Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Chaos, Solitons and Fractals 27 (2006) 121–126
CHAOS SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

Maximal operators and Fourier transforms of self-similar measures

Yong Lin [a,*], Huo-Jun Ruan [b]

[a] *Department of Mathematics, School of Information, Renmin University of China, Beijing 100872, China*
[b] *Department of Mathematics, Zhejiang University, Hangzhou 310027, China*

Accepted 28 January 2005

Abstract

A self-similar measure on $\mathbb{R}^n$ is defined to be a probability measure satisfying

$$\mu = \sum_{j=1}^{N} p_j \mu \circ S_j^{-1} + \sum_{j=1}^{M} q_j (\mu * \mu) \circ T_j^{-1},$$

where $S_j x = \rho_j R_j x + b_j$, $T_j x = \eta_j Q_j x + c_j$ are contractive similarities, $0 < \rho_j < 1$, $0 < \eta_j < \frac{1}{2}$, $0 < p_j < 1$, $0 < q_j < 1$, $\sum_{j=1}^{N} p_j + \sum_{j=1}^{M} q_j = 1$, R_j, Q_j are orthogonal matrix and $\mu * \mu$ is the convolution of two measures.

When $M = 0$, μ is a linear self-similar measure, we establish the asymptotic behavior of averages of the derivative of the Fourier transform of μ, such as

$$\int_{|x| \leqslant R} \left| \left(\frac{\partial}{\partial x} \right)^{\alpha} \hat{\mu}(x) \right|^2 \mathrm{d}x = \mathrm{O}(R^{n-\beta})$$

for any order derivation of $\hat{\mu}(x)$ as $R \to \infty$ under certain additional hypotheses.

When $M > 0$, μ is a nonlinear self-similar measure, we get some results of L^p boundedness for maximal operators of μ, from the pointwise asymptotic estimate of the Fourier transform of μ made by Strichartz.

1. Introduction

The Fourier transform, has been used to study problems of pointwise convergence for different kind of averages. This method was investigated by Stein, who considered the maximal spherical operators

$$\mathscr{M} f(x) = \sup_{t>0} \left| \int_{|y|=1} f(x - ty) \, \mathrm{d}\sigma(y) \right|, \tag{1.1}$$

* Corresponding author.
E-mail addresses: linyong01@ruc.edu.cn (Y. Lin), ruanhj@zju.edu.cn (H.-J. Ruan).

0960-0779/$ - see front matter © 2005 Elsevier Ltd. All rights reserved.
doi:10.1016/j.chaos.2005.02.038

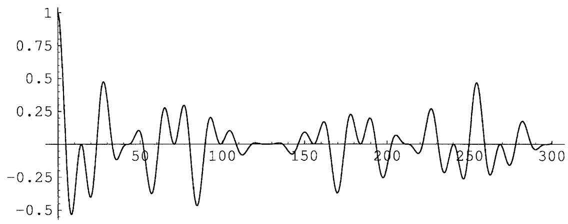

Fig. 1. The graph of $F(x) = \hat{\mu}(x)/e^{ix/2}$ where μ, is the Cantor measure with $\rho = \frac{1}{3}$.

where σ is the Lebesgue measure in the unit sphere S^{n-1}. Stein showed that $\mathscr{M}$ is bounded in $L^p(\mathbb{R}^n)$ if $p > n/(n-1)$, where f is initially taken to be in the class of Schwartz functions. Bourgain [1] has succeeded in extending this result to $\mathbb{R}^2$ and Greenleaf [6], Sogge and Stein [12], Rubio de Francia [4], Iosevich and Sawyer [8] extended this result to different cases. In this paper, we will extend Stein's result to (nonlinear) self-similar measures introduced by Hutchinson [7].

A (linear) self-similar measure on $\mathbb{R}^n$ was defined by Hutchinson to be a probability measure satisfying

$$\mu = \sum_{j=1}^{N} p_j \mu \circ S_j^{-1}, \tag{1.2}$$

where S_j are contractive similarities and weights p_j are probabilities. The following nonlinear self-similar measures were introduced by Glickenstein and Strichartz [5]. The identities have the form

$$\mu = \sum_{j=1}^{N} p_j \mu \circ S_j^{-1} + \sum_{j=1}^{M} q_j (\mu * \mu) \circ T_j^{-1}, \tag{1.3}$$

where $S_j x = \rho_j R_j x + b_j$, $T_j x = \eta_j Q_j x + c_j$ are contractive similarities, $0 < \rho_j < 1$, $0 < \eta_j < \frac{1}{2}$, $0 < p_j < 1$, $0 < q_j < 1$, $\sum_{j=1}^{N} p_j + \sum_{j=1}^{M} q_j = 1$, R_j, Q_j are orthogonal matrix and $\mu * \mu$, is the convolution of two measures.

A typical example is the Cantor measure where $N = 2$, $M = 0$, $p_1 = p_2 = \frac{1}{2}$, $S_1 x = \rho x$ and $S_2 x = \rho x + (1-\rho)$. The Fourier transform of the Cantor measure is

$$\hat{\mu}(x) = e^{ix/2} \prod_{k=1}^{\infty} \cos((1-\rho)\rho^{k-1} x/2).$$

The graph of $F(x) = \hat{\mu}(x)/e^{ix/2}$ with $\rho = \frac{1}{3}$ is shown for $0 \leqslant x \leqslant 300$ in Fig. 1.

This measure is also called convolved Bernoulli measure. It was conjectured that μ ought to be absolutely continuous for $\rho > \frac{1}{2}$. However, Erdös [2] in 1939 showed that if ρ^{-1} is a PV-number and $\rho \neq \frac{1}{2}$ (an algebraic integer $\alpha > 1$ is called a PV-number if all its conjugate roots have modulus strictly less that 1), then the Fourier transform $\hat{\mu}(x)$ does not tend to zero at infinity and thus μ is purely singular. It can be plainly seen, in Fig. 1, that $\hat{\mu}(x)$ exhibits a chaotic local behavior. So it is better to study the average asymptotic rate of $\hat{\mu}(x)$ which was studied extensively by Strichartz [13–15], Lau [9], Lau and Wang [10] and Fan and Lau [3]. Strichartz [13] showed that

$$\int_{|x| \leqslant R} |\hat{\mu}(x)|^2 \, dx = O(R^{n-\beta}) \tag{1.4}$$

under certain addition hypotheses, where β satisfies $\rho^{\beta} = \sum_{j=1}^{N} p_j^2$. In Section 2 of this paper, we will show that

$$\int_{|x| \leqslant R} \left| \left(\frac{\partial}{\partial x} \right)^{\alpha} \hat{\mu}(x) \right|^2 dx = O(R^{n-\beta}) \tag{1.5}$$

for any order derivation of $\hat{\mu}(x)$ under the same hypotheses of Strichartz.

For the maximal operator $\mathscr{M}$ when σ is the self-similar measure of (1.3), following the result of pointwise asymptotic behavior for the nonlinear self-similar measure, we get the L^p bounded result for $\mathscr{M}$ in (1.3) when $M > 0$. For the linear self-similar measure of (1.2), in some cases of Bernoulli convolution measure, we also get the L^p bounded result for $\mathscr{M}$.

2. The estimates of derivative of Fourier transforms

Let μ be a self-similar measure on $\mathbb{R}^n$, the Fourier transform of $\hat{\mu}(x)$ is defined as

$$\hat{\mu}(x) = \int e^{ix \cdot y} \, d\mu(y).$$

If μ is the linear self-similar measure of (1.2), then

$$\hat{\mu}(x) = \sum_{j=1}^{N} p_j e^{ib_j \cdot x} \hat{\mu}(\rho_j R_j^* x). \tag{2.1}$$

For any $K \in \mathbb{N}$, we define $\mathscr{J}_K = \{(j_1, j_2, \ldots, j_K) : j_\ell = 1, 2, \ldots, N \text{ for } \ell = 1, 2, \ldots, K\}$. Given $J \in \mathscr{J}_K$ we write $S_J = S_{j_1} \circ S_{j_2} \circ \cdots \circ S_{j_K}$. Then S_J is also a contractive similarity and has the form

$$S_J x = \rho_J R_J x + b_J,$$

where $\rho_J = \rho_{j_1} \rho_{j_2} \cdots \rho_{j_K}, R_J = R_{j_1} R_{j_2} \cdots R_{j_K}$ and

$$b_J = b_{j_1} + \rho_{j_1} R_{j_1} b_{j_2} + \cdots + \rho_{j_1} \cdots \rho_{j_{K-1}} R_{j_1} \cdots R_{j_{K-1}} b_{j_K}. \tag{2.2}$$

Similarly, we define $p_J = \prod_{\ell=1}^{K} p_{j_\ell}$. By iterating (2.1),

$$\hat{\mu}(x) = \sum_{J \in \mathscr{J}_K} p_J e^{ib_J \cdot x} \hat{\mu}(\rho_J R_J^* x). \tag{2.3}$$

Since μ is a probability measure, it is obvious that $\hat{\mu} \in C^\infty(\mathbb{R}^n)$ and $|\hat{\mu}(x)| \leqslant 1$. We need the so called open set condition for the self-similar set introduced by Hutchinson [7]: there exists a bounded open set U such that $S_j U \subseteq U$ for each S_j and the sets $S_j U$ are disjoint. We will also impose the hypothesis that all ρ_j are equal, say to ρ, which we call the equi-contractive condition. Then assume the open set condition and the equi-contractive condition, Strichartz [13] showed that there exists a constant $c > 0$ such that

$$|b_J - b_{J'}| \geqslant c \cdot \rho^K \quad \text{for } J \neq J' \text{ in } \mathscr{J}_K. \tag{2.4}$$

Theorem 2.1. *Assume the open set condition, the equi-contraction, and that the rotations R_j are either all equal or generate a finite group. Then for any differential monomial $\left(\frac{\partial}{\partial x}\right)^\alpha$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$, we have*

$$\int_{|x| \leqslant R} \left| \left(\frac{\partial}{\partial x}\right)^\alpha \hat{\mu}(x) \right|^2 dx = O(R^{n-\beta}), \tag{2.5}$$

where β satisfies $\rho^\beta = \sum_{j=1}^{N} p_j^2$.

Proof. Let $\mathscr{T}_\alpha = \{(t_1, \ldots, t_n) : 0 \leqslant t_\ell \leqslant \alpha_\ell \text{ for } \ell = 1, \ldots, n\}$. Denote $\binom{\alpha}{t} = \binom{\alpha_1}{t_1} \cdots \binom{\alpha_n}{t_n}$ for $t \in \mathscr{T}_\alpha$. From (2.3),

$$\left(\frac{\partial}{\partial x}\right)^\alpha \hat{\mu}(x) = \sum_{t \in \mathscr{T}_\alpha} \binom{\alpha}{t} \sum_{J \in \mathscr{J}_K} p_J \cdot \left(\frac{\partial}{\partial x}\right)^t e^{ib_J \cdot x} \cdot \left(\frac{\partial}{\partial x}\right)^{\alpha - t} \hat{\mu}(\rho^K R_J^* x).$$

Let

$$W_{\alpha,t,K}(x) = \sum_{J \in \mathscr{J}_K} p_J \cdot \left(\frac{\partial}{\partial x}\right)^t e^{ib_J \cdot x} \cdot \left(\frac{\partial}{\partial x}\right)^{\alpha - t} \hat{\mu}(\rho^K R_J^* x). \tag{2.6}$$

Then there exists a constant $\theta_1(\alpha) > 0$ such that

$$\left| \left(\frac{\partial}{\partial x}\right)^\alpha \hat{\mu}(x) \right|^2 \leqslant \theta_1(\alpha) \max_{t \in \mathscr{T}_\alpha} |W_{\alpha,t,K}(x)|^2. \tag{2.7}$$

Assume first that all R_j are equal. Then in (2.6), all the factors $\left(\frac{\partial}{\partial x}\right)^{\alpha-t} \hat{\mu}(\rho^K R_J^* x)$ are equal, so

$$|W_{\alpha,t,K(x)}| \leqslant \sup_{J \in \mathscr{J}_K} \left| \left(\frac{\partial}{\partial x}\right)^{\alpha - t} \hat{\mu}(\rho^K R_J^* x) \right| \cdot \left| \sum_{J \in \mathscr{J}_K} p_J \cdot \left(\frac{\partial}{\partial x}\right)^t e^{ib_J \cdot x} \right|. \tag{2.8}$$

Denote $R_J = (r_{J,uv})_{n \times n}$, then for any $x, y \in \mathbb{R}^n$ with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we have

$$R_J^* x \cdot y = \sum_{v=1}^{n} \left(\sum_{u=1}^{n} r_{J,uv} x_u \right) y_v = \sum_{u=1}^{n} \left(\sum_{v=1}^{n} r_{J,uv} y_v \right) x_u.$$

It follows that

$$\left(\frac{\partial}{\partial x}\right)^{\alpha-t}\hat{\mu}(\rho_J R_J^* x) = \left(\frac{\partial}{\partial x}\right)^{\alpha-t}\int e^{i(\rho^K R_J^* x)\cdot y}\,d\mu(y)$$
$$= (i\rho^K)^{|\alpha|-|t|}\int\left(\sum_{v=1}^n r_{J,1v}y_v\right)^{\alpha_1-t_1}\cdots\left(\sum_{v=1}^n r_{J,nv}y_v\right)^{\alpha_n-t_n} e^{i(\rho^K R_J^* x)\cdot y}\,d\mu(y), \tag{2.9}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $|t| = t_1 + \cdots + t_n$. Since the support of μ is compact, there exists a constant $\tau > 0$ such that the support of μ is contained in $B(0,\tau)$, the ball of radius τ centered at 0. Since R_j is orthogonal, it follows that $|r_{J,uv}| \leqslant 1$ for all u, v. Thus, note that $\rho < 1$, there exists a constant $\theta_2(\tau,\alpha)$ such that

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha-t}\hat{\mu}(\rho_J R_J^* x)\right| \leqslant \int_{y\in B(0,\tau)}\left(\sum_{v=1}^n |y_v|\right)^{|\alpha|-|t|} d\mu(y) \leqslant \theta_2(\tau,\alpha). \tag{2.10}$$

Let h be a function satisfying the following properties:

(i) $h \geqslant 0$,
(ii) $h(x) \geqslant c_1$ in $|x| \leqslant C_2$,
(iii) $\hat{h}(0) = 1$ and
(iv) $\hat{h}(x) = 0$ in $|x| \geqslant c_3$.

Note that it is easy to construct such a function for any $c_3 > 0$ provided c_2 and $c_1 > 0$ are chosen appropriately.

Denote $b_J = (b_{J,1}, \ldots, b_{J,n})$, from (2.2), $|b_{J,\ell}| \leqslant |b_J| = \frac{b^*}{1-\rho}$ for all $\ell = 1, \ldots, n$, where $b^* = \max\{|b_j| : 1 \leqslant j \leqslant N\}$. Note that

$$\left(\frac{\partial}{\partial x}\right)^t e^{ib_J\cdot x} = i^{t_1+\cdots+t_n} b_{J,1}^{t_1}\cdots b_{J,n}^{t_n} e^{ib_J\cdot x}, \tag{2.11}$$

we obtain that

$$\begin{aligned}
&\int_{|x|\leqslant c_2\rho^{-K}}\left|\sum_{J\in\mathscr{J}_K} p_J\left(\frac{\partial}{\partial x}\right)^t e^{ib_J\cdot x}\right|^2 dx \\
&= \int_{|x|\leqslant c_2\rho^{-K}}\left|\sum_{J\in\mathscr{J}_K} b_{J,1}^{t_1}\cdots b_{J,n}^{t_n} e^{ib_J\cdot x}\right|^2 dx \\
&\leqslant c_1^{-1}\int h(\rho^K x)\left|\sum_{J\in\mathscr{J}_K} b_{J,1}^{t_1}\cdots b_{J,n}^{t_n} e^{ib_J\cdot x}\right|^2 dx \\
&= c_1^{-1}\rho^{-Kn}\int \rho^{Kn}h(\rho^K x)\left|\sum_{J\in\mathscr{J}_K} b_{J,1}^{t_1}\cdots b_{J,n}^{t_n} e^{ib_J\cdot x}\right|^2 dx \\
&= c_1^{-1}\rho^{-Kn}\sum_{J\in\mathscr{J}_K}\sum_{J'\in\mathscr{J}_K} p_J\overline{p_{J'}} b_{J,1}^{t_1}\overline{b_{J,1}^{t_1}}\cdots b_{J,n}^{t_n}\overline{b_{J,n}^{t_n}}\hat{h}(\rho^{-K}(b_J - b_{J'})) \\
&= c_1^{-1}\rho^{-Kn}\sum_{J\in\mathscr{J}_K}|p_J|^2|b_{J,1}^{t_1}|^2\cdots|b_{J,n}^{t_n}|^2, \quad \text{by (2.4) and (iv) of } h \\
&\leqslant c_1^{-1}\theta_3\rho^{-Kn}\sum_{J\in\mathscr{J}_K}|p_J|^2, \quad \text{where } \theta_3 = \left(\frac{b^*}{1-\rho}\right)^{2|t|} \\
&= c_1^{-1}\theta_3\rho^{-Kn}\left(\sum_{j=1}^N p_j^2\right)^K \\
&= c_1^{-1}\theta_3\rho^{K(\beta-n)}
\end{aligned}$$

which implies that

$$\int_{|x|\leqslant R}\left|\sum_{J\in\mathscr{J}_K} p_J\left(\frac{\partial}{\partial x}\right)^t e^{ib_J\cdot x}\right|^2 dx = O(R^{n-\beta}).$$

Combining this with (2.7), (2.8) and (2.10), we obtain (2.5).

More generally, if the rotations R_j generate a finite group with say L elements, we can always write the sum in (2.1) as a sum of L sums in which the functions $\hat{\mu}(\rho_J R_J^* x)$ are the same. As a result, in (2.6), all the factors $\left(\frac{\partial}{\partial x}\right)^{\alpha - t} \hat{\mu}(\rho^K R_J^* x)$ are equal. Thus we can prove the theorem by the same argument. □

Remark 2.1. If $\alpha = (0,\ldots,0)$, Theorem 2.1 is the result of Strichartz [13].

Remark 2.2. β is a number between 0 and n. For the Cantor measure ($N = 2,\ M = 0,\ p_1 = p_2 = \frac{1}{2},\ \rho_1 = \rho_2 = \frac{1}{3},\ b_1 = 0,\ b_2 = \frac{2}{3}$), $\beta = \frac{\log 2}{\log 3}$.

3. Maximal operators of μ

Given a multiplier $m \in L^\infty(\mathbb{R}^n)$, we define the operator T^* by $T^*f(x) = \sup_{t>0}|T_t f(x)|$, where $(T_t f)^\wedge(\xi) = \hat{f}(\xi) m(t\xi)$. The following theorem of Rubio de Francia [4] is the main result we use to prove our theorems.

Theorem A. *Suppose that $m(\xi)$ is the Fourier transform of a compactly supported Borel measure σ, and*

$$|m(\xi)| \leqslant C|\xi|^{-a}, \quad \text{with } a > \frac{1}{2}. \tag{3.1}$$

Then T^* is bounded in $L^p(\mathbb{R}^n)$ for all $p > p_a = (2a + 1)/2a$.

It is easy to see that in the case of Theorem A, $T^*f(x) = \sup_{t>0}|\int f(x - ty)\,\mathrm{d}\sigma(y)|$. So T^* is the maximal operator (1.1) of σ we want to study. Thus, if we have the asymptotic behavior of the Fourier transform of self-similar measure μ, in ∞, we can get the L^p boundedness of maximal operators of μ.

For nonlinear self-similar measures in (2.3), Glichenstein and Strichartz [5] got the results we need.

Theorem B. *If μ is not degenerate (i.e., the support of μ is not in an affine hyperplane $\{x : x \cdot \omega = t\}$ for ω a unit vector), then*

(i) *If $N = 0$, then $|\hat{\mu}(x)| \leqslant \mathrm{e}^{-c|x|^\alpha}$, for large x for some positive constants c and α.*
(ii) *If $M > 0$, $N > 0$ and there is a constant $\epsilon > 0$ such that $\sum_{j=1}^N p_j \rho_j^{-\epsilon} < 1$, then there exists a positive constant c such that $|\hat{\mu}(x)| \leqslant c|x|^{-\epsilon}$.*

Now we can prove the L^p boundedness of maximal operators $\mathscr{M}$ for such measure μ. Now

$$\mathscr{M}f(x) = \sup_{t>0}\left|\int f(x - ty)\,\mathrm{d}\mu(y)\right|. \tag{3.2}$$

Theorem 3.1. *Under the hypothesis of Theorem* B, *we have*

(i) *In case* (i) *of Theorem* B, *the maximal operator in* (3.2) *is bounded in $L^p(\mathbb{R}^n)$ for all $p > 1$.*
(ii) *In case* (ii) *of Theorem* B *and suppose that there exists a constant $\epsilon > \frac{1}{2}$ satisfying $\sum_{j=1}^N p_j \rho_j^{-\epsilon} < 1$, then the maximal operator* (3.1) *is bounded in $L^p(\mathbb{R}^n)$ for $p \geqslant \frac{2\epsilon+1}{2\epsilon}$.*

Proof. In case (i) of Theorem B, for any $a > 1$, $|\hat{\mu}(x)| \leqslant c|x|^{-a}$ holds for large x. So (3.1) is satisfied for any $a > 1$. Thus $\mathscr{M}$ is bounded in $L^p(\mathbb{R}^n)$ for all $p > \frac{2a+1}{2a}$. Since $\frac{2a+1}{2a} \to 1$ when $a \to +\infty$, it follows that $\mathscr{M}$ is bounded in $L^p(\mathbb{R}^n)$ for all $p > 1$.

In case (ii) of Theorem B, (3.1) is satisfied for $a = \epsilon$, thus $\mathscr{M}$ is bounded in $L^p(\mathbb{R}^n)$ for all $p \geqslant \frac{2\epsilon+1}{2\epsilon}$.

For linear self-similar measure μ in (2.1), take $\mu = v_\lambda,\ N = 2,\ M = 0,\ p_1 = p_2 = \frac{1}{2},\ S_1 x = \lambda x + 1$ and $S_2 x = \lambda x - 1$. we need a result of Peres et al. [11] for Bernoulli convolution. □

Theorem C ([11], Proposition 6.1). *Let $1 < a < b < \infty$. Fix $k \geqslant 2$ and define*

$$r = \frac{1}{2}(b+1)^{-2}, \qquad A = 1 + (b+1)^2.$$

Suppose that

$$B < \frac{-\log[\cos(\pi r)]}{\log b}.$$

Then

$$\dim\{\lambda \in [b^{-1}, a^{-1}] : \hat{v}_\lambda(u) \neq O(u^{-B/k})\} \leqslant \frac{\log[eA^3 k]}{k \log a}.$$

To get a concrete numerical estimate, take $a = 2^{2^{-11}}$, $b = 2^{2^{-10}}$, $k = 34$, then $\frac{\log[eA^3k]}{k\log a} < 1$ and $\frac{-\log[\cos(\pi r)]}{k \log b} > 0.6$. Thus Theorem C implies

$$\dim\{\lambda \in [2^{2^{-10}}, 2^{2^{-11}}] : \hat{v}_\lambda(u) \neq O(u^{-0.6})\} < 1.$$

So the Lebesgue measure of λ in $[2^{2^{-10}}, 2^{2^{-11}}]$ such that $\hat{v}_\lambda(x) = O(x^{-0.6})$ is positive. By Theorem A, we have

Theorem 3.2. *The maximal operators $\mathscr{M}$ of μ is bounded in $L^p(\mathbb{R}^n)$ for all $p \geqslant \frac{2\times 0.6+1}{2\times 0.6} \approx 1.8$.*

Acknowledgments

This work was finished during the period when the authors visited the Morningside Center of Mathematics, Chinese Academy of Sciences. The authors thank Prof. Zhiying Wen for inviting them to visit the center. They would also wish to thank Prof. Lifeng Xi for his suggestion to consider this problem and thank Prof. Jacques Peyriére for his helpful discussions.

References

[1] Bourgain J. Averages in the plane over convex curves and maximal operators. J Anal Math 1986;47:69–85.
[2] Erdös P. On a family of symmetric Bernoulli convolutions. Amer J Math 1939;61:974–6.
[3] Fan AH, Lau K-S. Asymptotic behavior of multiperiodic functions $G(x) = \prod_{n=1}^{\infty} g(x/2^n)$. J Fourier Anal Appl 1998;4:129–50.
[4] Rubio de Francia JL. Maximal functions and Fourier transforms. Duke Math J 1986;53:395–404.
[5] Glickenstein D, Strichartz RS. Nonlinear self-similar measures and their Fourier transforms. Indiana Univ Math J 1996;45:205–20.
[6] Greenleaf A. Principal curvature and harmonic analysis. Indiana Univ Math J 1981;30:519–37.
[7] Hutchinson JE. Fractals and self-similarity. Indiana Univ Math J 1981;30:713–47.
[8] Iosevich A, Sawyer E. Maximal averages over surfaces. Adv Math 1997;132:46–119.
[9] Lau K-S. Fractal measures and mean p-variations. J Funct Anal 1992;108:427–57.
[10] Lau K-S, Wang JR. Mean quadratic variations and Fourier asymptotics of self-similar measures. Monatsh Math 1993;115:99–132.
[11] Peres Y, Schlag W, Solomyak B. Sixty years of Bernoulli convolutions. Fractal geometry and stochastics II. In: Bandt C, Graf S, Zähle M, editors. Progress in probability, 46. Birkhäuser; 2000. p. 39–65.
[12] Sogge CD, Stein EM. Averages over hypersurfaces, Smoothness of generalized Racon transforms. J Analyse Math 1990;54:165–88.
[13] Strichartz RS. Self-similar measures and their Fourier transforms I. Indiana Univ Math J 1990;39:797–817.
[14] Strichartz RS. Self-similar measures and their Fourier transforms II. Trans Amer Math Soc 1993;336:335–61.
[15] Strichartz RS. Self-similar measures and their Fourier transforms III. Indiana Univ Math J 1993;42:367–411.