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Lipschitz property of harmonic function on graphs ☆

Yong Lin [a], Lifeng Xi [b,*]

[a] Department of Mathematics, Information School, Renmin University of China, Beijing, 100872, PR China
[b] Institute of Mathematics, Zhejiang Wanli University, Ningbo, Zhejiang, 315100, PR China

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ABSTRACT

This paper proves the local Lipschitz property for harmonic (or positive subharmonic) functions on graphs. An example is also obtained to show that the global Lipschitz property of harmonic function on graphs does not hold.

1. Introduction

Suppose G is an infinite graph with vertices set V and edges set E. For two distinct vertices x and y, we write $x \sim y$ and say that x and y are neighbors, if there is an edge between x and y. For $x \in V$, let d_x denote the degree of x, i.e., the number of neighbors of x. A path σ, joining vertices x and y, is a sequence $x_0 = x, x_1, \ldots, x_n = y$ of vertices such that $x_i \sim x_{i+1}$ for any i, where $l(\sigma) = n$ is the length of σ. The graph G is said to be connected, if every pair of vertices can be joined by a path.

In this paper we always assume that G is connected and has uniformly bounded vertices degree $d = \sup_{x \in V} d_x < \infty$. In this case, we introduce a distance $\rho : V \times V \to \{0\} \cup \mathbb{N}$ by setting $\rho(x,x) = 0$ and for $x \neq y$, $\rho(x,y) = \inf l(\sigma)$, where infimum is taken over all paths σ joining x and y. For vertex $q \in V$ and real number $R \in \mathbb{N}$, we denote $B_R(q) = \{x \in G\colon\ \rho(x,q) \leqslant R\}$. Let $V^{\mathbb{R}} = \{u \mid u : V \to \mathbb{R}\}$. The Laplace operator Δ of graph G is defined by

$$\Delta u(x) = \frac{1}{d_x} \sum_{y \sim x} \big(u(y) - u(x)\big) \quad \text{for any } u \in V^{\mathbb{R}}.$$

We say the function is harmonic (resp. subharmonic) if $\Delta u = 0$ ($\Delta u \geqslant 0$). The square of absolute value of gradient of u is $\|\nabla u\|^2(x) = \sum_{y \sim x} (u(y) - u(x))^2$. Please refer to [2,3,8] for further details.

The aim of this paper is to study the Lipschitz property of harmonic functions on graphs. In [5], Koskela, Rajala and Shanmugalingam establish the local Lipschitz property of Cheeger-harmonic functions in certain metric spaces that endowed with a doubling measure supporting a $(1,2)$-Poincaré inequality and in addition supporting a corresponding Sobolev–Poincaré type inequality for the modification of the measure obtained via heat kernel. For Alexandrov spaces, Petrunin [6] also proved that the harmonic functions are locally Lipschitz. For Riemannian manifolds whose Ricci curvature is bounded from below, the local Lipschitz property follows from the Cheng–Yau gradient estimate [1].

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* Corresponding author.
E-mail addresses: linyong01@ruc.edu.cn (Y. Lin), xilf@zwu.edu.cn (L. Xi).

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On graphs, Lin and Yau [4] proved that the Ricci curvature in the sense of Bakry and Emery is uniformly bounded below by $-1/2$. Therefore it is expected that the local Lipschitz property holds on graph. In this paper, we prove that

Theorem 1. *Suppose u is a harmonic or positive subharmonic function on G, then for any $\lambda \in \mathbb{N}$, $q \in V$ and any $a, b \in B_\lambda(q)$,*

$$\left|u(a) - u(b)\right| \leqslant \frac{2}{3}\sqrt{d}\left[\frac{1}{\lambda^2}\sum_{z\in B_{6\lambda}(q)} u^2(z)\right]^{1/2} \cdot \rho(a,b), \tag{1.1}$$

where $d = \sup_{x\in V} d_x < \infty$.

Remark 1. If u is harmonic or positive subharmonic with $\sum_{z\in V} u^2(z) < \infty$, then u is constant by (1.1).

The paper is organized as follows. Theorem 1 is proved in Section 2. In Section 3, an example is obtained to illustrate that the global Lipschitz property does not holds for harmonic function on graphs.

2. Local Lipschitz property of harmonic functions

In this section, we will prove Theorem 1. The idea of our proof comes from [7]. Firstly, we introduce the notion of *cut-off function*.

Definition 1. Let $q \in V$ and R, r be integers with $R > r + 1$. A cut-off function φ associated to the region $C_{R,r}(q) = \{x \in V: r \leqslant \rho(x,q) \leqslant R\}$ is a function $\varphi: V \to [0,1]$ satisfying the following requirements:

(1) $\varphi(x) = 1$ when $\rho(x,q) \leqslant r$ and $\varphi(x) = 0$ when $\rho(x,q) \geqslant R$;
(2) For every vertex $x \in V$, $\|\nabla\varphi\|(x) \leqslant \frac{c}{R-r}$ with constant $c > 0$.

Let φ be a cut-off function associated to $C_{R,r}(q)$. Then for any $g \in V^{\mathbb{R}}$,

$$\sum_{B_R(q)}\sum_{y\sim x}\left[g^2(x) + g^2(y)\right]\left[\varphi(y) - \varphi(x)\right]^2 \leqslant 2\sum_{B_R(q)} g^2(x)\sum_{y\sim x}\left[\varphi(y) - \varphi(x)\right]^2 \quad [7], \tag{2.1}$$

$$\sum_{B_R(q)} d_x\varphi^2(x)\Delta g^2(x) = -\frac{1}{2}\sum_{x\in B_R(q)}\sum_{y\sim x}\left(\varphi^2(y) - \varphi^2(x)\right)\left(g^2(y) - g^2(x)\right). \tag{2.2}$$

To obtain (2.2), we notice that $\varphi(y) = 0$ for any y with $\rho(q,y) \geqslant R$, therefore

$$\begin{aligned}
&-\frac{1}{2}\sum_{x\in B_R(q)}\sum_{y\sim x}\left(\varphi^2(y) - \varphi^2(x)\right)\left(g^2(y) - g^2(x)\right)\\
&\quad = \frac{1}{2}\sum_{x\in B_R(q)}\varphi^2(x)\sum_{y\sim x}\left(g^2(y) - g^2(x)\right) - \frac{1}{2}\sum_{y\in B_R(q)}\varphi^2(y)\sum_{x\sim y}\left(g^2(y) - g^2(x)\right)\\
&\quad = \sum_{x\in B_R(q)}\varphi^2(x)\sum_{y\sim x}\left(g^2(y) - g^2(x)\right) = \sum_{B_R(q)} d_x\varphi^2(x)\Delta g^2(x).
\end{aligned}$$

Suppose $u(x)$ is a harmonic or positive subharmonic function on G, then

$$\|\nabla u\|^2(x) \leqslant d_x \cdot \Delta u^2(x). \tag{2.3}$$

To obtain (2.3), we notice that if $u(x)$ is harmonic or positive subharmonic, then

$$u(x)\left[\sum_{y\sim x} u(y)\right] \geqslant d_x \cdot u^2(x),$$

therefore, we have

$$\begin{aligned}
\|\nabla u\|^2(x) &= \sum_{y\sim x}\left[u(y) - u(x)\right]^2\\
&= \sum_{y\sim x} u^2(y) - 2u(x)\sum_{y\sim x} u(y) + \sum_{y\sim x} u^2(x)\\
&\leqslant \sum_{y\sim x} u^2(y) - 2d_x \cdot u^2(x) + d_x \cdot u^2(x) = \sum_{y\sim x}\left[u^2(y) - u^2(x)\right] = d_x \cdot \Delta u^2(x).
\end{aligned}$$

From (2.2) and (2.3), we have

$$\sum_{B_R(q)} \varphi^2(x) \sum_{y\sim x} \big[u(y)-u(x)\big]^2 \leqslant \sum_{B_R(q)} d_x \varphi^2(x) \Delta u^2(x)$$
$$= -\frac{1}{2} \sum_{B_R(q)} \sum_{y\sim x} \big(\varphi^2(y)-\varphi^2(x)\big)\big(u^2(y)-u^2(x)\big). \tag{2.4}$$

We square (2.4) and apply Cauchy–Schwartz inequality to get the following:

$$\begin{aligned}
&\bigg\{ \sum_{B_R(q)} \varphi^2(x) \sum_{y\sim x} \big[u(y)-u(x)\big]^2 \bigg\}^2 \\
&\leqslant \frac{1}{4} \bigg\{ \sum_{x\in C_{R,r}(q)} \sum_{y\sim x} \big(\varphi^2(y)-\varphi^2(x)\big)\big(u^2(y)-u^2(x)\big) \bigg\}^2 \\
&= \frac{1}{4} \bigg\{ \sum_{x\in C_{R,r}(q)} \sum_{y\sim x} \big[\big(\varphi(y)+\varphi(x)\big)\big(u(y)-u(x)\big)\big]\big[\big(\varphi(y)-\varphi(x)\big)\big(u(y)+u(x)\big)\big] \bigg\}^2 \\
&\leqslant \frac{1}{4} \bigg\{ \sum_{x\in C_{R,r}(q)} \Big(\sum_{y\sim x} \big(\varphi(y)+\varphi(x)\big)^2 \big(u(y)-u(x)\big)^2 \Big)^{1/2} \cdot \Big(\sum_{y\sim x} \big(\varphi(y)-\varphi(x)\big)^2 \big(u(y)+u(x)\big)^2 \Big)^{1/2} \bigg\}^2 \\
&\leqslant \bigg\{ \sum_{x\in C_{R,r}(q)} \sum_{y\sim x} \big(\varphi^2(y)+\varphi^2(x)\big)\big(u(y)-u(x)\big)^2 \bigg\} \cdot \bigg\{ \sum_{x\in C_{R,r}(q)} \sum_{y\sim x} \big(u^2(y)+u^2(x)\big)\big(\varphi(y)-\varphi(x)\big)^2 \bigg\} \\
&\leqslant \bigg\{ \sum_{x\in C_{R,r}(q)} \sum_{y\sim x} \big(\varphi^2(y)+\varphi^2(x)\big)\big(u(y)-u(x)\big)^2 \bigg\} \cdot \bigg\{ \sum_{x\in B_R(q)} \sum_{y\sim x} \big(u^2(y)+u^2(x)\big)\big(\varphi(y)-\varphi(x)\big)^2 \bigg\}.
\end{aligned}$$

It follows from (2.1) and (2) of Definition 1 that

$$\begin{aligned}
&\bigg\{ \sum_{B_R(q)} \varphi^2(x) \sum_{y\sim x} \big[u(y)-u(x)\big]^2 \bigg\}^2 \\
&\leqslant \bigg\{ 2 \sum_{x\in B_R(q)} u^2(x) \sum_{y\sim x} \big(\varphi(y)-\varphi(x)\big)^2 \bigg\} \cdot \bigg\{ \sum_{x\in C_{R,r}(q)} \sum_{y\sim x} \big(\varphi^2(y)+\varphi^2(x)\big)\big(u(y)-u(x)\big)^2 \bigg\} \\
&\leqslant \frac{2c^2}{(R-r)^2} \bigg\{ \sum_{x\in B_R(q)} u^2(x) \bigg\} \cdot \bigg\{ \sum_{x\in C_{R,r}(q)} \sum_{y\sim x} \big(\varphi^2(y)+\varphi^2(x)\big)\big(u(y)-u(x)\big)^2 \bigg\},
\end{aligned}$$

where

$$\sum_{x\in C_{R,r}(q)} \sum_{y\sim x} \big(\varphi^2(y)+\varphi^2(x)\big)\big(u(y)-u(x)\big)^2 = \sum_{x\in C_{R,r}(q)} \bigg[\sum_{y\sim x,\, y\in B_R(q)} + \sum_{y\sim x,\, y\notin B_R(q)} \bigg].$$

In fact, for $x \in B_R(q)$ and $y \notin B_R(q)$ with $y \sim x$, we have

$$\rho(x,q) = R, \qquad \rho(y,q) = R+1 \quad \text{and} \quad \varphi(x) = \varphi(y) = 0,$$

which implies $\sum_{y\sim x,\, y\notin B_R(q)} (\varphi^2(y)+\varphi^2(x))(u(y)-u(x))^2 = 0$, i.e.,

$$\begin{aligned}
&\sum_{x\in C_{R,r}(q)} \sum_{y\sim x} \big(\varphi^2(y)+\varphi^2(x)\big)\big(u(y)-u(x)\big)^2 \\
&= \sum_{x\in C_{R,r}(q)} \bigg[\sum_{y\sim x,\, y\in B_R(q)} \big(\varphi^2(y)+\varphi^2(x)\big)\big(u(y)-u(x)\big)^2 \bigg] \\
&\leqslant \sum_{\substack{(x,y)\in B_R(q)\times B_R(q) \\ x\sim y}} \big(\varphi^2(y)+\varphi^2(x)\big)\big(u(y)-u(x)\big)^2 \\
&\leqslant 2 \sum_{x\in B_R(q)} \varphi^2(x) \sum_{y\sim x,\, y\in B_R(q)} \big(u(y)-u(x)\big)^2
\end{aligned}$$

$$\leqslant 2 \sum_{x \in B_R(q)} \varphi^2(x) \sum_{y \sim x} \big(u(y) - u(x)\big)^2.$$

Therefore,

$$\left\{ \sum_{x \in B_R(q)} \varphi^2(x) \sum_{y \sim x} \big(u(y) - u(x)\big)^2 \right\}^2$$
$$\leqslant \frac{4c^2}{(R-r)^2} \left\{ \sum_{x \in B_R(q)} u^2(x) \right\} \cdot \left\{ \sum_{x \in B_R(q)} \varphi^2(x) \sum_{y \sim x} \big(u(y) - u(x)\big)^2 \right\}.$$

We have

$$\sum_{x \in B_R(q)} \varphi^2(x) \sum_{y \sim x} \big(u(y) - u(x)\big)^2 \leqslant \frac{4c^2}{(R-r)^2} \sum_{x \in B_R(q)} u^2(x). \tag{2.5}$$

Let $R = 2r$. Since $\varphi(x) = 1$ when $\rho(x, y) \leqslant r$, (2.5) implies

$$\sum_{x \in B_r(q)} \sum_{y \sim x} \big(u(y) - u(x)\big)^2 \leqslant \frac{4c^2}{r^2} \sum_{z \in B_{2r}(q)} u^2(z). \tag{2.6}$$

It means for any $w \in B_r(q)$,

$$\|\nabla u\|^2(w) \leqslant \frac{4c^2}{r^2} \sum_{z \in B_{2r}(q)} u^2(z). \tag{2.7}$$

Let $R = 2r = 6\lambda$.

Suppose vertices $a, b \in B_\lambda(q)$, then $\rho(a, b) \leqslant 2\lambda$. As a result, there are vertices $w_0 = a,\ w_1, \ldots, w_{\rho(x,y)} = b$ in $B_{3\lambda}(q) = B_r(q)$ such that $w_i \sim w_{i+1}$ for all i. It follows from the gradient estimate (2.7) that

$$\big|u(a) - u(b)\big| \leqslant 2c \left[\frac{1}{(3\lambda)^2} \sum_{z \in B_{6\lambda}(q)} u^2(z) \right]^{1/2} \cdot \rho(a, b), \tag{2.8}$$

for any $a, b \in B_\lambda(q)$. Here, we let

$$\varphi(x) = \begin{cases} 1, & \text{if } \rho(x,q) \leqslant r; \\ \frac{R-\rho(x,q)}{R-r}, & \text{if } r \leqslant \rho(x,q) \leqslant R; \\ 0, & \text{if } \rho(x,q) \geqslant R. \end{cases}$$

Then

$$\|\nabla \varphi\|(x) \leqslant \frac{\sqrt{d}}{R-r}, \tag{2.9}$$

where $d = \sup_{x \in V} d_x < \infty$. That means $c = \sqrt{d}$ in this case, which implies

$$\big|u(a) - u(b)\big| \leqslant \frac{2}{3}\sqrt{d} \left[\frac{1}{\lambda^2} \sum_{z \in B_{6\lambda}(q)} u^2(z) \right]^{1/2} \cdot \rho(a, b), \tag{2.10}$$

for any $a, b \in B_\lambda(q)$. This completes the proof of Theorem 1.

3. Non-global Lipschitz property of harmonic function

This section is devoted to construct a harmonic function which is not Lipschitz.

In the graph (see Fig. 1), there are vertices x, y, $\{x_{i_1 i_2 \cdots i_n}\colon\ n \geqslant 1 \text{ and } i_1 i_2 \cdots i_n \in \{1,2\}^n\}$ and $\{y_{i_1 i_2 \cdots i_n}\colon\ n \geqslant 1$ and $i_1 i_2 \cdots i_n \in \{1,2\}^n\}$, where each vertex has three neighbors.

The vertex x has 3 neighbors y, x_1 and x_2. By induction, for $n \geqslant 1$, $x_{i_1 \cdots i_{n-1} i_n}$ has 3 neighbors $x_{i_1 \cdots i_{n-1}}$ and $x_{i_1 \cdots i_n 1}$ and $x_{i_1 \cdots i_n 2}$, and $y_{i_1 \cdots i_{n-1} i_n}$ has 3 neighbors $y_{i_1 \cdots i_{n-1}}$ and $y_{i_1 \cdots i_n 1}$ and $y_{i_1 \cdots i_n 2}$.

Let $a = (3 + \sqrt{5})/2 > 1$, then

$$(a + 1/a)/3 = 1. \tag{3.1}$$

On this tree, we can define a harmonic function f which is not global Lipschitz (see Fig. 2).

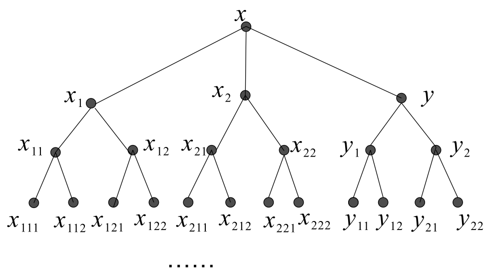

Fig. 1. A tree of degree 3.

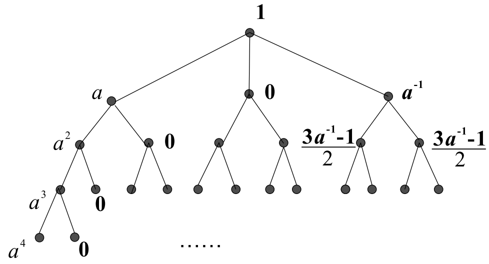

Fig. 2. Harmonic function on the tree.

Let $f(x)=1,\ f(x_1)=a,\ f(x_2)=0,\ f(y)=a^{-1},\ f(y_1)=f(y_2)=(3a^{-1}-1)/2$. For any $n\geqslant 1$, we write $[1]^n=\underbrace{1\cdots 1}_{n}$ and let

$$f(x_{[1]^n})=a^n \quad \text{and} \quad f(x_{[1]^n 2})=0. \tag{3.2}$$

On the other vertices of tree, we obtain values of f by induction, for example, assume that $f(x_{i_1\cdots i_k})$ and $f(x_{i_1\cdots i_{k-1}})$ have been defined, we let

$$f(x_{i_1\cdots i_k 1})=f(x_{i_1\cdots i_k 2})=\frac{3f(x_{i_1\cdots i_k})-f(x_{i_1\cdots i_{k-1}})}{2}; \tag{3.3}$$

similarly, assume that $f(y_{j_1\cdots j_k})$, $f(y_{j_1\cdots j_{k-1}})$ have been defined, we let

$$f(y_{j_1\cdots j_k 1})=f(y_{j_1\cdots j_k 2})=\frac{3f(y_{j_1\cdots j_k})-f(y_{j_1\cdots j_{k-1}})}{2}. \tag{3.4}$$

Here $d_x=3$ for any vertex x. By (3.1)–(3.4), we have

$$3f(x)=\sum_{y\sim x} f(y), \tag{3.5}$$

which implies that f is harmonic on the graph.

We notice that

$$\left|f(x_{[1]^n})-f(x_{[1]^{n-1}})\right|=(a-1)a^{n-1}\to\infty. \tag{3.6}$$

That means f is not a global Lipschitz function.

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