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Whitney's critical set in fractal

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Abstract

The problem is concerned about how large (e.g. the Hausdorff dimension) is Whitney's critical set contained in a given fractal. For this, we prove that the Moran arc, an arc containing a Moran set, is a Whitney's critical set. The excellent open set condition is defined, when the condition holds, the associated self-similar set contains a Whitney's critical subset of full dimension. As its application, the Sierpinski gasket and Koch curve have Whitney's critical subset of full dimension. Finally, we provide a self-similar tree which never contains any Whitney's critical set.

1. Introduction

In 1935, by constructing a C^1 function $f : R^2 \to R$ which is not constant but critical along an arc, Whitney [1] pointed out that smoothness hypothesis for the Morse–Sard Theorem – the basis of differential topology – could not be weakened. Here, an *arc* means the homeomorphic image of unit interval $[0,1]$. In his example, the critical set of f contains an arc of dimension $\log 4/\log 3$, and thus the image of critical set contains an interval which has positive Lebesgue measure. However the Sard theorem says that image of critical set has Lebesgue measure zero. Therefore the example shows that Morse–Sard theorem is invalid for function with low smoothness.

Definition 1. A closed connected set $A \subset R^n$ is said to be a Whitney's critical set ("Whitney set" in brief), if there is a C^1 function $f : R^n \to R$ such that $f|_A$ is not constant and f is critical on A, i.e., the restriction of gradient $\nabla f|_A \equiv 0$.

Remark 1. This phenomenon is very special: suppose γ is smooth arc and $\nabla f|_\gamma \equiv 0$ for some C^1 function $f : R^2 \to R$. Then for any $x, y \in \gamma$,

$$f(y) - f(x) = \int_{\gamma(x,y)} \nabla f|_\gamma = 0,$$

where $\gamma(x,y)$ is the subarc of γ connects that x and y. Then $f|_\gamma$ is constant, consequently γ is not a Whitney set.

Remark 2. Remark 1 shows that Whitney sets shall be fractals, and the following "calculus formula" is invalid in the category of fractals:

$$\int_A \partial\omega = \int_{\partial A} \omega,$$

where ω is a differential form, and A is a closed set with boundary ∂A. Then, it is very important for Whitney set to re-establish "intrinsic calculus on fractals" [2].

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Since 1935, many mathematicians such as Sard [3], Federer [4] and Norton [5] have worked on the structure of Whitney set by using Hausdorff measure. Because of the lack of approaches for critical set with low smoothness, it is very difficult to characterize the Whitney set geometrically in general case. The question was vaguely posed in Whitney's original paper [1], and can be stated as follows: given a function f, how far from rectifiable must a closed connected set be to be a critical set for f on which f is not constant?

In 1944, Choquet [6] proved that a planar arc, which is the graph of a continuous function, is not a Whitney set. In 1989, Norton [5] obtained a sufficient condition: if γ is a t-quasi-arc with $t < \dim_H \gamma$, then γ is a Whitney set. So far, the Whitney question is still open.

In this paper, we consider the following problem: given a connected set A, how large, e.g. the Hausdorff dimension, is the Whitney subset contained in A?

The paper is organized as follows.

In Section 2, we define the Moran arc, an arc containing a Moran set, and prove the following theorem.

Theorem 1. *The Moran arc is a Whitney set.*

In Section 3, we define the excellent open set condition for contractions and prove that:

Theorem 2. *If the excellent open set condition holds for contraction similitudes $\{S_i\}_{i=1}^m$, then the associated self-similar set Λ contains a Moran arc.*

Furthermore, if the similitudes $\{S_i\}_{i=1}^m$ have the same ratio, and the excellent open set condition with other separate condition holds, then the self-similar set Λ contains a Whitney subset Γ of full dimension, which means $\dim_H \Lambda = \dim_H \Gamma$.

This is Theorem 3.

In Section 4, applying Theorem 3, the Sierpinski gasket and Koch curve are discussed. We prove that:

Theorem 4. *The Sierpinski gasket is not a Whitney set, but contains a Whitney subset of full dimension.*

Theorem 5. *The Koch curve is a Whitney set.*

In Section 5, we find that Theorem 3 cannot be extended to arbitrary self-similar set. For example, we have:

Theorem 6. *There is a self-similar tree so that it could not contain any Whitney set.*

2. From Moran set to Moran arc and Whitney set

In this section, we will construct a certain fractal named Moran arc which is an arc containing a Moran set, and prove that the Moran arc is a Whitney set.

Some notations are introduced: For the Euclidean space $\mathbf{R}^n$ with metric d, and subsets $A, B \subset \mathbf{R}^n$, the shortest distance between two sets is defined by $d(A,B) = \inf_{x\in A, y\in B} d(x,y)$. The diameter of A is $|A| = \sup_{x,y\in A} d(x,y)$. Let $\mathrm{int}(A)$ denote the interior of A and, ∂A the boundary of A.

2.1. Moran set

Suppose $\{m_k\}_k$ is a sequence of integers, and $\{\varepsilon_k\}_k$ a monotone decreasing sequence of positive number with $\lim_{k\to\infty} \varepsilon_k = 0$.

Set the initial compact set $E \subset \mathbf{R}^n$, its interior $\mathrm{int}(E)$ contains m_1 disjoint compact sets $E_1, \ldots, E_{m_1}$.

Inductively, we have the sets $E_{i_1 i_2 \cdots i_k}$ $(1 \leqslant i_t \leqslant m_t)$ satisfying:

1. $E_{i_1 i_2 \cdots i_k} \subset \mathrm{int}(E_{i_1 i_2 \cdots i_{k-1}})$,
2. $d(E_{i_1 i_2 \cdots i_k}, \partial E_{i_1 i_2 \cdots i_{k-1}}) \geqslant \varepsilon_k$,
3. $d(E_{i_1 i_2 \cdots i_{k-1} i_k}, E_{i_1 i_2 \cdots i_{k-1} i'_k}) \geqslant \varepsilon_k$ when $i_k \neq i'_k$,
4. $\lim_{k\to\infty} \sup_{i_1 \cdots i_k} |E_{i_1 \cdots i_k}| = 0$.

Let $\Omega = \bigcap_k \bigcup_{i_1 i_2 \cdots i_k} E_{i_1 i_2 \cdots i_k}$. It is called a Moran set.

One can obtain a mass distribution μ with support Ω in the following way: the unit mass 1 is evenly subdivided to every $E_{i_1 i_2 \cdots i_k}$ in kth step so that

$$\mu(E_{i_1 i_2 \cdots i_k}) = \left(\prod_{i=1}^{k} m_i\right)^{-1}$$

for each word $i_1 \cdots i_k$ and $\mu(\Omega) = 1$.

Lemma 1

$$\dim_H \Omega \geqslant n \underline{\lim}_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \varepsilon_k^n)}.$$

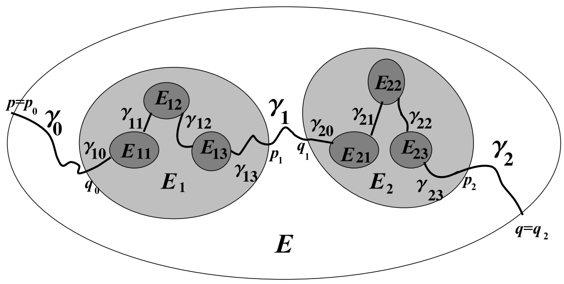
Proof. The conclusion is similar to Example 4.6 in [7], the only difference is dimension n of Euclidean space $\mathbf{R}^n$, where $n = 1$. The proof also follows similarly.

2.2. Moran arc

For notational convenience, let $\gamma = [x, y]$ denote the arc γ with end points x, y, and $\mathring{\gamma} = \gamma \setminus \{x, y\}$ the interior of γ. Following the notations for Moran set, some new assumptions are needed to define the Moran arc.

Assumptions. There are points $p, q \in \partial E$ and arcs $\gamma_0 = [p_0, q_0]$, $\gamma_1 = [p_1, q_1] \ldots,\ \gamma_{m_1} = [p_{m_1}, q_{m_1}]$ contained in Ω so that:

1. $p_0 = p,\ q_{m_1} = q$; if $1 \leqslant i \leqslant m_1$, then $p_i, q_{i-1} \in \partial E_i$.
2. $\mathring{\gamma}_0 \cup \cdots \cup \mathring{\gamma}_{m_1} \subset \mathrm{int}(E) \setminus \bigcup_{1 \leqslant i_1 \leqslant m_1} E_{i_1}$.
3. $d(\gamma_i, \gamma_{i'}) \geqslant \varepsilon_1$ if $i \neq i'$ (that means the arcs are disjoint).



By induction, the compact set $E_{i_1 \cdots i_k}$ contains $m_k + 1$ arcs

$$\gamma_{i_1 \cdots i_k t} = [p_{i_1 \cdots i_k t}, q_{i_1 \cdots i_k t}], \quad t = 0, 1, \ldots, m_k,$$

so that:

(I) $p_{i_1 \cdots i_{k-1} i_k 0} = q_{i_1 \cdots i_{k-1}(i_k - 1)},\ q_{i_1 \cdots i_k m_k} = p_{i_1 \cdots i_k}$.
When $1 \leqslant t \leqslant m_{k+1}$, $q_{i_1 \cdots i_k (t-1)}, p_{i_1 \cdots i_k t} \in \partial E_{i_1 \cdots i_k t}$.

(II) $\mathring{\gamma}_{i_1 \cdots i_k 0} \cup \cdots \cup \mathring{\gamma}_{i_1 \cdots i_k m_{k+1}} \subset \mathrm{int}(E_{i_1 \cdots i_k}) \setminus \bigcup_{1 \leqslant t \leqslant m_{k+1}} E_{i_1 \cdots i_k t}$.

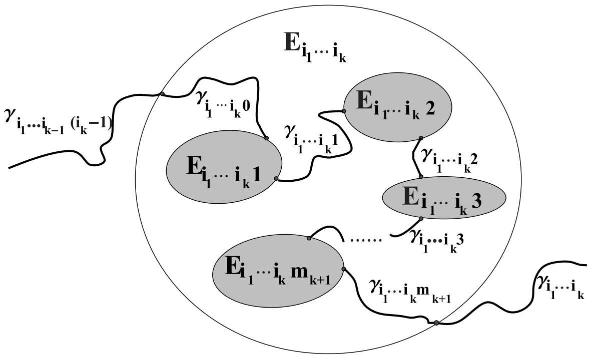
(III) $d(\gamma_{i_1 \cdots i_k t}, \gamma_{i_1 \cdots i_k t'}) \geqslant \varepsilon_{k+1}$ if $t \neq t'$,
$d(\gamma_{i_1 \cdots i_k t}, E_{i_1 \cdots i_k t'}) \geqslant \varepsilon_{k+1}$ if $t < t'$ or $t > t' + 1$,
$d(\gamma_{i_1 \cdots i_k}, \gamma_{i_1 \cdots i_k t}) \geqslant \varepsilon_{k+1}$ if $t \neq m_{k+1}$,

Define Moran arc as follows:

$$\Gamma = \Omega \cup \left(\bigcup_{\substack{k \geqslant 0,\ 0 \leqslant t \leqslant m_{k+1} \\ 1 \leqslant i_1 \leqslant m_1, \ldots, 1 \leqslant i_k \leqslant m_k}} \gamma_{i_1 \cdots i_k t} \right).$$

Above Moran arc is called Moran arc of type (m_k, ε_k).

Remark. In the original paper of Whitney [8], the famous example is a Moran arc of dimension log 4/log 3 with respect to the parameters $m_k \equiv 4$, $\varepsilon_k = 3^{-k}/12$.



2.3. Moran arc is Whitney set

We now use the special case of Whitney extension theorem as follows [8,9].

Lemma 2. *Suppose a compact set $A \subset \mathbf{R}^n$ and a function $g : A \to R$. If for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any points $x, y \in A$ with $0 < |x - y| < \delta$,*

$$\frac{|g(x) - g(y)|}{|x - y|} < \varepsilon. \tag{$*$}$$

Then there exists a C^1 extension $\overline{g} : \mathbf{R}^n \to R$ such that

$$\overline{g}|_A = g \quad \textit{and} \quad \nabla\overline{g}|_A \equiv 0,$$

where the grads

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Theorem 1. *For Moran arc Γ of type (m_k, ε_k), if*

$$\lim_{k\to\infty} \frac{1}{(m_1 \cdots m_k)\varepsilon_{k+1}} = 0, \tag{$**$}$$

then Γ is a Whitney set.

Proof. Since $\Omega \subset \Gamma$, the mass distribution μ supported on Ω is also a measure on Γ. Thus, $\mu(\gamma_{i_1\cdots i_k t}) = 0$. For the arc Γ, denote $\Gamma(x, y)$ the subarc lying between x and y. Fix an end point p, then for any $x \in \Gamma$, define a real valued function

$$g(x) = \mu(\Gamma(p, x)).$$

Now, for $x, y \in A$ with $0 < |x - y| < \delta$, one shall estimate

$$\frac{|g(x) - g(y)|}{|x - y|}.$$

From formula $(**)$, for any $\varepsilon > 0$, there exists an integer N such that when $k \geqslant N$,

$$\frac{1}{(m_1 \cdots m_k)\varepsilon_{k+1}} < \varepsilon.$$

Since $\varepsilon_k \downarrow 0$, we take $\delta = \varepsilon_{N+2}$. Suppose $E_{i_1\cdots i_k}$ is the minimal one containing both x and y. Now one shall distinguish three cases.

(1) If both x and y belong to one of $E_{i_1\cdots i_k i_{k+1}}$ or $\gamma_{i_1\cdots i_k t}$, by minimal property of $E_{i_1\cdots i_k}$, one can suppose $x, y \in \gamma_{i_1\cdots i_k t}$ for some t. Thus, $|g(x) - g(y)| = 0$ since $\mu(\gamma_{i_1\cdots i_k t}) = 0$, and inequality $(*)$ follows.

(2) If x, y belong to two disjoint components of $\{E_{i_1\cdots i_k i_{k+1}}\}_{i_{k+1}}$ and $\{\gamma_{i_1\cdots i_k t}\}_t$, respectively, then

$$|x - y| \geqslant \varepsilon_{k+1}.$$

When $|x - y| < \delta$, one has

$$\delta = \varepsilon_{N+2} \geqslant |x - y| \geqslant \varepsilon_{k+1},$$

which implies $k \geqslant N$. On the other hand,

$$|g(x) - g(y)| \leqslant \mu(E_{i_1\cdots i_k}) = \left(\prod_{i=1}^{k} m_i\right)^{-1},$$

Hence

$$\frac{|g(x) - g(y)|}{|x - y|} \leqslant \frac{1}{(m_1 \cdots m_k)\varepsilon_{k+1}} < \varepsilon,$$

The inequality $(*)$ follows from condition $(**)$.

(3) If x, y belong to two intersecting components of $\{E_{i_1\cdots i_k i_{k+1}}\}_{i_{k+1}}$ and $\{\gamma_{i_1\cdots i_k t}\}_t$, without loss of generality, one may assume

$$x \in E_{i_1\cdots i_k t}, \quad y \in \gamma_{i_1\cdots i_k t}.$$

One can discuss for different cases.

(3A) If $x \in \gamma_{i_1\cdots i_k t m_{k+2}}$, then $\mu(\gamma_{i_1\cdots i_k t m_{k+2}}) = \mu(\gamma_{i_1\cdots i_k t}) = 0$. Hence $|g(x) - g(y)| = 0$, and the inequality $(*)$ follows.

(3B) If $x \notin \gamma_{i_1\cdots i_k t m_{k+2}}$, then $d(x, y) \geqslant \varepsilon_{k+2}$. Similarly, $|x - y| < \delta$ implies

$$\delta = \varepsilon_{N+2} \geqslant |x - y| \geqslant \varepsilon_{k+2}.$$

Consequently $k \geqslant N$. In the same way,

$$|g(x) - g(y)| \leqslant \mu(E_{i_1\cdots i_k t}) = \left(\prod_{i=1}^{k+1} m_i\right)^{-1}.$$

Therefore,

$$\frac{|g(x) - g(y)|}{|x - y|} \leqslant \frac{1}{(m_1 \cdots m_{k+1})\varepsilon_{k+2}} < \varepsilon,$$

The inequality $(*)$ holds. □

3. Moran arc contained in self-similar set

Suppose $\{S_i\}_{i=1}^m$ is a set of contractions. For a word $a = j_1 \cdots j_k$ with $1 \leqslant j_i \leqslant m$, denote $|a|$ the length of word a, and write $S_a = S_{j_1} \circ \cdots \circ S_{j_k}$. Furthermore, when $\{S_i\}_{i=1}^m$ are similitudes with contraction ratios $\{\rho_i\}_{i=1}^m$, write $\rho_a = \rho_{j_1} \cdots \rho_{j_k}$.

Definition 2. We say that the *excellent open set condition* holds for contractions $\{S_i\}_{i=1}^m$, if:

(1) There is a compact set F with its interior $U = \mathrm{int}(F)$ so that

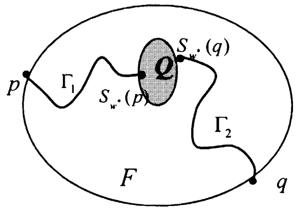
$$\bigcup_{i=1}^{m} S_i(U) \subset U$$

with this union disjoint.

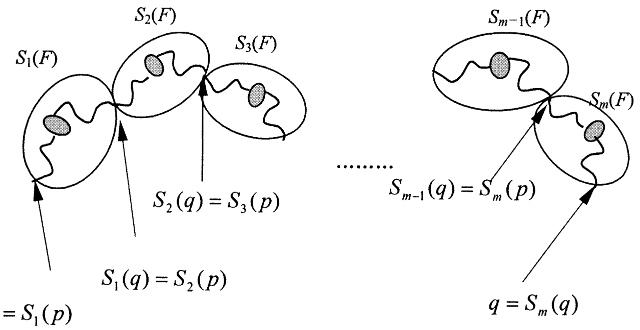
Denote by Λ the unique compact set invariant under the contractions $\{S_i\}_{i=1}^m$.

(2) There exists a finite word w^* such that $S_{w^*}(F) \subset U$, write $Q = S_{w^*}(F)$.

There are points $p,q \in \mathring{\Lambda}$ and arcs $\Gamma_1 = [p, S_{w*}(p)]$, $\Gamma_2 = [S_{w^*}(q), q] \subset \Lambda$ so that:
(2A) $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_1 \cup \Gamma_2 \subset U \setminus Q$,



(2B) $S_1(p) = p$, $S_1(q) = S_2(p), \ldots,$
$S_i(p) = S_{i-1}(q),\; S_i(q) = S_{i+1}(p), \ldots,$
$S_m(p) = S_{m-1}(q),\; S_m(q) = q.$

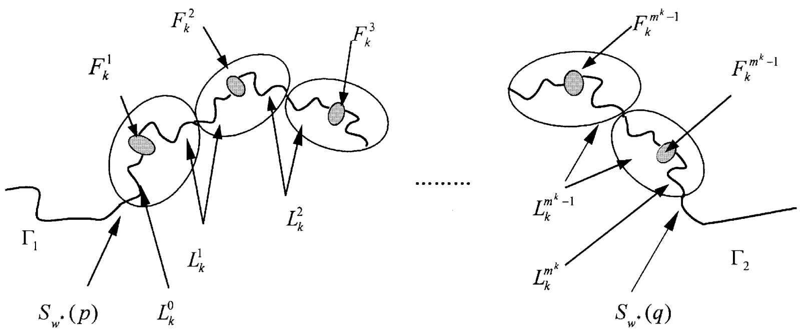


(3) Let $G = \Gamma_1 \cup \Gamma_2 \cup Q$. For each integer k, we consider the graph

$$G_k = \bigcup_{|a|=k} S_{w^*a}(G),$$

which is composed of $m^k + 1$ arcs (denoted by $L_k^0, L_k^1, \ldots, L_k^{m^k}$) and m^k small copies of F (denoted by $F_k^1, F_k^2, \ldots, F_k^{m^k}$).
The connected graph G_k is linked piecewise in the following order:

$$L_k^0, F_k^1, L_k^1, F_k^2, \ldots, F_k^{m^k}, L_k^{m^k},$$



There is a constant $\delta_k > 0$ so that some separate conditions hold:

(3A) $d(F_k^i, \partial Q) \geqslant \delta_k$, for $i = 1, \ldots, m^k$,
$d(L_k^i, \Gamma_1) \geqslant \delta_k$ when $i > 0$,
$d(L_k^i, \Gamma_2) \geqslant \delta_k$ when $i < m^k$,

(3B) $d(F_k^i, F_k^j) \geqslant \delta_k$, $d(L_k^i, L_k^j) \geqslant \delta_k$, when $i \neq j$,
$d(L_k^i, F_k^j) \geqslant \delta_k$ when $j < i$ or $j > i + 1$ (i.e., $L_k^i \cap F_k^j = \emptyset$).

Remark. The sequence $\{\delta_k\}_k$ in definition is called the separate parameter of the excellent open set condition.

Theorem 2. *Suppose the similitudes* $\{S_i\}_{i=1}^m$ *with contraction ratios* $\{\rho_i\}_{i=1}^m$, *and* $\Lambda \subset \mathbf{R}^n$ *the associated compact invariant set. If the excellent open set condition holds with separate parameter* $\{\delta_k\}_k$, *then* Λ *contains a Moran arc of type* (m_k, ε_k), *where* $m_k = m^{k-1}$, *and* $\varepsilon_k = \min(\rho_i)^{(k-1)^2/2}\delta_{k-1}$. *Furthermore, if*

$$\lim_{k\to\infty} \frac{1}{(m_1 \cdots m_k)\varepsilon_{k+1}} = 0,$$

then E contains a Whitney subset of Hausdorff dimension at least

$$n \varliminf_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \varepsilon_k^n)}.$$

Proof. By Theorem 1, we only need to show that Λ contains a Moran arc of type (m_k, ε_k). For $k \geqslant 1$, let $m_k = m^{k-1}$. For convenience, preserve the notations in Definition 2.

Define the Moran arc step by step as follows:

(1) Firstly, set the initial set $E = F$, $m_1 = 1$, $\gamma_0 = \Gamma_1$, $\gamma_1 = \Gamma_2$, and $E_1 = Q$.
(2) Secondly, $m_2 = m$, one can set

$$\gamma_{10} = L_1^0, \gamma_{11} = L_1^1, \ldots, \gamma_{1m_2} = L_1^m,$$

$$E_{11} = F_1^1, E_{22} = F_1^2, \ldots, E_{1m_2} = F_1^m.$$

(3) Inductively, for $k \geqslant 1$, suppose one has constructed $E_{i_1\cdots i_k} = S_{u_k}(F)$ with a word u_k of length

$$|u_k| = |w^*| + [1 + 2 + \cdots + (k-1)] \leqslant k^2/2 \quad \text{for large } k,$$

then let

$$\gamma_{i_1\cdots i_k j} = S_u(S_{w^*}^{-1}(L_k^j)) \quad \text{for } j = 0, \ldots, m_{k+1},$$

$$E_{i_1\cdots i_k i_{k+1}} = S_u(S_{w^*}^{-1}(F_k^{i_{k+1}})) \quad \text{for } i_{k+1} = 1, \ldots, m_{k+1},$$

where $E_{i_1\cdots i_k i_{k+1}} = S_{u_{k+1}}(F)$ with

$$|u_{k+1}| = |u_k| + k = |w^*| + [1 + 2 + \cdots + k].$$

From the definitions of $\{\varepsilon_k\}_k$ and $\{\delta_k\}_k$, one can set

$$\varepsilon_{k+1} = \min(\rho_i)^{k^2/2}\delta_k.$$

One can verify it for two typical cases as follows:

(I) When $j \neq j'$,

$$\begin{aligned} d(\gamma_{i_1\cdots i_k j}, \gamma_{i_1\cdots i_k j'}) &= d(S_{u_k}S_{w^*}^{-1}(L_k^j), S_{u_k}S_{w^*}^{-1}(L_k^{j'})) \\ &= \rho(S_{u_k}) d(S_{w^*}^{-1}(L_k^j), S_{w^*}^{-1}(L_k^{j'})) \\ &\geqslant \rho(S_{u_k}) d(L_k^j, L_k^{j'}) \\ &\geqslant \rho(S_{u_k})\delta_k \\ &\geqslant \min(\rho_i)^{|u_k|}\delta_k \\ &\geqslant \min(\rho_i)^{k^2/2}\delta_k \quad \text{for large } k. \end{aligned}$$

(II) When $t \neq 0$,

$$\begin{aligned} d(\gamma_{i_1\cdots i_{k-1}(i_k-1)}, \gamma_{i_1\cdots i_k t}) &\geqslant d(S_{w^*}^{-1}S_{u_k}(\Gamma_1), S_{w^*}^{-1}S_{u_k}(L_k^t)) \\ &\geqslant d(S_{u_k}(\Gamma_1), S_{u_k}(L_k^t)) \\ &\geqslant \rho(S_{u_k}) d(\Gamma_1, L_k^t) \\ &\geqslant \min(\rho_i)^{|u_k|}\delta_k \\ &\geqslant \min(\rho_i)^{k^2/2}\delta_k \quad \text{for large } k. \end{aligned}$$

A Moran arc of type (m_k, ε_k) is obtained, and the theorem follows from Theorem 1. □

Theorem 3. *Suppose the similitudes* $\{S_i\}_{i=1}^m$ *with same contraction ratios* ρ, *and* $\Lambda \subset \mathbf{R}^n$ *the associated compact invariant set with dimension* $\dim_H \Lambda > 1$. *If the excellent open set condition holds with separate parameter* $\{\delta_k\}_k$ *satisfying* $\delta_k \geqslant C\rho^k$ $(\forall k)$ *for some constant* C, *then* Λ *contains a Whitney subset* Γ *of full dimension, i.e.*, $\dim_H \Gamma = \dim_H \Lambda$.

Proof. It follows from the excellent open set condition that

$$\dim_H E = \frac{\log m}{\log 1/\rho} > 1.$$

Note that $\varepsilon_i = \rho^{(i-1)^2/2}\delta_{i-1} \geqslant C\rho^{(i-1)^2/2+(i-1)}$ and $m_i = m^{i-1}$. Then

$$\frac{1}{(m_1 \cdots m_k)\varepsilon_{k+1}} \leqslant \frac{1}{m^{[1+2+\cdots+(k-1)]} \cdot C\rho^{k^2/2+k}} \to 0,$$

since

$$\frac{\log m}{\log 1/\rho} > 1.$$

That means the Moran arc Γ is a Whitney set.

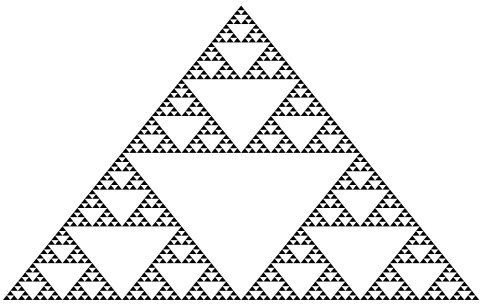
From Theorem 1, it follows that the Moran arc Γ has dimension

$$\begin{aligned} \dim_H \Gamma &\geqslant n \varliminf_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \varepsilon_k^n)} \\ &\geqslant n \varliminf_{k\to\infty} \frac{\log m^{[1+2+\cdots+(k-2)]}}{-[\log m^{k-1} C^n \rho^{n[(k-1)^2/2+(k-1)]}]} \\ &\geqslant \frac{\log m}{\log 1/\rho} = \dim_H E, \end{aligned}$$

which implies $\dim_H \Gamma = \dim_H \Lambda$. □

4. Applications to Sierpinski gasket and Koch curve

4.1. Sierpinski gasket

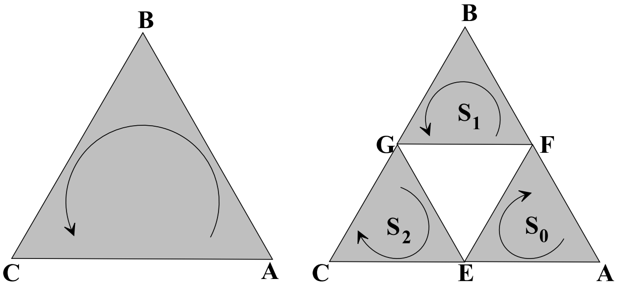


Sierpinski gasket

Define Sierpinski gasket by starting with an equilateral triangle ΔABC as follows. Let E, F, G be midpoints of three sides CA, AB, BC, respectively. Construct three similitudes S_1, S_2, S_3 with contraction ratios $\rho_1 = \rho_2 = \rho_3 = 1/2$ such that:

$S_1(\Delta ABC) = \Delta AEF$, $S_1(A) = A$, $S_1(B) = E$, $S_1(C) = F$,
$S_2(\Delta ABC) = \Delta FBG$, $S_2(A) = F$, $S_2(B) = B$, $S_2(C) = G$,
$S_3(\Delta ABC) = \Delta GEC$, $S_2(A) = G$, $S_2(B) = E$, $S_2(C) = C$.

In the following figure, after three similitudes, the arrow in ΔABC is transformed into three small arrows, respectively in three small triangles.



Let the Sierpinski gasket be the unique compact set invariant under similitudes $\{S_1, S_2, S_3\}$, exactly the same in usual sense, i.e., the Sierpinski gasket $H \cap H_j$, where $H_0 = \Delta ABC$ and sets $H_0 \supset H_1 \supset H_2 \cdots$ are defined inductively, the set H_{j+1} is obtained in the same way by removing the central triangle (with scale factor 1/2) of each triangle in H_j.

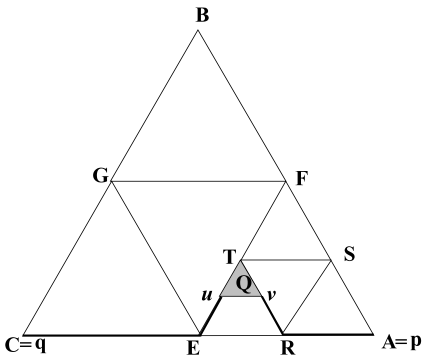
The Sierpinski gasket is not a Whitney set. Otherwise, suppose it is a Whitney set and f is the corresponding C^1 function $f : R^2 \to R$ such that $\nabla f|_H \equiv 0$. Then f shall be constant on every line segment of Sierpinski gasket. Since these line segments are connected and can approximate any point of the gasket, then f must be constant on the gasket. However it contradicts the definition of Whitney set where f is not constant restricted on the set.

One shall show that the excellent open set condition holds for the similitudes with respect to the Sierpinski gasket, and then it contains a Whitney subset of full dimension.

In the following figure:

1. Let $K = \Delta(ABC)$ be the compact set in Definition 2. Then for $U = \text{int}(K)$,

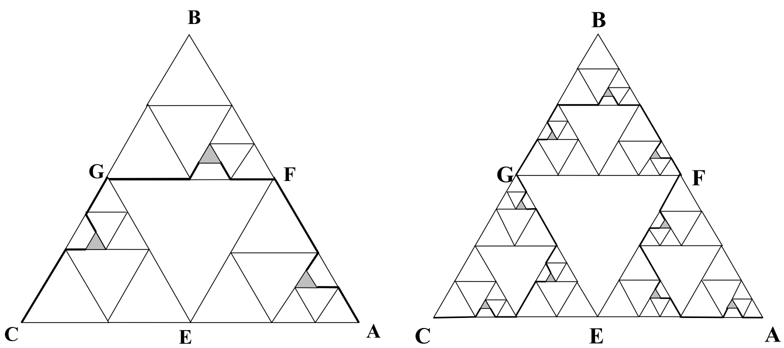
   $S_i(U) \subset U$.
2. Let $p = A$, $q = B$, and arcs $\Gamma_1 = AR \cup Rv$, $\Gamma_2 = uE \cup EC$ contained in Sierpinski gasket.
3. Let $Q = S_1 S_3(\Delta ABC) \subset U$.



R, S, T aremidpoints of EA, AF, FE, and *u, v* are

midpoints of TE, TR respectively.

4. $G = \Gamma_1 \cup \Gamma_2 \cup Q$, for any k, the graph $\{S_a(G)\}_{|a|=k}$ is connected. For example, $k = 2$ and $k = 3$ is presented in the following figure.



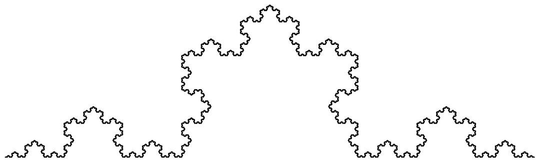
k=2 *k*=3

5. From a simple geometric estimation, one has $\delta_k \geqslant C_1 2^{-k}$ for some constant C_1.

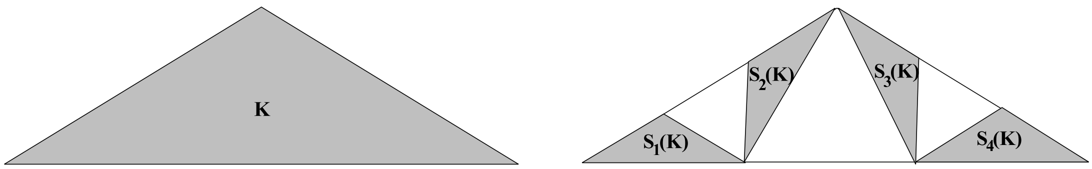
By Theorem 3, one has:

Theorem 4. *The Sierpinski gasket is not a Whitney set, but contains a Whitney subset of full dimension.*

4.2. Koch curve



For the Koch Curve, one can verify that excellent open set condition holds:



(1) The convex set K is an isosceles triangle with basic angle $\pi/6$. Let $U = \mathrm{int}(K)$. Then

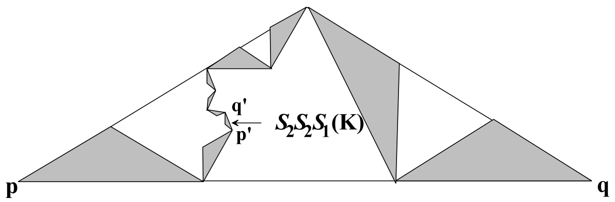
$$S_i(U) \subset U, \quad i = 1, 2, 3, 4,$$

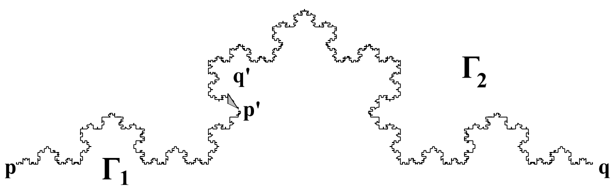
where S_i are similitudes with contraction ratio 1/3.

(2) Let $Q = S_2 S_2 S_1(K)$. Then $Q \subset U$.

Suppose p', q are end points of base of the isosceles triangle Q.

The Koch curve is an arc with end points p and q. Let Γ_1 be the subarc lying between p and p', and Γ_2 the subarc lying between q and q'.





(3) From a simple geometric estimation, one has $\delta_k \geqslant C_2 3^{-k}$ for some constant C_2.

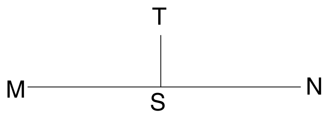
It follows from Theorem 3 that the Koch curve contains a Moran arc, which is a Whitney set of dimension $\log 4/\log 3$, with end points p and q. Since Koch curve is an arc with end points p and q, therefore the Whitney subset is exactly the Koch curve.

The above discussion can be stated in the following theorem.

Theorem 5. *The Koch curve is a Whitney set.*

5. Self-similar tree

In this section, we will construct a self-similar tree which never contains any Whitney set.



Self-similar mappings

In the figure above, the line segments $MS = SN = 1/2$, $TS \perp MN$, and TS is short enough.
The similitudes without reflection H_0, H_1, H_2 are defined by

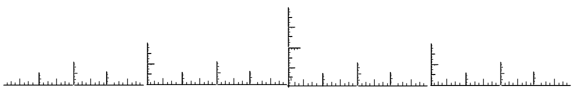
$$H_0(M) = M, \quad H_0(N) = S,$$

$$H_1(M) = T, \quad H_1(N) = S,$$

$$H_2(M) = S, \quad H_2(N) = N.$$

Let X be a self-similar tree generated by $\{H_0, H_1, H_2\}$, i.e.,

$$X = H_0(X) \cup H_1(X) \cup H_2(X).$$



Self-similar tree

Remark. TS is short enough so that $\{H_0, H_1, H_2\}$ satisfies the open-set condition. Let s be the Hausdorff dimension of the self-similar tree. Then

$$(1/2)^s + (TS)^s + (1/2)^s = 1$$

(cf. [10]) which means that $s > 1$, i.e., the Hausdorff dimension of self-similar tree exceeds 1.

5.1. Some observations

We say $H_{i_1 i_2 \cdots i_n}(X)$ a branch of rank n, where $H_{i_1 i_2 \cdots i_n} = H_{i_1} \circ H_{i_2} \circ \cdots \circ H_{i_n}$.

(1) For two points lying in distinct branches of rank 1, any connected set containing them must contain the point S, the touched point for these branches of rank 1.

(2) For any point x which lies in segment SN, SM or ST, we find that any connected set containing both S and x should contain segment Sx.

From the self-similarity of tree X, we can generalize the above observations to other branches.
These lead to the following lemma.

Lemma 3. *For any distinct points x_0, x_1, there exists a unique sequence of segments $\{A_i A_{i+1}\}_{i=-\infty}^{+\infty} \subset X$ such that $\lim_{i\to+\infty} A_i = x_1$, $\lim_{i\to-\infty} A_i = x_0$, and $\{A_i A_{i+1}\}_{i=-\infty}^{+\infty} \subset P$ for any connected set P containing both x_0 and x_1.*

Proof. Assume $x_0 = H_{i_1 i_2 \cdots i_n a \cdots}(X)$, $x_1 = H_{i_1 i_2 \cdots i_n b \cdots}(X)$ with $a, b \in \{0, 1, 2\}$ and $a \neq b$. Then let $A_0 = H_{i_1 i_2 \cdots i_n}(S)$, which is the touched point of the branches (rank n) containing x_0, x_1, respectively. Instead of x_0 by A_0, for the new pair x_0, x_1, we get another touched point denoted by A_1. Instead of x_0 by A_1, we get $A_2, \ldots$, therefore, we obtain a sequence $\{A_i\}_{i=1}^{+\infty}$ such that $\lim_{i\to+\infty} A_i = x_1$. Similarly, there is a sequence $\{A_i\}_{i=-1}^{-\infty}$ such that $\lim_{i\to+\infty} A_i = x_1$. Suppose an arbitrary connected set P contains both x_0 and x_1. Because of the self-similarity of X, observation (1) implies that $A_0 \in P$, and similarly the touched point $A_1 \in P$ since $A_0, x_1 \in P$. Furthermore, $\{A_i\}_{i=-\infty}^{+\infty} \subset P$, and segments $\{A_i A_{i+1}\}_{i=-\infty}^{+\infty} \subset P$ since $A_i, A_{i+1} \in P$ by a self-similarity version of observation (2). □

Theorem 6. *The self-similar tree X never contains any Whitney set.*

Proof. Suppose X contains some Whitney set P. Assume that a C^1 function f satisfies $\nabla f|_P \equiv 0$ and $f|_P$ is not constant. Given any two distinct points $x_0, x_1 \in P$, it follows from Lemma 11 that f shall be constant on segments $\{A_i A_{i+1}\}_{i=-\infty}^{+\infty} \subset P$ with $\lim_{i\to+\infty} A_i = x_1$ and $\lim_{i\to-\infty} A_i = x_0$. Consequently, $f(x_0) = f(x_1)$, since f is continuous. Because of the arbitrary choice of pair x_0, x_1, the function f shall be constant on P, this contradicts the assumption that P is a Whitney set. □

6. Further reading

[11,12].

Acknowledgements

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