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Existence of positive solutions to some nonlinear equations on locally finite graphs

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Abstract Let $G = (V, E)$ be a locally finite graph, whose measure $\mu(x)$ has positive lower bound, and Δ be the usual graph Laplacian. Applying the mountain-pass theorem due to Ambrosetti and Rabinowitz (1973), we establish existence results for some nonlinear equations, namely $\Delta u + hu = f(x, u)$, $x \in V$. In particular, we prove that if h and f satisfy certain assumptions, then the above-mentioned equation has strictly positive solutions. Also, we consider existence of positive solutions of the perturbed equation $\Delta u + hu = f(x, u) + \epsilon g$. Similar problems have been extensively studied on the Euclidean space as well as on Riemannian manifolds.

Keywords variational method, mountain-pass theorem, semi-linear equation on graphs

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1 Introduction

Let $G = (V, E)$ be a locally finite graph, where V denotes the vertex set and E denotes the edge set. We say that a graph is locally finite if for any $x \in V$, there are only finite y's such that $xy \in E$. For any edge $xy \in E$, we assume that its weight $w_{xy} > 0$ and that $w_{xy} = w_{yx}$. Let $\mu : V \to \mathbb{R}^+$ be a finite measure. For any function $u : V \to \mathbb{R}$, the μ-Laplacian (or Laplacian for short) of u is defined as

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x)).$$

Here and throughout this paper, $y \sim x$ stands for any vertex y with $xy \in E$. The associated gradient form reads

$$\Gamma(u, v)(x) = \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x))(v(y) - v(x)).$$

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Write $\Gamma(u)=\Gamma(u,u)$. We denote the length of its gradient by

$$|\nabla u|(x)=\sqrt{\Gamma(u)(x)}=\left(\frac{1}{2\mu(x)}\sum_{y\sim x}w_{xy}(u(y)-u(x))^2\right)^{1/2}.$$

For any function $g:V\to\mathbb{R}$, an integral of g over V is defined by

$$\int_V g d\mu=\sum_{x\in V}\mu(x)g(x).$$

Let $C_c(V)$ be the set of all functions with compact support, and $W^{1,2}(V)$ be the completion of $C_c(V)$ under the norm

$$\|u\|_{W^{1,2}(V)}=\left(\int_V(|\nabla u|^2+u^2)d\mu\right)^{1/2}.$$

Clearly, $W^{1,2}(V)$ is a Hilbert space with the inner product

$$\langle u,v\rangle=\int_V(\Gamma(u,v)+uv)d\mu,\quad \forall\, u,v\in W^{1,2}(V).$$

Let $h(x)\geqslant h_0>0$ for all $x\in V$. We define a space of functions

$$\mathscr{H}=\left\{u\in W^{1,2}(V):\int_V hu^2d\mu<+\infty\right\} \tag{1.1}$$

with a norm

$$\|u\|_{\mathscr{H}}=\left(\int_V(|\nabla u|^2+hu^2)d\mu\right)^{1/2}. \tag{1.2}$$

Obviously, $\mathscr{H}$ is also a Hilbert space with the inner product

$$\langle u,v\rangle_{\mathscr{H}}=\int_V(\Gamma(u,v)+huv)d\mu,\quad \forall\, u,v\in\mathscr{H}.$$

Let $h:V\to\mathbb{R}$ and $f:V\times\mathbb{R}\to\mathbb{R}$ be two functions. We say that $u:V\to\mathbb{R}$ is a *solution* of the equation

$$-\Delta u+hu=f(x,u) \tag{1.3}$$

if (1.3) holds for all $x\in V$. We shall prove the following.

Theorem 1.1. *Let $G=(V,E)$ be a locally finite graph. Assume that its weight satisfies $w_{xy}=w_{yx}$ for all $y\sim x\in V$, and that its measure $\mu(x)\geqslant\mu_{\min}>0$ for all $x\in V$. Let $h:V\to\mathbb{R}$ be a function satisfying the hypotheses*

(H_1) *there exists a constant $h_0>0$ such that $h(x)\geqslant h_0$ for all $x\in V$;*

(H_2) *$1/h\in L^1(V)$.*

Suppose that $f:V\times\mathbb{R}\to\mathbb{R}$ satisfies the following hypotheses:

(F_1) *$f(x,s)$ is continuous in s, $f(x,0)=0$, and for any fixed $M>0$, there exists a constant A_M such that $\max_{s\in[0,M]}f(x,s)\leqslant A_M$ for all $x\in V$;*

(F_2) *there exists a constant $\theta>2$ such that for all $x\in V$ and $s>0$,*

$$0<\theta F(x,s)=\theta\int_0^s f(x,t)dt\leqslant sf(x,s);$$

(F_3) $\limsup_{s\to0+}\frac{2F(x,s)}{s^2}<\lambda_1=\inf_{\int_V u^2d\mu=1}\int_V(|\nabla u|^2+hu^2)d\mu$.

Then (1.3) *has a strictly positive solution.*

We remark that it is enough to assume that $h(x)>0$ for all $x\in V$ in (H_1), which together with (H_2) implies that there exists a constant $h_0>0$ such that $h(x)\geqslant h_0$ for all $x\in V$.

There are other hypotheses on h and f such that (1.3) has a positive solution. In particular, we shall prove the following.

Theorem 1.2. *Let $G = (V, E)$ be a locally finite graph. Assume that its weight satisfies $w_{xy} = w_{yx}$ for all $y \sim x \in V$, and that its measure $\mu(x) \geqslant \mu_{\min} > 0$ for all $x \in V$. Let $h : V \to \mathbb{R}$ be a function satisfying* (H_1) *and*

(H_2') *$h(x) \to +\infty$ as $\mathrm{dist}(x, x_0) \to +\infty$ for some fixed $x_0 \in V$.*

Suppose that $f : V \times \mathbb{R} \to \mathbb{R}$ satisfies (F_2), (F_3), *and*

(F_1') *$f(x, 0) = 0$, $f(x, s) > 0$ for all $x \in V$ and all $s > 0$, and there exists some constant $L > 0$ such that*

$$|f(x,s) - f(x,t)| \leqslant L|s-t| \quad \text{for all} \quad x \in V$$

for all s, $t \in \mathbb{R}$; then (1.3) *has a strictly positive solution.*

We also consider the perturbation of (1.3), namely

$$-\Delta u + hu = f(x,u) + \epsilon g, \tag{1.4}$$

where $\epsilon > 0$, $g \in \mathscr{H}'$, the dual space of $\mathscr{H}$ defined by (1.1). Concerning this problem, we shall prove the following.

Theorem 1.3. *Let $G = (V, E)$ be a locally finite graph. Assume that its weight satisfies $w_{xy} = w_{yx}$ for all $y \sim x \in V$, and that its measure $\mu(x) \geqslant \mu_{\min} > 0$ for all $x \in V$. Let $h : V \to \mathbb{R}$ be a function satisfying* (H_1) *and* (H_2), *and $f : V \times \mathbb{R} \to \mathbb{R}$ be a function satisfying* (F_1)–(F_3). *Suppose that $g \in \mathscr{H}'$ satisfies $g(x) \geqslant 0$ for all $x \in V$ and $g \not\equiv 0$. Then there exists a constant $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$,* (1.4) *has two distinct strictly positive solutions.*

Theorem 1.4. *Let $G = (V, E)$ be a locally finite graph. Assume that its weight satisfies $w_{xy} = w_{yx}$ for all $y \sim x \in V$, and that its measure $\mu(x) \geqslant \mu_{\min} > 0$ for all $x \in V$. Let $h : V \to \mathbb{R}$ be a function satisfying* (H_1) *and* (H_2'), *and $f : V \times \mathbb{R} \to \mathbb{R}$ be a function satisfying* (F_1'), (F_2), *and* (F_3). *Suppose that $g \in \mathscr{H}'$ satisfies $g(x) \geqslant 0$ for all $x \in V$ and $g \not\equiv 0$. Then there exists a constant $\epsilon_1 > 0$ such that for any $0 < \epsilon < \epsilon_1$,* (1.4) *has two distinct strictly positive solutions.*

This kind of problems has been extensively studied in the Euclidean space; see for examples Alama and Li [4], Adimurthi [1], Adimurthi and Yadava [2], Adimurthi and Yang [3], Alves and Figueiredo [5], Cao [7], de Figueiredo et al. [8, 9], Ding and Ni [10], do Ó [11, 12], do Ó and de Souza [13], do Ó et al. [14], Jeanjean [18], Kryszewski and Szulkin [19], Panda [23], Yang [24, 25], and the references therein. For the Riemannian manifold case, we refer the reader to [15, 26–28].

The method of proving Theorems 1.1–1.4 is to use the critical point theory, in particular, the mountain-pass theorem. Though this idea has been used in the Euclidean space case and Riemannian manifold case, the Sobolev embedding in our setting is quite different from those cases. This let us assume different growth conditions on the nonlinear term $f(x, u)$. Our results closely resemble those of [3, 14, 24–26].

The remaining part of this paper is organized as follows. In Section 2, we prove two Sobolev embedding lemmas. In Sections 3 and 4, we prove Theorems 1.1 and 1.2, respectively. Theorems 1.3 and 1.4 are proved in Section 5. Finally, we give some concluding remarks in Section 6.

2 Sobolev embedding

Let $\mathscr{H}$ be defined by (1.1) and (1.2). To understand the function space $\mathscr{H}$, we have the following compact Sobolev embedding.

Lemma 2.1. *If $\mu(x) \geqslant \mu_{\min} > 0$ and h satisfies (H_1) and (H_2), then $\mathscr{H}$ is weakly pre-compact and $\mathscr{H}$ is compactly embedded into $L^q(V)$ for all $1 \leqslant q \leqslant +\infty$. Namely, if u_k is bounded in $\mathscr{H}$, then up to a subsequence, there exists some $u \in \mathscr{H}$ such that up to a subsequence, $u_k \rightharpoonup u$ weakly in $\mathscr{H}$ and $u_k \to u$ strongly in $L^q(V)$ for any fixed q with $1 \leqslant q \leqslant +\infty$.*

Proof. Suppose $\mu(x) \geqslant \mu_{\min} > 0$. It is easy to see that $W^{1,2}(V) \hookrightarrow L^\infty(V)$ continuously. Hence, interpolation implies that $W^{1,2}(V) \hookrightarrow L^q(V)$ continuously for all $2 \leqslant q \leqslant \infty$. Suppose u_k is bounded in $\mathscr{H}$. Since h satisfies (H_1), it holds that $\mathscr{H} \hookrightarrow W^{1,2}(V)$ continuously. Noting that $W^{1,2}(V)$ is reflexive (every Hilbert space is reflexive), we have up to a subsequence, $u_k \rightharpoonup u$ weakly in $\mathscr{H}$. In particular,

$$\lim_{k\to\infty}\int_V hu_k\varphi d\mu = \int_V hu\varphi d\mu, \quad \forall\, \varphi \in C_c(V).$$

This leads to $\lim_{k\to+\infty} u_k(x) = u(x)$ for any fixed $x \in V$. We now prove $u_k \to u$ in $L^q(V)$ for all $1 \leqslant q \leqslant \infty$. Since u_k is bounded in $\mathscr{H}$ and $u \in \mathscr{H}$, there exists some constant C_1 such that

$$\int_V h(u_k - u)^2 d\mu \leqslant C_1. \tag{2.1}$$

Let $x_0 \in V$ be fixed. For any $\epsilon > 0$, in view of (H_2), there exists some $R > 0$ such that

$$\int_{\mathrm{dist}(x,x_0)>R} \frac{1}{h} d\mu < \epsilon^2.$$

Hence, by Hölder's inequality,

$$\begin{aligned}\int_{\mathrm{dist}(x,x_0)>R} |u_k - u| d\mu &= \int_{\mathrm{dist}(x,x_0)>R} \frac{1}{\sqrt{h}}\sqrt{h}|u_k - u| d\mu \\ &\leqslant \bigg(\int_{\mathrm{dist}(x,x_0)>R} \frac{1}{h} d\mu\bigg)^{1/2}\bigg(\int_{\mathrm{dist}(x,x_0)>R} h|u_k - u|^2 d\mu\bigg)^{1/2} \\ &\leqslant \sqrt{C_1}\epsilon. \end{aligned}\tag{2.2}$$

Moreover, we have that up to a subsequence,

$$\lim_{k\to+\infty}\int_{\mathrm{dist}(x,x_0)\leqslant R} |u_k - u| d\mu = 0. \tag{2.3}$$

Combining (2.2) and (2.3), we conclude

$$\liminf_{k\to+\infty}\int_V |u_k - u| d\mu = 0.$$

In particular, it holds that up to a subsequence, $u_k \to u$ in $L^1(V)$. Since

$$\|u_k - u\|_{L^\infty(V)} \leqslant \frac{1}{\mu_{\min}}\int_V |u_k - u| d\mu,$$

we have for any $1 < q < +\infty$,

$$\int_V |u_k - u|^q d\mu \leqslant \frac{1}{\mu_{\min}^{q-1}}\bigg(\int_V |u_k - u| d\mu\bigg)^q.$$

Therefore, up to a subsequence, $u_k \to u$ in $L^q(V)$ for all $1 \leqslant q \leqslant +\infty$. □

Lemma 2.2. *If $\mu(x) \geqslant \mu_{\min} > 0$ and h satisfies* (H_1) *and* (H_2'), *then $\mathscr{H}$ is weakly pre-compact and $\mathscr{H}$ is compactly embedded into $L^q(V)$ for all $2 \leqslant q \leqslant +\infty$. Namely, if u_k is bounded in $\mathscr{H}$, then up to a subsequence, there exists some $u \in \mathscr{H}$ such that $u_k \rightharpoonup u$ weakly in $\mathscr{H}$ and $u_k \to u$ strongly in $L^q(V)$ for all $2 \leqslant q \leqslant +\infty$.*

Proof. We only stress the difference from Lemma 2.1. By (H_2'), $h(x) \to +\infty$ as $\mathrm{dist}(x, x_0) \to +\infty$, and there exists some $R > 0$ such that

$$h(x) \geqslant \frac{2C_1}{\epsilon} \quad \text{when} \quad \mathrm{dist}(x, x_0) > R.$$

This together with (2.1) gives

$$\int_{\mathrm{dist}(x,x_0)>R} (u_k - u)^2 d\mu \leqslant \frac{\epsilon}{2C_1}\int_{\mathrm{dist}(x,x_0)>R} h(u_k - u)^2 d\mu \leqslant \epsilon.$$

Moreover, it holds that up to a subsequence

$$\int_{\mathrm{dist}(x,x_0)\leqslant R}(u_k-u)^2d\mu\to 0.$$

Hence,

$$\liminf_{k\to+\infty}\int_V|u_k-u|^2d\mu=0.$$

Since the remaining part of the proof is completely analogous to that of Lemma 2.1, we omit the details here. □

3 Proof of Theorem 1.1

3.1 Weak solution

We first define a *weak solution* $u\in\mathscr{H}$ of (1.3). If it holds that

$$\int_V(\Gamma(u,\varphi)+hu\varphi)d\mu=\int_V f(x,u)\varphi d\mu,\quad \forall\,\varphi\in\mathscr{H},$$

then u is called a weak solution of (1.3). Note that $C_c(V)$ is the set of all functions on V with compact support and it is dense in $\mathscr{H}$. If u is a weak solution, then integration by parts gives

$$\int_V(-\Delta u+hu)\varphi d\mu=\int_V f(x,u)\varphi d\mu,\quad \forall\,\varphi\in C_c(V).\tag{3.1}$$

For any fixed $y\in V$, taking a test function $\varphi:V\to\mathbb{R}$ in (3.1) with

$$\varphi(x)=\begin{cases}-\Delta u(y)+h(y)u(y)-f(y,u(y)), & x=y,\\ 0, & x\neq y,\end{cases}$$

we have

$$-\Delta u(y)+h(y)u(y)-f(y,u(y))=0.$$

Since y is arbitrary, we conclude the following.

Proposition 3.1. *If $u\in\mathscr{H}$ is a weak solution of* (1.3)*, then u is also a point-wise solution of* (1.3)*.*

This proposition implies that we can use the variational method to solve (1.3).

3.2 A reduction

For the proof of Theorem 1.1, we shall make the following *reduction*: We can assume $f(x,s)\equiv 0$ for all $s\leqslant 0$. Moreover, we only need to find a nontrivial weak solution of (1.3).

For this purpose, we follow [12, 14] (see also [3, 24, 25]). Let

$$\widetilde{f}(x,s)=\begin{cases}0, & f(x,s)<0,\\ f(x,s), & f(x,s)\geqslant 0.\end{cases}$$

Let $u\in\mathscr{H}$ be a nontrivial weak solution of

$$-\Delta u+hu=\widetilde{f}(x,u)\quad\text{on}\quad V,\tag{3.2}$$

where h satisfies (H_1) and (H_2), and f satisfies (F_1)–(F_3). Here and in the sequel, we say that u is a nontrivial solution if $u\not\equiv 0$. Testing the above equation by the negative part of u, namely $u_-=\min\{u,0\}$, we have

$$\int_V(|\nabla u_-|^2+hu_-^2)d\mu=\int_V u_-\widetilde{f}(x,u)d\mu\leqslant 0.$$

In view of (H_1), we have by the above inequality that $u_- \equiv 0$. We *claim* that $u(x) > 0$ for all $x \in V$. In fact, if $u(x_0) = 0$ for some x_0, then one can see from (3.2) that $u(x) = 0$ for all $x \sim x_0$. Since x_0 is arbitrary, our claim follows immediately. This together with the hypothesis (F_2) leads to $f(x,u) \geqslant 0$. Hence $\widetilde{f}(x,u) = f(x,u)$ and u is a strictly positive solution of (1.3). Therefore, without loss of generality, we can assume $f(x,s) \equiv 0$ for all $s \leqslant 0$ in the proof of Theorem 1.1, and we only need to prove that (1.3) has a nontrivial weak solution.

3.3 Functional framework

We define a functional on $\mathscr{H}$ by

$$J(u) = \frac{1}{2}\int_V (|\nabla u|^2 + hu^2)d\mu - \int_V F(x,u)d\mu, \tag{3.3}$$

where h satisfies (H_1) and (H_2), $F(x,s) = \int_0^s f(x,t)dt$ is the primitive function of f, and f satisfies (F_1)–(F_3). We need to describe the geometry profile of J. Firstly we have the following.

Lemma 3.2. *There exists some nonnegative function $u \in \mathscr{H}$ such that $J(tu) \to -\infty$ as $t \to +\infty$.*

Proof. By (F_2), there exist the positive constants c_1 and c_2 such that $F(x,s) \geqslant c_1 s^\theta - c_2$ for all $(x,s) \in V \times [0,+\infty)$. Let $x_0 \in V$ be fixed. Take a function

$$u(x) = \begin{cases} 1, & x = x_0, \\ 0, & x \neq x_0. \end{cases}$$

Then we have

$$\begin{aligned} J(tu) &= \frac{t^2}{2}\sum_{x\sim x_0} \mu(x)|\nabla u|^2(x) + \frac{t^2}{2}\mu(x_0)h(x_0) - \mu(x_0)F(x_0,t) \\ &\leqslant \frac{t^2}{2}\sum_{x\sim x_0} \mu(x)|\nabla u|^2(x) + \frac{t^2}{2}\mu(x_0)h(x_0) - c_1 t^\theta \mu(x_0) + c_2\mu(x_0) \\ &\to -\infty \end{aligned}$$

as $t \to +\infty$, since $\theta > 2$ and V is locally finite. □

Secondly, we have the following.

Lemma 3.3. *There exist the positive constants δ and r such that $J(u) \geqslant \delta$ for all functions u with $\|u\|_{\mathscr{H}} = r$, where $\|\cdot\|_{\mathscr{H}}$ is defined as in* (1.2).

Proof. By (F_3), there exist the positive constants τ and ϱ such that if $|s| \leqslant \varrho$, then

$$F(x,s) \leqslant \frac{\lambda_1 - \tau}{2}s^2.$$

By (F_2), we have $F(x,s) > 0$ for all $s > 0$. Note also that $F(x,s) \equiv 0$ for all $s \leqslant 0$. It follows that if $|s| \geqslant \varrho$, then

$$F(x,s) \leqslant \frac{1}{\varrho^3}s^3 F(x,s).$$

For all $(x,s) \in V \times \mathbb{R}$, it holds that

$$F(x,s) \leqslant \frac{\lambda_1 - \tau}{2}s^2 + \frac{1}{\varrho^3}s^3 F(x,s).$$

In view of Lemma 2.1, for any function u with $\|u\|_{\mathscr{H}} \leqslant 1$, we have that $\|u\|_{L^\infty(V)} \leqslant C_2\|u\|_{\mathscr{H}}$ and $\|u\|_{L^3(V)} \leqslant C_3\|u\|_{\mathscr{H}}$ for the constants C_2 and C_3, and

$$\int_V u^3 F(x,u)d\mu \leqslant \Big(\max_{(x,s)\in V\times[0,C_2]} F(x,s)\Big)\int_V |u|^3 d\mu \leqslant C_4\|u\|^3_{\mathscr{H}},$$

where (F_1) is employed, and C_4 is some constant depending only on C_2, C_3, and A_{C_2}. Hence, we have for any u with $\|u\|_{\mathscr{H}} \leqslant 1$,

$$\begin{aligned} J(u) &\geqslant \frac{1}{2}\|u\|_{\mathscr{H}}^2 - \frac{\lambda_1-\tau}{2}\int_V u^2 d\mu - \frac{C_4}{\varrho^3}\|u\|_{\mathscr{H}}^3 \\ &\geqslant \left(\frac{1}{2} - \frac{\lambda_1-\tau}{2\lambda_1}\right)\|u\|_{\mathscr{H}}^2 - \frac{C_4}{\varrho^3}\|u\|_{\mathscr{H}}^3 \\ &= \left(\frac{\tau}{2\lambda_1} - \frac{C_4}{\varrho^3}\|u\|_{\mathscr{H}}\right)\|u\|_{\mathscr{H}}^2. \end{aligned}$$

Setting $r = \min\{1, \tau\varrho^3/(4\lambda_1 C_4)\}$, we have $J(u) \geqslant \tau r^2/(4\lambda_1)$ for all u with $\|u\|_{\mathscr{H}} = r$. This completes the proof of the lemma. □

Lemma 3.4. *If h satisfies* (H_1) *and* (H_2)*, f satisfies* (F_1) *and* (F_2)*, then J satisfies the* $(\mathrm{PS})_c$ *(Palais-Smale) condition for any $c \in \mathbb{R}$. Namely, if $(u_k) \subset \mathscr{H}$ is such that $J(u_k) \to c$ and $J'(u_k) \to 0$, then there exists some $u \in \mathscr{H}$ such that up to a subsequence, $u_k \to u$ in $\mathscr{H}$.*

Proof. Note that $J(u_k) \to c$ and $J'(u_k) \to 0$ as $k \to +\infty$ are equivalent to

$$\frac{1}{2}\|u_k\|_{\mathscr{H}}^2 - \int_V F(x, u_k) d\mu = c + o_k(1), \tag{3.4}$$

$$\left|\langle u_k, \varphi\rangle_{\mathscr{H}} - \int_V f(x, u_k)\varphi d\mu\right| = o_k(1)\|\varphi\|_{\mathscr{H}}, \quad \forall\, \varphi \in \mathscr{H}. \tag{3.5}$$

Here and in the sequel, $o_k(1) \to 0$ as $k \to +\infty$. Taking $\varphi = u_k$ in (3.5), we have

$$\|u_k\|_{\mathscr{H}}^2 = \int_V f(x, u_k) u_k d\mu + o_k(1)\|u_k\|_{\mathscr{H}}. \tag{3.6}$$

In view of (F_2), we have by combining (3.4) and (3.6) that

$$\begin{aligned} \|u_k\|_{\mathscr{H}}^2 &= 2\int_V F(x, u_k) d\mu + 2c + o_k(1) \\ &\leqslant \frac{2}{\theta}\int_V f(x, u_k) u_k d\mu + 2c + o_k(1) \\ &= \frac{2}{\theta}\|u_k\|_{\mathscr{H}}^2 + o_k(1)\|u_k\|_{\mathscr{H}} + 2c + o_k(1). \end{aligned}$$

Since $\theta > 2$, u_k is bounded in $\mathscr{H}$. By (H_1) and (H_2), the Sobolev embedding (see Lemma 2.1) implies that up to a subsequence, $u_k \rightharpoonup u$ weakly in $\mathscr{H}$, $u_k \to u$ in $L^q(V)$ for any $1 \leqslant q \leqslant +\infty$. It follows that

$$\left|\int_V f(x, u_k)(u_k - u) d\mu\right| \leqslant C\int_V |u_k - u| d\mu = o_k(1).$$

Replacing φ by $u_k - u$ in (3.5), we have

$$\langle u_k, u_k - u\rangle_{\mathscr{H}} = \int_V f(x, u_k)(u_k - u) d\mu + o_k(1)\|u_k - u\|_{\mathscr{H}} = o_k(1). \tag{3.7}$$

Moreover, since $u_k \rightharpoonup u$ weakly in $\mathscr{H}$, it holds that

$$\langle u, u_k - u\rangle_{\mathscr{H}} = o_k(1).$$

This together with (3.7) leads to $\|u_k - u\|_{\mathscr{H}} = o_k(1)$, or equivalently $u_k \to u$ in $\mathscr{H}$. □

3.4 Completion of the proof of Theorem 1.1

Proof of Theorem 1.1. By Lemmas 3.2–3.4, J satisfies all the hypotheses of the mountain-pass theorem: $J \in C^1(\mathscr{H}, \mathbb{R})$; $J(0) = 0$; $J(u) \geqslant \delta > 0$ when $\|u\|_{\mathscr{H}} = r$; $J(u^*) < 0$ for some $u^* \in \mathscr{H}$ with $\|u^*\|_{\mathscr{H}}$

$> r$; J satisfies the Palais-Smale condition. Using the mountain-pass theorem due to Ambrosetti and Rabinowitz [6], we conclude that

$$c = \min_{\gamma\in\Gamma}\max_{u\in\gamma} J(u)$$

is the critical point of J, where

$$\Gamma = \{\gamma \in C([0,1],\mathscr{H}) : \gamma(0) = 0, \gamma(1) = u^*\}.$$

In particular, there exists some $u \in \mathscr{H}$ such that $J(u) = c$. Clearly, the Euler-Lagrange equation of u is (1.3), or equivalently, u is a weak solution of (1.3). Since

$$J(u) = c \geqslant \delta > 0,$$

we have that $u \not\equiv 0$. Recalling the previous reduction (see Subsection 3.2), we finish the proof of the theorem. □

4 Proof of Theorem 1.2

The proof of Theorem 1.2 is analogous to that of Theorem 1.1. The difference is that (H_2) and (F_1) are replaced by (H_2') and (F_1'), respectively. Let $J : \mathscr{H} \to \mathbb{R}$ be defined by (3.3). The geometry of the functional J is described as below.

Lemma 4.1. *If h satisfies (H_1) and (H_2'), f satisfies (F_1') and (F_2), then J satisfies the $(\mathrm{PS})_c$ condition for any $c \in \mathbb{R}$. Namely, if $(u_k) \subset \mathscr{H}$ is such that $J(u_k) \to c$ and $J'(u_k) \to 0$, then there exists some $u \in \mathscr{H}$ such that up to a subsequence, $u_k \to u$ in $\mathscr{H}$.*

Proof. Similar to the proof of Lemma 3.4, it follows from $J(u_k) \to c$ and $J'(u_k) \to 0$ that (3.4) and (3.5) hold, and u_k is bounded in $\mathscr{H}$. By (H_1) and (H_2'), the Sobolev embedding (see Lemma 2.2) implies that $u_k \rightharpoonup u$ weakly in $\mathscr{H}$, $u_k \to u$ in $L^q(V)$ for any $2 \leqslant q \leqslant +\infty$. By (F_1'), we have

$$|f(x,u_k)| = |f(x,u_k) - f(x,0)| \leqslant L|u_k|.$$

Hence,

$$\begin{aligned}\left|\int_V f(x,u_k)(u_k-u)d\mu\right| &\leqslant L\int_V |u_k(u_k-u)|d\mu \\ &\leqslant L\bigg(\int_V u_k^2 d\mu\bigg)^{1/2}\bigg(\int_V |u_k-u|^2 d\mu\bigg)^{1/2} \\ &= o_k(1).\end{aligned}$$

Taking φ by $u_k - u$ in (3.5), we have

$$\langle u_k, u_k - u\rangle_{\mathscr{H}} = \int_V f(x,u_k)(u_k-u)d\mu + o_k(1)\|u_k-u\|_{\mathscr{H}} = o_k(1). \tag{4.1}$$

On the other hand, we have by $u_k \rightharpoonup u$ weakly in $\mathscr{H}$ that $\langle u, u_k - u\rangle_{\mathscr{H}} = o_k(1)$. This together with (4.1) leads to $u_k \to u$ in $\mathscr{H}$. □

Proof of Theorem 1.2. By Lemmas 3.2, 3.3 and 4.1, J satisfies all the hypotheses of the mountain-pass theorem: $J \in C^1(\mathscr{H},\mathbb{R})$; $J(0) = 0$; $J(u) \geqslant \delta > 0$ when $\|u\|_{\mathscr{H}} = r$; $J(u_1) < 0$ for some $u_1 \in \mathscr{H}$ with $\|u_1\|_{\mathscr{H}} > r$; J satisfies the Palais-Smale condition. Using the mountain-pass theorem due to Ambrosetti and Rabinowitz [6], we conclude that

$$c = \min_{\gamma\in\Gamma}\max_{u\in\gamma} J(u)$$

is the critical point of J, where $\Gamma = \{\gamma \in C([0,1],\mathscr{H}) : \gamma(0) = 0, \gamma(1) = u_1\}$. In particular, (1.3) has a weak solution $u \in \mathscr{H}$. Noting that $J(u) = c \geqslant \delta > 0$, we know that u is nontrivial. In view of the previous reduction (see Subsection 3.2), this completes the proof of the theorem. □

5 Positive solutions of the perturbed equation

In this section, we prove Theorems 1.3 and 1.4. In view of (1.4), when $\epsilon > 0$, $g \geqslant 0$ and $g \not\equiv 0$, similarly as in Section 3.2, we can assume $f(x,s) \equiv 0$ for all $s \in (-\infty, 0]$. Moreover, we only need to find two distinct weak solutions in each case. Indeed if u is a weak solution of (1.4) with $\epsilon > 0$, $g \geqslant 0$ and $g \not\equiv 0$, then obviously $u \not\equiv 0$, and thus the maximum principle implies that u is a strictly positive point-wise solution of (1.4).

5.1 Proof of Theorem 1.3

To prove Theorem 1.3, we define a functional on $\mathscr{H}$ by

$$J_\epsilon(u) = \frac{1}{2}\|u\|_{\mathscr{H}}^2 - \int_V F(x,u)d\mu - \epsilon\int_V gud\mu,$$

where $\epsilon > 0$ and $g \in \mathscr{H}'$. The geometric profile of J_ϵ is described by the following two lemmas.

Lemma 5.1. *For any $\epsilon > 0$, there exists some $u \in \mathscr{H}$ such that $J_\epsilon(tu) \to -\infty$ as $t \to +\infty$.*

Proof. The proof is an obvious analog of the proof of Lemma 3.2. □

Lemma 5.2. *There exists some $\epsilon_1 > 0$ such that if $0 < \epsilon < \epsilon_1$, there exist the constants $r_\epsilon > 0$ and $\delta_\epsilon > 0$ such that $J_\epsilon(u) \geqslant \delta_\epsilon$ for all $u \in \mathscr{H}$ with $\frac{1}{2}r_\epsilon \leqslant \|u\|_{\mathscr{H}} \leqslant 2r_\epsilon$. Furthermore, $r_\epsilon \to 0$ as $\epsilon \to 0$.*

Proof. By (F_2) and (F_3), we can find the positive constants τ and ϱ such that for all $(x,s) \in V \times \mathbb{R}$, it holds that

$$F(x,s) \leqslant \frac{\lambda_1 - \tau}{2}s^2 + \frac{s^3}{\varrho^3}F(x,s).$$

For any $u \in \mathscr{H}$ with $\|u\|_{\mathscr{H}} \leqslant 1$, we have by Lemma 2.1 that $\|u\|_{L^\infty(V)} \leqslant C$ for some constant C, and that there exists another constant (still denoted by C) such that

$$\begin{aligned} J_\epsilon(u) &\geqslant \frac{1}{2}\|u\|_{\mathscr{H}}^2 - \frac{\lambda_1 - \tau}{2\lambda_1}\|u\|_{\mathscr{H}}^2 - C\|u\|_{\mathscr{H}}^3 - \epsilon\|g\|_{\mathscr{H}'}\|u\|_{\mathscr{H}} \\ &= \|u\|_{\mathscr{H}}\left(\frac{\tau}{2\lambda_1}\|u\|_{\mathscr{H}} - C\|u\|_{\mathscr{H}}^2 - \epsilon\|g\|_{\mathscr{H}'}\right). \end{aligned} \tag{5.1}$$

Take

$$r_\epsilon = \sqrt{\epsilon}, \quad \delta_\epsilon = \frac{\tau\epsilon}{16\lambda_1}, \quad \epsilon_1 = \min\left\{\frac{1}{4}, \frac{\tau^2}{64\lambda_1^2(4C + \|g\|_{\mathscr{H}'})^2}\right\}.$$

Then if $0 < \epsilon < \epsilon_1$, we have $J_\epsilon(u) \geqslant \delta_\epsilon$ for all $u \in \mathscr{H}$ with $\frac{1}{2}r_\epsilon \leqslant \|u\|_{\mathscr{H}} \leqslant 2r_\epsilon$. Obviously $r_\epsilon \to 0$ as $\epsilon \to 0$. □

We now prove that J_ϵ satisfies the Palais-Smale condition.

Lemma 5.3. *Let $\epsilon \in \mathbb{R}$ be fixed. If h satisfies $(\mathrm{H}_1)$and (H_2), f satisfies (F_1) and (F_2), then J_ϵ satisfies the $(\mathrm{PS})_c$ condition for any $c \in \mathbb{R}$. Namely, if $(v_k) \subset \mathscr{H}$ is such that $J_\epsilon(v_k) \to c$ and $J_\epsilon'(v_k) \to 0$, then there exists some $v \in \mathscr{H}$ such that $v_k \to v$ in $\mathscr{H}$.*

Proof. Clearly, the hypotheses $J_\epsilon(u_k) \to c$ and $J_\epsilon'(u_k) \to 0$ are equivalent to the following:

$$\frac{1}{2}\int_V(|\nabla v_k|^2 + hv_k^2)d\mu - \int_V F(x,v_k)d\mu - \epsilon\int_V gv_kd\mu \to c \quad \text{as} \quad k \to +\infty, \tag{5.2}$$

$$\left|\int_V(\Gamma(v_k,\varphi) + hv_k\varphi)d\mu - \int_V f(x,v_k)\varphi d\mu - \epsilon\int_V g\varphi d\mu\right| \leqslant \epsilon_k\|\varphi\|_{\mathscr{H}}, \quad \forall\, \varphi \in \mathscr{H}, \tag{5.3}$$

where $\epsilon_k \to 0$ as $k \to +\infty$. Taking $\varphi = v_k$ in (5.3), we have

$$\|v_k\|_{\mathscr{H}}^2 = \int_V f(x,v_k)v_kd\mu + \epsilon\int_V gv_kd\mu + o_k(1)\|v_k\|_{\mathscr{H}}.$$

In view of (F_2), this together with (5.2) leads to

$$\begin{aligned}\|v_k\|^2_{\mathscr{H}} &= 2c + 2\int_V F(x,v_k)d\mu + 2\epsilon\int_V gv_k d\mu + o_k(1)\\ &\leqslant 2c + \frac{2}{\theta}\int_V f(x,v_k)v_k d\mu + 2\epsilon\int_V gv_k d\mu + o_k(1)\\ &= 2c + \frac{2}{\theta}\|v_k\|^2_{\mathscr{H}} + 2\epsilon\left(1-\frac{1}{\theta}\right)\int_V gv_k d\mu + o_k(1)\|v_k\|_{\mathscr{H}} + o_k(1)\\ &\leqslant 2c + \frac{2}{\theta}\|v_k\|^2_{\mathscr{H}} + \left(2\epsilon\left(1-\frac{1}{\theta}\right)\|g\|_{\mathscr{H}'} + o_k(1)\right)\|v_k\|_{\mathscr{H}} + o_k(1).\end{aligned}$$

Since $\theta > 2$, we can see from the above inequality that v_k is bounded in $\mathscr{H}$. By Lemma 2.1, there exists some $v \in \mathscr{H}$ such that up to a subsequence, $v_k \rightharpoonup v$ weakly in $\mathscr{H}$, and $v_k \to v$ strongly in $L^q(V)$ for any $1 \leqslant q \leqslant +\infty$. Taking $\varphi = v_k - v$ in (5.3), we have

$$\langle v_k, v_k - v\rangle_{\mathscr{H}} = \int_V f(x,v_k)(v_k - v)d\mu + \epsilon\int_V g(v_k - v)d\mu + o_k(1)\|v_k - v\|_{\mathscr{H}}. \tag{5.4}$$

Since $v_k \rightharpoonup v$ weakly in $\mathscr{H}$ and $g \in \mathscr{H}'$, it holds that

$$\lim_{k\to+\infty}\int_V g(v_k - v)d\mu = 0. \tag{5.5}$$

In view of (F_1), we can see that $|f(x,v_k)| \leqslant C$ for some constant C since v_k is uniformly bounded. Hence we estimate

$$\left|\int_V f(x,v_k)(v_k - v)d\mu\right| \leqslant C\int_V |v_k - v|d\mu = o_k(1). \tag{5.6}$$

Inserting (5.5) and (5.6) into (5.4), we obtain

$$\langle v_k, v_k - v\rangle_{\mathscr{H}} = o_k(1). \tag{5.7}$$

Moreover, it follows from $v_k \rightharpoonup v$ weakly in $\mathscr{H}$ that $\langle v, v_k - v\rangle_{\mathscr{H}} = o_k(1)$. This together with (5.7) leads to $v_k \to v$ in $\mathscr{H}$, and ends the proof of the lemma. □

For the first weak solution of (1.4), we have the following.

Proposition 5.4. *Let ϵ_1 be given as in Lemma 5.2. When $0 < \epsilon < \epsilon_1$, (1.4) has a mountain-pass type solution u_M verifying that $J_\epsilon(u_M) = c_M$, where $c_M > 0$ is a min-max value of J_ϵ.*

Proof. By Lemmas 5.1–5.3, J_ϵ satisfies all the hypotheses of the mountain-pass theorem: $J_\epsilon \in C^1(\mathscr{H},\mathbb{R})$; $J_\epsilon(0) = 0$; $J_\epsilon(u) \geqslant \delta_\epsilon > 0$ when $\|u\|_{\mathscr{H}} = r_\epsilon$; $J_\epsilon(\tilde{u}) < 0$ for some $\tilde{u} \in \mathscr{H}$ with $\|\tilde{u}\|_{\mathscr{H}} > r_\epsilon$. Using the mountain-pass theorem due to Ambrosetti and Rabinowitz [6], we conclude that

$$c_M = \min_{\gamma\in\Gamma}\max_{u\in\gamma} J_\epsilon(u)$$

is the critical point of J_ϵ, where

$$\Gamma = \{\gamma \in C([0,1],\mathscr{H}) : \gamma(0) = 0, \gamma(1) = \tilde{u}\}.$$

In particular, (1.4) has a weak solution $u_M \in \mathscr{H}$ verifying $J(u) = c_M \geqslant \delta_\epsilon > 0$. □

Lemma 5.5. *Assume h satisfies (H_1)and (H_2), $g \not\equiv 0$ and (F_1) holds. There exist $\tau_0 > 0$ and $v \in \mathscr{H}$ with $\|v\|_{\mathscr{H}} = 1$ such that $J_\epsilon(tv) < 0$ for all $0 < t < \tau_0$. Particularly,*

$$\inf_{\|u\|_{\mathscr{H}}\leqslant\tau_0} J_\epsilon(u) < 0.$$

Proof. We first *claim* that the equation

$$-\Delta v + hv = g \quad \text{in} \quad V \tag{5.8}$$

has a solution $v \in \mathscr{H}$. To see this, we minimize the functional

$$J_g(v) = \frac{1}{2}\int_V (|\nabla v|^2 + hv^2)dv_g - \int_V gvd\mu.$$

For any $v \in \mathscr{H}$, we have

$$\left|\int_V gvd\mu\right| \leqslant \|g\|_{\mathscr{H}'}\|v\|_{\mathscr{H}} \leqslant \frac{1}{4}\|v\|_{\mathscr{H}}^2 + \|g\|_{\mathscr{H}'}^2. \tag{5.9}$$

Hence J_g has a lower bound on $\mathscr{H}$. Denote

$$\lambda_g = \inf_{v\in\mathscr{H}} J_g(v).$$

Take $v_k \in \mathscr{H}$ such that $J_g(v_k) \to \lambda_g$. In view of (5.9), v_k is bounded in $\mathscr{H}$. Then by the Sobolev embedding (see Lemma 2.1), we can find some $v \in \mathscr{H}$ such that $v_k \rightharpoonup v$ weakly in $\mathscr{H}$. Hence,

$$J_g(v) \leqslant \liminf_{k\to+\infty} J_g(v_k) = \lambda_g,$$

and thus v is a minimizer of J_g. The Euler-Lagrange equation of v is exactly (5.8). Since $g \not\equiv 0$, it follows that

$$\int_V gvd\mu = \|v\|_{\mathscr{H}}^2 > 0. \tag{5.10}$$

Secondly, we consider the derivative of $J_\epsilon(tv)$ as follows:

$$\frac{d}{dt}J_\epsilon(tv) = t\|v\|_{\mathscr{H}}^2 - \int_V f(x,tv)vd\mu - \epsilon\int_V gvd\mu. \tag{5.11}$$

Since $f(x,0)=0$, we have by inserting (5.10) into (5.11)

$$\left.\frac{d}{dt}J_\epsilon(tv)\right|_{t=0} < 0.$$

This gives the desired result. □

The second weak solution of (1.4) can be found in the following way.

Proposition 5.6. *Let $\epsilon_1 > 0$ be given as in Lemma 5.2. Let ϵ, $0<\epsilon<\epsilon_1$, be fixed. Then there exists a function $u_0 \in \mathscr{H}$ with $\|u_0\|_{\mathscr{H}} \leqslant 2r_\epsilon$ such that*

$$J_\epsilon(u_0) = c_\epsilon = \inf_{\|u\|_{\mathscr{H}}\leqslant 2r_\epsilon} J_\epsilon(u),$$

where r_ϵ is given as in Lemma 5.2, and $c_\epsilon < 0$. Moreover, u_0 is a strictly positive solution of (1.4).

Proof. Let ϵ, $0<\epsilon<\epsilon_1$, be fixed. In view of (5.1), J_ϵ has a lower bound on the set

$$\mathcal{B}_{2r_\epsilon} = \{u \in \mathscr{H} : \|u\|_{\mathscr{H}} \leqslant 2r_\epsilon\}.$$

This together with Lemma 5.5 implies that

$$c_\epsilon = \inf_{\|u\|_{\mathscr{H}}\leqslant 2r_\epsilon} J_\epsilon(u) < 0.$$

Take a sequence of functions $(u_k) \subset \mathscr{H}$ such that $\|u_k\|_{\mathscr{H}} \leqslant 2r_\epsilon$ and $J_\epsilon(u_k) \to c_\epsilon$ as $k\to+\infty$. It follows from Lemma 2.1 that up to a subsequence, $u_k \rightharpoonup u_0$ weakly in $\mathscr{H}$ and $u_k \to u_0$ strongly in $L^q(V)$ for all $1 \leqslant q \leqslant +\infty$. In view of (F_1), there exists some constant C such that

$$|F(x,u_k) - F(x,u_0)| \leqslant C|u_k - u_0|,$$

which leads to

$$\lim_{k\to+\infty}\int_V F(x,u_k)d\mu = \int_V F(x,u_0)d\mu. \tag{5.12}$$

Since $u_k \rightharpoonup u_0$ weakly in $\mathscr{H}$, we obtain

$$\|u_0\|_{\mathscr{H}} \leqslant \limsup_{k\to+\infty} \|u_k\|_{\mathscr{H}} \tag{5.13}$$

and

$$\lim_{k\to+\infty} \int_V g u_k d\mu = \int_V g u_0 d\mu. \tag{5.14}$$

Combining (5.12)–(5.14), we obtain $\|u_0\|_{\mathscr{H}} \leqslant 2r_\epsilon$ and

$$J_\epsilon(u_0) \leqslant \limsup_{k\to+\infty} J_\epsilon(u_k) = c_\epsilon.$$

Therefore u_0 is the minimizer of J_ϵ on the set $\mathcal{B}_{2r_\epsilon}$. By Lemma 5.2, we conclude that

$$\|u_0\|_{\mathscr{H}} < r_\epsilon/2.$$

For any fixed $\varphi \in C_c(V)$, we define a smooth function $\zeta : \mathbb{R} \to \mathbb{R}$ by

$$\zeta(t) = J_\epsilon(u_0 + t\varphi).$$

Clearly, there exists a sufficiently small $\tau_1 > 0$ such that $u_0 + t\varphi \in \mathcal{B}_{2r_\epsilon}$ for all $t \in (-\tau_1, \tau_1)$. Hence $\zeta(0) = \min_{t\in(-\tau_1,\tau_1)} \zeta(t)$, and thus $\zeta'(0) = 0$, namely

$$\int_V (\Gamma(u_0, \varphi) + h u_0 \varphi) d\mu - \int_V f(x, u_0)\varphi d\mu - \epsilon \int_V g\varphi d\mu = 0.$$

This implies that u_0 is a weak solution of (1.4). This completes the proof of the proposition. □

Completion of the proof of Theorem 1.3. Let u_M and u_0 be two solutions of (1.4) given as in Propositions 5.4 and 5.6, respectively. Noting that $J_\epsilon(u_M) = c_M > 0$ and $J_\epsilon(u_0) = c_\epsilon < 0$, we finish the proof of Theorem 1.3. □

5.2 Proof of Theorem 1.4

Proof of Theorem 1.4. The proof is completely analogous to that of Theorem 1.3. We only stress their essential differences. During the process of finding the mountain-pass type solution, we use Lemma 2.2 instead of Lemma 2.1, and use (H_1), (H_2'), (F_1') and (F_2) to prove that J_ϵ satisfies the Palais-Smale condition. We only need to concern (5.6): By (F_1'), we have

$$|f(x, u_k)| = |f(x, u_k) - f(x, 0)| \leqslant L|u_k|,$$

which together with Hölder's inequality implies that

$$\left| \int_V f(x, u_k)(u_k - u) d\mu \right| \leqslant L\left(\int_V u_k^2 d\mu \right)^{1/2} \left(\int_V |u_k - u|^2 d\mu \right)^{1/2} = o_k(1).$$

While during the process of finding the solution of negative energy, we need to prove (5.12) by (F_1') instead of (F_1), namely

$$\begin{aligned}\left| \int_V (F(x, u_k) - F(x, u_0)) d\mu \right| &\leqslant L \int_V |u_k - u_0| \max\{|u_k|, |u_0|\} d\mu \\ &\leqslant L\left(\int_V (u_k^2 + u_0^2) d\mu \right)^{1/2} \left(\int_V |u_k - u_0|^2 d\mu \right)^{1/2} \\ &= o_k(1).\end{aligned}$$

We omit the details, but leave it to the interested readers. □

6 Concluding remarks

For a finite graph, since all Sobolev spaces are pre-compact, the variational method can be easily applied to existence of partial differential equations (see our recent works [16,17]). However, the situation becomes very subtle for infinite graphs, in which case there is no appropriate Sobolev embedding generally. This paper is an initiation in this regard. To inspire the interested readers, we give several remarks in the following.

Remark 6.1. The assumption $\mu(x) \geqslant \mu_{\min} > 0$ for all $x \in V$ is very strong since it implies that

$$W^{1,2}(V) \hookrightarrow L^{\infty}(V).$$

It is interesting to consider the general case $\mu(x) > 0$ for all $x \in V$. In the later case, existence results would severely depend on the Sobolev embedding

$$W^{1,2}(V) \hookrightarrow L^{p}(V)$$

for some $p > 2$. So far, we know little in this direction.

Remark 6.2. The hypothesis that h satisfies (H_1) and (H_2) (or $(\mathrm{H}_2'))$ guarantees that $E \hookrightarrow L^q(V)$ compactly for all $1 \leqslant q \leqslant \infty$. One can certainly choose other hypotheses instead of (H_1) and (H_2) (or (H_2')).

Remark 6.3. Let $G = (V, E)$ be a locally finite graph and $p > 2$. One can also consider the existence of nontrivial solutions of

$$-\Delta_p u + h|u|^{p-2}u = f(x,u) \quad \text{in} \quad V,$$

where Δ_p is the p-Laplacian operator defined by

$$\Delta_p u(x) = \frac{1}{2\mu(x)} \sum_{y \sim x} (|\nabla u|^{p-2}(y) + |\nabla u|^{p-2}(x)) w_{xy}(u(y) - u(x)).$$

Remark 6.4. It is possible to obtain the existence result for the elliptic system on a locally finite graph. This problem has been well understood in the Euclidean case. For references, we refer the reader to [20–22].

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