, $MT\mathrm{SO} = M\mathrm{SO}$, $MT\mathrm{Spin} = M\mathrm{Spin}$, $MT\mathrm{Pin}^+ = M\mathrm{Pin}^-$, $MT\mathrm{Pin}^- = M\mathrm{Pin}^+$. $\pi_d(\mathcal{B})$ is the d-th stable homotopy group of the spectrum $\mathcal{B}$.

$[\mathcal{B}, \Sigma^{n+1} I\mathbb{Z}]$ stands for the homotopy classes of maps from spectrum $\mathcal{B}$ to the $(n+1)$-th suspension of spectrum $I\mathbb{Z}$. The Anderson dual $I\mathbb{Z}$ is a spectrum that is the fiber of $I\mathbb{C} \to I\mathbb{C}^\times$ where $I\mathbb{C}(I\mathbb{C}^\times)$ is the Brown-Comenetz dual spectrum defined by

$$[X, I\mathbb{C}] = \mathrm{Hom}(\pi_0 X, \mathbb{C}), \tag{4.4}$$

$$[X, I\mathbb{C}^\times] = \mathrm{Hom}(\pi_0 X, \mathbb{C}^\times). \tag{4.5}$$

By the work of Freed-Hopkins [2], there is a 1:1 correspondence

$$\left\{ \begin{array}{c} \text{deformation classes of reflection positive} \\ \text{invertible } n\text{-dimensional extended topological} \\ \text{field theories with symmetry group } H_n \times \mathbb{G} \end{array} \right\} \cong [MT(H \times \mathbb{G}), \Sigma^{n+1} I\mathbb{Z}]_{\mathrm{tors}}. \tag{4.6}$$

There is an exact sequence

$$0 \to \mathrm{Ext}^1(\pi_n \mathcal{B}, \mathbb{Z}) \to [\mathcal{B}, \Sigma^{n+1} I\mathbb{Z}] \to \mathrm{Hom}(\pi_{n+1}\mathcal{B}, \mathbb{Z}) \to 0 \tag{4.7}$$

for any spectrum $\mathcal{B}$, especially for $MT(H \times \mathbb{G})$. The torsion part $[MT(H \times \mathbb{G}), \Sigma^{n+1} I\mathbb{Z}]_{\mathrm{tors}}$ is $\mathrm{Ext}^1((\pi_n MT(H \times \mathbb{G}))_{\mathrm{tors}}, \mathbb{Z}) = \mathrm{Hom}((\pi_n MT(H \times \mathbb{G}))_{\mathrm{tors}}, \mathrm{U}(1))$.

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[a] \tag{4.8}$$

where $|a| = 1$.

Theorem 1 (**Serre**, Ref. [24])**.**

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[\mathrm{Sq}^I x_2 | I \text{ admissible}, ex(I) < 2] = \mathbb{Z}_2[\mathrm{Sq}^{2^{i-1}} \cdots \mathrm{Sq}^2\mathrm{Sq}^1 x_2 | i \geq 0] \tag{4.9}$$

where x_2 is the generator of $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$.

Denote $\mathrm{Sq}^{2^{i-1}} \cdots \mathrm{Sq}^2\mathrm{Sq}^1 x_2 = x_{2^i+1}$.

Here $\mathrm{Sq}^I = \mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots$ and $I = (i_1, i_2, \dots)$ is admissible if $i_s \geq 2i_{s+1}$ for $s \geq 1$, $ex(I) = \sum_{s\geq 1}(i_s - 2i_{s+1})$.

Theorem 2 (Ref. [24])**.**

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_3, \mathbb{Z}_3) = F_{\mathbb{Z}_3}[a', b'] = \Lambda_{\mathbb{Z}_3}(a') \otimes \mathbb{Z}_3[b'] \tag{4.10}$$

where $|a'| = 1$ and $b' = \beta_{(3,3)} a'$.

Here $\beta_{(3,3)}$ is the Bockstein homomorphism in $\mathcal{A}_3$.

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3) = F_{\mathbb{Z}_3}[x_2', \beta_{(3,3)}x_2', Q_i x_2', \beta_{(3,3)}Q_i x_2', i \geq 1] = \mathbb{Z}_3[x_2', \beta_{(3,3)}Q_i x_2', i \geq 1] \otimes \Lambda_{\mathbb{Z}_3}(\beta_{(3,3)}x_2', Q_i x_2', i \geq 1)$$

where $|x_2'| = 2$ and Q_i is defined inductively by $Q_0 = \beta_{(3,3)}$, $Q_i = P^{3^{i-1}} Q_{i-1} - Q_{i-1} P^{3^{i-1}}$ for $i \geq 1$. Let $x_3' = \beta_{(3,3)} x_2'$, $x'_{2\cdot 3^i+1} = Q_i x_2'$, $x'_{2\cdot 3^i+2} = \beta_{(3,3)} Q_i x_2'$ for $i \geq 1$.

Here P^n is the n-th Steenrod power in $\mathcal{A}_3$.

$$\mathrm{H}^*(\mathrm{BPSU}(2), \mathbb{Z}_2) = \mathbb{Z}_2[w_2', w_3']. \tag{4.11}$$
$$\mathrm{H}^*(\mathrm{BPSU}(3), \mathbb{Z}_2) = \mathbb{Z}_2[c_2, c_3]. \tag{4.12}$$

Here w_i' is the i-th Stiefel-Whitney class $w_i(\mathrm{PSU}(2))$ of the universal principal PSU(2)-bundle over BPSU(2). Let p_i' be the i-th Pontryagin class $p_i(\mathrm{PSU}(2))$ of the universal principal PSU(2)-bundle over BPSU(2), then $p_1'(\mod 2) = w_2'^2$.

c_i is the i-th Chern class $c_i(\mathrm{PSU}(3))$ of the universal principal PSU(3)-bundle over BPSU(3).

Since $\frac{\mathrm{SU}(3)\times \mathrm{U}(1)}{\mathbb{Z}_3} = \mathrm{U}(3)$, $\mathrm{PSU}(3) = \mathrm{PU}(3)$.

Theorem 3 (Ref. [25])**.**

$$\mathrm{H}^*(\mathrm{BPSU}(3), \mathbb{Z}_3) = F_{\mathbb{Z}_3}[z_2, z_3, z_7, z_8, z_{12}]/J \tag{4.13}$$

where $|z_i| = i$, $J = (z_2z_3, z_2z_7, z_2z_8 + z_3z_7)$ is the ideal generated by $z_2z_3, z_2z_7, z_2z_8 + z_3z_7$ and $z_3 = \beta_{(3,3)}z_2$, $z_7 = P^1z_3$, $z_8 = \beta_{(3,3)}z_7$. Note that $c_2(\mod 3) = z_2^2$, $c_3(\mod 3) = z_2^3$.

In the following subsections, all bordism invariants are the pullback of cohomology classes along classifying maps $f : M \to X$ and $g : M \to \mathrm{B}H$.

4.2 Point

4.2.1 Ω_d^{O}

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\mathrm{O})_2^{\wedge} = \Omega_{t-s}^{\mathrm{O}}. \tag{4.14}$$

Here $\pi_{t-s}(M\mathrm{O})_2^{\wedge}$ is the 2-completion of the group $\pi_{t-s}(M\mathrm{O})$.

The mod 2 cohomology of Thom spectrum $M\mathrm{O}$ is

$$\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) = \mathcal{A}_2 \otimes \Omega^* \tag{4.15}$$

where $\Omega = \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]$ is the unoriented bordism ring, Ω^* is the $\mathbb{Z}_2$-linear dual of Ω.

On the other hand, $\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, w_3, \dots]U$ where U is the Thom class of the virtual bundle (of dimension 0) over BO which is the colimit of $E_n - n$ and E_n is the universal n-bundle over BO(n), w_i is the i-th Stiefel-Whitney class of the virtual bundle (of dimension 0) over BO. Note that the pullback of the virtual bundle (of dimension 0) over BO along the map $g : M \to \mathrm{BO}$ is just $TM - d$ where M is a d-dimensional manifold and TM is the tangent bundle of M, g is given by the O-structure on M. We will not distinguish w_i and $w_i(TM)$.

Here y_i are manifold generators, for example, $y_2 = \mathbb{RP}^2$, $y_4 = \mathbb{RP}^4$, y_5 is Wu manifold $\mathrm{SU}(3)/\mathrm{SO}(3)$. By Thom's result [7], two manifolds are unorientedly bordant if and only if they have identical sets of Stiefel-Whitney characteristic numbers. The nonvanishing Stiefel-Whitney numbers of $y_2 = \mathbb{RP}^2$ are w_2 and w_1^2, the nonvanishing Stiefel-Whitney numbers of $y_2^2 = \mathbb{RP}^2 \times \mathbb{RP}^2$ are w_2^2 and w_4, the nonvanishing Stiefel-Whitney numbers of $y_4 = \mathbb{RP}^4$ are w_1^4 and w_4, the only nonvanishing Stiefel-Whitney number of Wu manifold $\mathrm{SU}(3)/\mathrm{SO}(3)$ is w_2w_3.

So $y_2^* = w_1^2$ or w_2, $(y_2^2)^* = w_2^2$, $y_4^* = w_1^4$, $y_5^* = w_2w_3$, etc, where y_i^* is the $\mathbb{Z}_2$-linear dual of $y_i \in \Omega$.

Below we choose $y_2^* = w_1^2$ by default, this is reasonable since $\mathrm{Sq}^2(x_{d-2}) = (w_2 + w_1^2)x_{d-2}$ on d-manifold by Wu formula (6.62).

Hence we have the following theorem

i	Ω_i^{O}
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 4.

The bordism invariant of Ω_2^{O} is w_1^2.

The bordism invariants of Ω_4^{O} are w_1^4, w_2^2.

The bordism invariant of Ω_5^{O} is w_2w_3.

i	$\mathrm{TP}_i(\mathrm{O})$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 5.

The 2d topological term is w_1^2.

The 4d topological terms are w_1^4, w_2^2.

The 5d topological term is w_2w_3.

4.2.2 Ω_d^{SO}

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2),\mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\mathrm{SO})_2^{\wedge} = \Omega_{t-s}^{\mathrm{SO}}. \tag{4.16}$$

The mod 2 cohomology of Thom spectrum $M\mathrm{SO}$ is

$$\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2) = \mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1 \oplus \Sigma^4\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1 \oplus \Sigma^5\mathcal{A}_2 \oplus \cdots . \tag{4.17}$$

$$\cdots \longrightarrow \Sigma^3\mathcal{A}_2 \longrightarrow \Sigma^2\mathcal{A}_2 \longrightarrow \Sigma\mathcal{A}_2 \longrightarrow \mathcal{A}_2 \longrightarrow \mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1 \tag{4.18}$$

is an $\mathcal{A}_2$-resolution where the differentials d_1 are induced by Sq^1.

The E_2 page is shown in Figure 3.

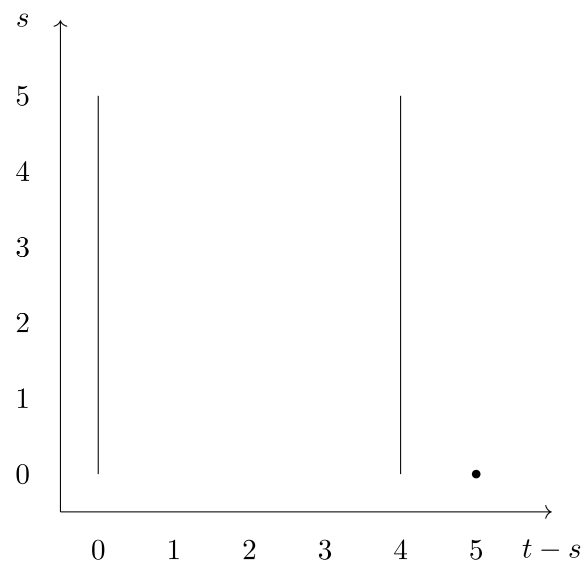


Figure 3: Ω_*^{SO}

Hence we have the following theorem

i	Ω_i^{SO}
0	$\mathbb{Z}$
1	0
2	0
3	0
4	$\mathbb{Z}$
5	$\mathbb{Z}_2$

Theorem 6.

The bordism invariant of Ω_4^{SO} is σ.

i	$\mathrm{TP}_i(\mathrm{SO})$
0	0
1	0
2	0
3	$\mathbb{Z}$
4	0
5	$\mathbb{Z}_2$

Here σ is the signature of a 4-manifold.

The bordism invariant of Ω_5^{SO} is w_2w_3.

Theorem 7.

Since $\sigma = \frac{p_1(TM)}{3}$, $p_1(TM) = \mathrm{dCS}_3^{(TM)}$, the 3d topological term is $\frac{1}{3}\mathrm{CS}_3^{(TM)}$.

The 5d topological term is w_2w_3.

4.2.3 Ω_d^{Spin}

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Spin},\mathbb{Z}_2),\mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\mathrm{Spin})_2^{\wedge} = \Omega_{t-s}^{\mathrm{Spin}}. \tag{4.19}$$

The mod 2 cohomology of Thom spectrum $M\mathrm{Spin}$ is

$$\mathrm{H}^*(M\mathrm{Spin},\mathbb{Z}_2) = \mathcal{A}_2 \otimes_{\mathcal{A}_2(1)} \{\mathbb{Z}_2 \oplus M\} \tag{4.20}$$

where M is a graded $\mathcal{A}_2(1)$-module with the degree i homogeneous part $M_i = 0$ for $i < 8$. Here $\mathcal{A}_2(1)$ stands for the subalgebra of $\mathcal{A}_2$ generated by Sq^1 and Sq^2. For $t - s < 8$, we can identify the E_2-page with

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathbb{Z}_2,\mathbb{Z}_2).$$

The E_2 page is shown in Figure 4.

Hence we have the following theorem

i	Ω_i^{Spin}
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}$
5	0

Theorem 8.

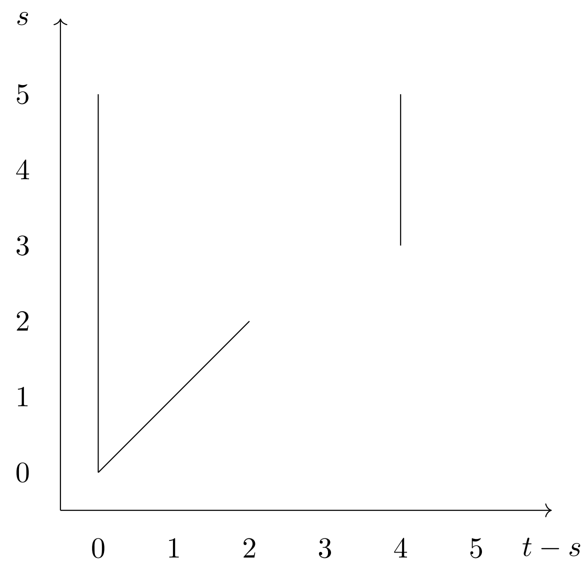


Figure 4: Ω_*^{Spin}

The bordism invariant of Ω_1^{Spin} is $\tilde{\eta}$.

Here $\tilde{\eta}$ is the "mod 2 index" of the 1d Dirac operator (#zero eigenvalues mod 2, no contribution from spectral asymmetry).

The bordism invariant of Ω_2^{Spin} is Arf (the Arf invariant).

The bordism invariant of Ω_4^{Spin} is $\frac{\sigma}{16}$.

i	$\mathrm{TP}_i(\mathrm{Spin})$
0	0
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	$\mathbb{Z}$
4	0
5	0

Theorem 9.

The 1d topological term is $\tilde{\eta}$.

The 2d topological term is Arf.

The 3d topological term is $\frac{1}{48}\mathrm{CS}_3^{(TM)}$.

4.2.4 $\Omega_d^{\mathrm{Pin}^+}$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^-, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\mathrm{Pin}^-)_2^\wedge = \Omega_{t-s}^{\mathrm{Pin}^+}. \tag{4.21}$$

$M\mathrm{Pin}^- = MT\mathrm{Pin}^+ \sim M\mathrm{Spin} \wedge S^1 \wedge MT\mathrm{O}(1)$.

For $t - s < 8$, we can identify the E_2-page with

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*-1}(MT\mathrm{O}(1), \mathbb{Z}_2), \mathbb{Z}_2).$$

By Thom's isomorphism,

$$\mathrm{H}^{*-1}(MT\mathrm{O}(1), \mathbb{Z}_2) = \mathbb{Z}_2[w_1]U \tag{4.22}$$

where U is the Thom class of the virtual bundle $-E_1$ over $\mathrm{BO}(1)$, E_1 is the universal 1-bundle over $\mathrm{BO}(1)$ and w_1 is the 1st Stiefel-Whitney class of E_1 over $\mathrm{BO}(1)$. The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MT\mathrm{O}(1), \mathbb{Z}_2)$ and the E_2 page are shown in Figure 5, 6.

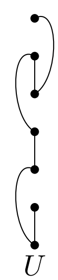


Figure 5: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MT\mathrm{O}(1), \mathbb{Z}_2)$

Hence we have the following theorem

i	$\Omega_i^{\mathrm{Pin}^+}$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_{16}$
5	0

Theorem 10.

The bordism invariant of $\Omega_2^{\mathrm{Pin}^+}$ is $w_1 \cup \tilde{\eta}$.

The bordism invariant of $\Omega_3^{\mathrm{Pin}^+}$ is $w_1 \cup \mathrm{Arf}$.

The bordism invariant of $\Omega_4^{\mathrm{Pin}^+}$ is η.

Here η is the usual Atiyah-Patodi-Singer eta-invariant of the 4d Dirac operator (="#zero eigenvalues + spectral asymmetry").

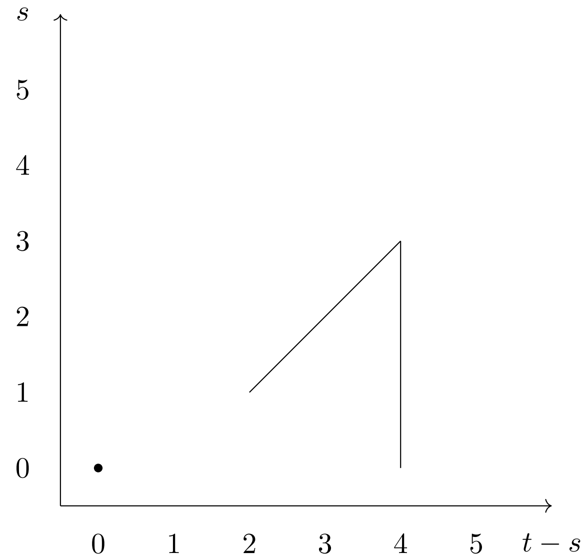


Figure 6: $\Omega_*^{\mathrm{Pin}^+}$

i	$\mathrm{TP}_i(\mathrm{Pin}^+)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_{16}$
5	0

Theorem 11.

The 2d topological term is $w_1 \cup \tilde{\eta}$.

The 3d topological term is $w_1 \cup \mathrm{Arf}$.

The 4d topological term is η.

4.2.5 $\Omega_d^{\mathrm{Pin}^-}$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\mathrm{Pin}^+)_2^\wedge = \Omega_{t-s}^{\mathrm{Pin}^-}. \tag{4.23}$$

$M\mathrm{Pin}^+ = MT\mathrm{Pin}^- \sim M\mathrm{Spin} \wedge S^{-1} \wedge M\mathrm{O}(1)$.

For $t - s < 8$, we can identify the E_2-page with

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*+1}(M\mathrm{O}(1), \mathbb{Z}_2), \mathbb{Z}_2).$$

By Thom's isomorphism,

$$\mathrm{H}^{*+1}(M\mathrm{O}(1), \mathbb{Z}_2) = \mathbb{Z}_2[w_1]U \tag{4.24}$$

where U is the Thom class of the universal 1-bundle E_1 over BO(1) and w_1 is the 1st Stiefel-Whitney class of E_1 over BO(1). The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+1}(M\mathrm{O}(1), \mathbb{Z}_2)$ and the E_2 page are shown in Figure 7, 8.

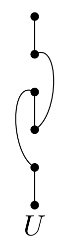


Figure 7: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+1}(M\mathrm{O}(1), \mathbb{Z}_2)$

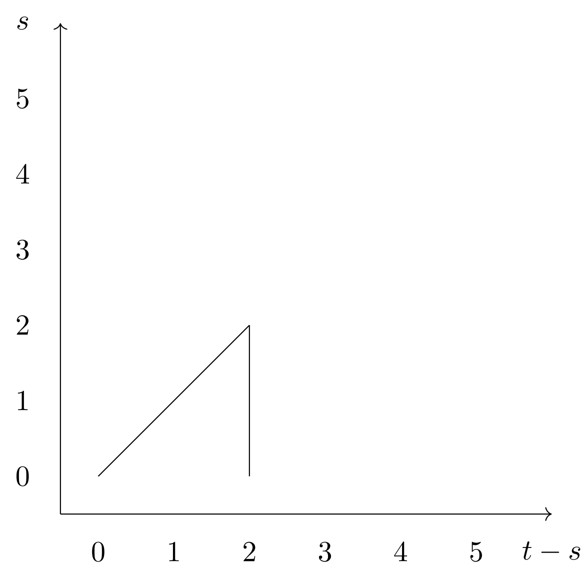


Figure 8: $\Omega_*^{\mathrm{Pin}^-}$

Hence we have the following theorem

i	$\Omega_i^{\mathrm{Pin}^-}$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	0
5	0

Theorem 12.

The bordism invariant of $\Omega_1^{\mathrm{Pin}^-}$ is $\tilde{\eta}$.

The bordism invariant of $\Omega_2^{\mathrm{Pin}^-}$ is ABK (the Arf-Brown-Kervaire invariant).

i	$\mathrm{TP}_i(\mathrm{Pin}^-)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	0
5	0

Theorem 13.

The 1d topological term is $\tilde{\eta}$.

The 2d topological term is ABK.

4.3 Atiyah-Hirzebruch spectral sequence

If $H = \mathrm{O}/\mathrm{SO}/\mathrm{Spin}/\mathrm{Pin}^{\pm}$, by the Atiyah-Hirzebruch spectral sequence, we have

$$\mathrm{H}_p(\mathrm{B}\mathbb{G}, \Omega_q^H) \Rightarrow \Omega_{p+q}^H(\mathrm{B}\mathbb{G}). \tag{4.25}$$

If $H = \mathrm{O}/\mathrm{Pin}^{\pm}$, since Ω_d^H are finite, $\Omega_d^{H\times\mathbb{G}} = \Omega_d^H(\mathrm{B}\mathbb{G})$ are also finite, so $\mathrm{TP}_d(H \times \mathbb{G}) = \Omega_d^{H\times\mathbb{G}}$ for $H = \mathrm{O}/\mathrm{Pin}^{\pm}$.

If $H = \mathrm{SO}/\mathrm{Spin}$,

$$\Omega_q^{\mathrm{SO}} = \begin{cases} \mathbb{Z} & q=0 \\ 0 & q=1 \\ 0 & q=2 \\ 0 & q=3 \\ \mathbb{Z} & q=4 \\ \mathbb{Z}_2 & q=5 \\ 0 & q=6 \end{cases}. \tag{4.26}$$

$$\Omega_q^{\mathrm{Spin}} = \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z}_2 & q=1 \\ \mathbb{Z}_2 & q=2 \\ 0 & q=3 \\ \mathbb{Z} & q=4 \\ 0 & q=5 \\ 0 & q=6 \end{cases}. \tag{4.27}$$

If $\mathrm{H}_p(\mathrm{B}\mathbb{G}, \mathbb{Z})$ are finite for $p > 0$, then $\Omega_6^H(\mathrm{B}\mathbb{G})$ is finite and $\mathrm{TP}_5(H \times \mathbb{G}) = \Omega_5^H(\mathrm{B}\mathbb{G})$ for $H = \mathrm{SO}/\mathrm{Spin}$.

If $\mathbb{G} = \mathrm{PSU}(2) = \mathrm{SO}(3)$, since $\mathrm{H}_2(\mathrm{BSO}(3), \mathbb{Z})$ and $\mathrm{H}_6(\mathrm{BSO}(3), \mathbb{Z})$ are finite, $\Omega_6^H(\mathrm{B}\mathbb{G})$ is also finite and $\mathrm{TP}_5(H \times \mathbb{G}) = \Omega_5^H(\mathrm{B}\mathbb{G})$ for $H = \mathrm{SO}/\mathrm{Spin}$.

If $\mathbb{G} = \mathrm{PSU}(3)$, then $\mathrm{H}_6(\mathrm{BPSU}(3), \mathbb{Z})$ contains a $\mathbb{Z}$ while $\mathrm{H}_2(\mathrm{BPSU}(3), \mathbb{Z})$ does not, so $\Omega_6^H(\mathrm{B}\mathbb{G})$ contains a $\mathbb{Z}$ and $\mathrm{TP}_5(H \times \mathbb{G}) = \Omega_5^H(\mathrm{B}\mathbb{G}) \times \mathbb{Z}$ for $H = \mathrm{SO}/\mathrm{Spin}$.

4.4 $\mathrm{B}^2 G_b : \mathrm{B}^2\mathbb{Z}_2, \mathrm{B}^2\mathbb{Z}_3$

4.4.1 $\Omega_d^{\mathbf{O}}(\mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O} \wedge (\mathrm{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\mathrm{O} \wedge (\mathrm{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2). \tag{4.28}$$

$$\begin{aligned} \mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) &= \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^* \otimes \mathbb{Z}_2[x_2, x_3, x_5, x_9, \dots] \\ &= \mathcal{A}_2 \oplus 2\Sigma^2\mathcal{A}_2 \oplus \Sigma^3\mathcal{A}_2 \oplus 4\Sigma^4\mathcal{A}_2 \oplus 4\Sigma^5\mathcal{A}_2 \oplus \cdots \end{aligned} \tag{4.29}$$

Here $\Sigma^n\mathcal{A}_2$ is the n-th iterated shift of the graded algebra $\mathcal{A}_2$.

Hence we have the following theorem (see 6.3.1 for detail)

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2^4$
5	$\mathbb{Z}_2^4$

Theorem 14.

The bordism invariants of $\Omega_2^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$ are x_2, w_1^2.

The bordism invariant of $\Omega_3^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$ is $x_3 = w_1x_2$.

The bordism invariants of $\Omega_4^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$ are $x_2^2, w_1^4, w_1^2x_2, w_2^2$.

The bordism invariants of $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$ are $x_2x_3, x_5, w_1^2x_3, w_2w_3$.

Theorem 15.

The 2d topological terms are x_2, w_1^2.

The 3d topological term is $x_3 = w_1x_2$.

i	$\mathrm{TP}_i(\mathrm{O}\times\mathrm{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2^4$
5	$\mathbb{Z}_2^4$

The 4d topological terms are $x_2^2, w_1^4, w_1^2x_2, w_2^2$.

The 5d topological terms are $x_2x_3, x_5, w_1^2x_3, w_2w_3$.

4.4.2 $\Omega_d^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t}=\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO}\wedge(\mathrm{B}^2\mathbb{Z}_2)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\pi_{t-s}(M\mathrm{SO}\wedge(\mathrm{B}^2\mathbb{Z}_2)_+)_2^{\wedge}=\Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2).\quad(4.30)$$

$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)=\mathbb{Z}_2[x_2,x_3,x_5,x_9,\dots]$ where x_2 is the generator of $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$, $x_3=\mathrm{Sq}^1x_2$, $x_5=\mathrm{Sq}^2\mathrm{Sq}^1x_2$, $x_9=\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1x_2$, etc, $\mathrm{Sq}^1x_2=x_3$, $\mathrm{Sq}^1x_3=0$, $\mathrm{Sq}^1(x_2^2)=0$, $\mathrm{Sq}^1(x_2x_3)=\mathrm{Sq}^1(x_5)=x_3^2$. We have used (6.60) and the Adem relations (6.77).

There is a differential d_2 corresponds to the $(2,4)$-Bockstein $\beta_{(2,4)}:\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_4)\to\mathrm{H}^{*+1}(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$ associated to $0\to\mathbb{Z}_2\to\mathbb{Z}_8\to\mathbb{Z}_4\to0$ [26]. See 6.5 for the definition of Bockstein homomorphisms.

Note that $\beta_{(2,4)}(\mathcal{P}_2(x_2))=\frac{1}{4}\delta(\mathcal{P}_2(x_2))=\frac{1}{4}\delta(x_2\cup x_2+x_2\underset{1}{\cup}\delta x_2)=\frac{1}{4}(2\delta x_2\cup x_2+\delta x_2\underset{1}{\cup}\delta x_2)=(\frac{1}{2}\delta x_2)\cup x_2+(\frac{1}{2}\delta x_2)\underset{1}{\cup}(\frac{1}{2}\delta x_2)=x_3\cup x_2+x_3\underset{1}{\cup}x_3=x_3\cup x_2+\mathrm{Sq}^2x_3=x_2x_3+x_5$. We have used the Steenrod's formula (6.12) and the definition $\mathrm{Sq}^kx_n=x_n\underset{n-k}{\cup}x_n$.

So there is a differential such that $d_2(x_2x_3+x_5)=x_2^2h_0^2$.

The E_2 page is shown in Figure 9.

Hence we have the following theorem (see 6.3.1 for detail)

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}\times\mathbb{Z}_4$
5	$\mathbb{Z}_2^2$

Theorem 16.

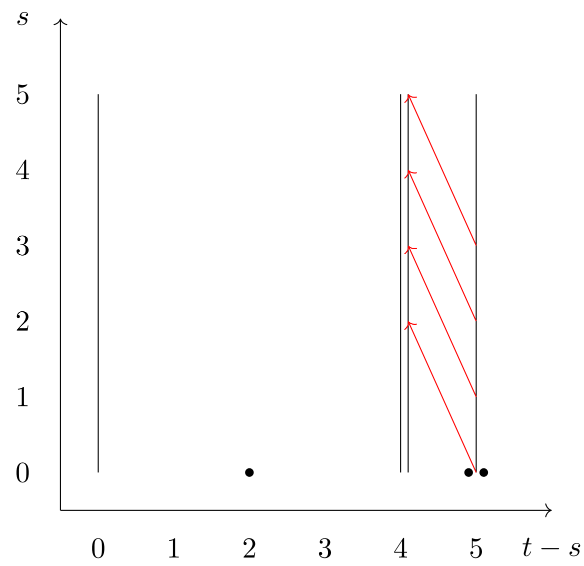


Figure 9: $\Omega_*^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$

The bordism invariant of $\Omega_2^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$ is x_2.

The bordism invariants of $\Omega_4^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$ are σ and $\mathcal{P}_2(x_2)$.

Here $\mathcal{P}_2(x_2)$ is the Pontryagin square of x_2.

The bordism invariants of $\Omega_5^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$ are $x_5 = x_2x_3$ and w_2w_3.

Here $x_2x_3+x_5 = \frac{1}{2}\tilde{w}_1\mathcal{P}_2(x_2)$ [**?**] where $\tilde{w}_1$ is the twisted first Stiefel-Whitney class of the tangent bundle, in particular, $w_1 = 0$ implies $\tilde{w}_1 = 0$, so $x_2x_3 = x_5$ on oriented 5-manifold.

i	$\mathrm{TP}_i(\mathrm{SO}\times\mathrm{B}\mathbb{Z}_2)$
0	0
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}$
4	$\mathbb{Z}_4$
5	$\mathbb{Z}_2^2$

Theorem 17.

The 2d topological term is x_2.

The 3d topological term is $\frac{1}{3}\mathrm{CS}_3^{(TM)}$.

The 4d topological term is $\mathcal{P}_2(x_2)$.

The 5d topological terms are $x_5 = x_2x_3$ and w_2w_3.

4.4.3 $\Omega_d^{\mathrm{Spin}}(\mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Spin} \wedge (\mathrm{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\mathrm{Spin} \wedge (\mathrm{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\mathrm{Spin}}(\mathrm{B}^2\mathbb{Z}_2). \quad (4.31)$$

For $t-s<8$, we can identify the E_2-page with

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2).$$

$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[x_2, x_3, x_5, x_9, \dots]$ where x_2 is the generator of $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = \mathrm{Sq}^1 x_2$, $x_5 = \mathrm{Sq}^2\mathrm{Sq}^1 x_2$, $x_9 = \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1 x_2$, etc, $\mathrm{Sq}^1 x_2 = x_3$, $\mathrm{Sq}^2 x_2 = x_2^2$, $\mathrm{Sq}^1 x_3 = 0$, $\mathrm{Sq}^2 x_3 = x_5$, $\mathrm{Sq}^1(x_2^2) = 0$, $\mathrm{Sq}^1(x_2x_3) = x_3^2$, $\mathrm{Sq}^1 x_5 = \mathrm{Sq}^2 x_2^2 = x_3^2$, $\mathrm{Sq}^2 x_5 = 0$. $\mathrm{Sq}^2(x_2x_3) = x_2^2x_3 + x_2x_5$. We have used (6.60) and the Adem relations (6.77).

There is a differential d_2 corresponds to the $(2,4)$-Bockstein $\beta_{(2,4)} : \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_4) \to \mathrm{H}^{*+1}(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$ associated to $0 \to \mathbb{Z}_2 \to \mathbb{Z}_8 \to \mathbb{Z}_4 \to 0$ [26]. See 6.5 for the definition of Bockstein homomorphisms.

Note that $\beta_{(2,4)}(\mathcal{P}_2(x_2)) = \frac{1}{4}\delta(\mathcal{P}_2(x_2)) = \frac{1}{4}\delta(x_2 \cup x_2 + x_2 \underset{1}{\cup} \delta x_2) = \frac{1}{4}(2\delta x_2 \cup x_2 + \delta x_2 \underset{1}{\cup} \delta x_2) = (\frac{1}{2}\delta x_2) \cup x_2 + (\frac{1}{2}\delta x_2) \underset{1}{\cup} (\frac{1}{2}\delta x_2) = x_3 \cup x_2 + x_3 \underset{1}{\cup} x_3 = x_3 \cup x_2 + \mathrm{Sq}^2 x_3 = x_2x_3 + x_5$. We have used the Steenrod's formula (6.12) and the definition $\mathrm{Sq}^k x_n = x_n \underset{n-k}{\cup} x_n$.

So there is a differential such that $d_2(x_2x_3 + x_5) = x_2^2 h_0^2$.

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$ and the E_2 page are shown in Figure 10, 11.

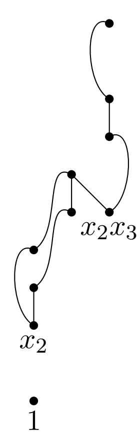


Figure 10: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$

Hence we have the following theorem (see 6.3.1 for detail)

Theorem 18.

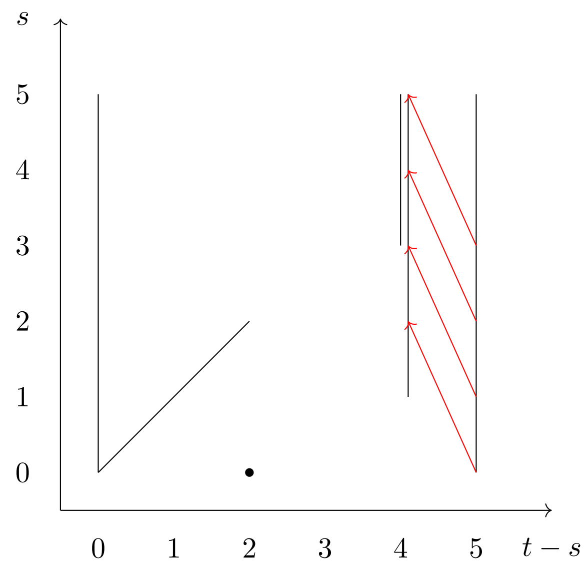


Figure 11: $\Omega_*^{\text{Spin}}(\text{B}^2\mathbb{Z}_2)$

i	$\Omega_i^{\text{Spin}}(\text{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2$
3	0
4	$\mathbb{Z}\times\mathbb{Z}_2$
5	0

By Wu formula (6.62), $x_2^2 = \text{Sq}^2(x_2) = (w_2(TM)+w_1(TM)^2)x_2 = 0$ on Spin 4-manifolds, $x_5 = \text{Sq}^2(x_3) = (w_2(TM)+w_1(TM)^2)x_3 = 0$ on Spin 5-manifolds, $\mathcal{P}_2(x_2) = x_2^2 = 0 \mod 2$ on Spin 4-manifolds.

The bordism invariants of $\Omega_2^{\text{Spin}}(\text{B}^2\mathbb{Z}_2)$ are x_2 and Arf.

The bordism invariants of $\Omega_4^{\text{Spin}}(\text{B}^2\mathbb{Z}_2)$ are $\frac{\sigma}{16}$ and $\frac{\mathcal{P}_2(x_2)}{2}$.

i	$\text{TP}_i(\text{Spin}\times\text{B}\mathbb{Z}_2)$
0	0
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}$
4	$\mathbb{Z}_2$
5	0

Theorem 19.

The 2d topological terms are x_2 and Arf.

The 3d topological term is $\frac{1}{48}\text{CS}_3^{(TM)}$.

The 4d topological term is $\frac{\mathcal{P}_2(x_2)}{2}$.

4.4.4 $\Omega_d^{\text{Pin}^+}(\text{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}_2}^{s,t}(\text{H}^*(M\text{Pin}^- \wedge (\text{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\text{Pin}^- \wedge (\text{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\text{Pin}^+}(\text{B}^2\mathbb{Z}_2). \quad (4.32)$$

$M\text{Pin}^- = MT\text{Pin}^+ \sim M\text{Spin} \wedge S^1 \wedge MT\text{O}(1)$.

For $t-s<8$, we can identify the E_2-page with

$$\text{Ext}_{\mathcal{A}_2(1)}^{s,t}(\text{H}^{*-1}(MT\text{O}(1), \mathbb{Z}_2) \otimes \text{H}^*(\text{B}^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2).$$

The $\mathcal{A}_2(1)$-module structure of $\text{H}^{*-1}(MT\text{O}(1), \mathbb{Z}_2) \otimes \text{H}^*(\text{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$ and the E_2 page are shown in Figure 12, 13.

Hence we have the following theorem (see 6.3.1 for detail)

i	$\Omega_i^{\text{Pin}^+}(\text{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_4 \times \mathbb{Z}_{16}$
5	$\mathbb{Z}_2^2$

Theorem 20.

The bordism invariants of $\Omega_2^{\text{Pin}^+}(\text{B}^2\mathbb{Z}_2)$ are x_2 and $w_1\tilde{\eta}$.

The bordism invariants of $\Omega_3^{\text{Pin}^+}(\text{B}^2\mathbb{Z}_2)$ are $w_1x_2 = x_3$ and $w_1\text{Arf}$.

The bordism invariants of $\Omega_4^{\text{Pin}^+}(\text{B}^2\mathbb{Z}_2)$ are $q_s(x_2)$ and η.

q_s is explained in the footnotes of Table 3.

The bordism invariants of $\Omega_5^{\text{Pin}^+}(\text{B}^2\mathbb{Z}_2)$ are x_2x_3 and $w_1^2x_3(=x_5)$.

i	$\text{TP}_i(\text{Pin}^+ \times \text{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_4 \times \mathbb{Z}_{16}$
5	$\mathbb{Z}_2^2$

Theorem 21.

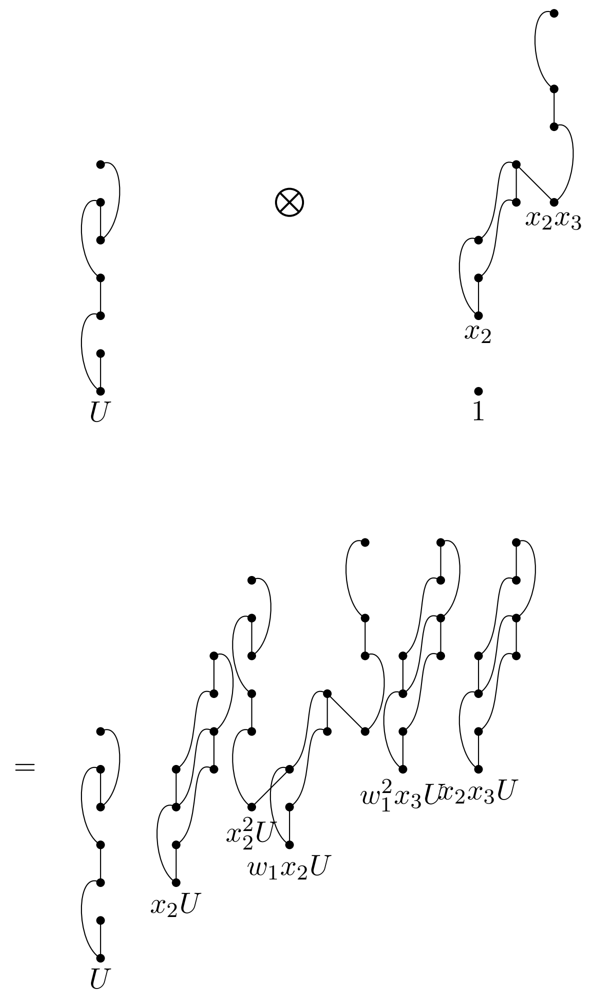


Figure 12: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MTO(1),\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$

The 2d topological terms are x_2 and $w_1\tilde{\eta}$.

The 3d topological terms are $w_1x_2 = x_3$ and $w_1\mathrm{Arf}$.

The 4d topological terms are $q_s(x_2)$ and η.

The 5d topological terms are x_2x_3 and $w_1^2x_3(= x_5)$.

4.4.5 $\Omega_d^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+\wedge(\mathrm{B}^2\mathbb{Z}_2)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow \pi_{t-s}(M\mathrm{Pin}^+\wedge(\mathrm{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_2). \quad (4.33)$$

$M\mathrm{Pin}^+ = MT\mathrm{Pin}^- \sim M\mathrm{Spin}\wedge S^{-1}\wedge MO(1)$.

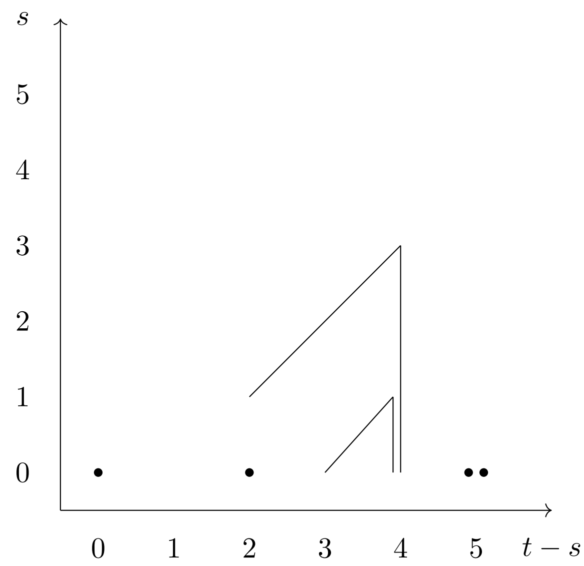


Figure 13: $\Omega_*^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_2)$

For $t-s<8$, we can identify the E_2-page with

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*+1}(M\mathrm{O}(1),\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2).$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+1}(M\mathrm{O}(1),\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$ and the E_2 page are shown in Figure 14, 15.

Hence we have the following theorem (see 6.3.1 for detail)

i	$\Omega_i^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2\times\mathbb{Z}_8$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2$
5	$\mathbb{Z}_2$

Theorem 22.

The bordism invariants of $\Omega_2^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_2)$ are x_2 and ABK.

The bordism invariant of $\Omega_3^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_2)$ is $w_1x_2=x_3$.

The bordism invariant of $\Omega_4^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_2)$ is $w_1^2x_2$.

The bordism invariant of $\Omega_5^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_2)$ is x_2x_3.

Theorem 23.

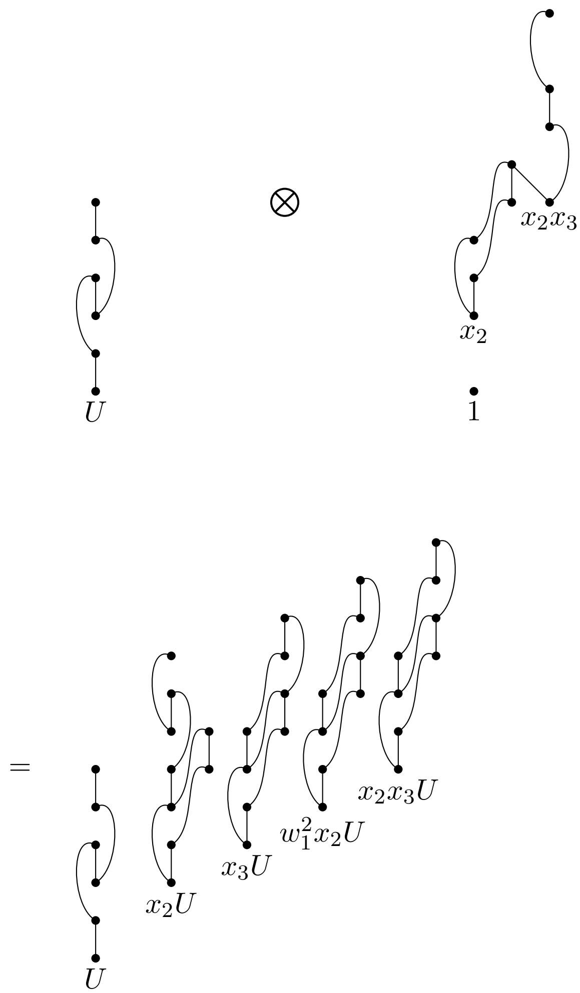


Figure 14: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+1}(M\mathrm{O}(1),\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$

i	$\mathrm{TP}_i(\mathrm{Pin}^-\times\mathrm{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2\times\mathbb{Z}_8$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2$
5	$\mathbb{Z}_2$

The 2d topological terms are x_2 and ABK.

The 3d topological term is $w_1x_2=x_3$.

The 4d topological term is $w_1^2x_2$.

The 5d topological term is x_2x_3.

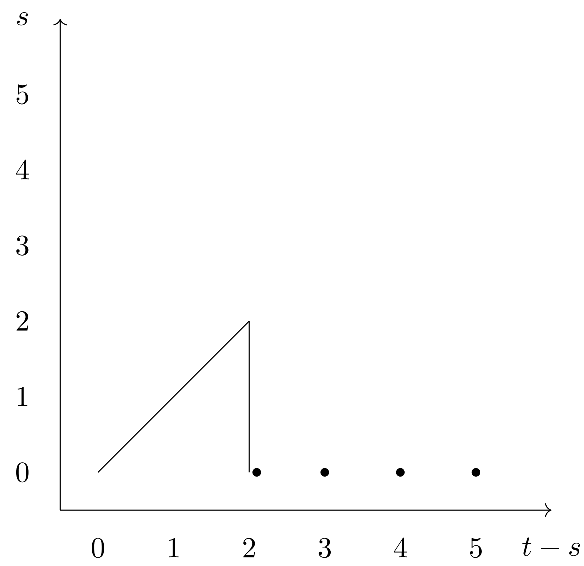


Figure 15: $\Omega_*^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_2)$

4.4.6 $\Omega_d^{\mathbf{O}}(\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O}\wedge(\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{4.34}$$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{O}\wedge(\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{4.35}$$

Since $M\mathrm{O}$ is the wedge sum of suspensions of the Eilenberg-MacLane spectrum $H\mathbb{Z}_2$, $\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_3)=0$, thus $\Omega_d^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)_3^\wedge=0$.

Since $\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathbb{Z}_2$, we have $\Omega_d^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)_2^\wedge=\Omega_d^{\mathrm{O}}$.

Hence $\Omega_d^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)=\Omega_d^{\mathrm{O}}$.

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 24.

The bordism invariant of $\Omega_2^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)$ is w_1^2.

The bordism invariants of $\Omega_4^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)$ are w_1^4, w_2^2.

The bordism invariant of $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)$ is w_2w_3.

i	$\mathrm{TP}_i(\mathrm{O}\times\mathrm{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 25.

The 2d topological term is w_1^2.

The 4d topological terms are w_1^4, w_2^2.

The 5d topological term is w_2w_3.

4.4.7 $\Omega_d^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO}\wedge(\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}. \tag{4.36}$$

Since $\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathbb{Z}_2$, we have $\Omega_d^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}=\Omega_d^{\mathrm{SO}}$.

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{SO}\wedge(\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)_3^{\wedge}. \tag{4.37}$$

The dual of $\mathcal{A}_3=\mathrm{H}^*(H\mathbb{Z}_3,\mathbb{Z}_3)$ is

$$\mathcal{A}_{3*}=\mathrm{H}_*(H\mathbb{Z}_3,\mathbb{Z}_3)=\Lambda_{\mathbb{Z}_3}(\tau_0,\tau_1,\dots)\otimes\mathbb{Z}_3[\xi_1,\xi_2,\dots] \tag{4.38}$$

where $\tau_i=(P^{3^{i-1}}\cdots P^3P^1\beta_{(3,3)})^*$ and $\xi_i=(P^{3^{i-1}}\cdots P^3P^1)^*$. Let $C=\mathbb{Z}_3[\xi_1,\xi_2,\dots]\subseteq\mathcal{A}_{3*}$, then

$$\mathrm{H}_*(M\mathrm{SO},\mathbb{Z}_3)=C\otimes\mathbb{Z}_3[z_1',z_2',\dots] \tag{4.39}$$

where $|z_k'|=4k$ for $k\neq\frac{3^t-1}{2}$.

$$\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_3)=(\mathbb{Z}_3[z_1',z_2',\dots])^*\otimes C^*=C^*\oplus\Sigma^8C^*\oplus\cdots \tag{4.40}$$

where $C^*=\mathcal{A}_3/(\beta_{(3,3)})$ and $(\beta_{(3,3)})$ is the two-sided ideal of $\mathcal{A}_3$ generated by $\beta_{(3,3)}$.

$$\cdots\longrightarrow\Sigma^2\mathcal{A}_3\oplus\Sigma^6\mathcal{A}_3\oplus\cdots\longrightarrow\Sigma\mathcal{A}_3\oplus\Sigma^5\mathcal{A}_3\oplus\cdots\longrightarrow\mathcal{A}_3\longrightarrow\mathcal{A}_3/(\beta_{(3,3)}) \tag{4.41}$$

is an $\mathcal{A}_3$-resolution of $\mathcal{A}_3/(\beta_{(3,3)})$ where the differentials d_1 are induced by $\beta_{(3,3)}$..

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_3)=\mathbb{Z}_3[x_2',x_8',\dots]\otimes\Lambda_{\mathbb{Z}_3}(x_3',x_7',\dots) \tag{4.42}$$

$\beta_{(3,3)}x_2'=x_3'$, $\beta_{(3,3)}x_2'^2=2x_2'x_3'$.

The E_2 page is shown in Figure 16.

Hence we have the following

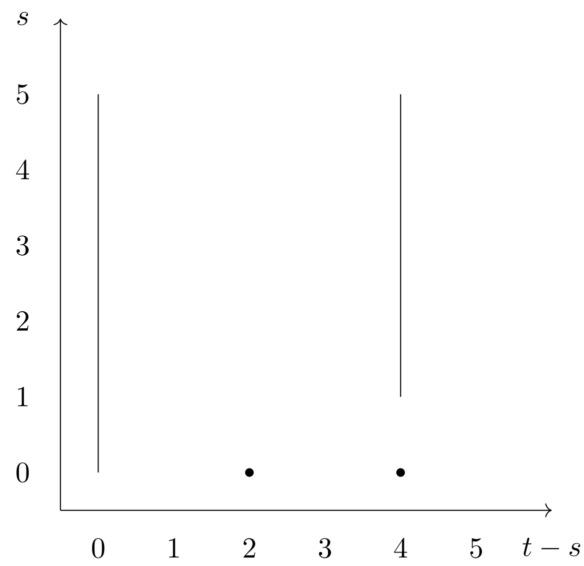


Figure 16: $\Omega_*^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)_3^\wedge$

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}$
1	0
2	$\mathbb{Z}_3$
3	0
4	$\mathbb{Z}\times\mathbb{Z}_3$
5	$\mathbb{Z}_2$

Theorem 26.

The bordism invariant of $\Omega_2^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)$ is x_2'.

The bordism invariants of $\Omega_4^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)$ are σ and $x_2'^2$.

The bordism invariant of $\Omega_5^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)$ is w_2w_3.

i	$\mathrm{TP}_i(\mathrm{SO}\times\mathrm{B}\mathbb{Z}_3)$
0	0
1	0
2	$\mathbb{Z}_3$
3	$\mathbb{Z}$
4	$\mathbb{Z}_3$
5	$\mathbb{Z}_2$

Theorem 27.

The 2d topological term is x_2'.

The 3d topological term is $\frac{1}{3}\mathrm{CS}_3^{(TM)}$.

The 4d topological term is $x_2'^2$.

The 5d topological term is w_2w_3.

4.4.8 $\Omega_d^{\mathrm{Spin}}(\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Spin}\wedge(\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Spin}}(\mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}. \tag{4.43}$$

Since $\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathbb{Z}_2$, we have $\Omega_d^{\mathrm{Spin}}(\mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}=\Omega_d^{\mathrm{Spin}}$.

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Spin}\wedge(\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{Spin}}(\mathrm{B}^2\mathbb{Z}_3)_3^{\wedge}. \tag{4.44}$$

Since there is a short exact sequence of groups

$$1\longrightarrow\mathbb{Z}_2\longrightarrow\mathrm{Spin}\longrightarrow\mathrm{SO}\longrightarrow 1, \tag{4.45}$$

we have a fibration

$$\begin{array}{ccc}\mathrm{B}\mathbb{Z}_2 & \longrightarrow & \mathrm{BSpin}\\ & & \downarrow\\ & & \mathrm{BSO}\end{array} \tag{4.46}$$

Take the localization at prime 3, we have a homotopy equivalence $\mathrm{BSpin}_{(3)}\sim\mathrm{BSO}_{(3)}$ since the localization of $\mathrm{B}\mathbb{Z}_2$ at 3 is trivial. Take the Thom spectra, we have a homotopy equivalence $M\mathrm{Spin}_{(3)}\sim M\mathrm{SO}_{(3)}$. Hence

$$\mathrm{H}^*(M\mathrm{Spin},\mathbb{Z}_3)=\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_3). \tag{4.47}$$

We have the following

i	$\Omega_i^{\mathrm{Spin}}(\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2\times\mathbb{Z}_3$
3	0
4	$\mathbb{Z}\times\mathbb{Z}_3$
5	0

Theorem 28.

The bordism invariants of $\Omega_2^{\mathrm{Spin}}(\mathrm{B}^2\mathbb{Z}_3)$ are Arf and x_2'.

The bordism invariants of $\Omega_4^{\mathrm{Spin}}(\mathrm{B}^2\mathbb{Z}_3)$ are $\frac{\sigma}{16}$ and $x_2'^2$.

Theorem 29.

The 2d topological terms are Arf and x_2'.

The 3d topological term is $\frac{1}{48}\mathrm{CS}_3^{(TM)}$.

The 4d topological term is $x_2'^2$.

i	$\mathrm{TP}_i(\mathrm{Spin}\times \mathrm{B}\mathbb{Z}_3)$
0	0
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2\times\mathbb{Z}_3$
3	$\mathbb{Z}$
4	$\mathbb{Z}_3$
5	0

4.4.9 $\Omega_d^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^-\wedge(\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}. \tag{4.48}$$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+\wedge(\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_3)_3^{\wedge}. \tag{4.49}$$

Since $MT\mathrm{Pin}^+ = M\mathrm{Pin}^- \sim M\mathrm{Spin}\wedge S^1\wedge MT\mathrm{O}(1)$ and $\mathrm{H}^*(MT\mathrm{O}(1),\mathbb{Z}_3)=0$, we have $\mathrm{H}^*(M\mathrm{Pin}^-,\mathbb{Z}_3)=0$, thus $\Omega_d^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)_3^{\wedge}=0$.

Since $\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathbb{Z}_2$, we have $\Omega_d^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}=\Omega_d^{\mathrm{Pin}^+}$.

Hence $\Omega_d^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)=\Omega_d^{\mathrm{Pin}^+}$.

i	$\Omega_i^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_{16}$
5	0

Theorem 30.

The bordism invariant of $\Omega_2^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)$ is $w_1\tilde{\eta}$.

The bordism invariant of $\Omega_3^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)$ is $w_1\mathrm{Arf}$.

The bordism invariant of $\Omega_4^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)$ is η.

i	$\mathrm{TP}_i(\mathrm{Pin}^+\times \mathrm{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_{16}$
5	0

Theorem 31.

The 2d topological term is $w_1\tilde{\eta}$.

The 3d topological term is $w_1\mathrm{Arf}$.

The 4d topological term is η.

4.4.10 $\Omega_d^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+\wedge(\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{4.50}$$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+\wedge(\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{4.51}$$

Since $MT\mathrm{Pin}^- = M\mathrm{Pin}^+ \sim M\mathrm{Spin}\wedge S^{-1}\wedge M\mathrm{O}(1)$ and $\mathrm{H}^*(M\mathrm{O}(1),\mathbb{Z}_3)=0$, we have $\mathrm{H}^*(M\mathrm{Pin}^+,\mathbb{Z}_3)=0$, thus $\Omega_d^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_3)_3^\wedge=0$. Since $\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathbb{Z}_2$, we have $\Omega_d^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_3)_2^\wedge=\Omega_d^{\mathrm{Pin}^-}$.

Hence $\Omega_d^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_3)=\Omega_d^{\mathrm{Pin}^-}$.

i	$\Omega_i^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	0
5	0

Theorem 32.

The bordism invariant of $\Omega_2^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_3)$ is ABK.

i	$\mathrm{TP}_i(\mathrm{Pin}^-\times\mathrm{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	0
5	0

Theorem 33.

The 2d topological term is ABK.

4.5 $\mathrm{B}G_a$: $\mathrm{BPSU}(2), \mathrm{BPSU}(3)$

4.5.1 $\Omega_d^{\mathrm{O}}(\mathrm{BPSU}(2))$

$$\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{BPSU}(2),\mathbb{Z}_2)=\mathcal{A}_2\oplus 2\Sigma^2\mathcal{A}_2\oplus\Sigma^3\mathcal{A}_2\oplus 4\Sigma^4\mathcal{A}_2\oplus 3\Sigma^5\mathcal{A}_2\oplus\cdots. \tag{4.52}$$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O}\wedge(\mathrm{BPSU}(2))_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{BPSU}(2))_2^\wedge \tag{4.53}$$

i	$\Omega_i^{\mathrm{O}}(\mathrm{BPSU}(2))$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2^4$
5	$\mathbb{Z}_2^3$

Theorem 34.

The bordism invariants of $\Omega_2^{\mathrm{O}}(\mathrm{BPSU}(2))$ are w_2', w_1^2.

The bordism invariant of $\Omega_3^{\mathrm{O}}(\mathrm{BPSU}(2))$ is $w_3'=w_1w_2'$.

The bordism invariants of $\Omega_4^{\mathrm{O}}(\mathrm{BPSU}(2))$ are $w_2'^2, w_1^4, w_1^2w_2', w_2^2$.

The bordism invariants of $\Omega_5^{\mathrm{O}}(\mathrm{BPSU}(2))$ are $w_2w_3, w_1^2w_3', w_2'w_3'$.

i	$\mathrm{TP}_i(\mathrm{O}\times\mathrm{PSU}(2))$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2^4$
5	$\mathbb{Z}_2^3$

Theorem 35.

The 2d topological terms are w_2', w_1^2.

The 3d topological term is $w_3'=w_1w_2'$.

The 4d topological terms are $w_2'^2, w_1^4, w_1^2w_2', w_2^2$.

The 5d topological terms are $w_2w_3, w_1^2w_3', w_2'w_3'$.

4.5.2 $\Omega_d^{\mathrm{SO}}(\mathrm{BPSU}(2))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO} \wedge (\mathrm{BPSU}(2))_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{BPSU}(2))_2^\wedge. \tag{4.54}$$

The E_2 page is shown in Figure 17.

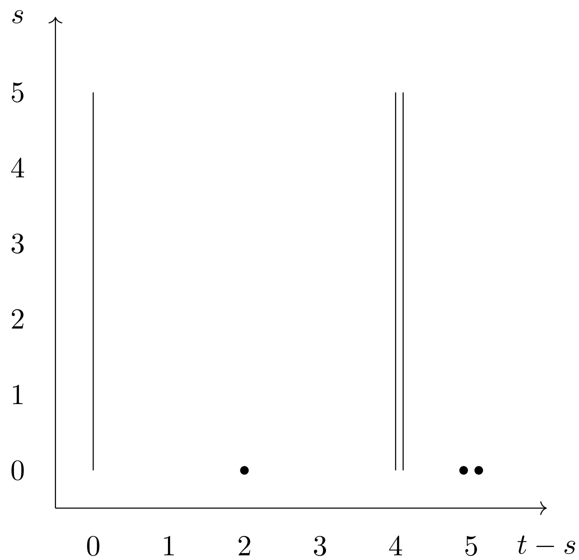


Figure 17: $\Omega_*^{\mathrm{SO}}(\mathrm{BPSU}(2))_2^\wedge$

i	$\Omega_i^{\mathrm{SO}}(\mathrm{BPSU}(2))$
0	$\mathbb{Z}$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}^2$
5	$\mathbb{Z}_2^2$

Theorem 36.

The bordism invariant of $\Omega_2^{\mathrm{SO}}(\mathrm{BPSU}(2))$ is w_2'.

The bordism invariants of $\Omega_4^{\mathrm{SO}}(\mathrm{BPSU}(2))$ are σ, p_1'.

The bordism invariants of $\Omega_5^{\mathrm{SO}}(\mathrm{BPSU}(2))$ are $w_2w_3, w_2'w_3'$.

The manifold generators of $\Omega_4^{\mathrm{SO}}(\mathrm{BPSU}(2))$ are $(\mathbb{CP}^2, 3)$ and $(\mathbb{CP}^2, L_{\mathbb{C}}+1)$ where n is the trivial real n-plane bundle and $L_{\mathbb{C}}$ is the tautological complex line bundle over $\mathbb{CP}^2$. Note that the principal SO(3)-bundle P associated to $L_{\mathbb{C}}+1$ is the induce bundle $P' \times_{\mathrm{SO}(2)} \mathrm{SO}(3)$ from P'

$$\begin{array}{ccc} S^1 = \mathrm{SO}(2) & \longrightarrow & S^5 \\ & & \downarrow \\ & & \mathbb{CP}^2 \end{array} \tag{4.55}$$

by the group homomorphism $\phi : \mathrm{SO}(2) \to \mathrm{SO}(3)$ which is the inclusion map, that means $P = P' \times_{\mathrm{SO}(2)} \mathrm{SO}(3) = (P' \times \mathrm{SO}(3))/\mathrm{SO}(2)$ which is the quotient of $P' \times \mathrm{SO}(3)$ by the right $\mathrm{SO}(2)$ action

$$(p, g)h = (ph, \phi(h^{-1})g). \tag{4.56}$$

$$p_1(L_{\mathbb{C}} + 1) = p_1(L_{\mathbb{C}}) = -c_2(L_{\mathbb{C}} \otimes_{\mathbb{R}} \mathbb{C}) = -c_2(L_{\mathbb{C}} \oplus \bar{L}_{\mathbb{C}}) = -c_1(L_{\mathbb{C}})c_1(\bar{L}_{\mathbb{C}}) = c_1(L_{\mathbb{C}})^2. \tag{4.57}$$

So

$$\int_{\mathbb{CP}^2} p_1(L_{\mathbb{C}} + 1) = 1. \tag{4.58}$$

i	$\mathrm{TP}_i(\mathrm{SO} \times \mathrm{PSU}(2))$
0	0
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}^2$
4	0
5	$\mathbb{Z}_2^2$

Theorem 37.

The 2d topological term is w_2'.

Since $p_1' = \mathrm{dCS}_3^{(\mathrm{SO}(3))}$, the 3d topological terms are $\frac{1}{3}\mathrm{CS}_3^{(TM)}$ and $\mathrm{CS}_3^{(\mathrm{SO}(3))}$.

The 5d topological terms are $w_2 w_3, w_2' w_3'$.

4.5.3 $\Omega_d^{\mathrm{Spin}}(\mathrm{BPSU}(2))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Spin} \wedge (\mathrm{BPSU}(2))_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Spin}}(\mathrm{BPSU}(2))_2^\wedge. \tag{4.59}$$

For $t - s < 8$,

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^*(\mathrm{BPSU}(2), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Spin}}(\mathrm{BPSU}(2))_2^\wedge. \tag{4.60}$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{BPSU}(2), \mathbb{Z}_2)$ and the E_2 page are shown in Figure 18, 19.

Theorem 38.

The bordism invariants of $\Omega_2^{\mathrm{Spin}}(\mathrm{BPSU}(2))$ are w_2' and Arf.

By Wu formula (6.62), $w_2'^2 = \mathrm{Sq}^2(w_2') = (w_2(TM) + w_1(TM)^2)w_2' = 0$ on Spin 4-manifolds, $p_1' = w_2'^2 = 0 \mod 2$ on Spin 4-manifolds.

The bordism invariants of $\Omega_4^{\mathrm{Spin}}(\mathrm{BPSU}(2))$ are $\frac{\sigma}{16}$ and $\frac{p_1'}{2}$.

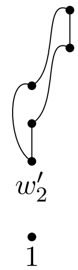


Figure 18: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{BPSU}(2), \mathbb{Z}_2)$

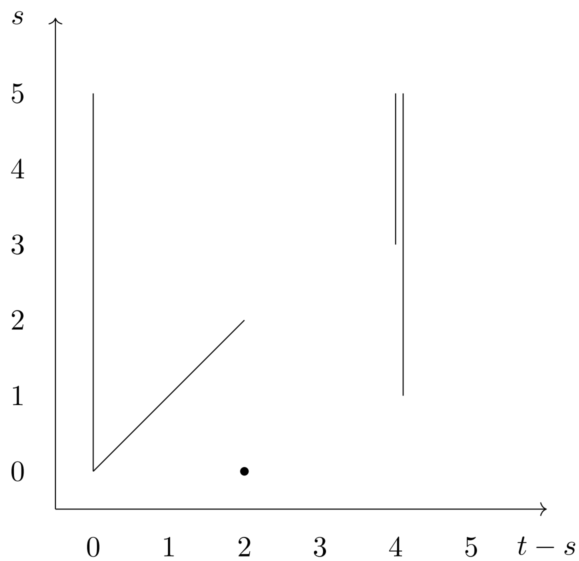


Figure 19: $\Omega_*^{\mathrm{Spin}}(\mathrm{BPSU}(2))_2^\wedge$

i	$\Omega_i^{\mathrm{Spin}}(\mathrm{BPSU}(2))$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2$
3	0
4	$\mathbb{Z}^2$
5	0

i	$\mathrm{TP}_i(\mathrm{Spin} \times \mathrm{PSU}(2))$
0	0
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}^2$
4	0
5	0

Theorem 39.

The 2d topological terms are w_2' and Arf.

The 3d topological terms are $\frac{1}{48}\mathrm{CS}_3^{(TM)}$ and $\frac{1}{2}\mathrm{CS}_3^{(\mathrm{SO}(3))}$.

4.5.4 $\Omega_d^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(MT\mathrm{Pin}^+ \wedge (\mathrm{BPSU}(2))_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))_2^\wedge. \tag{4.61}$$

$MT\mathrm{Pin}^+ = M\mathrm{Spin} \wedge S^1 \wedge MT\mathrm{O}(1)$.

For $t - s < 8$,

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*-1}(MT\mathrm{O}(1), \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{BPSU}(2), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))_2^\wedge. \tag{4.62}$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MT\mathrm{O}(1), \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{BPSU}(2), \mathbb{Z}_2)$ and the E_2 page are shown in Figure 20, 21.

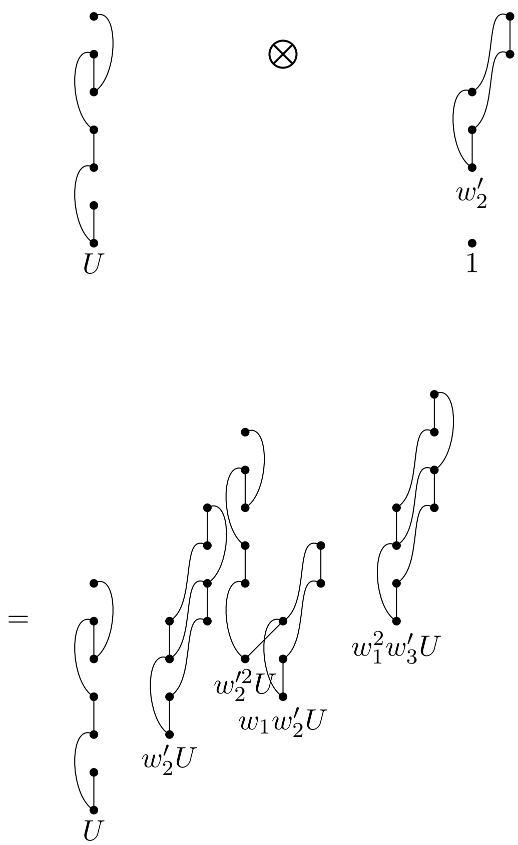


Figure 20: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MT\mathrm{O}(1), \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{BPSU}(2), \mathbb{Z}_2)$

Theorem 40.

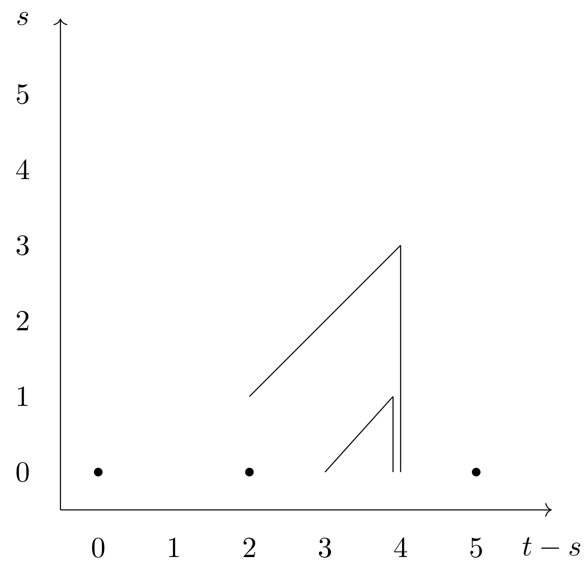


Figure 21: $\Omega_*^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))_2^\wedge$

i	$\Omega_i^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_4 \times \mathbb{Z}_{16}$
5	$\mathbb{Z}_2$

The bordism invariants of $\Omega_2^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))$ are w_2' and $w_1\tilde{\eta}$.

The bordism invariants of $\Omega_3^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))$ are $w_1w_2' = w_3'$ and $w_1\mathrm{Arf}$.

The bordism invariants of $\Omega_4^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))$ are $q_s(w_2')$ (this invariant has another form, see the footnotes of Table 3) and η.

The bordism invariant of $\Omega_5^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))$ is $w_1^2w_3'(= w_2'w_3')$.

i	$\mathrm{TP}_i(\mathrm{Pin}^+ \times \mathrm{PSU}(2))$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_4 \times \mathbb{Z}_{16}$
5	$\mathbb{Z}_2$

Theorem 41.

The 2d topological terms are w_2' and $w_1\tilde{\eta}$.

The 3d topological terms are $w_1w_2' = w_3'$ and $w_1\mathrm{Arf}$.

The 4d topological terms are $q_s(w_2')$ and η.

The 5d topological term is $w_1^2w_3'(=w_2'w_3')$.

4.5.5 $\Omega_d^{\mathrm{Pin}^-}(\mathrm{BPSU}(2))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(MT\mathrm{Pin}^-\wedge(\mathrm{BPSU}(2))_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{BPSU}(2))_2^\wedge. \tag{4.63}$$

$MT\mathrm{Pin}^-=M\mathrm{Spin}\wedge S^{-1}\wedge M\mathrm{O}(1)$.

For $t-s<8$,

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*+1}(M\mathrm{O}(1),\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BPSU}(2),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{BPSU}(2))_2^\wedge. \tag{4.64}$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+1}(M\mathrm{O}(1),\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BPSU}(2),\mathbb{Z}_2)$ and the E_2 page are shown in Figure 22, 23.

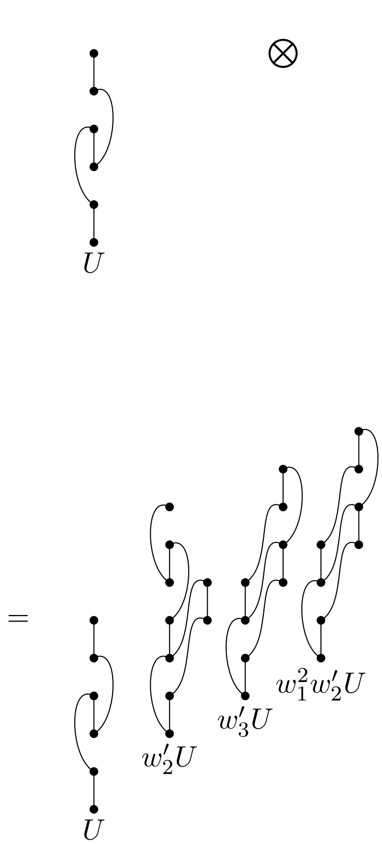


Figure 22: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+1}(M\mathrm{O}(1),\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BPSU}(2),\mathbb{Z}_2)$

Theorem 42.

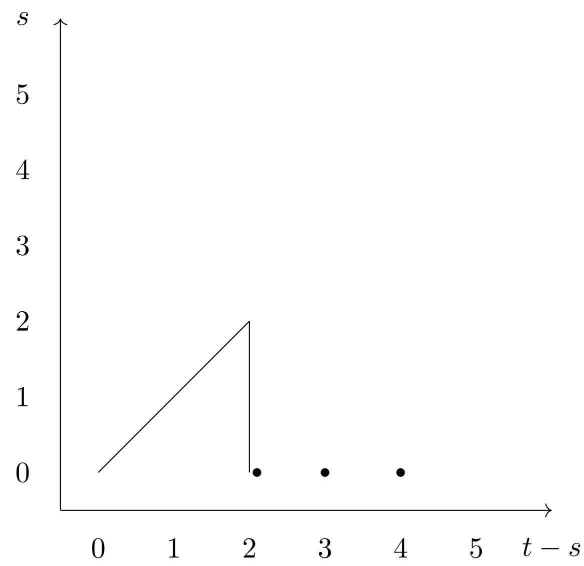


Figure 23: $\Omega_*^{\mathrm{Pin}^-}(\mathrm{BPSU}(2))_2^\wedge$

i	$\Omega_i^{\mathrm{Pin}^-}(\mathrm{BPSU}(2))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2 \times \mathbb{Z}_8$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2$
5	0

The bordism invariants of $\Omega_2^{\mathrm{Pin}^-}(\mathrm{BPSU}(2))$ are w_2' and ABK.

The bordism invariant of $\Omega_3^{\mathrm{Pin}^-}(\mathrm{BPSU}(2))$ is $w_1w_2' = w_3'$.

The bordism invariant of $\Omega_4^{\mathrm{Pin}^-}(\mathrm{BPSU}(2))$ is $w_1^2w_2'$.

i	$\mathrm{TP}_i(\mathrm{Pin}^- \times \mathrm{PSU}(2))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2 \times \mathbb{Z}_8$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2$
5	0

Theorem 43.

The 2d topological terms are w_2' and ABK.

The 3d topological term is $w_1w_2' = w_3'$.

The 4d topological term is $w_1^2w_2'$.

4.5.6 $\Omega_d^{\mathrm{O}}(\mathrm{BPSU}(3))$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_3)\otimes\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{BPSU}(3))_3^\wedge. \tag{4.65}$$

Since $\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_3)=0$, $\Omega_d^{\mathrm{O}}(\mathrm{BPSU}(3))_3^\wedge=0$.

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{BPSU}(3))_2^\wedge. \tag{4.66}$$

$$\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2)=\mathcal{A}_2\oplus\Sigma^2\mathcal{A}_2\oplus3\Sigma^4\mathcal{A}_2\oplus\Sigma^5\mathcal{A}_2\oplus\cdots. \tag{4.67}$$

i	$\Omega_i^{\mathrm{O}}(\mathrm{BPSU}(3))$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^3$
5	$\mathbb{Z}_2$

Theorem 44.

The bordism invariant of $\Omega_2^{\mathrm{O}}(\mathrm{BPSU}(3))$ is w_1^2.

The bordism invariants of $\Omega_4^{\mathrm{O}}(\mathrm{BPSU}(3))$ are $w_1^4, w_2^2, c_2(\mod 2)$.

The bordism invariant of $\Omega_5^{\mathrm{O}}(\mathrm{BPSU}(3))$ is w_2w_3.

i	$\mathrm{TP}_i(\mathrm{O}\times\mathrm{PSU}(3))$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^3$
5	$\mathbb{Z}_2$

Theorem 45.

The 2d topological term is w_1^2.

The 4d topological terms are $w_1^4, w_2^2, c_2(\mod 2)$.

The 5d topological term is w_2w_3.

4.5.7 $\Omega_d^{\mathrm{SO}}(\mathrm{BPSU}(3))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{BPSU}(3))_2^\wedge. \tag{4.68}$$

$$\mathrm{H}^*(\mathrm{BPSU}(3), \mathbb{Z}_2) = \mathbb{Z}_2[c_2, c_3]. \tag{4.69}$$

The E_2 page is shown in Figure 24.

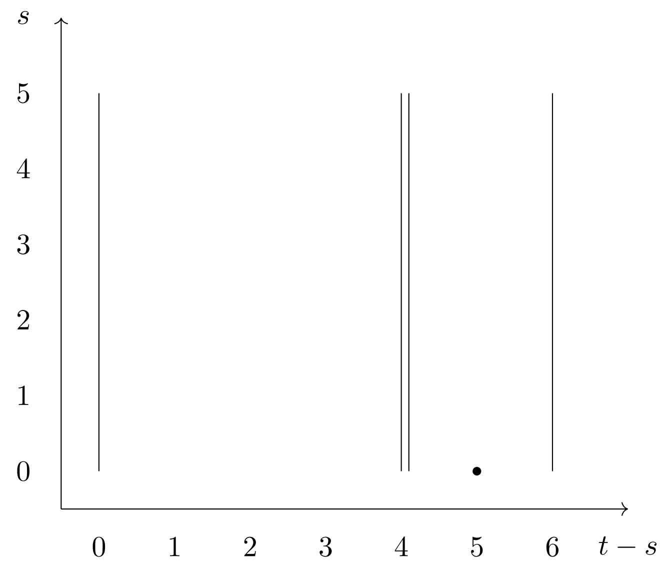


Figure 24: $\Omega_*^{\mathrm{SO}}(\mathrm{BPSU}(3))_2^\wedge$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{SO}, \mathbb{Z}_3) \otimes \mathrm{H}^*(\mathrm{BPSU}(3), \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{BPSU}(3))_3^\wedge. \tag{4.70}$$

$$\mathrm{H}^*(\mathrm{BPSU}(3), \mathbb{Z}_3) = (\mathbb{Z}_3[z_2, z_8, z_{12}] \otimes \Lambda_{\mathbb{Z}_3}(z_3, z_7))/(z_2 z_3, z_2 z_7, z_2 z_8 + z_3 z_7) \tag{4.71}$$

$\beta_{(3,3)} z_2 = z_3$, $\beta_{(3,3)} z_2^2 = 2 z_2 z_3 = 0$, $\beta_{(3,3)} z_2^3 = 0$.

The E_2 page is shown in Figure 25.

i	$\Omega_i^{\mathrm{SO}}(\mathrm{BPSU}(3))$
0	$\mathbb{Z}$
1	0
2	$\mathbb{Z}_3$
3	0
4	$\mathbb{Z}^2$
5	$\mathbb{Z}_2$
6	$\mathbb{Z}$

Theorem 46.

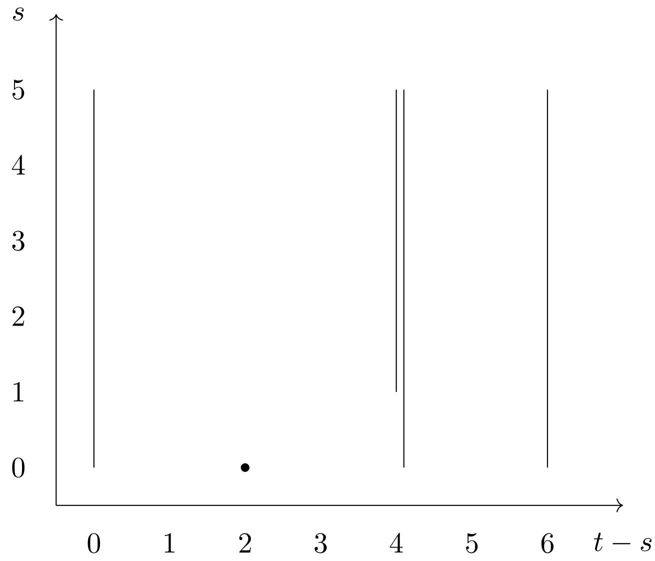


Figure 25: $\Omega_*^{\mathrm{SO}}(\mathrm{BPSU}(3))_3^\wedge$

The bordism invariant of $\Omega_2^{\mathrm{SO}}(\mathrm{BPSU}(3))$ is z_2.

The bordism invariants of $\Omega_4^{\mathrm{SO}}(\mathrm{BPSU}(3))$ are σ, c_2.

The bordism invariant of $\Omega_5^{\mathrm{SO}}(\mathrm{BPSU}(3))$ is $w_2 w_3$.

The bordism invariant of $\Omega_6^{\mathrm{SO}}(\mathrm{BPSU}(3))$ is c_3.

The manifold generators of $\Omega_4^{\mathrm{SO}}(\mathrm{BPSU}(3))$ are $(\mathbb{CP}^2, \mathbb{CP}^2 \times \mathrm{PSU}(3))$ and (S^4, H) where H is the induced bundle from the Hopf fibration H'

$$\begin{array}{ccc} S^3 = \mathrm{SU}(2) & \longrightarrow & S^7 \\ & & \downarrow \\ & & S^4 \end{array} \tag{4.72}$$

by the group homomorphism $\rho : \mathrm{SU}(2) \to \mathrm{PSU}(3)$ which is the composition of the inclusion map $\mathrm{SU}(2) \to \mathrm{SU}(3)$ and the quotient map $\mathrm{SU}(3) \to \mathrm{PSU}(3)$, that means $H = H' \times_{\mathrm{SU}(2)} \mathrm{PSU}(3) = (H' \times \mathrm{PSU}(3))/\mathrm{SU}(2)$ which is the quotient of $H' \times \mathrm{PSU}(3)$ by the right $\mathrm{SU}(2)$ action

$$(p, g)h = (ph, \rho(h^{-1})g). \tag{4.73}$$

Theorem 47.

The 2d topological term is z_2.

Since $c_2 = \mathrm{dCS}_3^{(\mathrm{PSU}(3))}$, the 3d topological terms are $\frac{1}{3}\mathrm{CS}_3^{(TM)}$ and $\mathrm{CS}_3^{(\mathrm{PSU}(3))}$.

Since $c_3 = \mathrm{dCS}_5^{(\mathrm{PSU}(3))}$, the 5d topological term are $\mathrm{CS}_5^{(\mathrm{PSU}(3))}$ and $w_2 w_3$.

i	$\mathrm{TP}_i(\mathrm{SO} \times \mathrm{PSU}(3))$
0	0
1	0
2	$\mathbb{Z}_3$
3	$\mathbb{Z}^2$
4	0
5	$\mathbb{Z} \times \mathbb{Z}_2$

4.5.8 $\Omega_d^{\mathrm{Spin}}(\mathrm{BPSU}(3))$

For $t - s < 8$,

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^*(\mathrm{BPSU}(3), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Spin}}(\mathrm{BPSU}(3))_2^\wedge. \tag{4.74}$$

The E_2 page is shown in Figure 26.

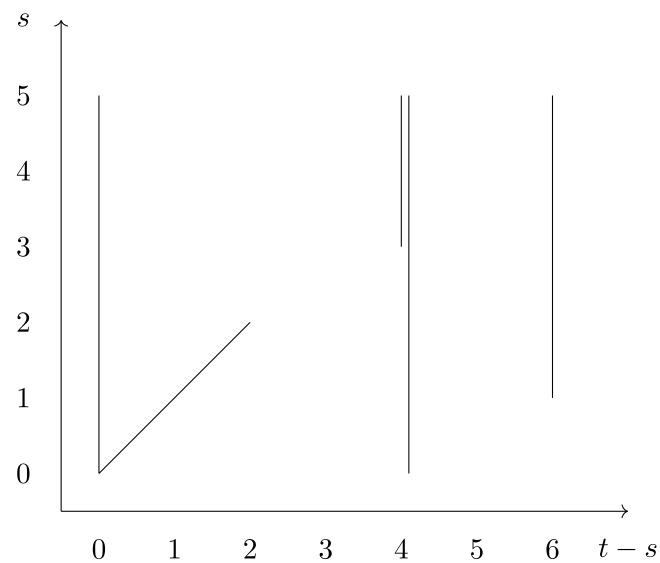


Figure 26: $\Omega_*^{\mathrm{Spin}}(\mathrm{BPSU}(3))_2^\wedge$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Spin}, \mathbb{Z}_3) \otimes \mathrm{H}^*(\mathrm{BPSU}(3), \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow \Omega_{t-s}^{\mathrm{Spin}}(\mathrm{BPSU}(3))_3^\wedge. \tag{4.75}$$

Since $\mathrm{H}^*(M\mathrm{Spin}, \mathbb{Z}_3) = \mathrm{H}^*(M\mathrm{SO}, \mathbb{Z}_3)$, the E_2 page is shown in Figure 27.

Theorem 48.

The bordism invariants of $\Omega_2^{\mathrm{Spin}}(\mathrm{BPSU}(3))$ are Arf and z_2.

The bordism invariants of $\Omega_4^{\mathrm{Spin}}(\mathrm{BPSU}(3))$ are $\frac{\sigma}{16}$ and c_2.

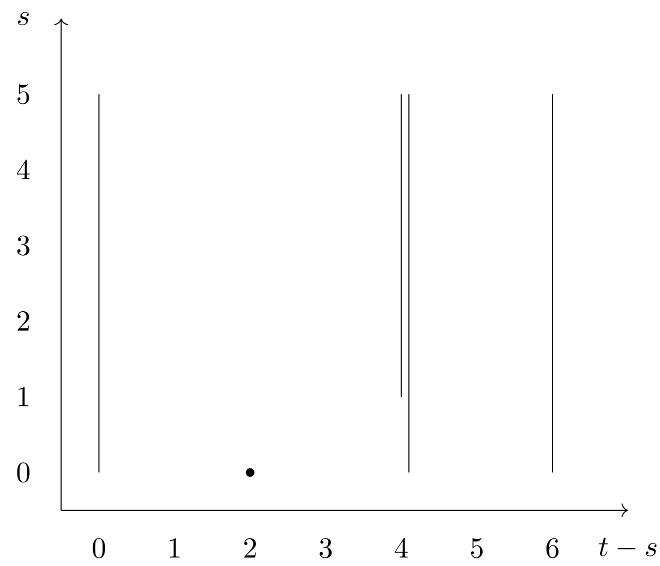


Figure 27: $\Omega_*^{\text{Spin}}(\text{BPSU}(3))_3^\wedge$

i	$\Omega_i^{\text{Spin}}(\text{BPSU}(3))$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2 \times \mathbb{Z}_3$
3	0
4	$\mathbb{Z}^2$
5	0
6	$\mathbb{Z}$

By Wu formula (6.62), $c_3 = \text{Sq}^2 c_2 = (w_2(TM) + w_1^2(TM))c_2 = 0 \mod 2$ on Spin 6-manifolds.

The bordism invariant of $\Omega_6^{\text{Spin}}(\text{BPSU}(3))$ is $\frac{c_3}{2}$.

i	$\text{TP}_i(\text{Spin} \times \text{PSU}(3))$
0	0
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2 \times \mathbb{Z}_3$
3	$\mathbb{Z}^2$
4	0
5	$\mathbb{Z}$

Theorem 49.

The 2d topological terms are Arf and z_2.

The 3d topological terms are $\frac{1}{48}\text{CS}_3^{(TM)}$ and $\text{CS}_3^{(\text{PSU}(3))}$.

The 5d topological term is $\frac{1}{2}\text{CS}_5^{(\text{PSU}(3))}$.

4.5.9 $\Omega_d^{\mathrm{Pin}^+}(\mathrm{BPSU}(3))$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^-,\mathbb{Z}_3)\otimes\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^+}(\mathrm{BPSU}(3))_3^\wedge. \tag{4.76}$$

Since $\mathrm{H}^*(M\mathrm{Pin}^-,\mathbb{Z}_3)=\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_3)=0$, $\Omega_{t-s}^{\mathrm{Pin}^+}(\mathrm{BPSU}(3))_3^\wedge=0$.

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^-,\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^+}(\mathrm{BPSU}(3))_2^\wedge. \tag{4.77}$$

For $t-s<8$,

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*-1}(MT\mathrm{O}(1),\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^+}(\mathrm{BPSU}(3))_2^\wedge. \tag{4.78}$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MT\mathrm{O}(1),\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2)$ and the E_2 page are shown in Figure 28, 29.

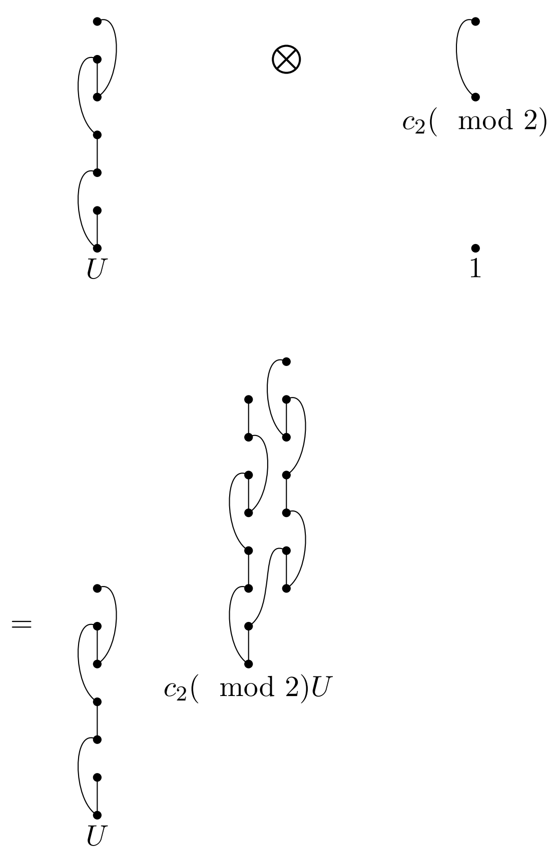


Figure 28: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MT\mathrm{O}(1),\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2)$

Theorem 50.

The bordism invariant of $\Omega_2^{\mathrm{Pin}^+}(\mathrm{BPSU}(3))$ is $w_1\tilde{\eta}$.

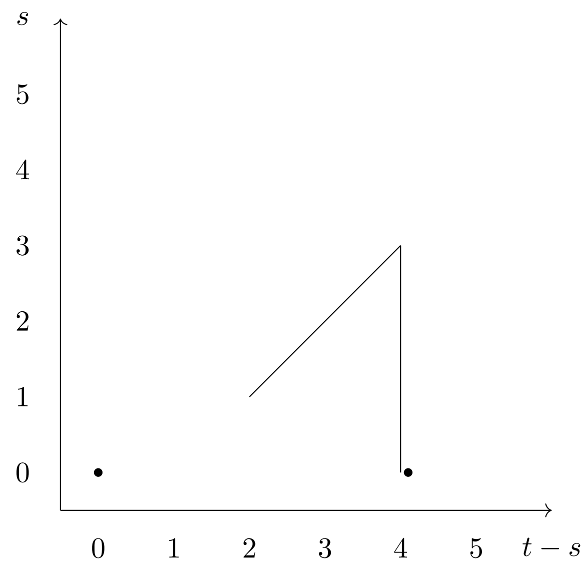


Figure 29: $\Omega_*^{\text{Pin}^+}(\text{BPSU}(3))_2^\wedge$

i	$\Omega_i^{\text{Pin}^+}(\text{BPSU}(3))$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$
5	0

The bordism invariant of $\Omega_3^{\text{Pin}^+}(\text{BPSU}(3))$ is $w_1\text{Arf}$.

The bordism invariants of $\Omega_4^{\text{Pin}^+}(\text{BPSU}(3))$ are $c_2(\mod 2)$ and η.

i	$\text{TP}_i(\text{Pin}^+ \times \text{PSU}(3))$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$
5	0

Theorem 51.

The 2d topological term is $w_1\tilde{\eta}$.

The 3d topological term is $w_1\text{Arf}$.

The 4d topological terms are $c_2(\mod 2)$ and η.

4.5.10 $\Omega_d^{\text{Pin}^-}(\text{BPSU}(3))$

$$\text{Ext}_{\mathcal{A}_3}^{s,t}(\text{H}^*(M\text{Pin}^+,\mathbb{Z}_3)\otimes\text{H}^*(\text{BPSU}(3),\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\text{Pin}^-}(\text{BPSU}(3))_3^\wedge. \tag{4.79}$$

Since $\text{H}^*(M\text{Pin}^+,\mathbb{Z}_3)=\text{H}^*(M\text{O},\mathbb{Z}_3)=0$, $\Omega_{t-s}^{\text{Pin}^-}(\text{BPSU}(3))_3^\wedge=0$.

$$\text{Ext}_{\mathcal{A}_2}^{s,t}(\text{H}^*(M\text{Pin}^+,\mathbb{Z}_2)\otimes\text{H}^*(\text{BPSU}(3),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\text{Pin}^-}(\text{BPSU}(3))_2^\wedge. \tag{4.80}$$

For $t-s<8$,

$$\text{Ext}_{\mathcal{A}_2(1)}^{s,t}(\text{H}^{*+1}(M\text{O}(1),\mathbb{Z}_2)\otimes\text{H}^*(\text{BPSU}(3),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\text{Pin}^-}(\text{BPSU}(3))_2^\wedge. \tag{4.81}$$

The $\mathcal{A}_2(1)$-module structure of $\text{H}^{*+1}(M\text{O}(1),\mathbb{Z}_2)\otimes\text{H}^*(\text{BPSU}(3),\mathbb{Z}_2)$ and the E_2 page are shown in Figure 30, 31.

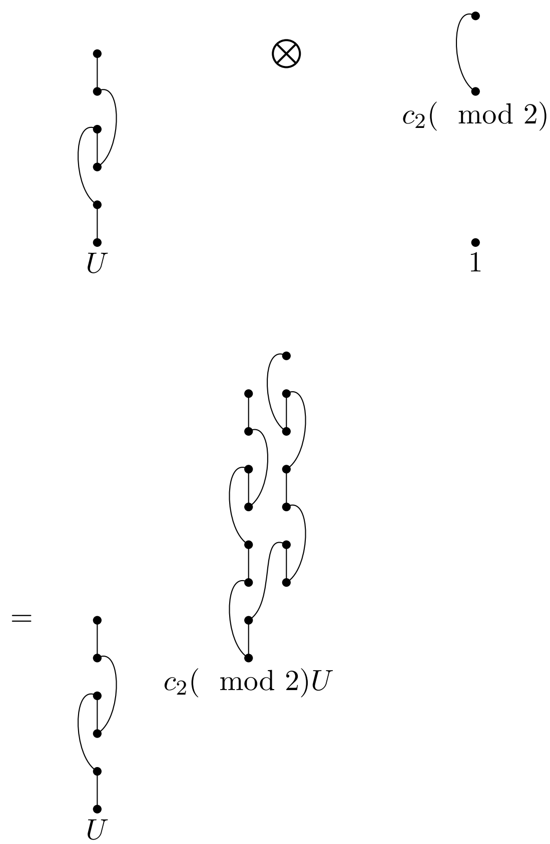


Figure 30: The $\mathcal{A}_2(1)$-module structure of $\text{H}^{*+1}(M\text{O}(1),\mathbb{Z}_2)\otimes\text{H}^*(\text{BPSU}(3),\mathbb{Z}_2)$

Theorem 52.

The bordism invariant of $\Omega_2^{\text{Pin}^-}(\text{BPSU}(3))$ is ABK.

The bordism invariant of $\Omega_4^{\text{Pin}^-}(\text{BPSU}(3))$ is $c_2(\mod 2)$.

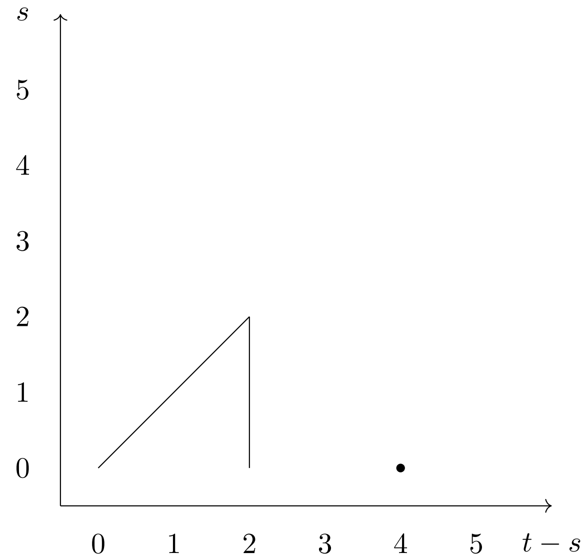


Figure 31: $\Omega_*^{\mathrm{Pin}^-}(\mathrm{BPSU}(3))_2^\wedge$

i	$\Omega_i^{\mathrm{Pin}^-}(\mathrm{BPSU}(3))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	$\mathbb{Z}_2$
5	0

i	$\mathrm{TP}_i(\mathrm{Pin}^- \times \mathrm{PSU}(3))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	$\mathbb{Z}_2$
5	0

Theorem 53.

The 2d topological term is ABK.

The 4d topological term is $c_2 \pmod 2$.

4.6 $(\mathrm{B}G_a, \mathrm{B}^2G_b) : (\mathrm{B}\mathbb{Z}_2, \mathrm{B}^2\mathbb{Z}_2), (\mathrm{B}\mathbb{Z}_3, \mathrm{B}^2\mathbb{Z}_3)$

4.6.1 $\Omega_d^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O} \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \\ \Rightarrow &\ \pi_{t-s}(M\mathrm{O} \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2). \end{aligned} \tag{4.82}$$

$$\begin{aligned} &\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^* \otimes \mathbb{Z}_2[a, x_2, x_3, x_5, x_9, \dots] \\ = &\ \mathcal{A}_2 \oplus \Sigma\mathcal{A}_2 \oplus 3\Sigma^2\mathcal{A}_2 \oplus 4\Sigma^3\mathcal{A}_2 \oplus 8\Sigma^4\mathcal{A}_2 \oplus 12\Sigma^5\mathcal{A}_2 \oplus \cdots \end{aligned} \tag{4.83}$$

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^8$
5	$\mathbb{Z}_2^{12}$

Theorem 54.

The bordism invariants of $\Omega_2^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are a^2, x_2, w_1^2.

The bordism invariants of $\Omega_3^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $x_3 = w_1x_2, ax_2, aw_1^2, a^3$.

The bordism invariants of $\Omega_4^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $w_1^4, w_2^2, a^4, a^2x_2, ax_3, x_2^2, w_1^2a^2, w_1^2x_2$.

The bordism invariants of $\Omega_5^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are

$$a^5, a^2x_3, a^3x_2, a^3w_1^2, ax_2^2, aw_1^4, ax_2w_1^2, aw_2^2, x_2x_3, w_1^2x_3, x_5, w_2w_3.$$

i	$\mathrm{TP}_i(\mathrm{O} \times \mathbb{Z}_2 \times \mathrm{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^8$
5	$\mathbb{Z}_2^{12}$

Theorem 55.

The 2d topological terms are a^2, x_2, w_1^2.

The 3d topological terms are $x_3 = w_1x_2, ax_2, aw_1^2, a^3$.

The 4d topological terms are $w_1^4, w_2^2, a^4, a^2x_2, ax_3, x_2^2, w_1^2a^2, w_1^2x_2$.

The 5d topological terms are

$$a^5, a^2x_3, a^3x_2, a^3w_1^2, ax_2^2, aw_1^4, ax_2w_1^2, aw_2^2, x_2x_3, w_1^2x_3, x_5, w_2w_3.$$

4.6.2 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO} \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \\ \Rightarrow \quad & \pi_{t-s}(M\mathrm{SO} \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2). \end{aligned} \tag{4.84}$$

$d_2(x_2x_3 + x_5) = x_2^2h_0^2$.

The E_2 page is shown in Figure 32.

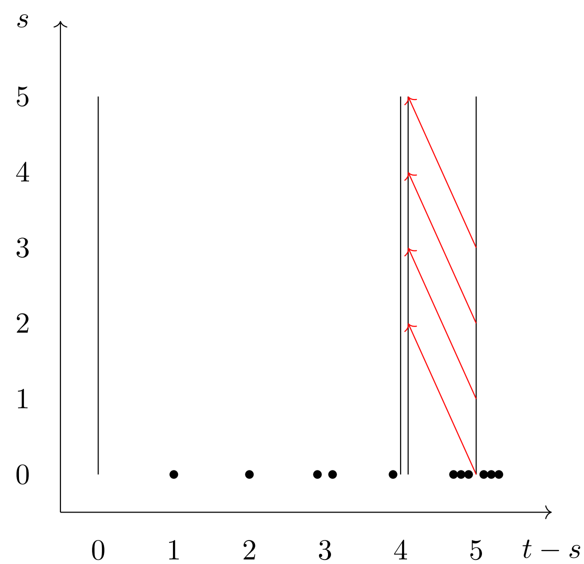


Figure 32: $\Omega_*^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4$
5	$\mathbb{Z}_2^6$

Theorem 56.

The bordism invariant of $\Omega_2^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ is x_2.

The bordism invariants of $\Omega_3^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are ax_2, a^3.

The bordism invariants of $\Omega_4^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are σ, $ax_3(= a^2x_2)$ and $\mathcal{P}_2(x_2)$.

The bordism invariants of $\Omega_5^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $ax_2^2, a^5, x_5, a^3x_2, w_2w_3, aw_2^2$.

i	$\mathrm{TP}_i(\mathrm{SO} \times \mathbb{Z}_2 \times \mathrm{B}\mathbb{Z}_2)$
0	0
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	$\mathbb{Z} \times \mathbb{Z}_2^2$
4	$\mathbb{Z}_2 \times \mathbb{Z}_4$
5	$\mathbb{Z}_2^6$

Theorem 57.

The 2d topological term is x_2.

The 3d topological terms are $\frac{1}{3}\mathrm{CS}_3^{(TM)}, ax_2, a^3$.

The 4d topological terms are $ax_3(= a^2x_2)$ and $\mathcal{P}_2(x_2)$.

The 5d topological terms are $ax_2^2, a^5, x_5, a^3x_2, w_2w_3, aw_2^2$.

4.6.3 $\Omega_d^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Spin} \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \\ \Rightarrow\ & \pi_{t-s}(M\mathrm{Spin} \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2). \end{aligned} \tag{4.85}$$

For $t - s < 8$,

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2). \tag{4.86}$$

$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[a, x_2, x_3, x_5, x_9, \dots]$ where $\mathrm{Sq}^1x_2 = x_3$, $\mathrm{Sq}^2x_2 = x_2^2$, $\mathrm{Sq}^1x_3 = 0$, $\mathrm{Sq}^2x_3 = x_5$, $\mathrm{Sq}^1x_5 = \mathrm{Sq}^2x_2^2 = x_3^2$, $\mathrm{Sq}^2x_5 = 0$.

$d_2(x_2x_3 + x_5) = x_2^2h_0^2$.

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$ and the E_2 page are shown in Figure 33, 34.

Theorem 58.

The bordism invariants of $\Omega_2^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $x_2, \mathrm{Arf}, a\tilde{\eta}$.

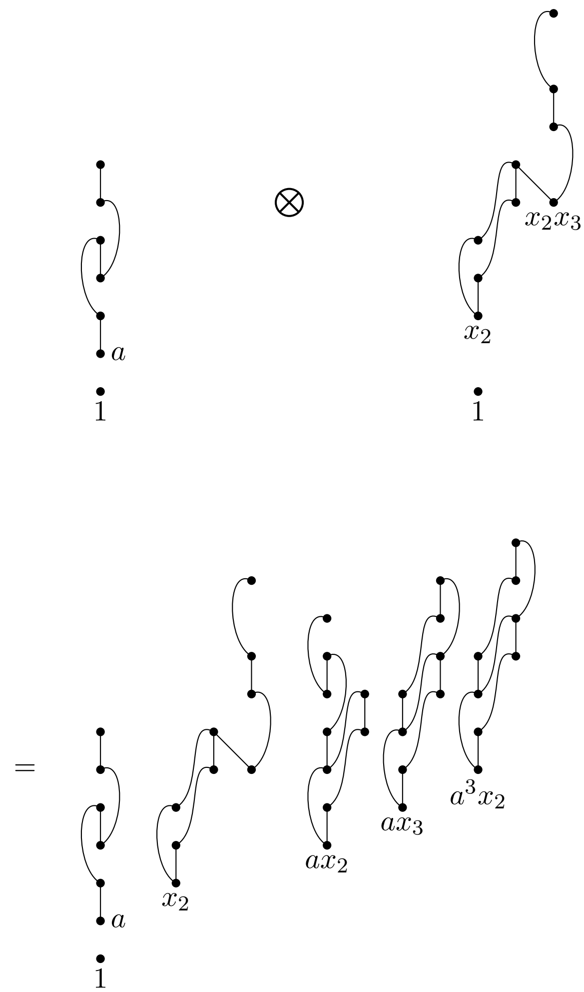


Figure 33: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$

i	$\Omega_i^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2^2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2 \times \mathbb{Z}_8$
4	$\mathbb{Z} \times \mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

The bordism invariants of $\Omega_3^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $ax_2, a\mathrm{ABK}$.

The bordism invariants of $\Omega_4^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $\frac{\sigma}{16}$, $ax_3(=a^2x_2)$ and $\frac{\mathcal{P}_2(x_2)}{2}$.

The bordism invariant of $\Omega_5^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ is a^3x_2.

Theorem 59.

The 2d topological terms are $x_2, \mathrm{Arf}, a\tilde{\eta}$.

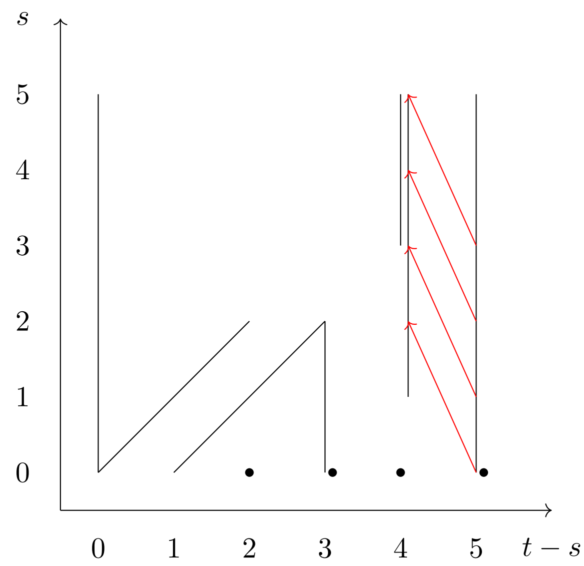


Figure 34: $\Omega_*^{\text{Spin}}(\text{B}\mathbb{Z}_2 \times \text{B}^2\mathbb{Z}_2)$

i	$\text{TP}_i(\text{Spin} \times \mathbb{Z}_2 \times \text{B}\mathbb{Z}_2)$
0	0
1	$\mathbb{Z}_2^2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_8$
4	$\mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

The 3d topological terms are $\frac{1}{48}\text{CS}_3^{(TM)}, ax_2, a\text{ABK}$.

The 4d topological terms are $ax_3 (= a^2x_2)$ and $\frac{\mathcal{P}_2(x_2)}{2}$.

The 5d topological term is a^3x_2.

4.6.4 $\Omega_d^{\text{Pin}^+}(\text{B}\mathbb{Z}_2 \times \text{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$\begin{aligned} &E_2^{s,t} = \text{Ext}_{\mathcal{A}_2}^{s,t}(\text{H}^*(M\text{Pin}^- \wedge (\text{B}\mathbb{Z}_2 \times \text{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \\ \Rightarrow\ &\pi_{t-s}(M\text{Pin}^- \wedge (\text{B}\mathbb{Z}_2 \times \text{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\text{Pin}^+}(\text{B}\mathbb{Z}_2 \times \text{B}^2\mathbb{Z}_2). \end{aligned} \quad (4.87)$$

$M\text{Pin}^- = MT\text{Pin}^+ \sim M\text{Spin} \wedge S^1 \wedge MT\text{O}(1)$.

For $t - s < 8$,

$$\text{Ext}_{\mathcal{A}_2(1)}^{s,t}(\text{H}^{*-1}(MT\text{O}(1), \mathbb{Z}_2) \otimes \text{H}^*(\text{B}\mathbb{Z}_2 \times \text{B}^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\text{Pin}^+}(\text{B}\mathbb{Z}_2 \times \text{B}^2\mathbb{Z}_2). \quad (4.88)$$

The $\mathcal{A}_2(1)$-module structure of $\text{H}^{*-1}(MT\text{O}(1), \mathbb{Z}_2) \otimes \text{H}^*(\text{B}\mathbb{Z}_2 \times \text{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$ and the E_2 page are shown in Figure 35, 36.

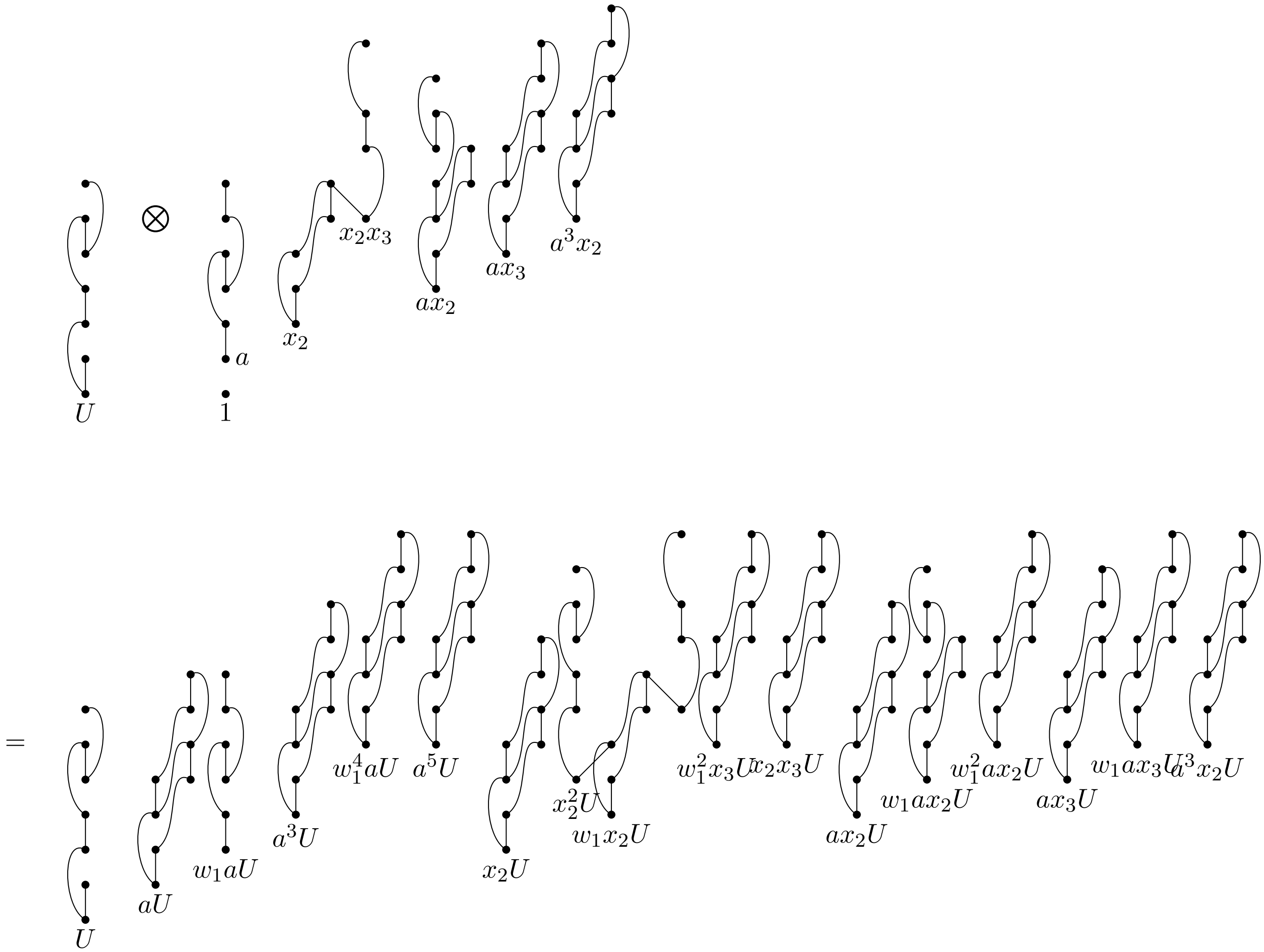


Figure 35: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MT\mathrm{O}(1),\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2\times\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$

i	$\Omega_i^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_2\times\mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^5$
4	$\mathbb{Z}_2^2\times\mathbb{Z}_4\times\mathbb{Z}_8\times\mathbb{Z}_{16}$
5	$\mathbb{Z}_2^7$

Theorem 60.

The bordism invariants of $\Omega_2^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_2\times\mathrm{B}^2\mathbb{Z}_2)$ are $w_1a=a^2, x_2, w_1\tilde{\eta}$.

The bordism invariants of $\Omega_3^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_2\times\mathrm{B}^2\mathbb{Z}_2)$ are $a^3, w_1x_2=x_3, ax_2, w_1a\tilde{\eta}, w_1\mathrm{Arf}$.

The bordism invariants of $\Omega_4^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_2\times\mathrm{B}^2\mathbb{Z}_2)$ are $ax_3, w_1ax_2(=a^2x_2+ax_3), q_s(x_2), w_1a(\mathrm{ABK}), \eta$.

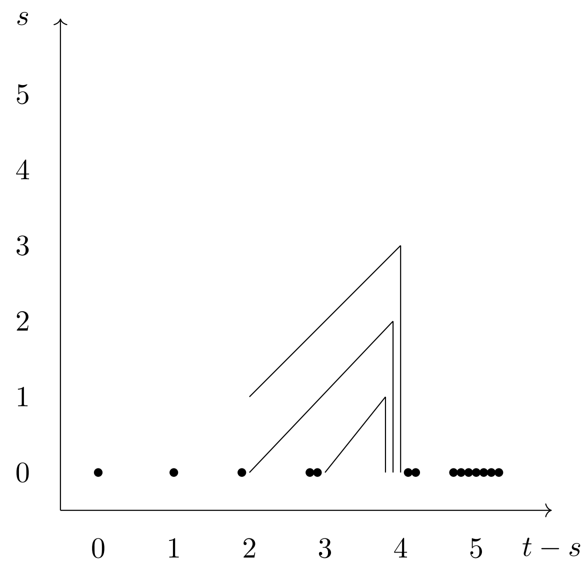


Figure 36: $\Omega_*^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$

The bordism invariants of $\Omega_5^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are

$$w_1^4 a, a^5(= w_1^2 a^3), w_1^2 x_3(= x_5), x_2 x_3, w_1^2 a x_2(= a x_2^2 + a^2 x_3), w_1 a x_3(= a^2 x_3), a^3 x_2.$$

i	$\mathrm{TP}_i(\mathrm{Pin}^+ \times \mathbb{Z}_2 \times \mathrm{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^5$
4	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \mathbb{Z}_{16}$
5	$\mathbb{Z}_2^7$

Theorem 61.

The 2d topological terms are $w_1 a = a^2, x_2, w_1\tilde{\eta}$.

The 3d topological terms are $a^3, w_1 x_2 = x_3, a x_2, w_1 a \tilde{\eta}, w_1 \mathrm{Arf}$.

The 4d topological terms are $a x_3, w_1 a x_2(= a^2 x_2 + a x_3), q_s(x_2), w_1 a(\mathrm{ABK}), \eta$.

The 5d topological terms are

$$w_1^4 a, a^5(= w_1^2 a^3), w_1^2 x_3(= x_5), x_2 x_3, w_1^2 a x_2(= a x_2^2 + a^2 x_3), w_1 a x_3(= a^2 x_3), a^3 x_2.$$

4.6.5 $\Omega_d^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+ \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \\ \Rightarrow \quad & \pi_{t-s}(M\mathrm{Pin}^+ \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2). \end{aligned} \quad (4.89)$$

$M\mathrm{Pin}^+ = MT\mathrm{Pin}^- \sim M\mathrm{Spin} \wedge S^{-1} \wedge M\mathrm{O}(1)$.

For $t-s<8$,

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*+1}(M\mathrm{O}(1),\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_2\times \mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2\times \mathrm{B}^2\mathbb{Z}_2). \quad (4.90)$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+1}(M\mathrm{O}(1),\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_2\times \mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$ and the E_2 page are shown in Figure 37, 38.

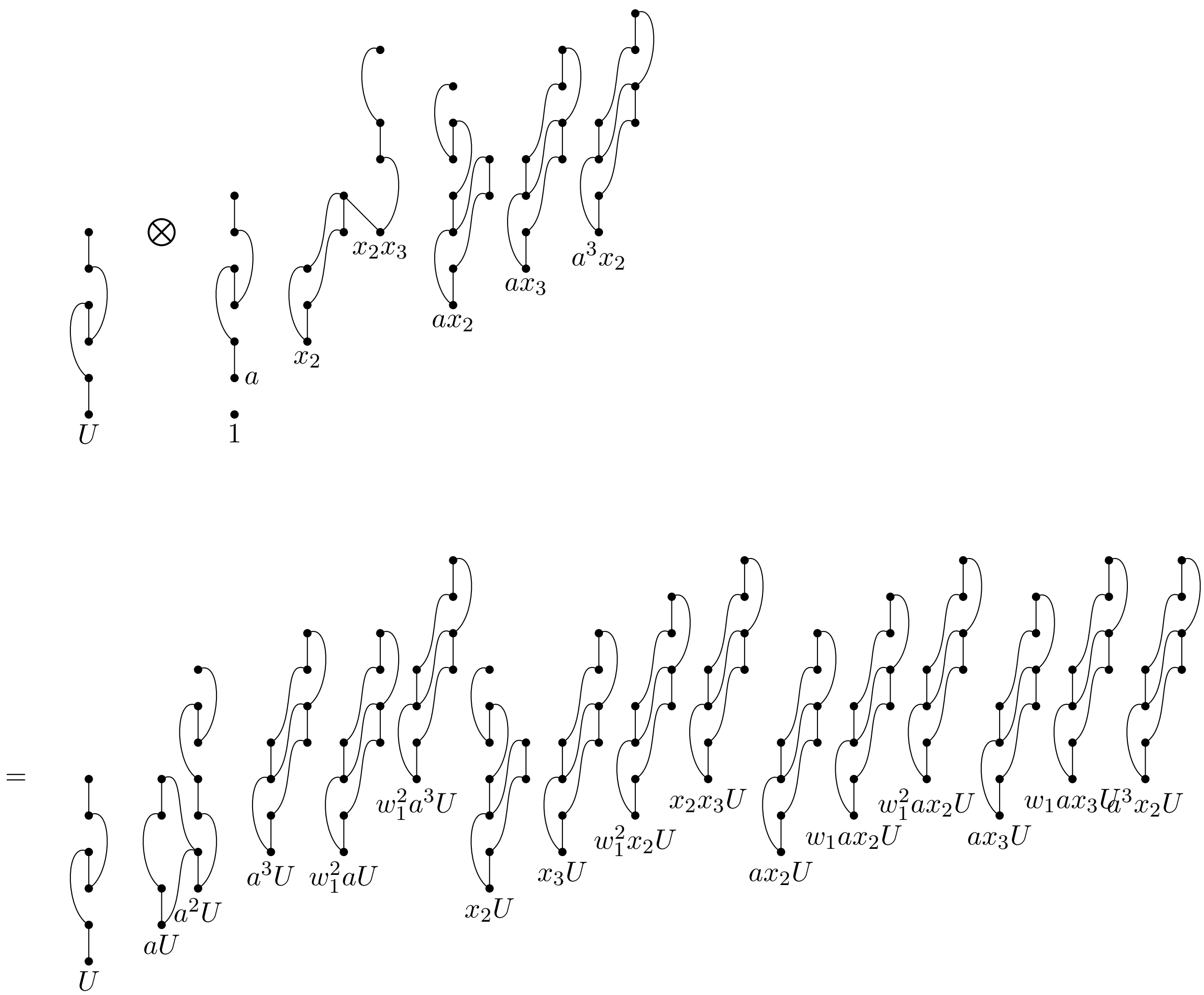


Figure 37: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+1}(M\mathrm{O}(1),\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_2\times \mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$

i	$\Omega_i^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2\times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2^2$
2	$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_8$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^3$
5	$\mathbb{Z}_2^5$

Theorem 62.

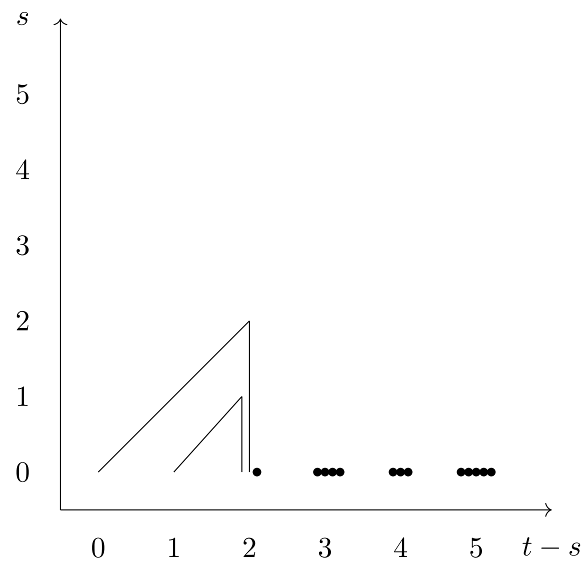


Figure 38: $\Omega_*^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$

The bordism invariants of $\Omega_2^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $x_2, f_s(a), \mathrm{ABK}$. ($f_s(a)$ is explained in the footnotes of Table 1.)

The bordism invariants of $\Omega_3^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $a^3, w_1^2 a, x_3 = w_1 x_2, a x_2$.

The bordism invariants of $\Omega_4^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $w_1^2 x_2, w_1 a x_2 (= a^2 x_2 + a x_3), a x_3$.

The bordism invariants of $\Omega_5^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $w_1^2 a^3, x_2 x_3, w_1^2 a x_2, w_1 a x_3 (= a^2 x_3), a^3 x_2$.

i	$\mathrm{TP}_i(\mathrm{Pin}^- \times \mathbb{Z}_2 \times \mathrm{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2^2$
2	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^3$
5	$\mathbb{Z}_2^5$

Theorem 63.

The 2d topological terms are $x_2, f_s(a), \mathrm{ABK}$.

The 3d topological terms are $a^3, w_1^2 a, x_3 = w_1 x_2, a x_2$.

The 4d topological terms are $w_1^2 x_2, w_1 a x_2 (= a^2 x_2 + a x_3), a x_3$.

The 5d topological terms are $w_1^2 a^3, x_2 x_3, w_1^2 a x_2, w_1 a x_3 (= a^2 x_3), a^3 x_2$.

4.6.6 $\Omega_d^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O} \wedge (\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}. \quad (4.91)$$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{O} \wedge (\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_3^{\wedge}. \quad (4.92)$$

Since $\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_3) = 0$, we have $\Omega_d^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_3^{\wedge} = 0$.

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_2) = \mathbb{Z}_2$, we have $\Omega_d^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_2^{\wedge} = \Omega_d^{\mathrm{O}}$.

Hence $\Omega_d^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3) = \Omega_d^{\mathrm{O}}$.

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 64.

The bordism invariant of $\Omega_2^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$ is w_1^2.

The bordism invariants of $\Omega_4^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$ are w_1^4, w_2^2.

The bordism invariant of $\Omega_5^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$ is w_2w_3.

i	$\mathrm{TP}_i(\mathrm{O} \times \mathbb{Z}_3 \times \mathrm{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 65.

The 2d topological term is w_1^2.

The 4d topological terms are w_1^4, w_2^2.

The 5d topological term is w_2w_3.

4.6.7 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO} \wedge (\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{4.93}$$

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_2) = \mathbb{Z}_2$, we have $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge = \Omega_d^{\mathrm{SO}}$.

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{SO} \wedge (\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{4.94}$$

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3) = \mathbb{Z}_3[b', x_2', x_8', \dots] \otimes \Lambda_{\mathbb{Z}_3}(a', x_3', x_7', \dots) \tag{4.95}$$

$\beta_{(3,3)}a' = b'$, $\beta_{(3,3)}x_2' = x_3'$, $\beta_{(3,3)}x_2'^2 = 2x_2'x_3'$.

The E_2 page is shown in Figure 39.

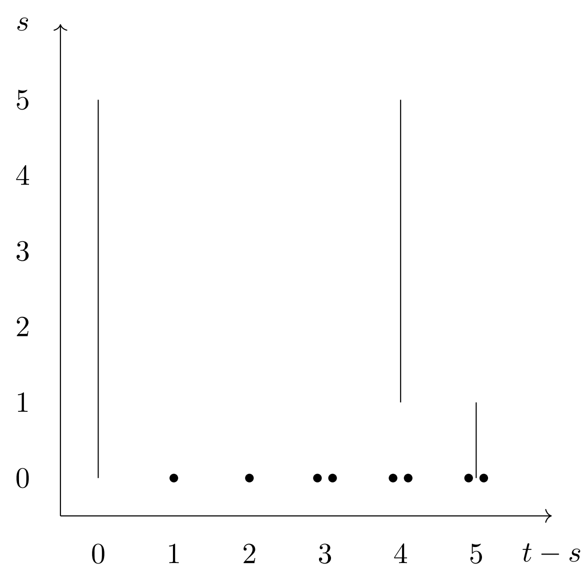


Figure 39: $\Omega_*^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge$

Hence we have the following

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_3$
2	$\mathbb{Z}_3$
3	$\mathbb{Z}_3^2$
4	$\mathbb{Z} \times \mathbb{Z}_3^2$
5	$\mathbb{Z}_2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_9$

Theorem 66.

The bordism invariant of $\Omega_2^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$ is x_2'.

The bordism invariants of $\Omega_3^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$ are $a'b', a'x_2'$.

The bordism invariants of $\Omega_4^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$ are σ, $a'x_3'(= b'x_2')$ and $x_2'^2$.

The bordism invariants of $\Omega_5^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$ are $w_2w_3, a'b'x_2', a'x_2'^2, \mathfrak{P}_3(b')$.

Here $\mathfrak{P}_3$ is the Postnikov square.

i	$\mathrm{TP}_i(\mathrm{SO} \times \mathbb{Z}_3 \times \mathrm{B}\mathbb{Z}_3)$
0	0
1	$\mathbb{Z}_3$
2	$\mathbb{Z}_3$
3	$\mathbb{Z} \times \mathbb{Z}_3^2$
4	$\mathbb{Z}_3^2$
5	$\mathbb{Z}_2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_9$

Theorem 67.

The 2d topological term is x_2'.

The 3d topological terms are $\frac{1}{3}\mathrm{CS}_3^{(TM)}, a'b', a'x_2'$.

The 4d topological terms are $a'x_3'(= b'x_2')$ and $x_2'^2$.

The 5d topological terms are $w_2w_3, a'b'x_2', a'x_2'^2, \mathfrak{P}_3(b')$.

4.6.8 $\Omega_d^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Spin} \wedge (\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{4.96}$$

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_2) = \mathbb{Z}_2$, we have $\Omega_d^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge = \Omega_d^{\mathrm{Spin}}$.

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Spin} \wedge (\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow \Omega_{t-s}^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{4.97}$$

Since

$$\mathrm{H}^*(M\mathrm{Spin}, \mathbb{Z}_3) = \mathrm{H}^*(M\mathrm{SO}, \mathbb{Z}_3),$$

we have the following

Theorem 68.

The bordism invariants of $\Omega_2^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$ are Arf and x_2'.

The bordism invariants of $\Omega_3^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$ are $a'b', a'x_2'$.

i	$\Omega_i^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2\times\mathbb{Z}_3$
2	$\mathbb{Z}_2\times\mathbb{Z}_3$
3	$\mathbb{Z}_3^2$
4	$\mathbb{Z}\times\mathbb{Z}_3^2$
5	$\mathbb{Z}_3^2\times\mathbb{Z}_9$

i	$\mathrm{TP}_i(\mathrm{Spin}\times\mathbb{Z}_3\times\mathrm{B}\mathbb{Z}_3)$
0	0
1	$\mathbb{Z}_2\times\mathbb{Z}_3$
2	$\mathbb{Z}_2\times\mathbb{Z}_3$
3	$\mathbb{Z}\times\mathbb{Z}_3^2$
4	$\mathbb{Z}_3^2$
5	$\mathbb{Z}_3^2\times\mathbb{Z}_9$

The bordism invariants of $\Omega_4^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$ are $\frac{\sigma}{16}$, $a'x_3'(=b'x_2')$ and $x_2'^2$.

The bordism invariants of $\Omega_5^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$ are $a'b'x_2', a'x_2'^2, \mathfrak{P}_3(b')$.

Theorem 69.

The 2d topological terms are Arf and x_2'.

The 3d topological terms are $\frac{1}{48}\mathrm{CS}_3^{(TM)}, a'b', a'x_2'$.

The 4d topological terms are $a'x_3'(=b'x_2')$ and $x_2'^2$.

The 5d topological terms are $a'b'x_2', a'x_2'^2, \mathfrak{P}_3(b')$.

4.6.9 $\Omega_d^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^-\wedge(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{4.98}$$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+\wedge(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{4.99}$$

Since $\mathrm{H}^*(M\mathrm{Pin}^-,\mathbb{Z}_3)=0$, we have $\Omega_d^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge=0$.

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathbb{Z}_2$, we have $\Omega_d^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge=\Omega_d^{\mathrm{Pin}^+}$.

Hence $\Omega_d^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)=\Omega_d^{\mathrm{Pin}^+}$.

Theorem 70.

i	$\Omega_i^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_{16}$
5	0

The bordism invariant of $\Omega_2^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$ is $w_1\tilde{\eta}$.

The bordism invariant of $\Omega_3^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$ is $w_1\mathrm{Arf}$.

The bordism invariant of $\Omega_4^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$ is η.

i	$\mathrm{TP}_i(\mathrm{Pin}^+\times\mathbb{Z}_3\times\mathrm{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_{16}$
5	0

Theorem 71.

The 2d topological term is $w_1\tilde{\eta}$.

The 3d topological term is $w_1\mathrm{Arf}$.

The 4d topological term is η.

4.6.10 $\Omega_d^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+\wedge(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{4.100}$$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+\wedge(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{4.101}$$

Since $\mathrm{H}^*(M\mathrm{Pin}^+,\mathbb{Z}_3)=0$, we have $\Omega_d^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge=0$.

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathbb{Z}_2$, we have $\Omega_d^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge=\Omega_d^{\mathrm{Pin}^-}$.

Hence $\Omega_d^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)=\Omega_d^{\mathrm{Pin}^-}$.

Theorem 72.

The bordism invariant of $\Omega_2^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$ is ABK.

i	$\Omega_i^{\text{Pin}^-}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	0
5	0

i	$\mathrm{TP}_i(\text{Pin}^- \times \mathbb{Z}_3 \times \mathrm{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	0
5	0

Theorem 73.

The 2d topological term is ABK.

4.7 $(\mathrm{B}G_a, \mathrm{B}^2G_b)$: $(\mathrm{BPSU}(2), \mathrm{B}^2\mathbb{Z}_2), (\mathrm{BPSU}(3), \mathrm{B}^2\mathbb{Z}_3)$

4.7.1 $\Omega_d^{\mathbf{O}}(\mathrm{BPSU}(2) \times \mathrm{B}^2\mathbb{Z}_2)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{BPSU}(2) \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{BPSU}(2) \times \mathrm{B}^2\mathbb{Z}_2). \tag{4.102}$$

$$\mathrm{H}^*(\mathrm{BPSU}(2), \mathbb{Z}_2) = \mathbb{Z}_2[w_2', w_3'], \tag{4.103}$$

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[x_2, x_3, x_5, x_9, \dots], \tag{4.104}$$

$$\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) = \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^*. \tag{4.105}$$

$$\begin{aligned} & \mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{BPSU}(2), \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) \\ = \; & \mathcal{A}_2 \oplus 3\Sigma^2\mathcal{A}_2 \oplus 2\Sigma^3\mathcal{A}_2 \oplus 7\Sigma^4\mathcal{A}_2 \oplus 8\Sigma^5\mathcal{A}_2 \oplus \cdots . \end{aligned} \tag{4.106}$$

i	$\Omega_i^{\mathrm{O}}(\mathrm{BPSU}(2) \times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_2^7$
5	$\mathbb{Z}_2^8$

Theorem 74.

The bordism invariants of $\Omega_2^{\mathrm{O}}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$ are w_2', x_2, w_1^2.

The bordism invariants of $\Omega_3^{\mathrm{O}}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$ are $x_3 = w_1x_2, w_3' = w_1w_2'$.

The bordism invariants of $\Omega_4^{\mathrm{O}}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$ are $w_1^4, w_2^2, x_2^2, w_2'^2, x_2w_1^2, w_2'w_1^2, w_2'x_2$.

The bordism invariants of $\Omega_5^{\mathrm{O}}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$ are $w_2'w_3', x_2w_3', w_1^2w_3', w_2'x_3, x_2x_3, w_1^2x_3, x_5, w_2w_3$.

i	$\mathrm{TP}_i(\mathrm{O}\times\mathrm{PSU}(2)\times\mathrm{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_2^7$
5	$\mathbb{Z}_2^8$

Theorem 75.

The 2d topological terms are w_2', x_2, w_1^2.

The 3d topological terms are $x_3 = w_1x_2, w_3' = w_1w_2'$.

The 4d topological terms are $w_1^4, w_2^2, x_2^2, w_2'^2, x_2w_1^2, w_2'w_1^2, w_2'x_2$.

The 5d topological terms are $w_2'w_3', x_2w_3', w_1^2w_3', w_2'x_3, x_2x_3, w_1^2x_3, x_5, w_2w_3$.

4.7.2 $\Omega_d^{\mathrm{SO}}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2). \tag{4.107}$$

$d_2(x_2x_3+x_5) = x_2^2h_0^2$.

The E_2 page is shown in Figure 40.

i	$\Omega_i^{\mathrm{SO}}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}$
1	0
2	$\mathbb{Z}_2^2$
3	0
4	$\mathbb{Z}^2\times\mathbb{Z}_2\times\mathbb{Z}_4$
5	$\mathbb{Z}_2^4$

Theorem 76.

The bordism invariants of $\Omega_2^{\mathrm{SO}}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$ are w_2', x_2.

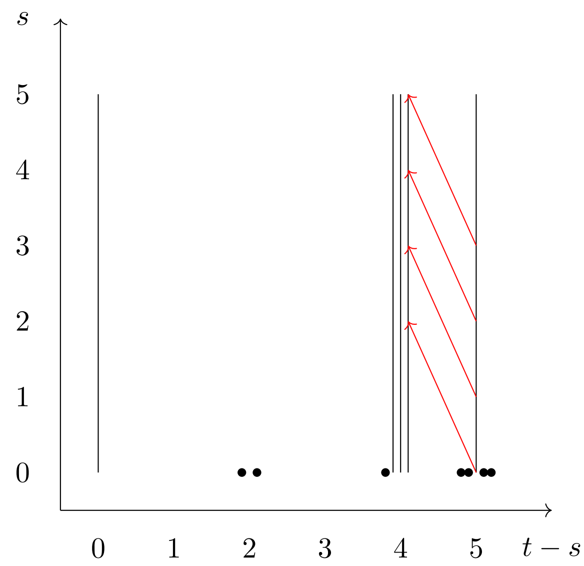


Figure 40: $\Omega_*^{\mathrm{SO}}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$

The bordism invariants of $\Omega_4^{\mathrm{SO}}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$ are σ, p_1', $w_2'x_2$ and $\mathcal{P}_2(x_2)$.

The bordism invariants of $\Omega_5^{\mathrm{SO}}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$ are $w_2'w_3', x_5, w_3'x_2(=w_2'x_3), w_2w_3$.

i	$\mathrm{TP}_i(\mathrm{SO}\times\mathrm{PSU}(2)\times\mathrm{B}\mathbb{Z}_2)$
0	0
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}^2$
4	$\mathbb{Z}_2\times\mathbb{Z}_4$
5	$\mathbb{Z}_2^4$

Theorem 77.

The 2d topological terms are w_2', x_2.

The 3d topological terms are $\frac{1}{3}\mathrm{CS}_3^{(TM)}$, $\mathrm{CS}_3^{(\mathrm{SO}(3))}$.

The 4d topological terms are $w_2'x_2$ and $\mathcal{P}_2(x_2)$.

The 5d topological terms are $w_2'w_3', x_5, w_3'x_2(=w_2'x_3), w_2w_3$.

4.7.3 $\Omega_d^{\mathrm{Spin}}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$

For $t-s<8$,

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^*(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Spin}}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2). \tag{4.108}$$

$d_2(x_2x_3 + x_5) = x_2^2 h_0^2$.

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{BPSU}(2) \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$ and the E_2 page is shown in Figure 41, 42.

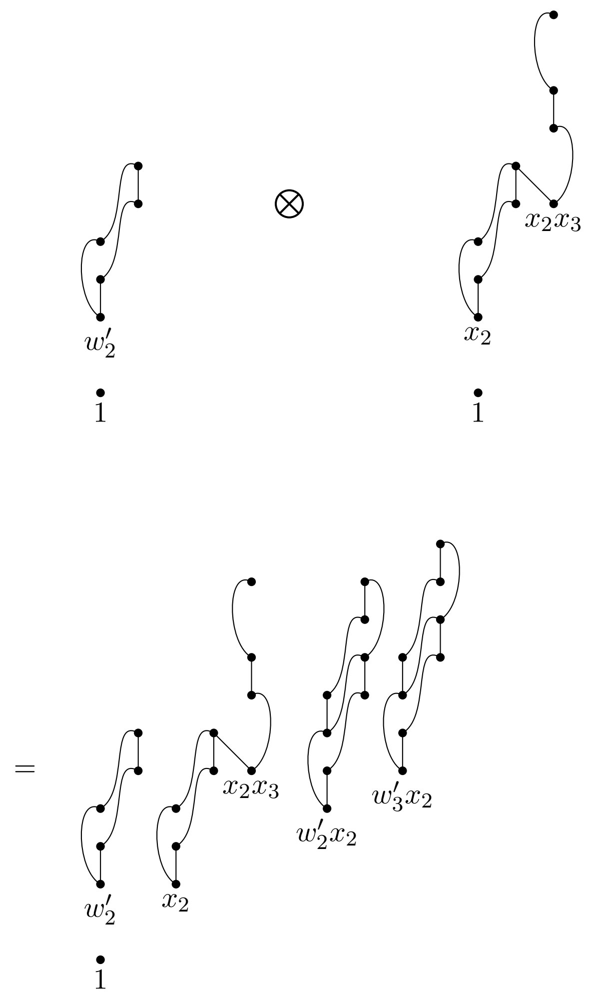


Figure 41: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{BPSU}(2) \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$

i	$\Omega_i^{\mathrm{Spin}}(\mathrm{BPSU}(2) \times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	0
4	$\mathbb{Z}^2 \times \mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 78.

The bordism invariants of $\Omega_2^{\mathrm{Spin}}(\mathrm{BPSU}(2) \times \mathrm{B}^2\mathbb{Z}_2)$ are w'_2, x_2, Arf.

The bordism invariants of $\Omega_4^{\mathrm{Spin}}(\mathrm{BPSU}(2) \times \mathrm{B}^2\mathbb{Z}_2)$ are $\frac{\sigma}{16}$, $\frac{p'_1}{2}$, w'_2x_2 and $\frac{\mathcal{P}_2(x_2)}{2}$.

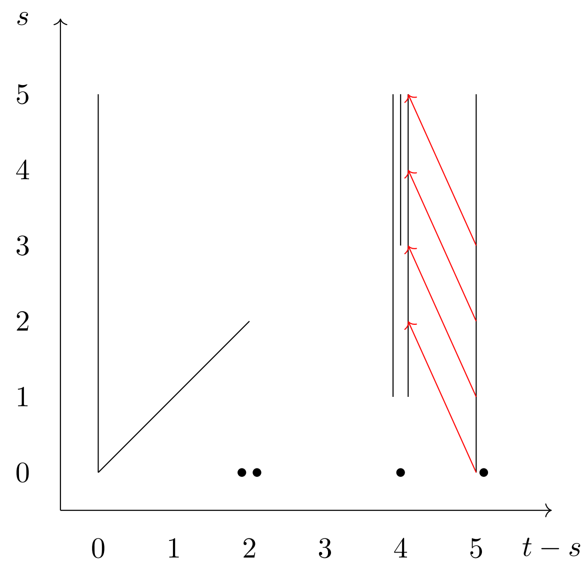


Figure 42: $\Omega_*^{\mathrm{Spin}}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$

The bordism invariant of $\Omega_5^{\mathrm{Spin}}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$ is $w_3'x_2(=w_2'x_3)$.

i	$\mathrm{TP}_i(\mathrm{Spin}\times \mathrm{PSU}(2)\times \mathrm{B}\mathbb{Z}_2)$
0	0
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}^2$
4	$\mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 79.

The 2d topological terms are w_2', x_2, Arf.

The 3d topological terms are $\frac{1}{48}\mathrm{CS}_3^{(TM)}$, $\frac{1}{2}\mathrm{CS}_3^{(\mathrm{SO}(3))}$.

The 4d topological terms are $w_2'x_2$ and $\frac{\mathcal{P}_2(x_2)}{2}$.

The 5d topological term is $w_3'x_2(=w_2'x_3)$.

4.7.4 $\Omega_d^{\mathrm{Pin}^+}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$

For $t-s<8$,

$$\begin{aligned}&\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*-1}(MT\mathrm{O}(1),\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2)\\ \Rightarrow\ &\Omega_{t-s}^{\mathrm{Pin}^+}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2).\end{aligned} \tag{4.109}$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MT\mathrm{O}(1),\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$ and the E_2 page are shown in Figure 43, 44.

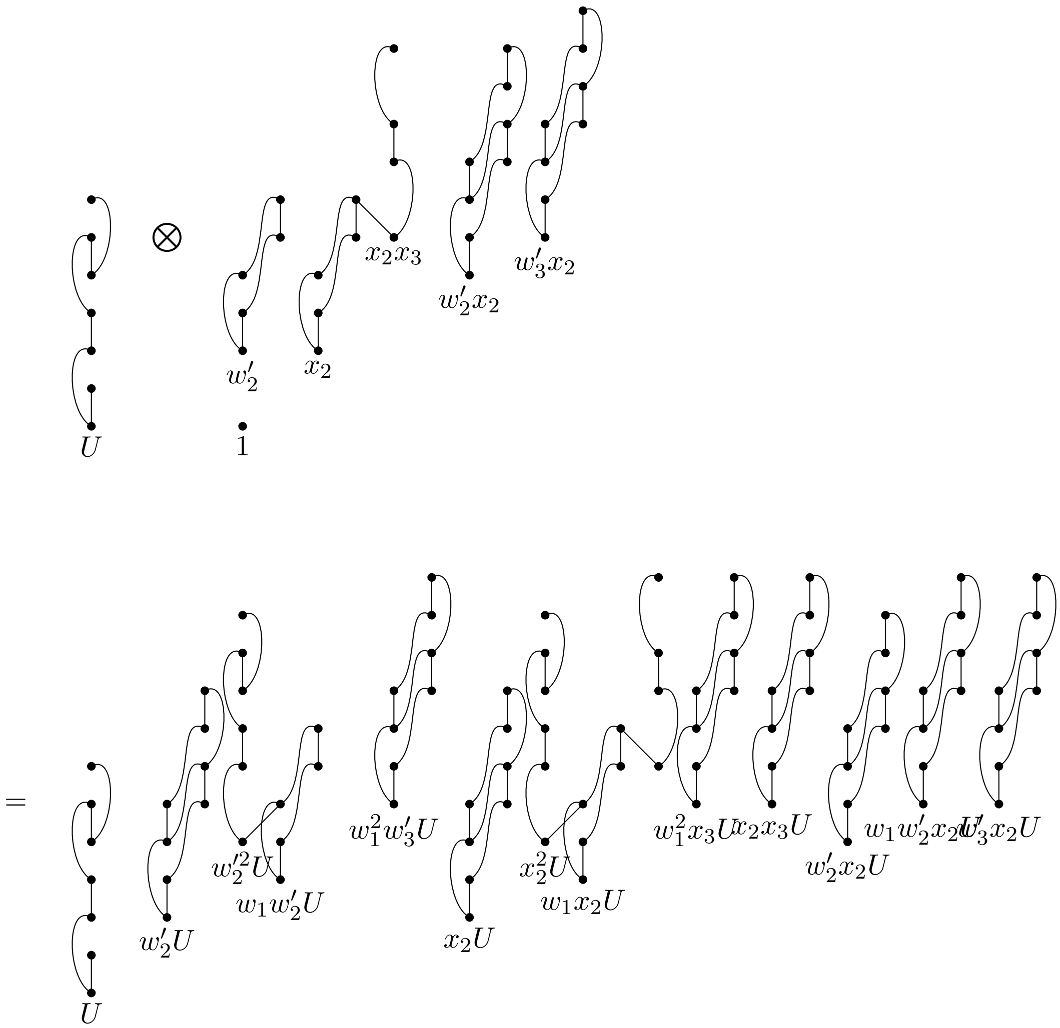


Figure 43: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MT\mathrm{O}(1),\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$

i	$\Omega_i^{\mathrm{Pin}^+}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^3$
4	$\mathbb{Z}_4^2\times\mathbb{Z}_{16}\times\mathbb{Z}_2$
5	$\mathbb{Z}_2^5$

Theorem 80.

The bodism invariants of $\Omega_2^{\mathrm{Pin}^+}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$ are $w_2', x_2, w_1\tilde{\eta}$.

The bodism invariants of $\Omega_3^{\mathrm{Pin}^+}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$ are $w_1w_2'=w_3', w_1x_2=x_3, w_1\mathrm{Arf}$.

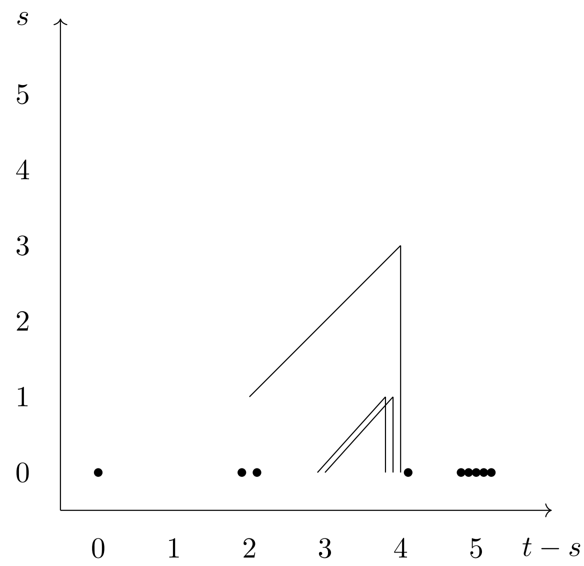


Figure 44: $\Omega_*^{\text{Pin}^+}(\text{BPSU}(2)\times\text{B}^2\mathbb{Z}_2)$

The bodism invariants of $\Omega_4^{\text{Pin}^+}(\text{BPSU}(2)\times\text{B}^2\mathbb{Z}_2)$ are $q_s(w_2'), q_s(x_2), \eta, w_2'x_2$.

The bodism invariants of $\Omega_5^{\text{Pin}^+}(\text{BPSU}(2)\times\text{B}^2\mathbb{Z}_2)$ are

$$w_1^2w_3'(=w_2'w_3'), w_1^2x_3(=x_5), x_2x_3, w_3'x_2, w_1w_2'x_2(=w_2'x_3+w_3'x_2).$$

i	$\text{TP}_i(\text{Pin}^+\times\text{PSU}(2)\times\text{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^3$
4	$\mathbb{Z}_4^2\times\mathbb{Z}_{16}\times\mathbb{Z}_2$
5	$\mathbb{Z}_2^5$

Theorem 81.

The 2d topological terms are $w_2', x_2, w_1\tilde{\eta}$.

The 3d topological terms are $w_1w_2'=w_3', w_1x_2=x_3, w_1\text{Arf}$.

The 4d topological terms are $q_s(w_2'), q_s(x_2), \eta, w_2'x_2$.

The 5d topological terms are

$$w_1^2w_3'(=w_2'w_3'), w_1^2x_3(=x_5), x_2x_3, w_3'x_2, w_1w_2'x_2(=w_2'x_3+w_3'x_2).$$

4.7.5 $\Omega_d^{\text{Pin}^-}(\text{BPSU}(2) \times \text{B}^2\mathbb{Z}_2)$

For $t - s < 8$,

$$\text{Ext}_{\mathcal{A}_2(1)}^{s,t}(\text{H}^{*+1}(M\text{O}(1), \mathbb{Z}_2) \otimes \text{H}^*(\text{BPSU}(2) \times \text{B}^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\text{Pin}^-}(\text{BPSU}(2) \times \text{B}^2\mathbb{Z}_2). \quad (4.110)$$

The $\mathcal{A}_2(1)$-module structure of $\text{H}^{*+1}(M\text{O}(1).\mathbb{Z}_2) \otimes \text{H}^*(\text{BPSU}(2) \times \text{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$ and the E_2 page are shown in Figure 45, 46.

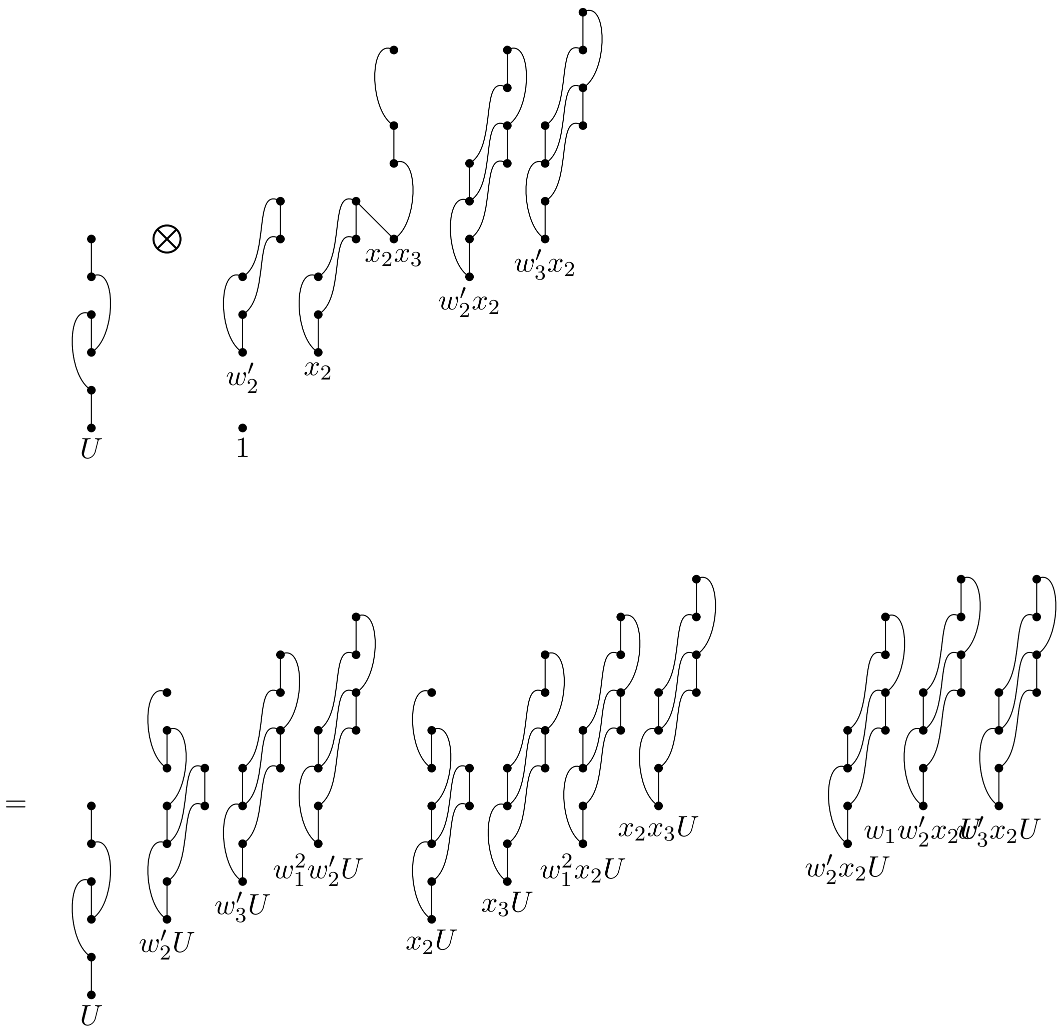


Figure 45: The $\mathcal{A}_2(1)$-module structure of $\text{H}^{*+1}(M\text{O}(1), \mathbb{Z}_2) \otimes \text{H}^*(\text{BPSU}(2) \times \text{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$

Theorem 82.

The bordism invariants of $\Omega_2^{\text{Pin}^-}(\text{BPSU}(2) \times \text{B}^2\mathbb{Z}_2)$ are w_2', x_2, ABK.

The bordism invariants of $\Omega_3^{\text{Pin}^-}(\text{BPSU}(2) \times \text{B}^2\mathbb{Z}_2)$ are $w_1w_2' = w_3', w_1x_2 = x_3$.

The bordism invariants of $\Omega_4^{\text{Pin}^-}(\text{BPSU}(2) \times \text{B}^2\mathbb{Z}_2)$ are $w_1^2w_2', w_1^2x_2, w_2'x_2$.

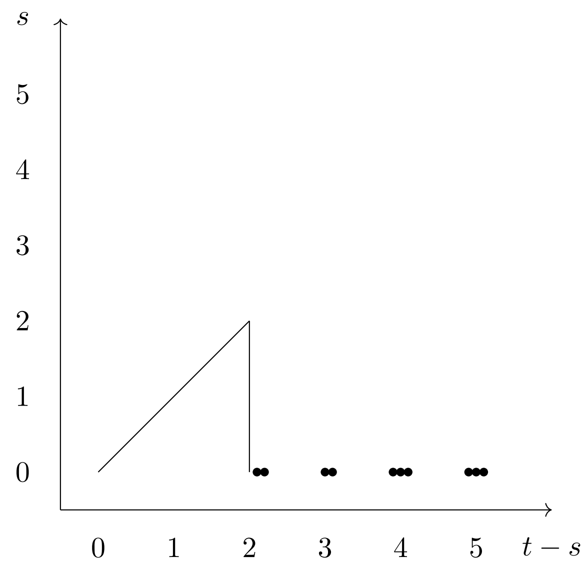


Figure 46: $\Omega_*^{\mathrm{Pin}^-}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$

i	$\Omega_i^{\mathrm{Pin}^-}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2\times\mathbb{Z}_8$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_2^3$
5	$\mathbb{Z}_2^3$

The bordism invariants of $\Omega_5^{\mathrm{Pin}^-}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$ are $x_2x_3, w_3'x_2, w_1w_2'x_2(=w_2'x_3+w_3'x_2)$.

i	$\mathrm{TP}_i(\mathrm{Pin}^-\times\mathrm{PSU}(2)\times\mathrm{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2\times\mathbb{Z}_8$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_2^3$
5	$\mathbb{Z}_2^3$

Theorem 83.

The 2d topological terms are w_2', x_2, ABK.

The 3d topological terms are $w_1w_2'=w_3', w_1x_2=x_3$.

The 4d topological terms are $w_1^2w_2', w_1^2x_2, w_2'x_2$.

The 5d topological terms are $x_2x_3, w_3'x_2, w_1w_2'x_2(=w_2'x_3+w_3'x_2)$.

4.7.6 $\Omega_d^{\mathbf{O}}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{O} \wedge (\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)_3^{\wedge}. \tag{4.111}$$

Since $\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_3) = 0$, $\Omega_d^{\mathrm{O}}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)_3^{\wedge} = 0$.

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O} \wedge (\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}. \tag{4.112}$$

Since $\mathrm{H}^*(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_2) = \mathrm{H}^*(\mathrm{BPSU}(3), \mathbb{Z}_2)$, $\Omega_d^{\mathrm{O}}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)_2^{\wedge} = \Omega_d^{\mathrm{O}}(\mathrm{BPSU}(3))_2^{\wedge}$.

i	$\Omega_i^{\mathrm{O}}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^3$
5	$\mathbb{Z}_2$

Theorem 84.

The bordism invariant of $\Omega_2^{\mathrm{O}}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)$ is w_1^2.

The bordism invariants of $\Omega_4^{\mathrm{O}}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)$ are $w_1^4, w_2^2, c_2(\mod 2)$.

The bordism invariant of $\Omega_5^{\mathrm{O}}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)$ is w_2w_3.

i	$\mathrm{TP}_i(\mathrm{O} \times \mathrm{PSU}(3) \times \mathrm{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^3$
5	$\mathbb{Z}_2$

Theorem 85.

The 2d topological term is w_1^2.

The 4d topological terms are $w_1^4, w_2^2, c_2(\mod 2)$.

The 5d topological term is w_2w_3.

4.7.7 $\Omega_d^{\mathrm{SO}}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO} \wedge (\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}. \tag{4.113}$$

Since $\mathrm{H}^*(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2)$, $\Omega_d^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge=\Omega_d^{\mathrm{SO}}(\mathrm{BPSU}(3))_2^\wedge$.

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{SO}\wedge(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{4.114}$$

$\beta_{(3,3)}x_2'=x_3'$, $\beta_{(3,3)}z_2=z_3$, $\beta_{(3,3)}x_2'^2=2x_2'x_3'$, $\beta_{(3,3)}(x_2'z_2)=x_2'z_3+x_3'z_2$, $\beta_{(3,3)}(x_2'z_3)=x_3'z_3=-\beta_{(3,3)}(x_3'z_2)$.

The E_2 page is shown in Figure 47.

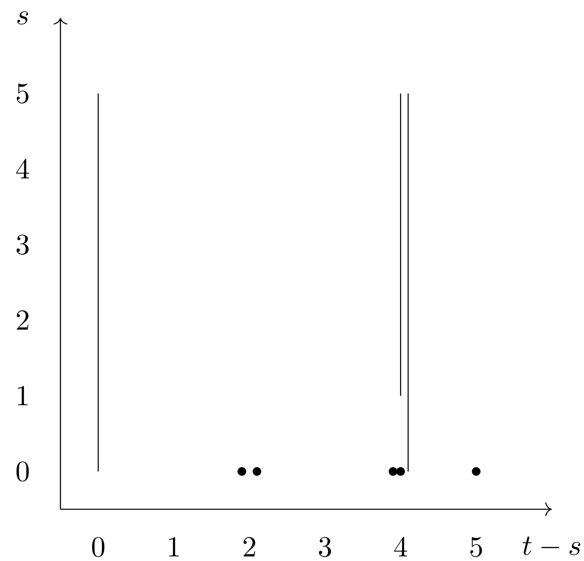


Figure 47: $\Omega_*^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge$

i	$\Omega_i^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}$
1	0
2	$\mathbb{Z}_3^2$
3	0
4	$\mathbb{Z}^2\times\mathbb{Z}_3^2$
5	$\mathbb{Z}_2\times\mathbb{Z}_3$

Theorem 86.

The bordism invariants of $\Omega_2^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$ are x_2', z_2.

The bordism invariants of $\Omega_4^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$ are σ, c_2, $x_2'^2$ and $x_2'z_2$.

The bordism invariants of $\Omega_5^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$ are w_2w_3, $z_2x_3'(=-z_3x_2')$.

Theorem 87.

The 2d topological terms are x_2', z_2.

The 3d topological terms are $\frac{1}{3}\mathrm{CS}_3^{(TM)}$, $\mathrm{CS}_3^{(\mathrm{PSU}(3))}$.

i	$\mathrm{TP}_i(\mathrm{SO}\times\mathrm{PSU}(3)\times\mathrm{B}\mathbb{Z}_3)$
0	0
1	0
2	$\mathbb{Z}_3^2$
3	$\mathbb{Z}^2$
4	$\mathbb{Z}_3^2$
5	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_3$

The 4d topological terms are $x_2'^2$ and $x_2'z_2$.

The 5d topological terms are $\mathrm{CS}_5^{(\mathrm{PSU}(3))}, w_2w_3, z_2x_3'(=-z_3x_2')$.

4.7.8 $\Omega_d^{\mathrm{Spin}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Spin}\wedge(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Spin}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{4.115}$$

Since $\mathrm{H}^*(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2)$, $\Omega_d^{\mathrm{Spin}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge=\Omega_d^{\mathrm{Spin}}(\mathrm{BPSU}(3))_2^\wedge$.

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Spin}\wedge(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{Spin}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{4.116}$$

Since $\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_3)=\mathrm{H}^*(M\mathrm{Spin},\mathbb{Z}_3)$, $\Omega_d^{\mathrm{Spin}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge=\Omega_d^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge$.

i	$\Omega_i^{\mathrm{Spin}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2\times\mathbb{Z}_3^2$
3	0
4	$\mathbb{Z}^2\times\mathbb{Z}_3^2$
5	$\mathbb{Z}_3$

Theorem 88.

The bordism invariants of $\Omega_2^{\mathrm{Spin}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$ are Arf, x_2', z_2.

The bordism invariants of $\Omega_4^{\mathrm{Spin}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$ are $\frac{\sigma}{16}$, c_2, $x_2'^2$ and $x_2'z_2$.

The bordism invariant of $\Omega_5^{\mathrm{Spin}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$ is $z_2x_3'(=-z_3x_2')$.

Theorem 89.

The 2d topological terms are Arf, x_2', z_2.

The 3d topological terms are $\frac{1}{48}\mathrm{CS}_3^{(TM)}$, $\mathrm{CS}_3^{(\mathrm{PSU}(3))}$.

i	$\mathrm{TP}_i(\mathrm{Spin}\times\mathrm{PSU}(3)\times\mathrm{B}\mathbb{Z}_3)$
0	0
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2\times\mathbb{Z}_3^2$
3	$\mathbb{Z}^2$
4	$\mathbb{Z}_3^2$
5	$\mathbb{Z}\times\mathbb{Z}_3$

The 4d topological terms are $x_2'^2$ and $x_2'z_2$.

The 5d topological terms are $\frac{1}{2}\mathrm{CS}_5^{(\mathrm{PSU}(3))}$, $z_2x_3'(=-z_3x_2')$.

4.7.9 $\Omega_d^{\mathrm{Pin}^+}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^-\wedge(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^+}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_3^{\wedge}. \tag{4.117}$$

Since $\mathrm{H}^*(M\mathrm{Pin}^-,\mathbb{Z}_3)=0$, $\Omega_d^{\mathrm{Pin}^+}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_3^{\wedge}=0$.

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^-\wedge(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^+}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}. \tag{4.118}$$

Since $\mathrm{H}^*(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2)$, $\Omega_d^{\mathrm{Pin}^+}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}=\Omega_d^{\mathrm{Pin}^+}(\mathrm{BPSU}(3))_2^{\wedge}$.

i	$\Omega_i^{\mathrm{Pin}^+}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2\times\mathbb{Z}_{16}$
5	0

Theorem 90.

The bordism invariant of $\Omega_2^{\mathrm{Pin}^+}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$ is $w_1\tilde{\eta}$.

The bordism invariant of $\Omega_3^{\mathrm{Pin}^+}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$ is $w_1\mathrm{Arf}$.

The bordism invariants of $\Omega_4^{\mathrm{Pin}^+}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$ are $c_2(\mod 2)$ and η.

Theorem 91.

The 2d topological term is $w_1\tilde{\eta}$.

The 3d topological term is $w_1\mathrm{Arf}$.

The 4d topological terms are $c_2(\mod 2)$ and η.

i	$\mathrm{TP}_i(\mathrm{Pin}^+ \times \mathrm{PSU}(3) \times \mathrm{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$
5	0

4.7.10 $\Omega_d^{\mathrm{Pin}^-}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+ \wedge (\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow \Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{4.119}$$

Since $\mathrm{H}^*(M\mathrm{Pin}^+, \mathbb{Z}_3) = 0$, $\Omega_d^{\mathrm{Pin}^-}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge = 0$.

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+ \wedge (\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{4.120}$$

Since $\mathrm{H}^*(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_2) = \mathrm{H}^*(\mathrm{BPSU}(3), \mathbb{Z}_2)$, $\Omega_d^{\mathrm{Pin}^-}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge = \Omega_d^{\mathrm{Pin}^-}(\mathrm{BPSU}(3))_2^\wedge$.

i	$\Omega_i^{\mathrm{Pin}^-}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	$\mathbb{Z}_2$
5	0

Theorem 92.

The bordism invariant of $\Omega_2^{\mathrm{Pin}^-}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)$ is ABK.

The bordism invariant of $\Omega_4^{\mathrm{Pin}^-}(\mathrm{BPSU}(3) \times \mathrm{B}^2\mathbb{Z}_3)$ is $c_2(\mod 2)$.

i	$\mathrm{TP}_i(\mathrm{Pin}^- \times \mathrm{PSU}(3) \times \mathrm{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	$\mathbb{Z}_2$
5	0

Theorem 93.

The 2d topological term is ABK.

The 4d topological term is $c_2(\mod 2)$.

5 More computation of O/SO bordism groups

5.1 Summary

$\Omega_d^H(-)$	BO(3)	BO(4)	BO(5)	$\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3))$	$\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4))$
2 SO	$\mathbb{Z}_2$: w_2'	$\mathbb{Z}_2$: w_2'	$\mathbb{Z}_2$: w_2'	$\mathbb{Z}_3$: z_2[11]	$\mathbb{Z}_4$: x_2[12]
2 O	$\mathbb{Z}_2^3$: w_1^2, $w_1'^2, w_2'$	$\mathbb{Z}_2^3$: w_1^2, $w_1'^2, w_2'$	$\mathbb{Z}_2^3$: w_1^2, $w_1'^2, w_2'$	$\mathbb{Z}_2^2$: w_1^2, a^2[13]	$\mathbb{Z}_2^2$: w_1^2, a^2, $\tilde{x}_2$[14]

Table 9: $2d$ bordism groups-1.

$\Omega_d^H(-)$	$\mathrm{B}^2\mathbb{Z}_4$	$\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3$	$\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_{18} \times \mathrm{B}^2\mathbb{Z}_3$
2 SO	$\mathbb{Z}_4$: x_2[15]	$\mathbb{Z}_2$: x_2[16]	$\mathbb{Z}_3$: x_2'[17]	$\mathbb{Z}_2$: x_2[18]	$\mathbb{Z}_3$: x_2'[19]
2 O	$\mathbb{Z}_2^2$: $w_1^2, \tilde{x}_2$[20]	$\mathbb{Z}_2^3$: $w_1^2, \tilde{b}$,[21] x_2[22]	$\mathbb{Z}_2^2$: w_1^2, a^2[23]	$\mathbb{Z}_2^3$: $w_1^2, \tilde{b}$,[24] x_2[25]	$\mathbb{Z}_2^2$: w_1^2, a^2[26]

Table 10: $2d$ bordism groups-2.

$\Omega_d^H(-)$	BO(3)	BO(4)	BO(5)	$\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3))$	$\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4))$
3 SO	$\mathbb{Z}_2^2$: $w_1'^3$, $w_1'w_2'$ = w_3'	$\mathbb{Z}_2^2$: $w_1'^3$, $w_1'w_2'$ = w_3'	$\mathbb{Z}_2^2$: $w_1'^3$, $w_1'w_2'$ = w_3'	$\mathbb{Z}_2$: a^3	$\mathbb{Z}_2^2$: $a^3, a\tilde{x}_2$
3 O	$\mathbb{Z}_2^4$: $w_1'w_1^2, w_1'^3$, $w_1'w_2', w_3'$	$\mathbb{Z}_2^4$: $w_1'w_1^2, w_1'^3$, $w_1'w_2', w_3'$	$\mathbb{Z}_2^4$: $w_1'w_1^2, w_1'^3$, $w_1'w_2', w_3'$	$\mathbb{Z}_2^2$: a^3, aw_1^2	$\mathbb{Z}_2^4$: a^3, aw_1^2, $x_3, a\tilde{x}_2$[27]

Table 11: $3d$ bordism groups-1.

[11]Here $z_2 = w_2(\mathrm{PSU}(3)) \in \mathrm{H}^2(\mathrm{BPSU}(3), \mathbb{Z}_3)$ is the generalized Stiefel-Whitney class.
[12]Here $x_2 = w_2(\mathrm{PSU}(4)) \in \mathrm{H}^2(\mathrm{BPSU}(4), \mathbb{Z}_4)$ is the generalized Stiefel-Whitney class.
[13]Here $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2)$.
[14]$\tilde{x}_2 = x_2 \mod 2 = w_2(\mathrm{PSU}(4)) \mod 2$.
[15]Here $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_4)$.
[16]Here $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$.
[17]Here $x_2' \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3)$.
[18]Here $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$.
[19]Here $x_2' \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3)$.
[20]Here $\tilde{x}_2 = x_2 \mod 2$, $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_4)$.
[21]Here $\tilde{b} = b \mod 2$, $b \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_4)$.
[22]Here $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$.
[23]Here $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2)$.
[24]Here $\tilde{b} = b \mod 2$, $b \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$.
[25]Here $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$.
[26]Here $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2)$.
[27]Here $x_3 = \beta_{(2,4)}x_2$, $\tilde{x}_2 = x_2 \mod 2$, $x_2 = w_2(\mathrm{PSU}(4)) \in \mathrm{H}^2(\mathrm{BPSU}(4), \mathbb{Z}_4)$ is the generalized Stiefel-Whitney class.

$\Omega_d^H(-)$	$\mathrm{B}^2\mathbb{Z}_4$	$\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3$	$\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_{18} \times \mathrm{B}^2\mathbb{Z}_3$
3 SO	0	$\mathbb{Z}_4\times\mathbb{Z}_2$: $ab, \tilde{a}x_2$[28]	$\mathbb{Z}_3^2\times\mathbb{Z}_2$: $a'b', a'x_2'$, a^3[29]	$\mathbb{Z}_8\times\mathbb{Z}_2$: $ab, \tilde{a}x_2$[30]	$\mathbb{Z}_9 \times \mathbb{Z}_3\times\mathbb{Z}_2$: $a'b', \tilde{a}'x_2'$, a^3[31]
3 O	$\mathbb{Z}_2$: x_3[32]	$\mathbb{Z}_2^4$: $\tilde{a}b, x_3$, $\tilde{a}x_2, \tilde{a}w_1^2$[33]	$\mathbb{Z}_2^2$: a^3, aw_1^2	$\mathbb{Z}_2^4$: $\tilde{a}b, x_3$, $\tilde{a}x_2, \tilde{a}w_1^2$[34]	$\mathbb{Z}_2^2$: a^3, aw_1^2

Table 12: 3d bordism groups-2.

$\Omega_d^H(-)$	BO(3)	BO(4)	BO(5)
4 SO	$\mathbb{Z}^2 \times \mathbb{Z}_2$: σ, p_1', $w_1'^2 w_2'$	$\mathbb{Z}^2 \times \mathbb{Z}_2^2$: σ, p_1', $w_1'^2 w_2', w_4'$	$\mathbb{Z}^2 \times \mathbb{Z}_2^2$: σ, p_1', $w_1'^2 w_2', w_4'$
4 O	$\mathbb{Z}_2^8$: w_1^4, w_2^2, $w_1^2 w_1'^2, w_1^2 w_2'$, $w_1' w_3', w_1'^2 w_2'$, $w_1'^4, w_2'^2$	$\mathbb{Z}_2^8$: w_1^4, w_2^2, $w_1^2 w_1'^2, w_1^2 w_2'$, $w_1' w_3', w_1'^2 w_2'$, $w_1'^4, w_2'^2$, w_4'	$\mathbb{Z}_2^8$: w_1^4, w_2^2, $w_1^2 w_1'^2, w_1^2 w_2'$, $w_1' w_3', w_1'^2 w_2'$, $w_1'^4, w_2'^2$, w_4'

Table 13: 4d bordism groups-1.

$\Omega_d^H(-)$	$\mathrm{B}^2\mathbb{Z}_4$	$\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3$	$\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_{18} \times \mathrm{B}^2\mathbb{Z}_3$
4 SO	$\mathbb{Z} \times \mathbb{Z}_8$: $\sigma, \mathcal{P}_2(x_2)$[35]	$\mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_2$: $\sigma, \mathcal{P}_2(x_2)$, $\tilde{b}x_2$[36]	$\mathbb{Z}\times\mathbb{Z}_3^2$: $\sigma, a'x_3' =$ $b'x_2'$, $x_2'^2$[37]	$\mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_2$: $\sigma, \mathcal{P}_2(x_2)$, $\tilde{b}x_2$[38]	$\mathbb{Z}\times\mathbb{Z}_3^2$: $\sigma, \tilde{b}'x_2'$, $x_2'^2$[39]
4 O	$\mathbb{Z}_2^4$: w_1^4, w_2^2, $w_1^2\tilde{x}_2, \tilde{x}_2^2$[40]	$\mathbb{Z}_2^8$: $\tilde{a}x_3, \tilde{b}x_2$, $\tilde{b}^2, x_2^2$, w_1^4, w_2^2, $\tilde{b}w_1^2, x_2w_1^2$[41]	$\mathbb{Z}_2^4$: w_1^4, w_2^2, $a^4, a^2w_1^2$[42]	$\mathbb{Z}_2^8$: $\tilde{a}x_3, \tilde{b}x_2$, $\tilde{b}^2, x_2^2$, w_1^4, w_2^2, $\tilde{b}w_1^2, x_2w_1^2$[43]	$\mathbb{Z}_2^4$: w_1^4, w_2^2, $a^4, a^2w_1^2$[44]

Table 14: 4d bordism groups-2.

[28] $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_4)$, $b \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_4)$, $\tilde{a} = a \mod 2$, $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$.

[29] $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2)$, $a' \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_3, \mathbb{Z}_3)$, $b' = \beta_{(3,3)}a'$, $x_2' \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3)$.

[30] $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$, $b \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$, $\tilde{a} = a \mod 2$, $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$.

[31] $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2)$, $a' \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_9, \mathbb{Z}_9)$, $\tilde{a}' = a' \mod 3$, $b' \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_9, \mathbb{Z}_9)$, $x_2' \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3)$.

[32] Here $x_3 = \beta_{(2,4)}x_2$, $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_4)$.

[33] $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_4)$, $b \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_4)$, $\tilde{a} = a \mod 2$, $\tilde{b} = b \mod 2$, $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = \mathrm{Sq}^1 x_2$.

[34] $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$, $b \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$, $\tilde{a} = a \mod 2$, $\tilde{b} = b \mod 2$, $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = \mathrm{Sq}^1 x_2$.

[35] $\mathcal{P}_2(x_2)$ is the Pontryagin square of $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_4)$.

[36] $\mathcal{P}_2(x_2)$ is the Pontryagin square of $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$, $\tilde{b} = b \mod 2$, $b \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_4)$.

[37] $a' \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_3, \mathbb{Z}_3)$, $b' = \beta_{(3,3)}a'$, $x_2' \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3)$, $x_3' = \beta(3,3)x_2'$.

[38] $\mathcal{P}_2(x_2)$ is the Pontryagin square of $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$, $\tilde{b} = b \mod 2$, $b \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$.

[39] $a' \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_9, \mathbb{Z}_9)$, $b' \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_9, \mathbb{Z}_9)$, $\tilde{b}' = b' \mod 3 = \beta_{(3,9)}a'$, $x_2' \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3)$.

[40] $\tilde{x}_2 = x_2 \mod 2$, $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_4)$.

[41] $\tilde{a} = a \mod 2$, $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_4)$, $\tilde{b} = b \mod 2 = \beta_{(2,4)}a$, $b \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_4)$, $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = \mathrm{Sq}^1 x_2$.

[42] $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2)$.

[43] $\tilde{a} = a \mod 2$, $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$, $\tilde{b} = b \mod 2 = \beta_{(2,8)}a$, $b \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$, $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = \mathrm{Sq}^1 x_2$.

[44] $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2)$.

$\Omega_d^H(-)$	BO(3)	BO(4)	BO(5)
5 SO	$\mathbb{Z}_2^6$: $w_2w_3, w_2^2w_1'$, $w_2'w_3', w_1'w_2'^2$, $w_1'^2w_3'$ = $w_1'^3w_2', w_1'^5$	$\mathbb{Z}_2^6$: $w_2w_3, w_2^2w_1'$, $w_2'w_3', w_1'w_2'^2$, $w_1'^2w_3'$ = $w_1'^3w_2', w_1'^5$	$\mathbb{Z}_2^7$: $w_2w_3, w_2^2w_1'$, $w_2'w_3', w_1'w_2'^2$, $w_1'^2w_3'$ = $w_1'^3w_2', w_1'^5$, $w_1'w_4' = w_5'$
5 O	$\mathbb{Z}_2^{11}$: $w_2w_3, w_2^2w_1'$, $w_1^4w_1', w_1^2w_1'^3$, $w_1^2w_1'w_2', w_1^2w_3'$, $w_2'w_3', w_1'w_2'^2$, $w_1'^2w_3', w_1'^3w_2'$, $w_1'^5$	$\mathbb{Z}_2^{12}$: $w_2w_3, w_2^2w_1'$, $w_1^4w_1', w_1^2w_1'^3$, $w_1^2w_1'w_2', w_1^2w_3'$, $w_2'w_3', w_1'w_2'^2$, $w_1'^2w_3', w_1'^3w_2'$, $w_1'^5, w_1'w_4'$	$\mathbb{Z}_2^{13}$: $w_2w_3, w_2^2w_1'$, $w_1^4w_1', w_1^2w_1'^3$, $w_1^2w_1'w_2', w_1^2w_3'$, $w_2'w_3', w_1'w_2'^2$, $w_1'^2w_3', w_1'^3w_2'$, $w_1'^5, w_1'w_4'$, w_5'

Table 15: $5d$ bordism groups-1.

$\Omega_d^H(-)$	$\mathrm{B}^2\mathbb{Z}_4$	$\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3$	$\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_{18} \times \mathrm{B}^2\mathbb{Z}_3$
5 SO	$\mathbb{Z}_2^2$: w_2w_3, x_5	$\mathbb{Z}_4^3 \times \mathbb{Z}_2^3$: $a\mathcal{P}_2(x_2), ab^2$, $a(\sigma \bmod 4), x_5$ = x_2x_3, $\tilde{a}\tilde{b}x_2, w_2w_3$	$\mathbb{Z}_2^3 \times \mathbb{Z}_3^2 \times \mathbb{Z}_9$: a^5, aw_2^2, $w_2w_3, a'b'x_2'$ $a'x_2'^2, \mathfrak{P}_3(b')$[45]	$\mathbb{Z}_4 \times \mathbb{Z}_8^2 \times \mathbb{Z}_2^3$: $(a \bmod 4)\mathcal{P}_2(x_2), ab^2$, $a(\sigma \bmod 8), x_5$ = x_2x_3, $\tilde{a}\tilde{b}x_2, w_2w_3$	$\mathbb{Z}_2^3 \times \mathbb{Z}_3^3 \times \mathbb{Z}_{27}$: a^5, aw_2^2, $w_2w_3, \tilde{a}'(\sigma \bmod 3)$, $\tilde{a}'\tilde{b}'x_2', \tilde{a}'x_2'^2$, $\mathfrak{P}_3(b')$[46]
5 O	$\mathbb{Z}_2^4$: $w_2w_3, w_1^2x_3$, $\tilde{x}_2x_3, x_5$[47]	$\mathbb{Z}_2^{12}$: $\tilde{a}x_2^2, \tilde{b}x_3$, $x_2x_3, \tilde{a}\tilde{b}^2$, $x_5, \tilde{a}\tilde{b}x_2$, $w_2w_3, \tilde{a}w_2^2$, $\tilde{a}w_1^4, \tilde{a}\tilde{b}w_1^2$, $x_3w_1^2$ = $w_1^3x_2, \tilde{a}x_2w_1^2$[48]	$\mathbb{Z}_2^5$: $a^5, a^3w_1^2$, aw_1^4, aw_2^2, w_2w_3[49]	$\mathbb{Z}_2^{12}$: $\tilde{a}x_2^2, \tilde{b}x_3$, $x_2x_3, \tilde{a}\tilde{b}^2$, $x_5, \tilde{a}\tilde{b}x_2$, $w_2w_3, \tilde{a}w_2^2$, $\tilde{a}w_1^4, \tilde{a}\tilde{b}w_1^2$, $x_3w_1^2$ = $w_1^3x_2, \tilde{a}x_2w_1^2$[50]	$\mathbb{Z}_2^5$: $a^5, a^3w_1^2$, aw_1^4, aw_2^2, w_2w_3[51]

Table 16: $5d$ bordism groups-2.

[45] $a' \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_3, \mathbb{Z}_3)$, $b' = \beta_{(3,3)}a' \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_3, \mathbb{Z}_3)$, $x_2' \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3)$, $\mathfrak{P}_3(b')$ is the Postnikov square of b'.

[46] $\tilde{a}' = a' \bmod 3$, $a' \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_9, \mathbb{Z}_9)$, $\tilde{b}' = b' \bmod 3 = \beta_{(3,9)}a'$, $b' \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_9, \mathbb{Z}_9)$, $x_2' \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3)$, $\mathfrak{P}_3(b')$ is the Postnikov square of b'.

[47] $\tilde{x}_2 = x_2 \bmod 2$, $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_4)$, $x_3 = \beta_{(2,4)}x_2$, $x_5 = \mathrm{Sq}^2x_3$.

[48] $\tilde{a} = a \bmod 2$, $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_4)$, $\tilde{b} = b \bmod 2 = \beta_{(2,4)}a$, $b \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_4)$, $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = \mathrm{Sq}^1x_2$, $x_5 = \mathrm{Sq}^2x_3$.

[49] $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2)$.

[50] $\tilde{a} = a \bmod 2$, $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$, $\tilde{b} = b \bmod 2 = \beta_{(2,8)}a$, $b \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$, $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = \mathrm{Sq}^1x_2$, $x_5 = \mathrm{Sq}^2x_3$.

[51] $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2)$.

5.2 $B^2\mathbb{Z}_4$

5.2.1 $\Omega_d^{\mathrm{O}}(B^2\mathbb{Z}_4)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_4,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_4) \tag{5.1}$$

$$\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)=\mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^* \tag{5.2}$$

where $y_2^*=w_1^2$, $(y_2^2)^*=w_2^2$, $y_4^*=w_1^4$, $y_5^*=w_2w_3$, etc.

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_4,\mathbb{Z}_2)=\mathbb{Z}_2[\tilde{x}_2,x_3,x_5,x_9,\dots] \tag{5.3}$$

where $\tilde{x}_2=x_2 \mod 2$, $x_2\in\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_4,\mathbb{Z}_4)$, $x_3=\beta_{(2,4)}x_2$, $x_5=\mathrm{Sq}^2x_3$, $x_9=\mathrm{Sq}^4x_5$, etc.

$$\begin{aligned}\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_4,\mathbb{Z}_2) &= \mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^*\otimes\mathbb{Z}_2[\tilde{x}_2,x_3,x_5,x_9,\dots] \\ &= \mathcal{A}_2\oplus 2\Sigma^2\mathcal{A}_2\oplus\Sigma^3\mathcal{A}_2\oplus 4\Sigma^4\mathcal{A}_2\oplus 4\Sigma^5\mathcal{A}_2\oplus\cdots\end{aligned} \tag{5.4}$$

Hence we have the following theorem

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_4)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2^4$
5	$\mathbb{Z}_2^4$

Theorem 94.

The 2d bordism invariants are $w_1^2,\tilde{x}_2$.

The 3d bordism invariant is x_3.

The 4d bordism invariants are $w_1^4,w_2^2,w_1^2\tilde{x}_2,\tilde{x}_2^2$.

The 5d bordism invariants are $w_2w_3,w_1^2x_3,\tilde{x}_2x_3,x_5$.

5.2.2 $\Omega_d^{\mathrm{SO}}(B^2\mathbb{Z}_4)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_4,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_4) \tag{5.5}$$

$$\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)=\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^4\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^5\mathcal{A}_2\oplus\cdots \tag{5.6}$$

$\mathrm{Sq}^1\tilde{x}_2=2\beta_{(2,4)}x_2=0$, $\beta_{(2,4)}(x_2)=\frac{1}{4}\delta x_2=x_3$, $\beta_{(2,4)}(x_2^2)=2x_2x_3=2\tilde{x}_2x_3=0$, $\mathrm{Sq}^1(\tilde{x}_2^2)=2\beta_{(2,4)}(x_2^2)=0$, $\beta_{(2,8)}\mathcal{P}_2(x_2)=\frac{1}{8}\delta\mathcal{P}_2(x_2)=\frac{1}{8}\delta(x_2\cup x_2+x_2\underset{1}{\cup}\delta x_2)=\frac{1}{8}(2\delta x_2\cup x_2+\delta x_2\underset{1}{\cup}\delta x_2)=x_3\cup x_2+2x_3\underset{1}{\cup}x_3=x_3\cup x_2+2\mathrm{Sq}^2x_3=\tilde{x}_2x_3+2x_5=\tilde{x}_2x_3$, $\mathrm{Sq}^1x_3=0$, $\mathrm{Sq}^1(\tilde{x}_2x_3)=0$, $\mathrm{Sq}^1x_5=\mathrm{Sq}^1\mathrm{Sq}^2\beta_{(2,4)}x_2=\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1(\frac{1}{2}\tilde{x}_2)=\mathrm{Sq}^2((\frac{1}{2}\tilde{x}_2)^2)=(\mathrm{Sq}^1(\frac{1}{2}\tilde{x}_2))^2=(\beta_{(2,4)}x_2)^2=x_3^2$.

$d_2(x_3)=\tilde{x}_2h_0^2$, $d_3(\tilde{x}_2x_3)=\tilde{x}_2^2h_0^3$.

The E_2 page is shown in Figure 48.

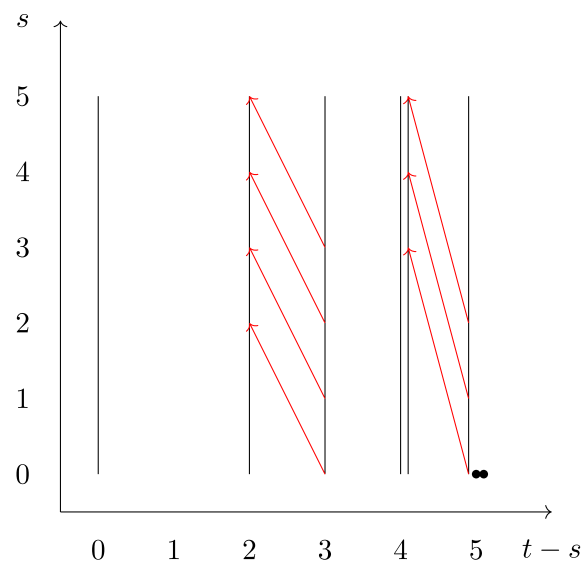


Figure 48: $\Omega_*^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_4)$

Hence we have the following theorem

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_4)$
0	$\mathbb{Z}$
1	0
2	$\mathbb{Z}_4$
3	0
4	$\mathbb{Z}\times\mathbb{Z}_8$
5	$\mathbb{Z}_2^2$

Theorem 95.

The 2d bordism invariant is x_2.

The 4d bordism invariants are $\sigma,\mathcal{P}_2(x_2)$.

The 5d bordism invariants are w_2w_3,x_5.

5.3 B**O**(3)

5.3.1 $\Omega_d^{\mathbf{O}}(\mathrm{B}\mathbf{O}(3))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BO}(3),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{BO}(3)) \tag{5.7}$$

$$\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)=\mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^* \tag{5.8}$$

where $y_2^*=w_1^2$, $(y_2^2)^*=w_2^2$, $y_4^*=w_1^4$, $y_5^*=w_2w_3$, etc.

$$\mathrm{H}^*(\mathrm{BO}(3),\mathbb{Z}_2)=\mathbb{Z}_2[w_1',w_2',w_3'] \tag{5.9}$$

$$\begin{aligned}\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BO}(3),\mathbb{Z}_2) &= \mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^*\otimes\mathbb{Z}_2[w_1',w_2',w_3'] \\ &= \mathcal{A}_2\oplus\Sigma\mathcal{A}_2\oplus3\Sigma^2\mathcal{A}_2\oplus4\Sigma^3\mathcal{A}_2\oplus8\Sigma^4\mathcal{A}_2\oplus11\Sigma^5\mathcal{A}_2\oplus\cdots\end{aligned} \tag{5.10}$$

Hence we have the following theorem

i	$\Omega_i^{\mathrm{O}}(\mathrm{BO}(3))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^8$
5	$\mathbb{Z}_2^{11}$

Theorem 96.

The 2d bordism invariants are $w_1^2, w_1'^2, w_2'$.

The 3d bordism invariant are $w_1'w_1^2, w_1'^3, w_1'w_2', w_3'$.

The 4d bordism invariants are $w_1^4, w_2^2, w_1^2w_1'^2, w_1^2w_2', w_1'w_3', w_1'^2w_2', w_1'^4, w_2'^2$.

The 5d bordism invariants are $w_2w_3, w_2^2w_1', w_1^4w_1', w_1^2w_1'^3, w_1^2w_1'w_2', w_1^2w_3', w_2'w_3', w_1'w_2'^2, w_1'^2w_3', w_1'^3w_2', w_1'^5$.

5.3.2 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbf{O}(3))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BO}(3),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{BO}(3)) \tag{5.11}$$

$$\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)=\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^4\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^5\mathcal{A}_2\oplus\cdots \tag{5.12}$$

$$\mathrm{H}^*(\mathrm{BO}(3),\mathbb{Z}_2)=\mathbb{Z}_2[w_1',w_2',w_3'] \tag{5.13}$$

where $\mathrm{Sq}^1w_2'=w_1'w_2'+w_3'$, $\mathrm{Sq}^1w_3'=w_1'w_3'$.

The E_2 page is shown in Figure 49.

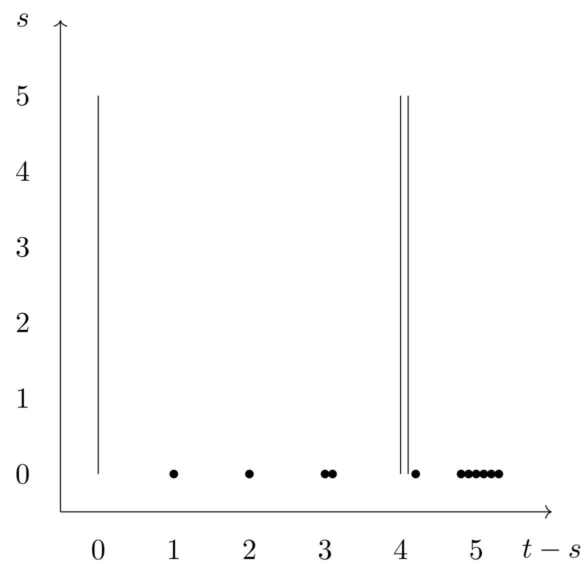


Figure 49: $\Omega_*^{\mathrm{SO}}(\mathrm{BO}(3))$

Hence we have the following theorem

i	$\Omega_i^{\mathrm{SO}}(\mathrm{BO}(3))$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}^2\times\mathbb{Z}_2$
5	$\mathbb{Z}_2^6$

Theorem 97.

The 2d bordism invariant is w_2'.

The 3d bordism invariants are $w_1'^3, w_1'w_2'=w_3'$.

The 4d bordism invariants are $\sigma, p_1', w_1'^2w_2'$.

The 5d bordism invariants are $w_2w_3, w_2^2w_1', w_2'w_3', w_1'w_2'^2, w_1'^2w_3'=w_1'^3w_2', w_1'^5$.

5.4 B**O**(4)

5.4.1 $\Omega_d^{\mathbf{O}}(\mathrm{B}\mathbf{O}(4))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BO}(4),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{BO}(4)) \tag{5.14}$$

$$\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)=\mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^* \tag{5.15}$$

where $y_2^*=w_1^2$, $(y_2^2)^*=w_2^2$, $y_4^*=w_1^4$, $y_5^*=w_2w_3$, etc.

$$\mathrm{H}^*(\mathrm{BO}(4),\mathbb{Z}_2)=\mathbb{Z}_2[w_1',w_2',w_3',w_4'] \tag{5.16}$$

$$\begin{aligned}\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BO}(4),\mathbb{Z}_2) &= \mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^*\otimes\mathbb{Z}_2[w_1',w_2',w_3',w_4'] \\ &= \mathcal{A}_2\oplus\Sigma\mathcal{A}_2\oplus3\Sigma^2\mathcal{A}_2\oplus4\Sigma^3\mathcal{A}_2\oplus9\Sigma^4\mathcal{A}_2\oplus12\Sigma^5\mathcal{A}_2\oplus\cdots\end{aligned} \tag{5.17}$$

Hence we have the following theorem

i	$\Omega_i^{\mathrm{O}}(\mathrm{BO}(4))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^9$
5	$\mathbb{Z}_2^{12}$

Theorem 98.

The 2d bordism invariants are $w_1^2,w_1'^2,w_2'$.

The 3d bordism invariant are $w_1'w_1^2,w_1'^3,w_1'w_2',w_3'$.

The 4d bordism invariants are $w_1^4,w_2^2,w_1^2w_1'^2,w_1^2w_2',w_1'w_3',w_1'^2w_2',w_1'^4,w_2'^2,w_4'$.

The 5d bordism invariants are

$$w_2w_3,w_2^2w_1',w_1^4w_1',w_1^2w_1'^3,w_1^2w_1'w_2',w_1^2w_3',w_2'w_3',w_1'w_2'^2,w_1'^2w_3',w_1'^3w_2',w_1'^5,w_1'w_4'.$$

5.4.2 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbf{O}(4))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{BO}(4),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{BO}(4)) \tag{5.18}$$

$$\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)=\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^4\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^5\mathcal{A}_2\oplus\cdots \tag{5.19}$$

$$\mathrm{H}^*(\mathrm{BO}(4),\mathbb{Z}_2)=\mathbb{Z}_2[w_1',w_2',w_3',w_4'] \tag{5.20}$$

where $\mathrm{Sq}^1w_2'=w_1'w_2'+w_3'$, $\mathrm{Sq}^1w_3'=w_1'w_3'$, $\mathrm{Sq}^1w_4'=w_1'w_4'$.

The E_2 page is shown in Figure 50.

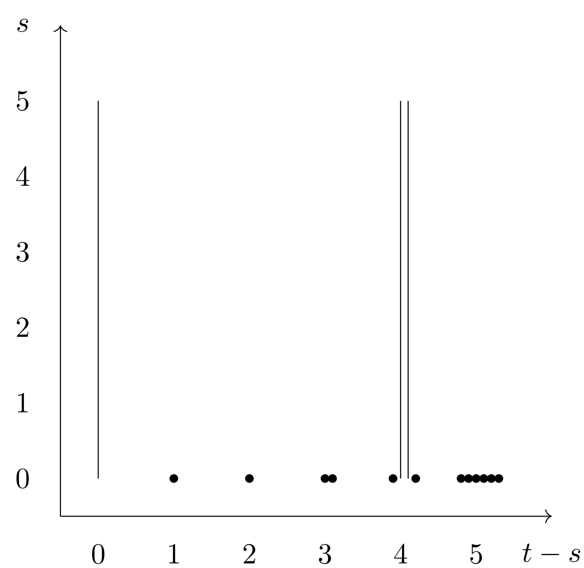


Figure 50: $\Omega_*^{\mathrm{SO}}(\mathrm{BO}(4))$

Hence we have the following theorem

i	$\Omega_i^{\mathrm{SO}}(\mathrm{BO}(4))$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}^2\times\mathbb{Z}_2^2$
5	$\mathbb{Z}_2^6$

Theorem 99.

The 2d bordism invariant is w_2'.

The 3d bordism invariants are $w_1'^3, w_1'w_2' = w_3'$.

The 4d bordism invariants are $\sigma, p_1', w_1'^2 w_2', w_4'$.

The 5d bordism invariants are $w_2w_3, w_2^2 w_1', w_2'w_3', w_1'w_2'^2, w_1'^2 w_3' = w_1'^3 w_2', w_1'^5$.

5.5 B**O**(5)

5.5.1 $\Omega_d^{\mathbf{O}}(\mathrm{B}\mathbf{O}(5))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BO}(5),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{BO}(5)) \tag{5.21}$$

$$\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)=\mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^* \tag{5.22}$$

where $y_2^*=w_1^2$, $(y_2^2)^*=w_2^2$, $y_4^*=w_1^4$, $y_5^*=w_2w_3$, etc.

$$\mathrm{H}^*(\mathrm{BO}(5),\mathbb{Z}_2)=\mathbb{Z}_2[w_1',w_2',w_3',w_4',w_5'] \tag{5.23}$$

$$\begin{aligned}\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BO}(5),\mathbb{Z}_2) &= \mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^*\otimes\mathbb{Z}_2[w_1',w_2',w_3',w_4',w_5'] \\ &= \mathcal{A}_2\oplus\Sigma\mathcal{A}_2\oplus3\Sigma^2\mathcal{A}_2\oplus4\Sigma^3\mathcal{A}_2\oplus9\Sigma^4\mathcal{A}_2\oplus13\Sigma^5\mathcal{A}_2\oplus\cdots\end{aligned} \tag{5.24}$$

Hence we have the following theorem

i	$\Omega_i^{\mathrm{O}}(\mathrm{BO}(5))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^9$
5	$\mathbb{Z}_2^{13}$

Theorem 100.

The 2d bordism invariants are $w_1^2, w_1'^2, w_2'$.

The 3d bordism invariant are $w_1'w_1^2, w_1'^3, w_1'w_2', w_3'$.

The 4d bordism invariants are $w_1^4, w_2^2, w_1^2 w_1'^2, w_1^2 w_2', w_1' w_3', w_1'^2 w_2', w_1'^4, w_2'^2, w_4'$.

The 5d bordism invariants are

$$w_2 w_3, w_2^2 w_1', w_1^4 w_1', w_1^2 w_1'^3, w_1^2 w_1' w_2', w_1^2 w_3', w_2' w_3', w_1' w_2'^2, w_1'^2 w_3', w_1'^3 w_2', w_1'^5, w_1' w_4', w_5'.$$

5.5.2 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbf{O}(5))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BO}(5),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{BO}(5)) \tag{5.25}$$

$$\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)=\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^4\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^5\mathcal{A}_2\oplus\cdots \tag{5.26}$$

$$\mathrm{H}^*(\mathrm{BO}(5),\mathbb{Z}_2)=\mathbb{Z}_2[w_1',w_2',w_3',w_4',w_5'] \tag{5.27}$$

where $\mathrm{Sq}^1 w_2' = w_1' w_2' + w_3'$, $\mathrm{Sq}^1 w_3' = w_1' w_3'$, $\mathrm{Sq}^1 w_4' = w_1' w_4' + w_5'$.

The E_2 page is shown in Figure 51.

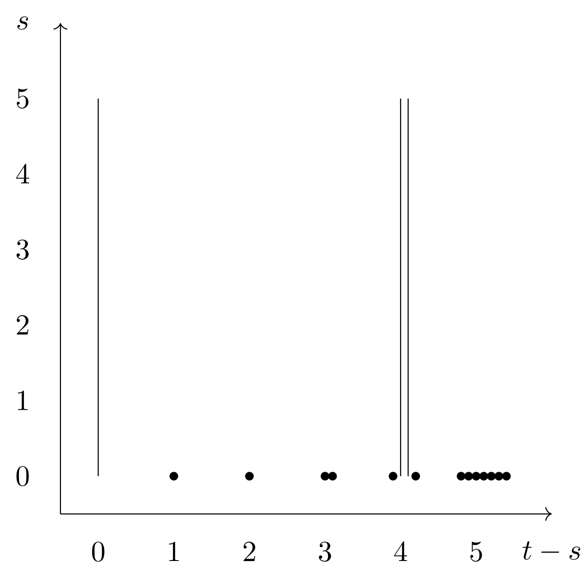


Figure 51: $\Omega_*^{\mathrm{SO}}(\mathrm{BO}(5))$

Hence we have the following theorem

Theorem 101.

The 2d bordism invariant is w_2'.

i	$\Omega_i^{\text{SO}}(\text{BO}(5))$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}^2 \times \mathbb{Z}_2^2$
5	$\mathbb{Z}_2^7$

The 3d bordism invariants are $w_1'^3, w_1'w_2' = w_3'$.

The 4d bordism invariants are $\sigma, p_1', w_1'^2 w_2', w_4'$.

The 5d bordism invariants are $w_2w_3, w_2^2 w_1', w_2'w_3', w_1'w_2'^2, w_1'^2 w_3' = w_1'^3 w_2', w_1'^5, w_1'w_4' = w_5'$.

5.6 $\text{B}\mathbb{Z}_{2n} \times \text{B}^2\mathbb{Z}_n$

5.6.1 $\Omega_d^{\text{O}}(\text{B}\mathbb{Z}_4 \times \text{B}^2\mathbb{Z}_2)$

$$\text{H}^*(\text{B}\mathbb{Z}_4, \mathbb{Z}_4) = \mathbb{Z}_4[a, b]/(a^2 = 2b) \tag{5.28}$$

where $a \in \text{H}^1(\text{B}\mathbb{Z}_4, \mathbb{Z}_4)$, $b \in \text{H}^2(\text{B}\mathbb{Z}_4, \mathbb{Z}_4)$.

$$\text{H}^*(\text{B}\mathbb{Z}_4, \mathbb{Z}_2) = \Lambda_{\mathbb{Z}_2}(\tilde{a}) \otimes \mathbb{Z}_2[\tilde{b}] \tag{5.29}$$

where $\tilde{a} = a \mod 2 \in \text{H}^1(\text{B}\mathbb{Z}_4, \mathbb{Z}_2)$, $\tilde{b} = b \mod 2 \in \text{H}^2(\text{B}\mathbb{Z}_4, \mathbb{Z}_2)$.

$$\text{H}^*(\text{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[x_2, x_3, x_5, x_9, \dots] \tag{5.30}$$

$$\text{Ext}_{\mathcal{A}_2}^{s,t}(\text{H}^*(M\text{O}, \mathbb{Z}_2) \otimes \text{H}^*(\text{B}\mathbb{Z}_4 \times \text{B}^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\text{O}}(\text{B}\mathbb{Z}_4 \times \text{B}^2\mathbb{Z}_2) \tag{5.31}$$

$$\text{H}^*(M\text{O}, \mathbb{Z}_2) = \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^* \tag{5.32}$$

where $y_2^* = w_1^2$, $(y_2^2)^* = w_2^2$, $y_4^* = w_1^4$, $y_5^* = w_2w_3$, etc.

$$\begin{aligned} & \text{H}^*(M\text{O}, \mathbb{Z}_2) \otimes \text{H}^*(\text{B}\mathbb{Z}_4 \times \text{B}^2\mathbb{Z}_2, \mathbb{Z}_2) \\ = \; & \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^* \otimes \Lambda_{\mathbb{Z}_2}(\tilde{a}) \otimes \mathbb{Z}_2[\tilde{b}] \otimes \mathbb{Z}_2[\tilde{x}_2, x_3, x_5, x_9, \dots] \\ = \; & \mathcal{A}_2 \oplus \Sigma\mathcal{A}_2 \oplus 3\Sigma^2\mathcal{A}_2 \oplus 4\Sigma^3\mathcal{A}_2 \oplus 8\Sigma^4\mathcal{A}_2 \oplus 12\Sigma^5\mathcal{A}_2 \oplus \cdots \end{aligned} \tag{5.33}$$

Hence we have the following theorem

Theorem 102. The bordism groups are

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^8$
5	$\mathbb{Z}_2^{12}$

The 2d bordism invariants are $\tilde{b}, x_2, w_1^2$.

The 3d bordism invariants are $\tilde{a}\tilde{b}, x_3, \tilde{a}x_2, \tilde{a}w_1^2$.

The 4d bordism invariants are $\tilde{a}x_3, \tilde{b}x_2, \tilde{b}^2, x_2^2, w_1^4, w_2^2, \tilde{b}w_1^2, x_2w_1^2$.

The 5d bordism invariants are $\tilde{a}x_2^2, \tilde{b}x_3, x_2x_3, \tilde{a}\tilde{b}^2, x_5, \tilde{a}\tilde{b}x_2, w_2w_3, \tilde{a}w_2^2, \tilde{a}w_1^4, \tilde{a}\tilde{b}w_1^2, x_3w_1^2 = w_1^3x_2, \tilde{a}x_2w_1^2$.

Note

$$x_3 = w_1x_2 \tag{5.34}$$

except $x_2x_3 = \frac{1}{2}w_1x_2^2$.

5.6.2 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2) \tag{5.35}$$

$$\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2) = \mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1 \oplus \Sigma^4\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1 \oplus \Sigma^5\mathcal{A}_2 \oplus \cdots \tag{5.36}$$

$\beta_{(2,4)}a = \tilde{b}$, $\mathrm{Sq}^1x_2 = x_3$, $\mathrm{Sq}^1(\tilde{a}x_2) = \tilde{a}x_3$, $\mathrm{Sq}^1(\tilde{b}x_2) = \tilde{b}x_3$, $\beta_{(2,4)}(ab) = \tilde{b}^2$, $\beta_{(2,4)}(\mathcal{P}_2(x_2)) = x_2x_3 + x_5$, $\mathrm{Sq}^1(x_2x_3) = \mathrm{Sq}^1x_5 = x_3^2$, $\mathrm{Sq}^1(\tilde{a}\tilde{b}x_2) = \tilde{a}\tilde{b}x_3$, $\beta_{(2,4)}(a\mathcal{P}_2(x_2)) = \tilde{b}x_2^2 + \tilde{a}(x_2x_3 + x_5)$, $\beta_{(2,4)}(ab^2) = \tilde{b}^3$, $\beta_{(2,4)}(a(\sigma \mod 4)) = \tilde{b}w_2^2$.

$d_2(\tilde{b}) = \tilde{a}h_0^2$, $d_2(\tilde{b}^2) = \tilde{a}\tilde{b}h_0^2$, $d_2(x_2x_3 + x_5) = x_2^2h_0^2$, $d_2(\tilde{b}x_2^2 + \tilde{a}(x_2x_3 + x_5)) = \tilde{a}x_2^2h_0^2$, $d_2(\tilde{b}^3) = \tilde{a}\tilde{b}^2h_0^2$, $d_2(\tilde{b}w_2^2) = \tilde{a}w_2^2h_0^2$.

The E_2 page is shown in Figure 52.

Hence we have the following theorem

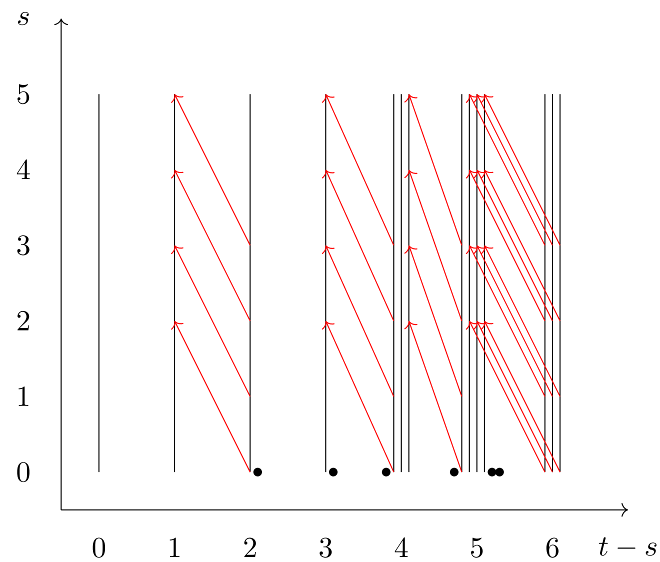


Figure 52: $\Omega_*^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2)$

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_4$
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_4 \times \mathbb{Z}_2$
4	$\mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_2$
5	$\mathbb{Z}_4^3 \times \mathbb{Z}_2^3$

Theorem 103. The bordism groups are

The 2d bordism invariant is x_2.

The 3d bordism invariants are ab and $\tilde{a}x_2$.

The 4d bordism invariants are σ, $\mathcal{P}_2(x_2)$ and $\tilde{b}x_2$.

The 5d bordism invariants are $a\mathcal{P}_2(x_2)$, ab^2, $a(\sigma \mod 4)$, $x_5 = x_2x_3$, $\tilde{a}\tilde{b}x_2$ and w_2w_3.

5.6.3 $\Omega_d^{\mathbf{O}}(\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_2) = \mathrm{H}^*(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[a] \tag{5.37}$$

where $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2)$.

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_3) \otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge \tag{5.38}$$

Since $\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_3)=0$, we have $\Omega_d^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge=0$.

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge \tag{5.39}$$

$$\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)=\mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^* \tag{5.40}$$

where $y_2^*=w_1^2$, $(y_2^2)^*=w_2^2$, $y_4^*=w_1^4$, $y_5^*=w_2w_3$, etc.

$$\begin{aligned}\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2) &= \mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^*\otimes\mathbb{Z}_2[a] \\ &= \mathcal{A}_2\oplus\Sigma\mathcal{A}_2\oplus2\Sigma^2\mathcal{A}_2\oplus2\Sigma^3\mathcal{A}_2\oplus4\Sigma^4\mathcal{A}_2\oplus5\Sigma^5\mathcal{A}_2\oplus\cdots\end{aligned} \tag{5.41}$$

Hence we have the following theorem

Theorem 104. The bordism groups are

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_2^4$
5	$\mathbb{Z}_2^5$

The 2d bordism invariants are w_1^2,a^2.

The 3d bordism invariants are a^3,aw_1^2.

The 4d bordism invariants are $a^4,a^2w_1^2,w_1^4,w_2^2$.

The 5d bordism invariants are $a^5,a^3w_1^2,aw_1^4,aw_2^2,w_2w_3$.

5.6.4 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO}\wedge(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{5.42}$$

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2)$, we have $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge=\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2)$.

The E_2 page is shown in Figure 53.

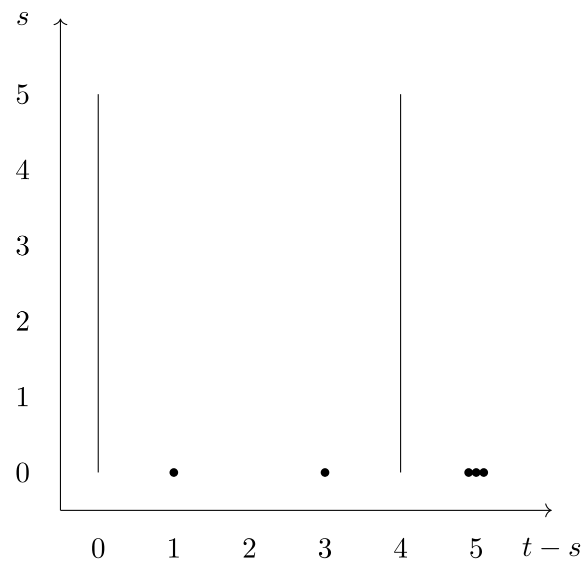


Figure 53: $\Omega_*^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2)$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{SO} \wedge (\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \quad (5.43)$$

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3) = \mathrm{H}^*(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3)$, we have $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge = \Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge$.

Hence we have the following theorem

Theorem 105. The bordism groups are

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_3 \times \mathbb{Z}_2$
2	$\mathbb{Z}_3$
3	$\mathbb{Z}_3^2 \times \mathbb{Z}_2$
4	$\mathbb{Z} \times \mathbb{Z}_3^2$
5	$\mathbb{Z}_2^3 \times \mathbb{Z}_3^2 \times \mathbb{Z}_9$

The 2d bordism invariant is x_2'.

The 3d bordism invariants are $a'b', a'x_2', a^3$.

The 4d bordism invariants are σ, $a'x_3'(= b'x_2')$ and $x_2'^2$.

The 5d bordism invariants are $a^5, aw_2^2, w_2w_3, a'b'x_2', a'x_2'^2, \mathfrak{P}_3(b')$.

Here $\mathfrak{P}_3$ is the Postnikov square.

5.7 $\mathrm{B}\mathbb{Z}_{2n^2} \times \mathrm{B}^2\mathbb{Z}_n$

5.7.1 $\Omega_d^{\mathbf{O}}(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2)$

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8) = \mathbb{Z}_8[a, b]/(a^2 = 4b) \tag{5.44}$$

where $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$, $b \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$.

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_2) = \Lambda_{\mathbb{Z}_2}(\tilde{a}) \otimes \mathbb{Z}_2[\tilde{b}] \tag{5.45}$$

where $\tilde{a} = a \mod 2 \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_2)$, $\tilde{b} = b \mod 2 \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_2)$.

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[x_2, x_3, x_5, x_9, \dots] \tag{5.46}$$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2) \tag{5.47}$$

$$\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) = \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^* \tag{5.48}$$

where $y_2^* = w_1^2$, $(y_2^2)^* = w_2^2$, $y_4^* = w_1^4$, $y_5^* = w_2w_3$, etc.

$$\begin{aligned} & \mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) \\ = \;& \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^* \otimes \Lambda_{\mathbb{Z}_2}(\tilde{a}) \otimes \mathbb{Z}_2[\tilde{b}] \otimes \mathbb{Z}_2[\tilde{x}_2, x_3, x_5, x_9, \dots] \\ = \;& \mathcal{A}_2 \oplus \Sigma\mathcal{A}_2 \oplus 3\Sigma^2\mathcal{A}_2 \oplus 4\Sigma^3\mathcal{A}_2 \oplus 8\Sigma^4\mathcal{A}_2 \oplus 12\Sigma^5\mathcal{A}_2 \oplus \cdots \end{aligned} \tag{5.49}$$

Hence we have the following theorem

Theorem 106. The bordism groups are

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^8$
5	$\mathbb{Z}_2^{12}$

The 2d bordism invariants are $\tilde{b}, x_2, w_1^2$.

The 3d bordism invariants are $\tilde{a}\tilde{b}, x_3, \tilde{a}x_2, \tilde{a}w_1^2$.

The 4d bordism invariants are $\tilde{a}x_3, \tilde{b}x_2, \tilde{b}^2, x_2^2, w_1^4, w_2^2, \tilde{b}w_1^2, x_2w_1^2$.

The 5d bordism invariants are $\tilde{a}x_2^2, \tilde{b}x_3, x_2x_3, \tilde{a}\tilde{b}^2, x_5, \tilde{a}\tilde{b}x_2, w_2w_3, \tilde{a}w_2^2, \tilde{a}w_1^4, \tilde{a}\tilde{b}w_1^2, x_3w_1^2 = w_1^3x_2, \tilde{a}x_2w_1^2$.

Note

$$x_3 = w_1x_2 \tag{5.50}$$

except $x_2x_3 = \frac{1}{2}w_1x_2^2$.

5.7.2 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2) \tag{5.51}$$

$$\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2) = \mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1 \oplus \Sigma^4\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1 \oplus \Sigma^5\mathcal{A}_2 \oplus \cdots \tag{5.52}$$

$\beta_{(2,8)}a = \tilde{b}$, $\mathrm{Sq}^1x_2 = x_3$, $\mathrm{Sq}^1(\tilde{a}x_2) = \tilde{a}x_3$, $\mathrm{Sq}^1(\tilde{b}x_2) = \tilde{b}x_3$, $\beta_{(2,8)}(ab) = \tilde{b}^2$, $\beta_{(2,4)}(\mathcal{P}_2(x_2)) = x_2x_3 + x_5$, $\mathrm{Sq}^1(x_2x_3) = \mathrm{Sq}^1x_5 = x_3^2$, $\mathrm{Sq}^1(\tilde{a}\tilde{b}x_2) = \tilde{a}\tilde{b}x_3$, $\beta_{(2,4)}((a \mod 4)\mathcal{P}_2(x_2)) = 2\beta_{(2,8)}(a)x_2^2 + \tilde{a}(x_2x_3 + x_5) = 2\tilde{b}x_2^2 + \tilde{a}(x_2x_3 + x_5) = \tilde{a}(x_2x_3 + x_5)$, $\beta_{(2,8)}(ab^2) = \tilde{b}^3$, $\beta_{(2,8)}(a(\sigma \mod 8)) = \tilde{b}w_2^2$.

$d_3(\tilde{b}) = \tilde{a}h_0^2$, $d_3(\tilde{b}^2) = \tilde{a}\tilde{b}h_0^2$, $d_2(x_2x_3 + x_5) = x_2^2h_0^2$, $d_2(\tilde{a}(x_2x_3 + x_5)) = \tilde{a}x_2^2h_0^2$, $d_3(\tilde{b}^3) = \tilde{a}\tilde{b}^2h_0^2$, $d_3(\tilde{b}w_2^2) = \tilde{a}w_2^2h_0^2$.

The E_2 page is shown in Figure 54.

Hence we have the following theorem

Theorem 107. The bordism groups are

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_8$
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_8 \times \mathbb{Z}_2$
4	$\mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_2$
5	$\mathbb{Z}_4 \times \mathbb{Z}_8^2 \times \mathbb{Z}_2^3$

The 2d bordism invariant is x_2.

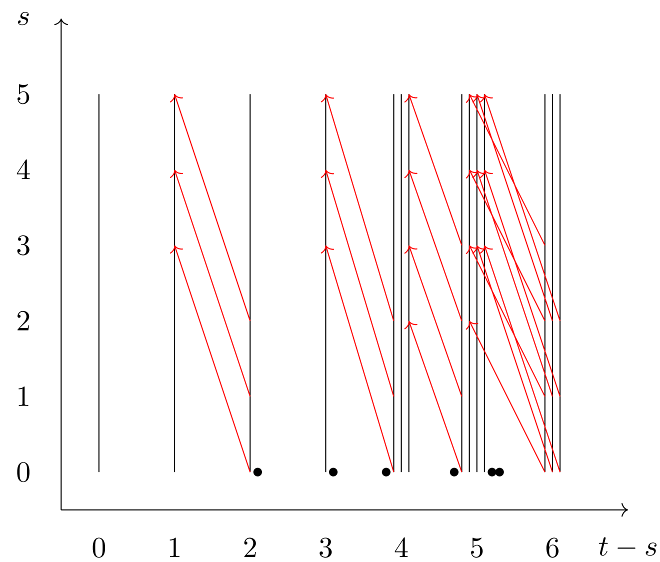


Figure 54: $\Omega_*^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_8\times\mathrm{B}^2\mathbb{Z}_2)$

The 3d bordism invariants are ab and $\tilde{a}x_2$.

The 4d bordism invariants are σ, $\mathcal{P}_2(x_2)$ and $\tilde{b}x_2$.

The 5d bordism invariants are $(a \mod 4)\mathcal{P}_2(x_2)$, ab^2, $a(\sigma \mod 8)$, $x_5=x_2x_3$, $\tilde{a}\tilde{b}x_2$ and w_2w_3.

5.7.3 $\Omega_d^{\mathbf{O}}(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2)=\mathbb{Z}_2[a] \tag{5.53}$$

where $a\in\mathrm{H}^1(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2)$.

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_3)\otimes\mathrm{H}^*(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)_3^{\wedge} \tag{5.54}$$

Since $\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_3)=0$, we have $\Omega_d^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)_3^{\wedge}=0$.

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)_2^{\wedge} \tag{5.55}$$

$$\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)=\mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^* \tag{5.56}$$

where $y_2^*=w_1^2$, $(y_2^2)^*=w_2^2$, $y_4^*=w_1^4$, $y_5^*=w_2w_3$, etc.

$$\begin{aligned}\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2) &= \mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^*\otimes\mathbb{Z}_2[a] \\ &= \mathcal{A}_2\oplus\Sigma\mathcal{A}_2\oplus 2\Sigma^2\mathcal{A}_2\oplus 2\Sigma^3\mathcal{A}_2\oplus 4\Sigma^4\mathcal{A}_2\oplus 5\Sigma^5\mathcal{A}_2\oplus\cdots\end{aligned} \tag{5.57}$$

Hence we have the following theorem

Theorem 108. The bordism groups are

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_2^4$
5	$\mathbb{Z}_2^5$

The 2d bordism invariants are w_1^2, a^2.

The 3d bordism invariants are a^3, aw_1^2.

The 4d bordism invariants are $a^4, a^2w_1^2, w_1^4, w_2^2$.

The 5d bordism invariants are $a^5, a^3w_1^2, aw_1^4, aw_2^2, w_2w_3$.

5.7.4 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO}\wedge(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{5.58}$$

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2)$, we have $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge=\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2)$.

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{SO}\wedge(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{5.59}$$

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_3)=\mathrm{H}^*(\mathrm{B}\mathbb{Z}_9\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_3)$, we have $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge=\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_9\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge$.

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_9,\mathbb{Z}_9)=\Lambda_{\mathbb{Z}_9}(a')\otimes\mathbb{Z}_9[b']. \tag{5.60}$$

where $a'\in\mathrm{H}^1(\mathrm{B}\mathbb{Z}_9,\mathbb{Z}_9)$, $b'\in\mathrm{H}^2(\mathrm{B}\mathbb{Z}_9,\mathbb{Z}_9)$.

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_9,\mathbb{Z}_3)=\Lambda_{\mathbb{Z}_3}(\tilde{a}')\otimes\mathbb{Z}_3[\tilde{b}']. \tag{5.61}$$

where $\tilde{a}' = a' \mod 3$, $\tilde{b}' = b' \mod 3$, $\tilde{b}' = \beta_{(3,9)}(a')$.

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3) = \mathbb{Z}_3[x_2', x_8', \dots] \otimes \Lambda_{\mathbb{Z}_3}(x_3', x_7', \dots) \tag{5.62}$$

$\beta_{(3,3)}(\tilde{a}') = 3\beta_{(3,9)}(a') = 3\tilde{b}' = 0$, $\beta_{(3,3)}(x_2') = x_3'$, $\beta_{(3,3)}(x_2'^2) = 2x_2'x_3'$, $\beta_{(3,9)}(a'b') = \tilde{b}'^2$, $\beta_{(3,9)}(a'b'^2) = \tilde{b}'^3$, $\beta_{(3,3)}(\tilde{a}'x_2') = \tilde{a}'x_3'$, $\beta_{(3,3)}(\tilde{b}'x_2') = \tilde{b}'x_3'$, $\beta_{(3,3)}(\tilde{a}'\tilde{b}'x_2') = \tilde{a}'\tilde{b}'x_3'$, $\beta_{(3,3)}(\tilde{a}'x_2'^2) = 2\tilde{a}'x_2'x_3'$.

There is a differential d_2 corresponding to the $(3,9)$-Bockstein [26].

$d_2(\tilde{b}') = \tilde{a}'h_0'^2$, $d_2(\tilde{b}'^2) = \tilde{a}'\tilde{b}'h_0'^2$, $d_2(\tilde{b}'^3) = \tilde{a}'\tilde{b}'^2h_0'^2$.

The E_2 page is shown in Figure 55.

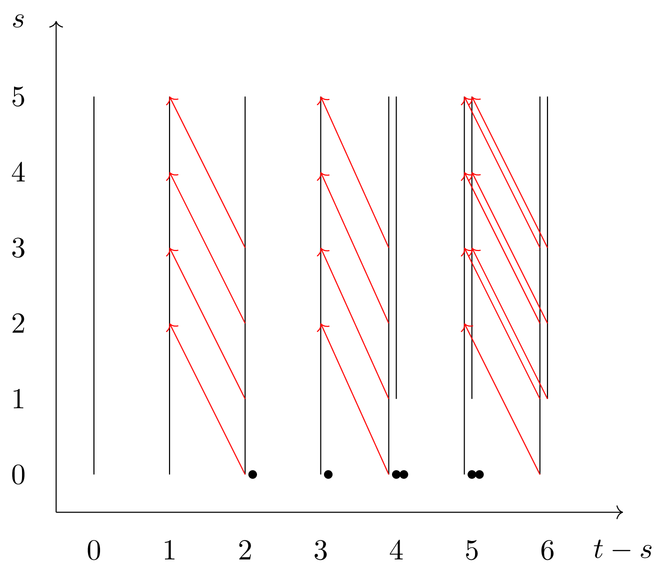


Figure 55: $\Omega_*^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_9 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge$

Hence we have the following theorem

Theorem 109. The bordism groups are

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_{18} \times \mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_9 \times \mathbb{Z}_2$
2	$\mathbb{Z}_3$
3	$\mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_2$
4	$\mathbb{Z} \times \mathbb{Z}_3^2$
5	$\mathbb{Z}_2^3 \times \mathbb{Z}_3^3 \times \mathbb{Z}_{27}$

The 2d bordism invariant is x_2'.

The 3d bordism invariants are $a'b', \tilde{a}'x_2', a^3$.

The 4d bordism invariants are $\sigma, \tilde{b}'x_2', x_2'^2$.

The 5d bordism invariants are $a^5, aw_2^2, w_2w_3, \tilde{a}'(\sigma \mod 3), \tilde{a}'\tilde{b}'x_2', \tilde{a}'x_2'^2, \mathfrak{P}_3(b')$.

Here $\mathfrak{P}_3$ is the Postnikov square.

5.8 $\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(N))$

For $N > 2$, the outer automorphism group of $\mathrm{PSU}(N)$ is $\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathrm{PSU}(N)$ via complex conjugation.

5.8.1 $\Omega_3^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)),\mathbb{Z}_2),\mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3))) \tag{5.63}$$

$$\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2) = \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^* \tag{5.64}$$

where $y_2^* = w_1^2$, $(y_2^2)^* = w_2^2$, $y_4^* = w_1^4$, $y_5^* = w_2w_3$, etc.

We have a fibration

$$\mathrm{BPSU}(3) \to \mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)) \to \mathrm{B}\mathbb{Z}_2. \tag{5.65}$$

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2) = \mathbb{Z}_2[a] \tag{5.66}$$

$$\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2) = \mathbb{Z}_2[c_2, c_3] \tag{5.67}$$

By Serre spectral sequence, we have

$$\mathrm{H}^p(\mathrm{B}\mathbb{Z}_2, \mathrm{H}^q(\mathrm{BPSU}(3),\mathbb{Z}_2)) \Rightarrow \mathrm{H}^{p+q}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)),\mathbb{Z}_2). \tag{5.68}$$

The relevant piece is shown in Figure 56.

Hence $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)),\mathbb{Z}_2) = \mathrm{H}^*(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2)$ for $* \le 3$.

$$\begin{aligned} & \mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2) \\ = \;& \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^* \otimes \mathbb{Z}_2[a] \\ = \;& \mathcal{A}_2 \oplus \Sigma\mathcal{A}_2 \oplus 2\Sigma^2\mathcal{A}_2 \oplus 2\Sigma^3\mathcal{A}_2 \oplus \cdots \end{aligned} \tag{5.69}$$

3	0	0	0	0
2	0	0	0	0
1	0	0	0	0
0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	0	1	2	3

Figure 56: Serre spectral sequence for $(\mathrm{B}\mathbb{Z}_2, \mathrm{BPSU}(3))$ with coefficients $\mathbb{Z}_2$

Hence we have the following theorem

Theorem 110. The bordism groups are

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2^2$

The 1d bordism invariant is a.

The 2d bordism invariants are a^2, w_1^2.

The 3d bordism invariants are a^3, aw_1^2.

5.8.2 $\Omega_3^{\mathrm{SO}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)))_2^{\wedge} \quad (5.70)$$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{SO}, \mathbb{Z}_3) \otimes \mathrm{H}^*(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)), \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)))_3^{\wedge} \quad (5.71)$$

$$\mathrm{H}^*(\mathrm{BPSU}(3), \mathbb{Z}_3) = (\mathbb{Z}_3[z_2, z_8, z_{12}] \otimes \Lambda_{\mathbb{Z}_3}(z_3, z_7))/(z_2z_3, z_2z_7, z_2z_8 + z_3z_7) \quad (5.72)$$

By Serre spectral sequence, we have

$$\mathrm{H}^p(\mathrm{B}\mathbb{Z}_2, \mathrm{H}^q(\mathrm{BPSU}(3), \mathbb{Z}_3)) \Rightarrow \mathrm{H}^{p+q}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)), \mathbb{Z}_3). \quad (5.73)$$

The relevant piece is shown in Figure 57.

3	$\mathbb{Z}_3$	0	0	0
2	$\mathbb{Z}_3$	0	0	0
1	0	0	0	0
0	$\mathbb{Z}_3$	0	0	0
	0	1	2	3

Figure 57: Serre spectral sequence for $(\mathrm{B}\mathbb{Z}_2, \mathrm{BPSU}(3))$ with coefficients $\mathbb{Z}_3$

Hence $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)), \mathbb{Z}_3) = \mathrm{H}^*(\mathrm{BPSU}(3), \mathbb{Z}_3)$ for $* \leq 3$.

Combining this with previous results, we have the following theorem

Theorem 111. The bordism groups are

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)))$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_3$
3	$\mathbb{Z}_2$

The 1d bordism invariant is a.

The 2d bordism invariant is z_2.

The 3d bordism invariant is a^3.

5.8.3 $\Omega_3^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4))) \tag{5.74}$$

$$\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) = \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^* \tag{5.75}$$

where $y_2^* = w_1^2$, $(y_2^2)^* = w_2^2$, $y_4^* = w_1^4$, $y_5^* = w_2 w_3$, etc.

We have a fibration

$$\mathrm{BPSU}(4) \to \mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)) \to \mathrm{B}\mathbb{Z}_2. \tag{5.76}$$

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[a] \tag{5.77}$$

We have a fibration

$$\mathrm{BSU}(4) \to \mathrm{BPSU}(4) \to \mathrm{B}^2\mathbb{Z}_4 \tag{5.78}$$

$$\mathrm{H}^*(\mathrm{BSU}(4), \mathbb{Z}_2) = \mathbb{Z}_2[c_2, c_3, c_4] \tag{5.79}$$

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_2) = \mathbb{Z}_2[\tilde{x}_2, x_3, x_5, x_9, \dots] \tag{5.80}$$

By Serre spectral sequence, we have

$$\mathrm{H}^p(\mathrm{B}^2\mathbb{Z}_4, \mathrm{H}^q(\mathrm{BSU}(4), \mathbb{Z}_2)) \Rightarrow \mathrm{H}^{p+q}(\mathrm{BPSU}(4), \mathbb{Z}_2). \tag{5.81}$$

The relevant piece is shown in Figure 58.

3	0	0	0	0
2	0	0	0	0
1	0	0	0	0
0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	0	1	2	3

Figure 58: Serre spectral sequence for $(\mathrm{B}^2\mathbb{Z}_4, \mathrm{BSU}(4))$

Hence $\mathrm{H}^*(\mathrm{BPSU}(4), \mathbb{Z}_2) = \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_2)$ for $* \le 3$.

Again by Serre spectral sequence, we have

$$\mathrm{H}^p(\mathrm{B}\mathbb{Z}_2, \mathrm{H}^q(\mathrm{BPSU}(4), \mathbb{Z}_2)) \Rightarrow \mathrm{H}^{p+q}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)), \mathbb{Z}_2). \tag{5.82}$$

3	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	0	0	0	0	0
0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	0	1	2	3	4

Figure 59: Serre spectral sequence for $(\mathrm{B}\mathbb{Z}_2, \mathrm{BPSU}(4))$

The relevant piece is shown in Figure 59.

There are no differentials,

$$\mathrm{H}^n(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)), \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & n=0 \\ \mathbb{Z}_2 & n=1 \\ \mathbb{Z}_2^2 & n=2 \\ \mathbb{Z}_2^3 & n=3 \end{cases} \tag{5.83}$$

$$\begin{aligned} & \mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)), \mathbb{Z}_2) \\ = \ & \mathcal{A}_2 \oplus \Sigma\mathcal{A}_2 \oplus 3\Sigma^2\mathcal{A}_2 \oplus 4\Sigma^3\mathcal{A}_2 \oplus \cdots \end{aligned} \tag{5.84}$$

Hence we have the following theorem

Theorem 112. The bordism groups are

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^4$

The 1d bordism invariant is a.

The 2d bordism invariants are $a^2, \tilde{x}_2, w_1^2$.

The 3d bordism invariants are $a^3, x_3, a\tilde{x}_2, aw_1^2$.

Here $\tilde{x}_2 = x_2 \mod 2$, $x_2 \in \mathrm{H}^2(\mathrm{BPSU}(4), \mathbb{Z}_4)$, $x_3 = \beta_{(2,4)}x_2$.

5.8.4 $\Omega_3^{\mathrm{SO}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)),\mathbb{Z}_2),\mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4))) \tag{5.85}$$

$d_2(x_3) = \tilde{x}_2 h_0^2$.

The E_2 page is shown in Figure 60.

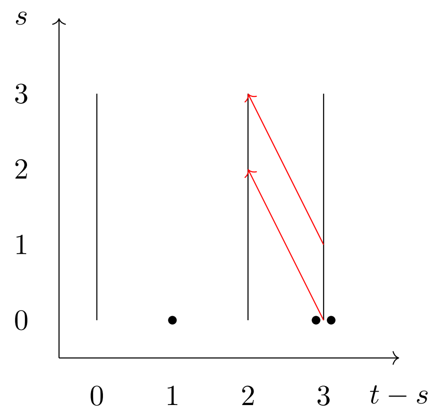


Figure 60: $\Omega_*^{\mathrm{SO}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)))$

Hence we have the following theorem

Theorem 113. The bordism groups are

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)))$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_4$
3	$\mathbb{Z}_2^2$

The 1d bordism invariant is a.

The 2d bordism invariant is x_2.

The 3d bordism invariants are $a^3, a\tilde{x}_2$.

6 Background

For more information, see [14, 27–29].

6.1 Cohomology theory

6.1.1 Cup product

Let X be a topological space, an n-simplex of X is a map $\sigma : \Delta^n \to X$ where

$$\Delta^n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} | t_0 + t_1 + \cdots + t_n = 1, t_i \geq 0\}, \tag{6.1}$$

it is denoted by $[v_0, \dots, v_n]$ where v_i are vertices of Δ^n.

n-simplexes of X generates an abelian group $C_n(X)$, the elements of $C_n(X)$ are called n-chains. Δ^{n-1} embeds in Δ^n in the canonical way, define $\partial : C_n(X) \to C_{n-1}(X)$ by

$$\partial(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}. \tag{6.2}$$

It is easy to verify that $\partial^2 = 0$, so $(C_\bullet(X), \partial)$ is a chain complex.

Let G be an abelian group, let $C^n(X, G) := \mathrm{Hom}(C_n(X), G)$, the elements of $C^n(X, G)$ are called n-cochains with coefficients G. Define $\delta : C^n(X, G) \to C^{n+1}(X, G)$ by $\delta(\alpha)(\sigma) = \alpha(\partial(\sigma))$, then $\delta^2 = 0$, so $(C^\bullet(X, G), \delta)$ is a cochain complex.

$\mathrm{H}^n(X, G)$ is defined to be $\frac{\mathrm{Ker}\delta : C^n(X,G) \to C^{n+1}(X,G)}{\mathrm{Im}\delta : C^{n-1}(X,G) \to C^n(X,G)}$. It is an abelian group, called the n-th cohomology group of X with coefficients G, the elements of the abelian group $Z^n(X, G) := \mathrm{Ker}\delta : C^n(X, G) \to C^{n+1}(X, G)$ are called n-cocycles, the elements of $B^n(X, G) := \mathrm{Im}\delta : C^{n-1}(X, G) \to C^n(X, G)$ are called n-coboundaries.

By abusing the notation, we also use $[v_0, \dots, v_n]$ to denote an n-chain.

If G is additionally a ring R, then we can define a cup product such that $\mathrm{H}^*(X, R)$ is a graded ring. First we define the cup product of two cochains:

$$\begin{aligned} C^n(X, R) \times C^m(X, R) &\to C^{n+m}(X, R) \\ (\alpha, \beta) &\mapsto \alpha \cup \beta \end{aligned} \tag{6.3}$$

$$\alpha \cup \beta([v_0, \dots, v_{n+m}]) := \alpha([v_0, \dots, v_n]) \cdot \beta([v_n, \dots, v_{n+m}]) \tag{6.4}$$

where $\cdot$ is the multiplication in R.

The cup product satisfies

$$\delta(\alpha \cup \beta) = (\delta\alpha) \cup \beta + (-1)^n \alpha \cup (\delta\beta) \tag{6.5}$$

$\alpha \cup \beta$ is a cocycle if both α and β are cocycles. If both α and β are cocycles, then $\alpha \cup \beta$ is a coboundary if one of α and β is a coboundary. So the cup product is also an operation on cohomology groups $\cup : \mathrm{H}^n(X, R) \times \mathrm{H}^m(X, R) \to \mathrm{H}^{n+m}(X, R)$. The cup product of two cocycles satisfies

$$\alpha \cup \beta = (-1)^{nm} \beta \cup \alpha + \text{coboundary} \tag{6.6}$$

For the convenience of defining higher cup product, we use the notation $i \to j$ for the consecutive sequence from i to j

$$i \to j \equiv i, i+1, \cdots, j-1, j. \tag{6.7}$$

We also denote an n-chain by $(0 \to n)$. We use $\langle \alpha, \sigma \rangle$ to denote the value of $\alpha(\sigma)$ for n-cochain α and n-chain σ.

Let f_m be an m-cochain, h_n be an n-cochain, we define higher cup product $f_m \underset{k}{\cup} h_n$ which yields an $(m+n-k)$-cochain:

$$\begin{aligned} &\langle f_m \underset{k}{\cup} h_n, (0,1,\cdots,m+n-k)\rangle \\ =& \sum_{0\le i_0<\cdots<i_k\le n+m-k} (-1)^p \langle f_m, (0\to i_0, i_1 \to i_2, \cdots)\rangle \times \langle h_n, (i_0 \to i_1, i_2 \to i_3, \cdots)\rangle, \end{aligned} \tag{6.8}$$

and $f_m \underset{k}{\cup} h_n = 0$ for $k > m$ or n or $k < 0$. Here $i \to j$ is the sequence $i, i+1, \cdots, j-1, j$, and p is the number of transpositions (it is not unique but its parity is unique) in the decomposition of the permutation to bring the sequence

$$0 \to i_0, i_1 \to i_2, \cdots; i_0+1 \to i_1 - 1, i_2 + 1 \to i_3 - 1, \cdots \tag{6.9}$$

to the sequence

$$0 \to m+n-k. \tag{6.10}$$

For example

$$\begin{aligned} &\langle f_m \underset{1}{\cup} h_n, (0,1,\cdots,m+n-1)\rangle = \sum_{i=0}^{m-1} (-1)^{(m-i)(n+1)} \times \\ &\langle f_m, (0 \to i, i+n \to m+n-1)\rangle \langle h_n, (i \to i+n)\rangle. \end{aligned} \tag{6.11}$$

We can see that $\underset{0}{\cup} = \cup$. Unlike cup product at $k = 0$, the higher cup product of two cocycles may not be a cocycle.

Steenrod studied the higher cup product of cochains and found a formula [?, Theorem 5.1]:

$$\delta(u \underset{i}{\cup} v) = (-1)^{p+q-i} u \underset{i-1}{\cup} v + (-1)^{pq+p+q} v \underset{i-1}{\cup} u + \delta u \underset{i}{\cup} v + (-1)^p u \underset{i}{\cup} \delta v \tag{6.12}$$

where u is a p-cochain, v is a q-cochain.

Also Steenrod defined Steenrod square using higher cup product:

$$\mathrm{Sq}^{n-k}(z_n) \equiv z_n \underset{k}{\cup} z_n \tag{6.13}$$

6.1.2 Universal coefficient theorem and Künneth formula

If X is a topological space, R is a principal ideal domain ($\mathbb{Z}$ or a field), G is an R-module, then the homology version of universal coefficient theorem is

$$\mathrm{H}_n(X, G) = \mathrm{H}_n(X, R) \otimes_R G \oplus \mathrm{Tor}_1^R(\mathrm{H}_{n-1}(X, R), G). \tag{6.14}$$

The cohomology version of universal coefficient theorem is

$$\mathrm{H}^n(X,G) = \mathrm{Hom}_R(\mathrm{H}_n(X,R),G) \oplus \mathrm{Ext}^1_R(\mathrm{H}_{n-1}(X,R),G). \tag{6.15}$$

We will abbreviate $\mathrm{Tor}_1^{\mathbb{Z}}$ by Tor, $\mathrm{Ext}^1_{\mathbb{Z}}$ by Ext.

If X and X' are topological spaces, R is a principle ideal domain and G, G' are R-modules such that $\mathrm{Tor}_1^R(G,G') = 0$. We also require either
(1) $\mathrm{H}_n(X;\mathbb{Z})$ and $\mathrm{H}_n(X';\mathbb{Z})$ are finitely generated, or
(2) G' and $H_n(X';\mathbb{Z})$ are finitely generated.

The homology version of Künneth formula is

$$\begin{aligned} &\mathrm{H}_d(X \times X', G \otimes_R G') \\ &\simeq \Big[\oplus_{k=0}^d \mathrm{H}_k(X,G) \otimes_R \mathrm{H}_{d-k}(X',G')\Big] \oplus \Big[\oplus_{k=0}^{d-1} \mathrm{Tor}_1^R(\mathrm{H}^k(X,G), \mathrm{H}_{d-k-1}(X',G'))\Big]. \end{aligned} \tag{6.16}$$

The cohomology version of Künneth formula is

$$\begin{aligned} &\mathrm{H}^d(X \times X', G \otimes_R G') \\ &\simeq \Big[\oplus_{k=0}^d \mathrm{H}^k(X,G) \otimes_R \mathrm{H}^{d-k}(X',G')\Big] \oplus \Big[\oplus_{k=0}^{d+1} \mathrm{Tor}_1^R(\mathrm{H}^k(X,G), \mathrm{H}^{d-k+1}(X',G'))\Big]. \end{aligned} \tag{6.17}$$

Note that $\mathbb{Z}$ and $\mathbb{R}$ are principal ideal domains, while $\mathbb{R}/\mathbb{Z}$ is not. Also, $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ are not finitely generate R-modules if $R = \mathbb{Z}$.

Special cases: 1. $R = G' = \mathbb{Z}$.

In this case, the condition $\mathrm{Tor}_1^R(G,G') = \mathrm{Tor}_1^{\mathbb{Z}}(G,\mathbb{Z}) = 0$ is always satisfied. G can be $\mathbb{R}/\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Z}_n$ *etc* . So we have

$$\begin{aligned} &\mathrm{H}^d(X \times X', G) \\ &\simeq \Big[\oplus_{k=0}^d \mathrm{H}^k(X,G) \otimes_{\mathbb{Z}} \mathrm{H}^{d-k}(X';\mathbb{Z})\Big] \oplus \Big[\oplus_{k=0}^{d+1} \mathrm{Tor}(\mathrm{H}^k(X,G), \mathrm{H}^{d-k+1}(X';\mathbb{Z}))\Big]. \end{aligned} \tag{6.18}$$

Take X to be the space of one point in (6.18), and use

$$\mathrm{H}^n(X,G)) = \begin{cases} G, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \tag{6.19}$$

to reduce (6.18) to

$$\mathrm{H}^d(X,G) \simeq \mathrm{H}^d(X;\mathbb{Z}) \otimes_{\mathbb{Z}} G \oplus \mathrm{Tor}(\mathrm{H}^{d+1}(X;\mathbb{Z}), G). \tag{6.20}$$

where X' is renamed as X. This is also called the universal coefficient theorem which can be used to calculate $\mathrm{H}^*(X,G)$ from $\mathrm{H}^*(X;\mathbb{Z})$ and the module G. Here $\mathrm{Tor} = \mathrm{Tor}_1^{\mathbb{Z}}$.

Homology version of (6.20) is just the universal coefficient theorem for homology with $R = \mathbb{Z}$.

2. $R = G = G' = \mathbb{F}$ is a field, $\mathrm{Tor}_1^R(G, G') = 0$.

$$\mathrm{H}^*(X \times X', \mathbb{F}) = \mathrm{H}^*(X, \mathbb{F}) \otimes \mathrm{H}^*(X', \mathbb{F}), \tag{6.21}$$

This is called the Künneth formula.

There is also a relative version of Künneth formula [27, Theorem 3.18]:

$$\tilde{\mathrm{H}}^*(X \wedge X', \mathbb{F}) = \tilde{\mathrm{H}}^*(X, \mathbb{F}) \otimes \tilde{\mathrm{H}}^*(X', \mathbb{F}). \tag{6.22}$$

Here $X \wedge X'$ is the smash product, $\tilde{\mathrm{H}}$ is the reduced cohomology.

6.2 Bordism theory

As the pioneer of bordism theory, Thom [7] studied when the disjoint union of two closed n-manifolds is the boundary of a compact $(n+1)$-manifold, he found that this relation is an equivalence relation on the set of closed n-manifolds, moreover, the disjoint union operation defines an abelian group structure on the set of equivalence classes. This group is called the unoriented bordism group, it is denoted by Ω_n^{O}. Furthermore, Thom found that the Cartesian product defines a graded ring structure on $\Omega_*^{\mathrm{O}} := \bigoplus_{n \geq 0} \Omega_n^{\mathrm{O}}$, which is called the unoriented bordism ring. Thom also found that the bordism invariants of Ω_n^{O} are the Stiefel-Whitney numbers, namely, two manifolds are unorientedly bordant if and only if they have identical sets of Stiefel-Whitney characteristic numbers. This yields many interesting consequences. For example, $\mathbb{RP}^2$ is not a boundary while $\mathbb{RP}^3$ is, also $\mathbb{CP}^2$ and $\mathbb{RP}^2 \times \mathbb{RP}^2$ are unorientedly bordant.

Many generalizations are made to bordism theory so far.

For example, if we consider the manifolds which are equipped with an H-structure and a map to a fixed topological space X, then we can define an abelian group

Definition 114.

$$\begin{aligned}\Omega_n^H(X) := \{&(M, f) | M \text{ is a closed} \\ &n\text{-manifold with } H\text{-structure, } f : M \to X \text{ is a map}\}/\text{bordism}.\end{aligned} \tag{6.23}$$

where bordism is an equivalence relation, namely, (M, f) and (M', f') are bordant if there exists a compact $n+1$-manifold N with H-structure and a map $h : N \to X$ such that the boundary of N is the disjoint union of M and M', the H-structures on M and M' are induced from the H-structure on N and $h|_M = f$, $h|_{M'} = f'$.

We follow the definition of H-structure given in [**?**].

Definition 115. If H is a group with a group homomorphism $\rho : H \to \mathrm{O}$, V is a vector bundle over M with a metric, then an H-structure on V is a principal H-bundle P over M, together with an isomorphism of bundles $P \times_H \mathrm{O} \xrightarrow{\sim} \mathcal{B}_{\mathrm{O}}(V)$ where $P \times_H \mathrm{O}$ is the quotient $(P \times \mathrm{O})/H$ where H acts freely on right of $P \times \mathrm{O}$ by

$$(p, g) \cdot h = (p \cdot h, \rho(h)^{-1} g), \quad p \in P, \quad g \in \mathrm{O}, \quad h \in H$$

and $\mathcal{B}_{\mathrm{O}}(V)$ is the orthonormal frame bundle of V.

In particular, if $V = TM$, then an H-structure on TM is also called a tangential H-structure (or an H-structure) on M. Here we assume the H-structures are defined on the tangent bundles instead of normal bundles.

Below we consider manifolds with a metric.

Any manifold admits an O-structure, a manifold M admits an SO-structure if and only if $w_1(TM) = 0$, a manifold admits a Spin structure if and only if $w_1(TM) = w_2(TM) = 0$, a manifold admits a Pin^+ structure if and only if $w_2(TM) = 0$, a manifold admits a Pin^- structure if and only if $w_2(TM) + w_1(TM)^2 = 0$.

In particular, when $X = \mathrm{B}^2\mathbb{Z}_n$, $f : M \to \mathrm{B}^2\mathbb{Z}_n$ is a cohomology class in $\mathrm{H}^2(M, \mathbb{Z}_n)$. When $X = \mathrm{B}G$, with G is a Lie group or a finite group (viewed as a Lie group with discrete topology), then $f : M \to \mathrm{B}G$ is a principal G-bundle over M.

To explain our notation, here $\mathrm{B}G$ is a classifying space of G, and $\mathrm{B}^2\mathbb{Z}_n$ is a higher classifying space (Eilenberg-MacLane space $K(\mathbb{Z}_n, 2)$) of $\mathbb{Z}_n$.

In the particular case that $H = \mathrm{O}$ and X is a point, this definition 114 coincides with Thom's original definition.

In this paper, we study the cases in which $H = \mathrm{O}/\mathrm{SO}/\mathrm{Spin}/\mathrm{Pin}^{\pm}$, and X is a higher classifying space, or more complicated cases.

We first introduce several concepts which are important for bordism theory:

Thom space: Let $V \to Y$ be a real vector bundle, and fix a Euclidean metric. The Thom space $\mathrm{Thom}(Y; V)$ is the quotient $D(V)/S(V)$ where $D(V)$ is the unit disk bundle and $S(V)$ is the unit sphere bundle. Thom spaces satisfy

$$\begin{aligned} \mathrm{Thom}(X \times Y; V \times W) &= \mathrm{Thom}(X; V) \wedge \mathrm{Thom}(Y; W), \\ \mathrm{Thom}(X, V \oplus \underline{\mathbb{R}^n}) &= \Sigma^n \mathrm{Thom}(X; V), \\ \mathrm{Thom}(X, \underline{\mathbb{R}^n}) &= \Sigma^n X_+ \end{aligned} \tag{6.24}$$

where $V \to X$ and $W \to Y$ are real vector bundles, $\underline{\mathbb{R}^n}$ is the trivial real vector bundle of dimension n, Σ is the suspension, X_+ is the disjoint union of X and a point.

We follow the definition of Thom spectrum and Madsen-Tillmann spectrum given in [**?**].

Thom spectrum [7]: MH is the Thom spectrum of the group H, its 0-th space is the colimit of $\Omega^n MH(n)$, where $MH(n) = \mathrm{Thom}(\mathrm{B}H(n); V_n)$, and V_n is the induced vector bundle (of dimension n) by the map $\mathrm{B}H(n) \to \mathrm{BO}(n)$.

In other words, $MH = \text{Thom}(\text{B}H; V)$, where V is the induced virtual bundle (of dimension 0) by the map $\text{B}H \to \text{BO}$.

Madsen-Tillmann spectrum [8]: MTH is the Madsen-Tillmann spectrum of the group H, it is the colimit of $\Sigma^n MTH(n)$, where $MTH(n) = \text{Thom}(\text{B}H(n); -V_n)$, and V_n is the induced vector bundle (of dimension n) by the map $\text{B}H(n) \to \text{BO}(n)$. The 0-th space of the virtual Thom spectrum $MTH(n)$ is the colimit of $\Omega^{n+q}\text{Thom}(\text{B}H(n, n+q), Q_q)$ where $\text{B}H(n, n+q)$ is the pullback

$$\begin{array}{ccc} \text{B}H(n,n+q) & \dashrightarrow & \text{B}H(n) \\ \downarrow & & \downarrow \\ Gr_n(\mathbb{R}^{n+q}) & \longrightarrow & \text{BO}(n) \end{array} \tag{6.25}$$

and there is a direct sum $\underline{\mathbb{R}^{n+q}} = V_n \oplus Q_q$ of vector bundles over $Gr_n(\mathbb{R}^{n+q})$ and, by pullback, over $\text{B}H(n, n+q)$ where $\underline{\mathbb{R}^{n+q}}$ is the trivial real vector bundle of dimension $n+q$.

In other words, $MTH = \text{Thom}(\text{B}H; -V)$, where V is the induced virtual bundle (of dimension 0) by the map $\text{B}H \to \text{BO}$.

Here Ω is the loop space, Σ is the suspension.

Note: "T" in MTH denotes that the H-structures are on tangent bundles instead of normal bundles.

(Co)bordism theory is a generalized (co)homology theory which is represented by a spectrum by the Brown representability theorem.

In fact, it is represented by Thom spectrum due to the Pontryagin-Thom isomorphism:

$$\begin{aligned} \pi_n(MTH) &= \Omega_n^H \text{ the cobordism group of } n\text{-manifolds with tangential } H\text{-structure}, \\ \pi_n(MH) &= \Omega_n^{\nu H} \text{ the cobordism group of } n\text{-manifolds with normal } H\text{-structure} \end{aligned} \tag{6.26}$$

In the case when tangential H-structure is the same as normal H'-structure, the relevant Thom spectra are weakly equivalent. In particular, $MT\text{O} \simeq M\text{O}$, $MT\text{SO} \simeq M\text{SO}$, $MT\text{Spin} \simeq M\text{Spin}$, $MT\text{Pin}^+ \simeq M\text{Pin}^-$, $MT\text{Pin}^- \simeq M\text{Pin}^+$.

$\text{Pin}^\pm$ cobordism groups are not rings, though they are modules over the Spin cobordism ring.

By the generalized Pontryagin-Thom construction, for X a topological space, then the group of H-bordism classes of H-manifolds in X is isomorphic to the generalized homology of X with coefficients in MTH:

$$\Omega_d^H(X) = \pi_d(MTH \wedge X_+) = MTH_d(X) \tag{6.27}$$

where $\pi_d(MTH \wedge X_+)$ is the d-th stable homotopy group of the spectrum $MTH \wedge X_+$. The d-th stable homotopy group of a spectrum M is

$$\pi_d(M) = \text{colim}_{k\to\infty}\pi_{d+k}M_k. \tag{6.28}$$

So the computation of the bordism group $\Omega_d^H(X)$ is the same as the computation of the stable homotopy group of the spectrum $MTH \wedge X_+$ which can be computed by Adams spectral sequence method.

Next, we introduce the Thom isomorphism [7]: Let $p : E \to B$ be a real vector bundle of rank n. Then there is an isomorphism, called Thom isomorphism

$$\Phi : \mathrm{H}^k(B, \mathbb{Z}_2) \to \tilde{\mathrm{H}}^{k+n}(T(E), \mathbb{Z}_2) \tag{6.29}$$

where $\tilde{\mathrm{H}}$ is the reduced cohomology, $T(E) = \mathrm{Thom}(E; B)$ is the Thom space and

$$\Phi(b) = p^*(b) \cup U \tag{6.30}$$

where U is the Thom class. We can define the i-th Stiefel-Whitney class of the vector bundle $p : E \to B$ by

$$w_i(p) = \Phi^{-1}(\mathrm{Sq}^i U) \tag{6.31}$$

where Sq is the Steenrod square.

6.3 Spectral sequences

In this paper, we use three kinds of spectral sequence: Adams spectral sequence, Atiyah-Hirzebruch spectral sequence, Serre spectral sequence.

6.3.1 Adams spectral sequence

The Adams spectral sequence is a spectral sequence introduced by Adams in [**?**], it is of the form

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_p}^{s,t}(\mathrm{H}^*(Y, \mathbb{Z}_p), \mathbb{Z}_p) \Rightarrow \pi_{t-s}(Y)_p^\wedge. \tag{6.32}$$

We need consider $Y = MTH \wedge X_+$ and focus on $p = 2$.

We introduce the notions used in Adams spectral sequence:

p-completion: For any finitely generated abelian group G, $G_p^\wedge = \lim_n G/p^n G$ is the p-completion of G. If G is finite, then $G_p^\wedge$ is the Sylow p-subgroup of G. If $G = \mathbb{Z}$, $G_p^\wedge$ is the ring of p-adic integers.

Steenrod algebra: The mod p Steenrod algebra is $\mathcal{A}_p := [H\mathbb{Z}_p, H\mathbb{Z}_p]_{-*}$ where $H\mathbb{Z}_p$ is the mod p Eilenberg-MacLane spectrum. Every cohomology ring $\mathrm{H}^*(X, \mathbb{Z}_p) = [X, H\mathbb{Z}_p]_{-*}$ is an $\mathcal{A}_p$-module.

For $p = 2$, the generators of $\mathcal{A}_2$ are Steenrod squares Sq^n. The subalgebra $\mathcal{A}_2(1)$ of $\mathcal{A}_2$ generated by Sq^1 and Sq^2 looks like Figure 61.

Each dot stands for a $\mathbb{Z}_2$, all relations are from Adem relations (6.77).

For odd primes p, the generators of $\mathcal{A}_p$ are the Bockstein homomorphism $\beta_{(p,p)}$ and Steenrod powers P_p^n.

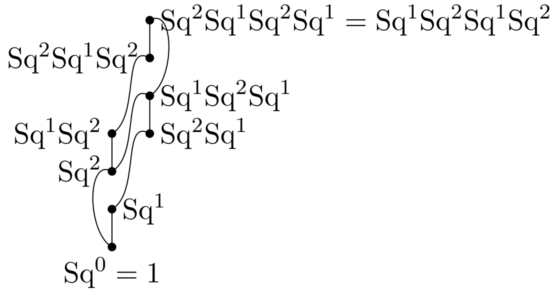


Figure 61: $\mathcal{A}_2(1)$

Ext functor: Let $R = \mathcal{A}_p$ or $\mathcal{A}_2(1)$. $\mathrm{Ext}_R^{s,t}$ is the internal degree t part of the s-th derived functor of Hom_R^*.

In general, we can find a projective R-resolution $P_\bullet$ of L to compute $\mathrm{Ext}_R^i(L, \mathbb{Z}_p)$, $\mathrm{Ext}_R^i(L, \mathbb{Z}_p) = \mathrm{H}^i(\mathrm{Hom}_R(P_\bullet, \mathbb{Z}_p))$ (the i-th cohomology of the chain complex $\mathrm{Hom}_R(P_\bullet, \mathbb{Z}_p)$).

In Adams chart, the horizontal axis is degree $t-s$ and the vertical axis is degree s. The differential $d_r^{s,t} : E_r^{s,t} \to E_r^{s+r,t+r-1}$ is an arrow starting at the bidegree $(t-s, s)$ with direction $(-1, r)$. $E_{r+1}^{s,t} := \frac{\mathrm{Ker} d_r^{s,t}}{\mathrm{Im} d_r^{s-r,t-r+1}}$ for $r \geq 2$. There exists N such that $E_{N+k} = E_N$ for $k > 0$, denote $E_\infty := E_N$.

We explain how to read the result from the Adams chart: In the E_∞ page, one dot indicates a $\mathbb{Z}_p$, an vertical line connecting n dots indicates a $\mathbb{Z}_{p^n}$, when $n = \infty$, the line indicates a $\mathbb{Z}$.

In the $H = \mathrm{O}$ cases, $M\mathrm{O}$ is the wedge sum of suspensions of the Eilenberg-MacLane spectrum $H\mathbb{Z}_2$, $\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2)$ is the direct sum of suspensions of the Steenrod algebra $\mathcal{A}_2$.

$\mathrm{H}^*(M\mathrm{O} \wedge X_+, \mathbb{Z}_2) = \mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) \otimes \mathrm{H}^*(X, \mathbb{Z}_2)$ is also the direct sum of suspensions of the Steenrod algebra $\mathcal{A}_2$. We have used the Künneth formula (6.22). Let $L = \mathrm{H}^*(M\mathrm{O} \wedge X_+, \mathbb{Z}_2)$, then $P_0 = L$, $P_s = 0$ for $s > 0$ gives a projective $\mathcal{A}_2$-resolution of L.

Since

$$\mathrm{Ext}_{\mathcal{A}_p}^{s,t}(\Sigma^r \mathcal{A}_p, \mathbb{Z}_p) = \begin{cases} \mathrm{Hom}_{\mathcal{A}_p}^t(\Sigma^r \mathcal{A}_p, \mathbb{Z}_p) = \mathbb{Z}_p & \text{if } t = r, s = 0 \\ 0 & \text{else} \end{cases}, \tag{6.33}$$

all dots are concentrated in $s = 0$ in the Adams chart of $\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O} \wedge X_+, \mathbb{Z}_2), \mathbb{Z}_2)$, there are no differentials, $E_2 = E_\infty$, $\Omega_d^{\mathrm{O}}(X)$ is a $\mathbb{Z}_2$-vector space.

Example: $\Omega_d^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$.

Since

$$\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) = \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^* \tag{6.34}$$

and

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[x_2, x_3, x_5, x_9, \dots]. \tag{6.35}$$

We need only count the dimension of $\mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^* \otimes \mathbb{Z}_2[x_2, x_3, x_5, x_9, \dots]$ at each degree as a $\mathbb{Z}_2$-vector space. Then we get the bordism groups and the bordism invariants.

In the $H = \mathrm{SO}$ cases, the localization of $M\mathrm{SO}$ at the prime 2 is

$$M\mathrm{SO}_{(2)} = H\mathbb{Z}_{(2)} \vee \Sigma^4 H\mathbb{Z}_{(2)} \vee \Sigma^5 H\mathbb{Z}_2 \vee \cdots \tag{6.36}$$

where $H\mathbb{Z}$ is the Eilenberg-MacLane spectrum and $\mathrm{H}^*(H\mathbb{Z}, \mathbb{Z}_2) = \mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1$.

$$\cdots \longrightarrow \Sigma^3\mathcal{A}_2 \longrightarrow \Sigma^2\mathcal{A}_2 \longrightarrow \Sigma\mathcal{A}_2 \longrightarrow \mathcal{A}_2 \longrightarrow \mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1 \tag{6.37}$$

is an $\mathcal{A}_2$-resolution (denoted by $P_\bullet$) where the differentials d_1 are induced by Sq^1.

When X is a point, the Adams chart of $\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO}, \mathbb{Z}_2), \mathbb{Z}_2)$ is shown in Figure 62. For general X, $P_\bullet \otimes \mathrm{H}^*(X, \mathbb{Z}_2)$ is a projective $\mathcal{A}_2$-resolution of $\mathrm{H}^*(H\mathbb{Z}, \mathbb{Z}_2) \otimes \mathrm{H}^*(X, \mathbb{Z}_2)$ (since $P_\bullet$ is actually a free $\mathcal{A}_2$-resolution), the differentials d_1 are induced by Sq^1.

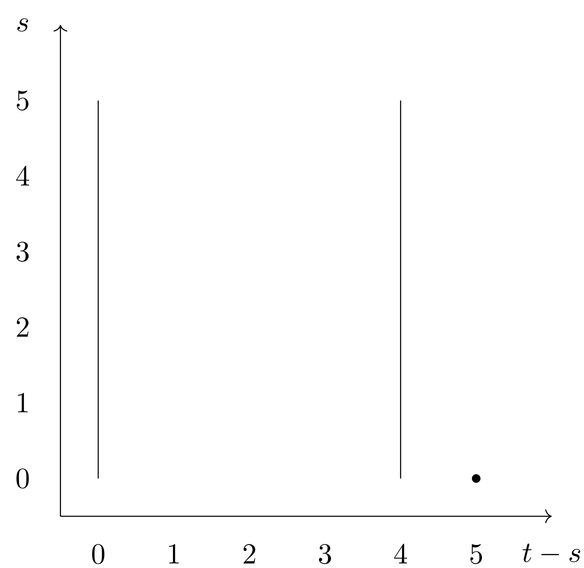


Figure 62: Adams chart of $\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO}, \mathbb{Z}_2), \mathbb{Z}_2)$

The localization of $M\mathrm{SO}$ at the prime 3 is the wedge sum of suspensions of the Brown-Peterson spectrum BP ($M\mathrm{SO}_{(3)} = BP \vee \Sigma^8 BP \vee \cdots$) and $\mathrm{H}^*(BP, \mathbb{Z}_3) = \mathcal{A}_3/(\beta_{(3,3)})$ where $(\beta_{(3,3)})$ is the two-sided ideal generated by $\beta_{(3,3)}$.

$$\cdots \longrightarrow \Sigma^2\mathcal{A}_3 \oplus \Sigma^6\mathcal{A}_3 \oplus \cdots \longrightarrow \Sigma\mathcal{A}_3 \oplus \Sigma^5\mathcal{A}_3 \oplus \cdots \longrightarrow \mathcal{A}_3 \longrightarrow \mathcal{A}_3/(\beta_{(3,3)}) \tag{6.38}$$

is an $\mathcal{A}_3$-resolution of $\mathcal{A}_3/(\beta_{(3,3)})$ (denoted by $P'_\bullet$) where the differentials d_1 are induced by $\beta_{(3,3)}$.

When X is a point, the Adams chart of $\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{SO}, \mathbb{Z}_3), \mathbb{Z}_3)$ is shown in Figure 63. For general X, $P'_\bullet \otimes \mathrm{H}^*(X, \mathbb{Z}_3)$ is a projective $\mathcal{A}_3$-resolution of $\mathrm{H}^*(BP, \mathbb{Z}_3) \otimes \mathrm{H}^*(X, \mathbb{Z}_3)$ (since $P'_\bullet$ is actually a free $\mathcal{A}_3$-resolution), the differentials d_1 are induced by $\beta_{(3,3)}$.

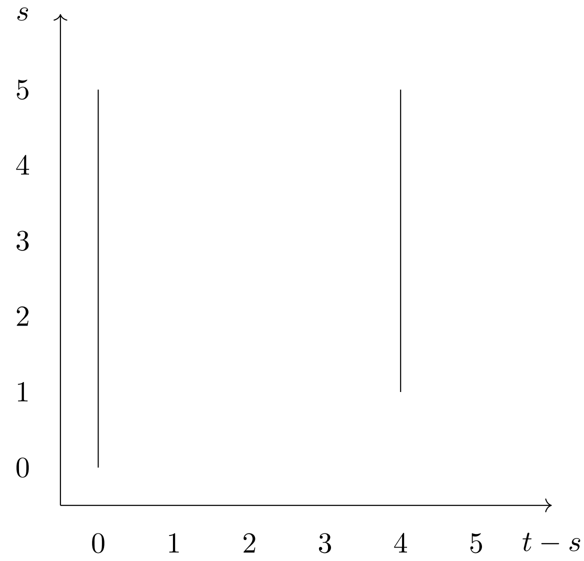


Figure 63: Adams chart of $\text{Ext}_{\mathcal{A}_3}^{s,t}(\text{H}^*(M\text{SO},\mathbb{Z}_3),\mathbb{Z}_3)$

There may be differentials d_n corresponding to the Bockstein homomorphism $\beta_{(p,p^n)}$ [26] for both $p=2$ and $p=3$. See 6.5 for the definition of Bockstein homomorphisms. Since $M\text{SO}_{(3)}=M\text{Spin}_{(3)}$, $\Omega_d^{\text{SO}}(X)_3^{\wedge}=\Omega_d^{\text{Spin}}(X)_3^{\wedge}$.

Example: $\Omega_d^{\text{SO}}(\text{B}^2\mathbb{Z}_2)$.

Since

$$\text{H}^*(\text{B}^2\mathbb{Z}_2,\mathbb{Z}_2)=\mathbb{Z}_2[x_2,x_3,x_5,x_9,\dots] \tag{6.39}$$

where $x_3=\text{Sq}^1x_2$, $\text{Sq}^1(x_2x_3)=\text{Sq}^1x_5=x_3^2$.

We shift Figure 62 the same times as the dimension of $\text{H}^*(\text{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$ at each degree as a $\mathbb{Z}_2$-vector space. We obtain the E_1 page for $\Omega_*^{\text{SO}}(\text{B}^2\mathbb{Z}_2)$, the differentials d_1 are induced by Sq^1, as shown in Figure 64.

Then take the differentials d_2 into account, we obtain the E_2 page for $\Omega_*^{\text{SO}}(\text{B}^2\mathbb{Z}_2)$, as shown in Figure 9.

In the $H=\text{Spin}/\text{Pin}^{\pm}$ cases, since the mod 2 cohomology of the Thom spectrum $M\text{Spin}$ is

$$\text{H}^*(M\text{Spin},\mathbb{Z}_2)=\mathcal{A}_2\otimes_{\mathcal{A}_2(1)}\{\mathbb{Z}_2\oplus M\} \tag{6.40}$$

where M is a graded $\mathcal{A}_2(1)$-module with the degree i homogeneous part $M_i=0$ for $i<8$, when we compute $\Omega_d^H(X)_2^{\wedge}$, we are reduced to compute $\text{Ext}_{\mathcal{A}_2(1)}^{s,t}(L,\mathbb{Z}_2)$ for $t-s<8$, where L is some $\mathcal{A}_2(1)$-module (our cases are some mod 2 cohomology $\text{H}^*(-,\mathbb{Z}_2)$).

Example 1: $L=\mathcal{A}_2(1)$, $\text{Ext}_{\mathcal{A}_2(1)}^{s,t}(L,\mathbb{Z}_2)=\text{Hom}_{\mathcal{A}_2(1)}(\mathcal{A}_2(1),\mathbb{Z}_2)=\text{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2,\mathbb{Z}_2)=\mathbb{Z}_2$ if $t=s=0$ and 0 else.

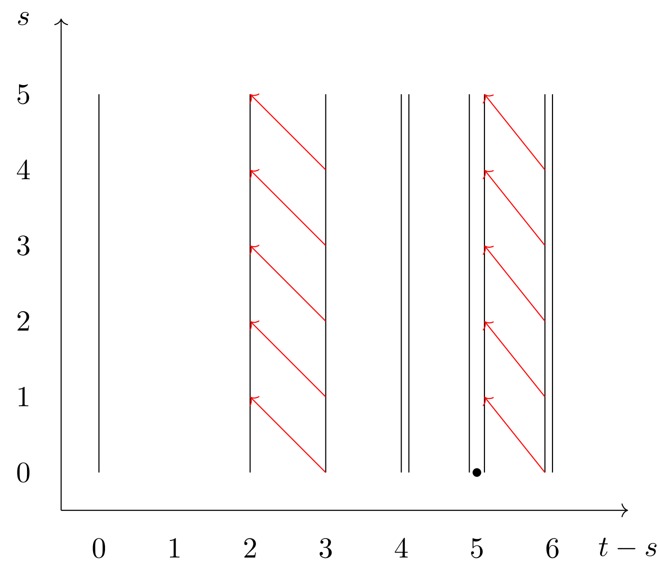


Figure 64: E_1 page for $\Omega_*^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$

Example 2: $L = \mathbb{Z}_2$, the $\mathcal{A}_2(1)$-resolution of L is

$$\cdots \to \Sigma^3\mathcal{A}_2(1) \oplus \Sigma^7\mathcal{A}_2(1) \to \Sigma^2\mathcal{A}_2(1) \oplus \Sigma^4\mathcal{A}_2(1) \to \Sigma\mathcal{A}_2(1) \oplus \Sigma^2\mathcal{A}_2(1) \to \mathcal{A}_2(1) \to \mathbb{Z}_2. \quad (6.41)$$

The Adams chart looks like Figure 65.

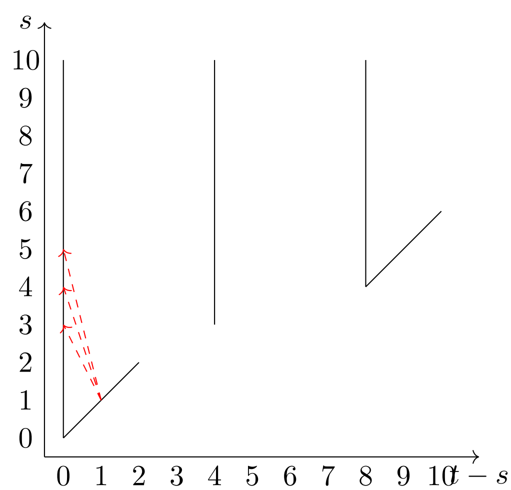


Figure 65: Adams chart of $\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2)$. The dashed arrows indicate the possible differentials.

The only possible differentials are $d_r(h_1) = h_0^{r+1}$ where $h_0 \in \mathrm{Ext}_{\mathcal{A}_2(1)}^{1,1}(\mathbb{Z}_2\mathbb{Z}_2)$, $h_1 \in \mathrm{Ext}_{\mathcal{A}_2(1)}^{1,2}(\mathbb{Z}_2\mathbb{Z}_2)$. If there were such a differential d_r for $r \geq 2$, then since $h_0h_1 = 0$, $0 = d_r(h_0h_1) = h_0^{r+2}$ which is not true. Hence $E_2 = E_\infty$.

This is in fact real Bott periodicity ($\pi_* ko$).

Our computation is based on the following fact:

Lemma 116. Given a short exact sequence of $\mathcal{A}_2(1)$-modules

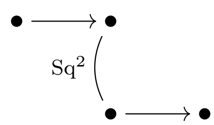
$$0 \to L_1 \to L_2 \to L_3 \to 0, \tag{6.42}$$

then for any t, there is a long exact sequence

$$\begin{aligned} \cdots &\to \mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(L_3, \mathbb{Z}_2) \to \mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(L_2, \mathbb{Z}_2) \to \mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(L_1, \mathbb{Z}_2) \\ &\xrightarrow{d_1} \mathrm{Ext}^{s+1,t}_{\mathcal{A}_2(1)}(L_3, \mathbb{Z}_2) \to \mathrm{Ext}^{s+1,t}_{\mathcal{A}_2(1)}(L_2, \mathbb{Z}_2) \to \cdots \end{aligned} \tag{6.43}$$

After using this fact repeatedly, we obtain the E_2 page.

Example 3:



(6.44)

is a short exact sequence where the left dot is L_1, the middle part is L_2, the right dot is L_3.

The Adams chart looks like Figure 66.

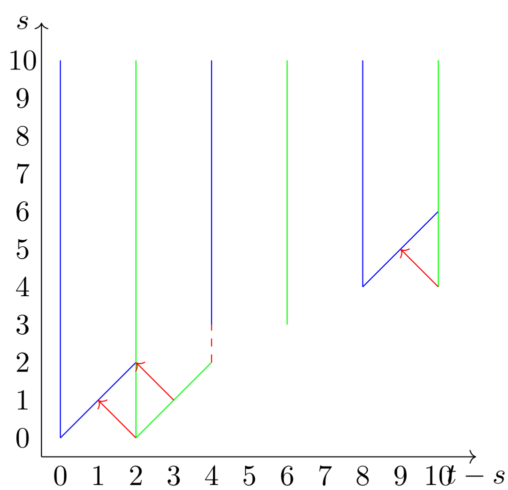
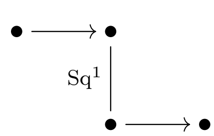


Figure 66: Adams chart of $\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(L_2, \mathbb{Z}_2)$. The arrows indicate the differential d_1, the dashed line indicates the extension.

Example 4:



(6.45)

is a short exact sequence where the left dot is L'_1, the middle part is L'_2, the right dot is L'_3.

The Adams chart looks like Figure 67.

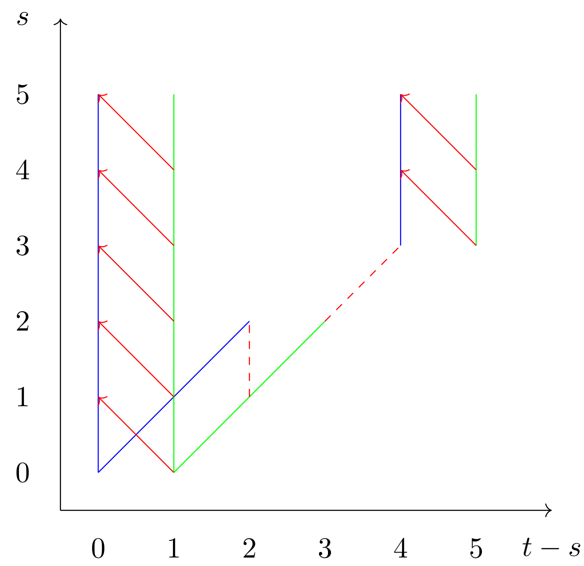


Figure 67: Adams chart of $\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(L_2', \mathbb{Z}_2)$. The arrows indicate the differential d_1, the dashed line indicates the extension.

Example: $\Omega_d^{\mathrm{Spin}}(\mathrm{B}^2\mathbb{Z}_2)$.

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$ is shown in Figure 10. The dot at the bottom is a $\mathbb{Z}_2$ which has been discussed before. Now we consider the part above the bottom dot. We will use Lemma 116 several times. Two steps are shown in Figure 68, 69.

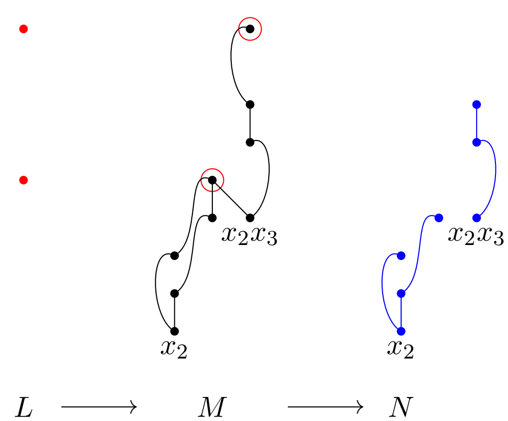


Figure 68: First step to get the E_2 page of $\Omega_*^{\mathrm{Spin}}(\mathrm{B}^2\mathbb{Z}_2)$.

We will proceed in the reversed order.

First, we apply Lemma 116 to the short exact sequence of $\mathcal{A}_2(1)$-modules: $0 \to P \to N \to Q \to 0$ in the second step (as shown in Figure 69), the Adams chart of $\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(N, \mathbb{Z}_2)$ is shown in Figure 70.

Next, we apply Lemma 116 to the short exact sequence of $\mathcal{A}_2(1)$-modules: $0 \to L \to M \to$

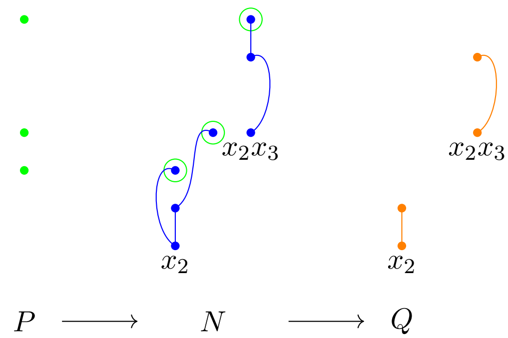


Figure 69: Second step to get the E_2 page of $\Omega_*^{\text{Spin}}(\mathrm{B}^2\mathbb{Z}_2)$.

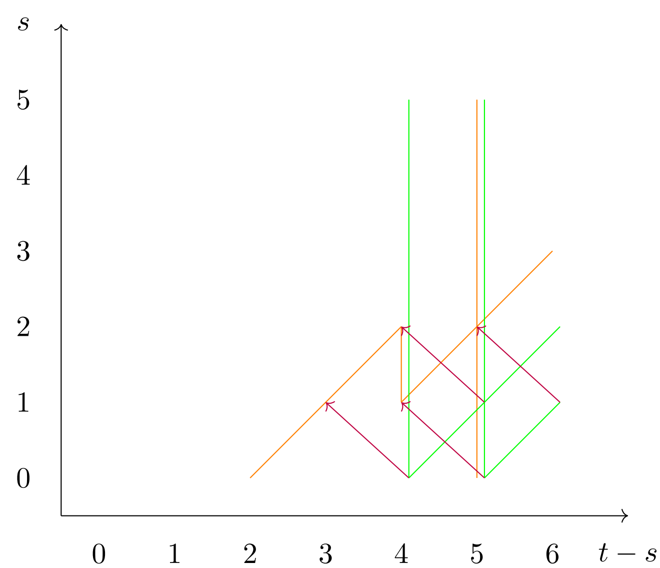


Figure 70: Adams chart of $\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(N,\mathbb{Z}_2)$. The arrows indicate the differential d_1.

$N \to 0$ in the first step (as shown in Figure 68), the Adams chart of $\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(M,\mathbb{Z}_2)$ is shown in Figure 71.

Then take the differentials d_2 into account, we obtain the E_2 page for $\Omega_*^{\text{Spin}}(\mathrm{B}^2\mathbb{Z}_2)$, as shown in Figure 11.

6.3.2 Serre spectral sequence

Given a fibration $F \to E \to B$, the Serre spectral sequence is the following:

$$E_2^{p,q} = \mathrm{H}^p(B, H^q(F,\mathbb{Z})) \Rightarrow \mathrm{H}^{p+q}(E,\mathbb{Z}) \tag{6.46}$$

This can be used in computing the integral cohomology group of the total space of a nontrivial fibration.

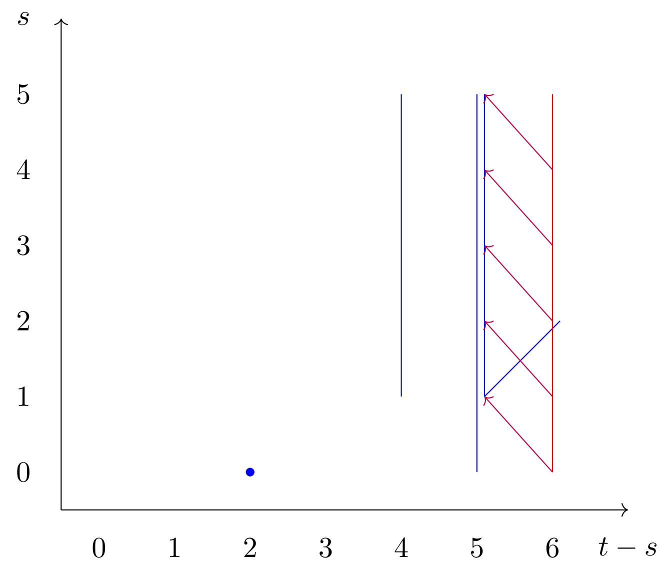


Figure 71: Adams chart of $\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(M, \mathbb{Z}_2)$. The arrows indicate the differential d_1.

There is also a homology version:

$$E^2_{p,q} = \mathrm{H}_p(B, H_q(F, \mathbb{Z})) \Rightarrow \mathrm{H}_{p+q}(E, \mathbb{Z}) \tag{6.47}$$

6.3.3 Atiyah-Hirzebruch spectral sequence

The Atiyah-Hirzebruch spectral sequence can be viewed as a generalization of the Serre spectral sequence. Given a fibration $F \to E \to B$, the Atiyah-Hirzebruch spectral sequence is the following:

$$E^2_{p,q} = \mathrm{H}_p(B, h_q(F, \mathbb{Z})) \Rightarrow h_{p+q}(E, \mathbb{Z}) \tag{6.48}$$

where h_* is an extraordinary homology theory. For example, h_* can be the bordism theory Ω^H_*. In particular, if the fiber F is a point, then the Atiyah-Hirzebruch spectral sequence is of the form:

$$\mathrm{H}_p(X, \Omega^H_q) \Rightarrow \Omega^H_{p+q}(X) \tag{6.49}$$

6.4 Characteristic classes

6.4.1 Introduction to characteristic classes

Characteristic classes are cohomology classes of the base space of a vector bundle. Stiefel-Whitney classes are defined for real vector bundles, Chern classes are defined for complex vector bundles, Pontryagin classes are defined for real vector bundles. All characteristic classes are natural with respect to bundle maps. Characteristic classes of a principal bundle are defined to be the characteristic classes of the associated vector bundle of the principal bundle.

Given a real vector bundle $V \to M$ and a complex vector bundle $E \to M$, the i-th Stiefel-Whitney class of V is $w_i(V) \in \mathrm{H}^i(M, \mathbb{Z}_2)$, the i-th Chern class of E is $c_i(E) \in \mathrm{H}^{2i}(M, \mathbb{Z})$, the i-th Pontryagin class of V is $p_i(V) \in \mathrm{H}^{4i}(M, \mathbb{Z})$.

Pontryagin classes are closely related to Chern classes via complexification:

$$p_i(V) = (-1)^i c_{2i}(V \otimes_{\mathbb{R}} \mathbb{C}) \in \mathrm{H}^{4i}(M, \mathbb{Z}) \tag{6.50}$$

where $V \otimes_{\mathbb{R}} \mathbb{C} \to M$ is the complexification of the real vector bundle $V \to M$.

The relation between Pontryagin classes and Stiefel-Whitney classes is

$$p_i(V) = w_{2i}(V)^2 \mod 2. \tag{6.51}$$

For a manifold M, the integrals over M of characteristic classes of a vector bundle over M (the pairing of the characteristic classes with the fundamental class of M) are called characteristic numbers.

Let E_n be the universal n-bundle over $\mathrm{BO}(n)$, the colimit of $E_n - n$ is a virtual bundle E (of dimension 0) over BO, the pullback of E along the map $g : M \to \mathrm{BO}$ given by the O-structure on M is just $TM - d$ where M is a d-manifold and TM is the tangent bundle of M. By the naturality of characteristic classes, the pullback of the characteristic classes of E is the characteristic classes of TM.

Chern-Simons form: By Chern-Weil theory, Chern classes (and Pontryagin classes) can also be defined as a closed differential form (in de Rham cohomology). By Poincaré Lemma, they are exact locally:

$$c_n = \mathrm{dCS}_{2n-1} \tag{6.52}$$

where d is the exterior differential operator, CS_{2n-1} is called the Chern-Simons $2n-1$-form.

Whitney sum formula: Let $w(V) = 1 + w_1(V) + w_2(V) + \cdots \in \mathrm{H}^*(M, \mathbb{Z}_2)$ be the total Stiefel-Whitney class, $c(E) = 1 + c_1(E) + c_2(E) + \cdots \in \mathrm{H}^*(M, \mathbb{Z})$ be the total Chern class, $p(V) = 1 + p_1(V) + p_2(V) + \cdots \in \mathrm{H}^*(M, \mathbb{Z})$ be the total Pontryagin class, then

$$w(V \oplus V') = w(V)w(V'), \tag{6.53}$$

$$c(E \oplus E') = c(E)c(E'), \tag{6.54}$$

$$2p(V \oplus V') = 2p(V)p(V'). \tag{6.55}$$

That is, the total Stiefel-Whitney class and the total Chern class are multiplicative with respect to Whitney sum of vector bundles, the total Pontryagin class is multiplicative modulo 2-torsion with respect to Whitney sum of vector bundles.

6.4.2 Wu formulas

The total Stiefel-Whitney class $w = 1 + w_1 + w_2 + \cdots$ is related to the total Wu class $u = 1 + u_1 + u_2 + \cdots$ through the total Steenrod square:

$$w = \mathrm{Sq}(u), \quad \mathrm{Sq} = 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \cdots . \tag{6.56}$$

Therefore, $w_n = \sum_{i=0}^{n} \mathrm{Sq}^i(u_{n-i})$. The Steenrod squares satisfy:

$$\mathrm{Sq}^i(x_j) = 0,\ i > j,\ \ \mathrm{Sq}^j(x_j) = x_j x_j,\ \ \mathrm{Sq}^0 = 1, \tag{6.57}$$

for any $x_j \in \mathrm{H}^j(M^d; \mathbb{Z}_2)$. Thus

$$u_n = w_n + \sum_{i=1, 2i \le n} \mathrm{Sq}^i(u_{n-i}). \tag{6.58}$$

This allows us to compute u_n iteratively, using Wu formula

$$\begin{aligned} &\mathrm{Sq}^i(w_j) = 0, \ \ i > j, \qquad \mathrm{Sq}^i(w_i) = w_i w_i, \\ &\mathrm{Sq}^i(w_j) = w_i w_j + \sum_{k=1}^{i} \binom{j-i-1+k}{k} w_{i-k} w_{j+k}, \ \ i < j, \end{aligned} \tag{6.59}$$

and the Steenrod relation

$$\mathrm{Sq}^n(xy) = \sum_{i=0}^{n} \mathrm{Sq}^i(x) \mathrm{Sq}^{n-i}(y). \tag{6.60}$$

We find

$$\begin{aligned} &u_0 = 1, \qquad u_1 = w_1, \qquad u_2 = w_1^2 + w_2, \\ &u_3 = w_1 w_2, \qquad u_4 = w_1^4 + w_2^2 + w_1 w_3 + w_4, \\ &u_5 = w_1^3 w_2 + w_1 w_2^2 + w_1^2 w_3 + w_1 w_4. \end{aligned} \tag{6.61}$$

On the tangent bundle of M^d, the corresponding Wu class and the Steenrod square satisfy

$$\mathrm{Sq}^{d-j}(x_j) = u_{d-j} x_j, \text{ for any } x_j \in \mathrm{H}^j(M^d; \mathbb{Z}_2). \tag{6.62}$$

This is also called Wu formula.

6.5 Bockstein homomorphisms

In general, given a chain complex $C_\bullet$ and a short exact sequence of abelian groups:

$$0 \to A' \to A \to A'' \to 0, \tag{6.63}$$

we have a short exact sequence of cochain complexes:

$$0 \to \mathrm{Hom}(C_\bullet, A') \to \mathrm{Hom}(C_\bullet, A) \to \mathrm{Hom}(C_\bullet, A'') \to 0. \tag{6.64}$$

Hence we obtain a long exact sequence of cohomology groups:

$$\cdots \to \mathrm{H}^n(C_\bullet, A') \to \mathrm{H}^n(C_\bullet, A) \to \mathrm{H}^n(C_\bullet, A'') \xrightarrow{\partial} \mathrm{H}^{n+1}(C_\bullet, A') \to \cdots , \tag{6.65}$$

the connecting homomorphism ∂ is called Bockstein homomorphism.

For example, $\beta_{(n,m)} : \mathrm{H}^*(-, \mathbb{Z}_m) \to \mathrm{H}^{*+1}(-, \mathbb{Z}_n)$ is the Bockstein homomorphism associated to the extension $\mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_{nm} \to \mathbb{Z}_m$.

Let $\rho_{(nm,m)} : \mathrm{H}^*(-, \mathbb{Z}_{nm}) \to \mathrm{H}^*(-, \mathbb{Z}_m)$ be the mod m reduction map, then $\beta_{(n,m)}\rho_{(nm,m)} = 0$ by the long exact sequence. In particular, $\beta_{(2,2)}\rho_{(4,2)} = 0$.

Relations between the Bocksteins: If we have a chain complex $C_\bullet$ and a commutative diagram of abelian groups with exact rows:

$$\begin{CD} 0 @>>> C' @>>> C @>>> C'' @>>> 0 , \\ @. @VVV @VVV @VVV @. \\ 0 @>>> A' @>>> A @>>> A'' @>>> 0 \end{CD} \tag{6.66}$$

then we have a commutative diagram of cochain complexes with exact rows:

$$\begin{CD} 0 @>>> \mathrm{Hom}(C_\bullet, C') @>>> \mathrm{Hom}(C_\bullet, C) @>>> \mathrm{Hom}(C_\bullet, C'') @>>> 0 , \\ @. @VVV @VVV @VVV @. \\ 0 @>>> \mathrm{Hom}(C_\bullet, A') @>>> \mathrm{Hom}(C_\bullet, A) @>>> \mathrm{Hom}(C_\bullet, A'') @>>> 0 \end{CD} \tag{6.67}$$

By the naturality of the connecting homomorphism [30, Theorem 6.13], we have a commutative diagram of abelian groups with exact rows:

$$\begin{CD} \cdots @>>> \mathrm{H}^n(C_\bullet, C') @>>> \mathrm{H}^n(C_\bullet, C) @>>> \mathrm{H}^n(C_\bullet, C'') @>{\partial}>> \mathrm{H}^{n+1}(C_\bullet, C') @>>> \cdots \\ @. @VVV @VVV @VVV @VVV @. \\ \cdots @>>> \mathrm{H}^n(C_\bullet, A') @>>> \mathrm{H}^n(C_\bullet, A) @>>> \mathrm{H}^n(C_\bullet, A'') @>{\partial'}>> \mathrm{H}^{n+1}(C_\bullet, A') @>>> \cdots \end{CD} \tag{6.68}$$

There are commutative diagrams:

$$\begin{CD} \mathbb{Z}_n @>{\cdot m}>> \mathbb{Z}_{nm} @>{\text{mod } m}>> \mathbb{Z}_m \\ @| @VV{\cdot k}V @VV{\cdot k}V \\ \mathbb{Z}_n @>{\cdot km}>> \mathbb{Z}_{knm} @>{\text{mod } km}>> \mathbb{Z}_{km} \end{CD} \tag{6.69}$$

$$\begin{CD} \mathbb{Z}_{kn} @>{\cdot m}>> \mathbb{Z}_{knm} @>{\text{mod } m}>> \mathbb{Z}_m \\ @V{\text{mod } n}VV @V{\text{mod } nm}VV @| \\ \mathbb{Z}_n @>{\cdot m}>> \mathbb{Z}_{nm} @>{\text{mod } m}>> \mathbb{Z}_{km} \end{CD} \tag{6.70}$$

By (6.68), we have the following commutative diagrams:

$$\begin{array}{ccc} \mathrm{H}^*(-,\mathbb{Z}_m) & \xrightarrow{\beta_{(n,m)}} & \mathrm{H}^{*+1}(-,\mathbb{Z}_n) \\ \downarrow{\cdot k} & & \| \\ \mathrm{H}^*(-,\mathbb{Z}_{km}) & \xrightarrow{\beta_{(n,km)}} & \mathrm{H}^{*+1}(-,\mathbb{Z}_n) \end{array} \tag{6.71}$$

$$\begin{array}{ccc} \mathrm{H}^*(-,\mathbb{Z}_m) & \xrightarrow{\beta_{(kn,m)}} & \mathrm{H}^{*+1}(-,\mathbb{Z}_{kn}) \\ \| & & \downarrow{\bmod n} \\ \mathrm{H}^*(-,\mathbb{Z}_m) & \xrightarrow{\beta_{(n,m)}} & \mathrm{H}^{*+1}(-,\mathbb{Z}_n) \end{array} \tag{6.72}$$

Hence we have

$$\beta_{(n,m)} = \beta_{(n,km)} \cdot k, \tag{6.73}$$

$$\rho_{(kn,n)}\beta_{(kn,m)} = \beta_{(n,m)}, \tag{6.74}$$

By definition,

$$\beta_{(2,2^n)} = \frac{1}{2^n}\delta \mod 2 \tag{6.75}$$

where δ is the coboundary map.

Moreover, $\mathrm{Sq}^1 = \beta_{(2,2)}$.

By (6.74), $\beta_{(2,4)} = \rho_{(4,2)}\beta_{(4,4)}$, thus $\beta_{(2,2)}\beta_{(2,4)} = \beta_{(2,2)}\rho_{(4,2)}\beta_{(4,4)} = 0$.

Similarly, $\beta_{(2,8)} = \rho_{(4,2)}\beta_{(4,8)}$, thus $\beta_{(2,2)}\beta_{(2,8)} = \beta_{(2,2)}\rho_{(4,2)}\beta_{(4,8)} = 0$, etc.

Combining this with the Adem relation $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$, we obtain the important formula:

$$\mathrm{Sq}^1\beta_{(2,2^n)} = 0 \tag{6.76}$$

6.6 Useful fomulas

Adem relations:

$$\mathrm{Sq}^a\mathrm{Sq}^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j}\mathrm{Sq}^j \tag{6.77}$$

for $0 < a < 2b$. In particular, we have $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$, $\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1 = \mathrm{Sq}^2\mathrm{Sq}^2$.

Recall that

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[a] \tag{6.78}$$

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[x_2, x_3, x_5, x_9, \dots] \tag{6.79}$$

$$\mathrm{H}^*(\mathrm{BPSU}(2), \mathbb{Z}_2) = \mathbb{Z}_2[w_2', w_3'] \tag{6.80}$$

Combining (6.62) and (6.60), we have

$$\begin{aligned}
a^2 = \mathrm{Sq}^1 a &= w_1 a \text{ in 2d} && (6.81)\\
x_3 = \mathrm{Sq}^1 x_2 &= w_1 x_2 \text{ in 3d} && (6.82)\\
w_3' = \mathrm{Sq}^1 w_2' &= w_1 w_2' \text{ in 3d} && (6.83)\\
\mathrm{Sq}^1(a x_2) &= a^2 x_2 + a x_3 = w_1 a x_2 \text{ in 4d} && (6.84)\\
\mathrm{Sq}^2(a x_2) &= a x_2^2 + a^2 x_3 = (w_2 + w_1^2) a x_2 \text{ in 5d} && (6.85)\\
x_5 = \mathrm{Sq}^2 x_3 &= (w_2 + w_1^2) x_3 \text{ in 5d} && (6.86)\\
w_2' w_3' = \mathrm{Sq}^2(w_3') &= (w_2 + w_1^2) w_3' \text{ in 5d} && (6.87)\\
\mathrm{Sq}^1(w_2 x_2) &= (w_1 w_2 + w_3) x_2 + w_2 x_3 = w_1 w_2 x_2 \Rightarrow w_3 x_2 = w_2 x_3 \text{ in 5d} && (6.88)\\
\mathrm{Sq}^1(w_1^2 x_2) &= w_1^2 x_3 = w_1^3 x_2 \text{ in 5d} && (6.89)\\
\mathrm{Sq}^3 x_2 &= w_1 w_2 x_2 = 0 \text{ in 5d} && (6.90)\\
\mathrm{Sq}^1(w_2 w_2') &= (w_1 w_2 + w_3) w_2' + w_2 w_3' = w_1 w_2 w_2' \Rightarrow w_3 w_2' = w_2 w_3' \text{ in 5d} && (6.91)\\
\mathrm{Sq}^1(w_1^2 w_2') &= w_1^2 w_3' = w_1^3 w_2' \text{ in 5d} && (6.92)\\
\mathrm{Sq}^3 w_2' &= w_1 w_2 w_2' = 0 \text{ in 5d} && (6.93)\\
\mathrm{Sq}^1(x_2^2) &= w_1 x_2^2 = 2 x_2 x_3 = 0 \text{ in 5d} && (6.94)\\
\mathrm{Sq}^1(w_2'^2) &= w_1 w_2'^2 = 2 w_2' w_3' = 0 \text{ in 5d} && (6.95)\\
\mathrm{Sq}^1(w_2' x_2) &= w_3' x_2 + w_2' x_3 = w_1 w_2' x_2 \text{ in 5d} && (6.96)
\end{aligned}$$

where w_i is the i-th Stiefel-Whitney class of the tangent bundle of M, all cohomology classes are pulled back to M.

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References

[1] Z. Wan and J. Wang, *in preparation, to appear* (2018).

[2] D. S. Freed and M. J. Hopkins, *Reflection positivity and invertible topological phases*, *ArXiv e-prints* (2016 Apr.), [arXiv:1604.06527].

[3] C. N. Yang and R. L. Mills, *Conservation of Isotopic Spin and Isotopic Gauge Invariance*, *Phys. Rev.* **96** 191–195 (1954 Oct).

[4] A. Kapustin, *Symmetry Protected Topological Phases, Anomalies, and Cobordisms: Beyond Group Cohomology*, arXiv:1403.1467.

[5] A. Kapustin, R. Thorngren, A. Turzillo and Z. Wang, *Fermionic Symmetry Protected Topological Phases and Cobordisms*, *JHEP* **12** 052 (2015), [arXiv:1406.7329].

[6] X.-G. Wen, *Zoo of quantum-topological phases of matter*, *ArXiv e-prints* (2016 Oct.), [arXiv:1610.03911].

[7] R. Thom, *Quelques propriétés globales des variétés différentiables*, *Commentarii Mathematici Helvetici* **28** 17–86 (1954).

[8] S. Galatius, I. Madsen, U. Tillmann and M. Weiss, *The homotopy type of the cobordism category*, *Acta Math.* **202** 195–239 (2009), [arXiv:math/0605249].

[9] D. Gaiotto, A. Kapustin, N. Seiberg and B. Willett, *Generalized Global Symmetries*, *JHEP* **02** 172 (2015), [arXiv:1412.5148].

[10] M. Guo, P. Putrov and J. Wang, *Time reversal, SU(N) Yang-Mills and cobordisms: Interacting topological superconductors/insulators and quantum spin liquids in 3 + 1 D*, *Annals of Physics* **394** 244–293 (2018 July), [arXiv:1711.11587].

[11] C. Crdova, T. T. Dumitrescu and K. Intriligator, *Exploring 2-Group Global Symmetries*, arXiv:1802.04790.

[12] F. Benini, C. Crdova and P.-S. Hsin, *On 2-Group Global Symmetries and their Anomalies*, arXiv:1803.09336.

[13] C. Delcamp and A. Tiwari, *From gauge to higher gauge models of topological phases*, *JHEP* **10** 049 (2018), [arXiv:1802.10104].

[14] C. Zhu, T. Lan and X.-G. Wen, *Topological non-linear σ-model, higher gauge theory, and a realization of all 3+1D topological orders for boson systems*, *ArXiv e-prints* (2018 Aug.), [arXiv:1808.09394].

[15] A. Kapustin and R. Thorngren, *Fermionic SPT phases in higher dimensions and bosonization*, *JHEP* **10** 080 (2017), [arXiv:1701.08264].

[16] J. Wang, K. Ohmori, P. Putrov, Y. Zheng, Z. Wan, M. Guo et al., *Tunneling topological vacua via extended operators: (Spin-)TQFT spectra and boundary deconfinement in various dimensions*, *Progress of Theoretical and Experimental Physics* **2018** 053A01 (2018 May), [arXiv:1801.05416].

[17] K. Yonekura, *On the cobordism classification of symmetry protected topological phases*, *ArXiv e-prints* (2018 Mar.), [arXiv:1803.10796].

[18] I. García-Etxebarria and M. Montero, *Dai-Freed anomalies in particle physics*, *ArXiv e-prints* (2018 July), [arXiv:1808.00009].

[19] C. Cordova and T. T. Dumitrescu, *Candidate Phases for SU(2) Adjoint QCD_4 with Two Flavors from $\mathcal{N} = 2$ Supersymmetric Yang-Mills Theory*, *ArXiv e-prints* (2018 June), [arXiv:1806.09592].

[20] A. Clément, *Integral Cohomology of Finite Postnikov Towers*, *Doctoral Thesis* (2002).

[21] S. Eilenberg and S. MacLane, *On the groups $H(\Pi, n)$. II. Methods of computation*, *Ann. Math.* **60** 49–139 (1954).

[22] A. Kapustin and R. Thorngren, *Topological Field Theory on a Lattice, Discrete Theta-Angles and Confinement*, *ArXiv e-prints* (2013 Aug.), [arXiv:1308.2926].

[23] A. Kapustin and R. Thorngren, *Higher symmetry and gapped phases of gauge theories*, *ArXiv e-prints* (2013 Sept.), [arXiv:1309.4721].

[24] H. Tamanoi, *$\mathcal{Q}$-subalgebras, Milnor basis, and cohomology of Eilenberg-MacLane spaces*, *Journal of Pure and Applied Algebra* **137** 153–198 (1999).

[25] A. Kono, M. Mimura and N. Shimada, *Cohomology of classifying spaces of certain associative H-spaces*, *J. Math. Kyoto Univ.* **15** 607–617 (1975).

[26] J. P. May and R. J. Milgram, *The Bockstein and the Adams spectral sequences*, *Proceedings of the American Mathematical Society* 128–130 (1981).

[27] A. Hatcher, *Algebraic topology*. Cambridge University Press, 2002.

[28] X.-G. Wen, *Construction of bosonic symmetry-protected-trivial states and their topological invariants via $G \times SO(\infty)$ nonlinear σ models*, *Physical Review B* **91** 205101 (2015).

[29] X.-G. Wen, *Exactly soluble local bosonic cocycle models, statistical transmutation, and simplest time-reversal symmetric topological orders in 3+ 1 dimensions*, *Physical Review B* **95** 205142 (2017).

[30] J. J. Rotman, *An introduction to homological algebra*. Springer Science & Business Media, 2008.