Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2018,**38B**(3):843–856
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

BLOW-UP PROBLEMS FOR NONLINEAR PARABOLIC EQUATIONS ON LOCALLY FINITE GRAPHS*

Yong LIN (林勇) *Yiting WU* (吴艺婷)†
Department of Mathematics, Renmin University of China, Beijing 100872, China
E-mail: *linyong01@ruc.edu.cn; yitingly@126.com*

Abstract Let $G = (V, E)$ be a locally finite connected weighted graph, and Δ be the usual graph Laplacian. In this article, we study blow-up problems for the nonlinear parabolic equation $u_t = \Delta u + f(u)$ on G. The blow-up phenomenons for $u_t = \Delta u + f(u)$ are discussed in terms of two cases: (i) an initial condition is given; (ii) a Dirichlet boundary condition is given. We prove that if f satisfies appropriate conditions, then the corresponding solutions will blow up in a finite time.

Key words Blow-up; parabolic equations; locally finite graphs; differential inequalities

2010 MR Subject Classification 35B44; 35K55; 35R02; 26D10

1 Introduction and Main Results

As is known to us, many structures in our real life can be represented by a connected graph whose vertices represent nodes, and whose edges represent their links, such as the internet, brain, organizations, and so on. In recent years, the investigations of discrete weighted Laplacians and various equations on graphs have attracted attention from many authors (see [1, 4, 5, 9–12, 15, 16, 18, 20] and references therein). There have been some works on dealing with blow-up phenomenons of equations on graphs; for example, Xin et al [20] investigated the blow-up properties of the Dirichlet boundary value problem for $u_t = \Delta u + u^p$ $(p > 0)$ on a finite graph. However, as far as we know, the blow-up phenomenon on a locally finite graph has not been studied in the literature. The main concern of this article is to discuss blow-up phenomenons for the nonlinear parabolic equation $u_t = \Delta u + f(u)$ on a locally finite graph. This equation is the mathematical model of heat diffusion and can be used to model solid fuel ignition [2]. The function $f(u)$ is typically a nonlinear function, such as u^p $(p > 1)$. The main purpose of this article is to study blow-up phenomenons for the nonlinear parabolic equation in terms of the following two cases: (i) an initial condition is given; (ii) a Dirichlet boundary condition is given.

*Received April 14, 2017; revised July 30, 2017. The first author is supported by the National Science Foundation of China (11671401); the second author is supported by the Fundamental Research Funds for the Central Universities, and the Research Funds of Renmin University of China (17XNH106).

†Corresponding author

Let $G=(V,E)$ be a locally finite connected graph, where V denotes the vertex set of G and E denotes the edge set of G. For any $T>0$, a function $u=u(t,x)$ is said to be a solution of (1.1) (or (1.2)) in $[0,T]\times V$, if equation (1.1) (or (1.2)) is satisfied by u in $[0,T]\times V$, meanwhile, u is bounded and continuous with respect to t in $[0,T]\times V$. A solution u of (1.1) (or (1.2)) in $[0,+\infty)\times V$ is a function whose restriction to $[0,T]\times V$ is a solution of (1.1) (or (1.2)) in $[0,T]\times V$ for any $T>0$. Moreover, we say that a solution u blows up in a finite time T, if there exists $x\in V$ such that

$$|u(t,x)|\to+\infty \quad \text{as} \quad t\to T^{-}.$$

Focussing on the research goals mentioned above, in this article, we first deal with the blow-up phenomenon of the following Cauchy problem on G

$$\begin{cases}\dfrac{\partial}{\partial t}u(t,x)=\Delta u(t,x)+f(u(t,x)), & (t,x)\in(0,+\infty)\times V,\\ u(0,x)=a(x), & x\in V,\end{cases} \tag{1.1}$$

where Δ is μ-Laplacian on G.

Given a non-empty finite subset $\Omega\subset V$, the boundary of Ω is defined by

$$\partial\Omega:=\{x\in\Omega:\ \exists\ y\in\Omega^{c} \text{ such that } y \text{ is adjacent to } x\}$$

and the interior of Ω is defined by $\Omega^{\circ}:=\Omega\backslash\partial\Omega$. In this article, we assume that Ω° is non-empty.

Next, we consider the blow-up phenomenon arising from the following discrete nonlinear parabolic equations on G

$$\begin{cases}\dfrac{\partial}{\partial t}u(t,x)=\Delta_{\Omega}u(t,x)+f(u(t,x)), & (t,x)\in(0,+\infty)\times\Omega^{\circ},\\ u(0,x)=a(x), & x\in\Omega^{\circ},\\ u(t,x)=0, & (t,x)\in[0,+\infty)\times\partial\Omega,\end{cases} \tag{1.2}$$

where Δ_{Ω} is Dirichlet Laplacian on Ω°.

The earlier blow-up results on parabolic equations on $\mathbb{R}$ are due to Kaplan [13] and Fujita [6, 7]. For the finite time blow-up, Osgood [17] gave a criterion, namely, the nonlinear term on the right-hand side of equation $u_t=\Delta u+f(u)$ must satisfy

$$\int_r^{\infty}\frac{\mathrm{d}\tau}{f(\tau)}<\infty \quad \text{for some } r>0.$$

In this article, we consider blow-up problems for the nonlinear parabolic equation $u_t=\Delta u+f(u)$ on a locally finite graph G. We establish our results under assumptions that f satisfies the following properties:

(H1) f is continuous in $[0,+\infty)$;

(H2) $f(0)=0$ and $f(\tau)>0$ for all $\tau>0$;

(H3) f is convex in $[0,+\infty)$;

(H4) $\int_r^{\infty}\frac{\mathrm{d}\tau}{f(\tau)}<\infty$ for some $r>0$.

Our main results are stated in Theorems 1.1 and 1.2 below.

Theorem 1.1 Let G be a locally finite connected graph and have polynomial volume growth of degree $m>0$. Suppose that f satisfies the assumptions (H1)–(H4); $a(x)$ given by

(1.1) is bounded, non-negative, and not trivial in V. Set

$$F(r)=\int_r^{+\infty}\frac{\mathrm{d}\tau}{f(\tau)}.$$

If there exists a real number $\theta\in(0,1)$ such that

$$F(t^{-1})\le t^{\frac{\theta}{m}} \tag{1.3}$$

for sufficiently large t, then the non-negative solution u of (1.1) blows up in a finite time.

Theorem 1.2 Let G be a locally finite connected graph. Suppose that f satisfies the assumptions (H1)–(H4) with $f\in C^1(\mathbb{R})$, $a(x)$ given by (1.2) is bounded, non-negative, and not trivial in Ω°. If $f(\tau)-\lambda_1\tau>0$ for $\tau\ge\kappa$, then the solution u of (1.2) blows up in a finite time, where $\kappa=\sum\limits_{x\in\Omega^\circ}\mu(x)a(x)\phi_1(x)$, $\lambda_1=\lambda_1(\Omega)$ is the smallest eigenvalue of $-\Delta_\Omega$, and $\phi_1(x)$ is the eigenfunction corresponding to $\lambda_1(\Omega)$.

Remark 1.3 In particular, if we choose $f(u)=u^{1+\alpha}$ $(\alpha>0)$ in Theorem 1.1, then

$$F(u)=\int_u^{+\infty}\frac{\mathrm{d}\tau}{\tau^{1+\alpha}}=\frac{1}{\alpha}u^{-\alpha}.$$

Theorem 1.1 shows that the solution of (1.1) blows up if the following condition (*c1*) is satisfied.

(*c1*). There exists a real number $\theta\in(0,1)$ such that for sufficiently large t,

$$\frac{1}{\alpha}t^{\alpha}\le t^{\frac{\theta}{m}},\quad \text{that is,}\quad t^{\frac{m\alpha-\theta}{m}}\le\alpha,$$

where m and α are positive constants.

It is easy to find that for sufficiently large t and positive constants m, α, θ, we have

$$(c1)\Longleftrightarrow\begin{cases}\exists\,\theta\in(0,1)\text{ such that }0<m\alpha<\theta, & \text{for }\alpha<1\\ \exists\,\theta\in(0,1)\text{ such that }0<m\alpha\le\theta, & \text{for }\alpha\ge1\end{cases}$$
$$\Longleftrightarrow 0<m\alpha<1.$$

The assertion in Remark 1.3 leads to the following result, which was obtained by Lin and Wu in an earlier article [15].

Corollary 1.4 Let G be a locally finite connected graph and have polynomial volume growth of degree $m>0$. If $0<m\alpha<1$, then the solution of the following semilinear heat equation

$$\begin{cases}u_t=\Delta u+u^{1+\alpha} & \text{in }(0,+\infty)\times V,\\ u(0,x)=a(x) & \text{in }V\end{cases}$$

blows up for any bounded, non-negative, and non-trivial initial value, where $\alpha>0$.

Remark 1.5 Let $f(u)=u^{1+\alpha}$ $(\alpha>0)$, then Theorem 1.2 shows that the solution of (1.2) blows up if

$$\sum_{x\in\Omega^\circ}\mu(x)a(x)\phi_1(x)\ge\lambda_1^{\frac{1}{\alpha}}.$$

The remaining parts of this article are organized as follows. In Section 2, we introduce some concepts, notations, and known results which are essential to prove the main results of this article. In Sections 3 and 4, we give the proofs of Theorems 1.1 and 1.2, respectively.

2 Preliminaries

Let $G = (V, E)$ denote a locally finite connected graph. In this article, we consider weighted graphs, that is, we allow the edges and vertices on G to be weighted. Let $\omega : V \times V \to [0, \infty)$ be an edge weight function satisfying $\omega_{xy} = \omega_{yx}$ for all $x, y \in V$ and $\omega_{xy} > 0$ if and only if x is adjacent to y (also denoted by $x \sim y$). Furthermore, let $\mu : V \to (0, \infty)$ be a positive weight function on vertices of G and satisfy $\mu_0 := \inf_{x\in V} \mu(x) > 0$. In particular, all the graphs in our concern are assumed to satisfy

$$D_\mu := \sup_{x\in V} \frac{m(x)}{\mu(x)} < \infty,$$

where $m(x) := \sum_{y\sim x} \omega_{xy}$.

2.1 The Laplacian on graphs

A function on a graph is understood as a function defined on its vertex set. We use the notation $C(V)$ to denote the set of real functions on V. For any $1 \le p < \infty$, we denote by

$$\ell^p(V, \mu) = \left\{ h \in C(V) : \sum_{x\in V} \mu(x)|h(x)|^p < \infty \right\}$$

the set of ℓ^p integrable functions on V with respect to μ. For $p = \infty$, let

$$\ell^\infty(V, \mu) = \left\{ h \in C(V) : \sup_{x\in V} |h(x)| < \infty \right\}.$$

The integral of a function $h \in \ell^1(V, \mu)$ is defined by

$$\int_V h\mathrm{d}\mu = \sum_{x\in V} \mu(x)h(x).$$

For any function $h \in C(V)$, the μ-Laplacian Δ of h is defined by

$$\Delta h(x) = \frac{1}{\mu(x)} \sum_{y\sim x} \omega_{xy}(h(y) - h(x)).$$

It can be checked that $D_\mu < \infty$ is equivalent to the μ-Laplacian Δ being bounded on $\ell^p(V, \mu)$ for all $p \in [1, \infty]$ (see [12]).

Given a finite subset $\Omega \subset V$, we denote by $C(\Omega^\circ)$ the set of real functions on Ω°. For any function $h \in C(\Omega^\circ)$, the Dirichlet Laplacian Δ_Ω on Ω° is defined as follows: First we extend h to the whole V by setting $h \equiv 0$ outside Ω° and then set

$$\Delta_\Omega h = (\Delta h)|_{\Omega^\circ}.$$

Thus, for any $x \in \Omega^\circ$, we have

$$\Delta_\Omega h(x) = \frac{1}{\mu(x)} \sum_{y\sim x} \omega_{xy}(h(y) - h(x)),$$

where $h(y) = 0$ whenever $y \notin \Omega^\circ$. A simple calculation shows that $-\Delta_\Omega$ is a positive self-adjoint operator (see [8, 18]). We arrange the eigenvalues of $-\Delta_\Omega$ in increasing order, that is, $\lambda_1(\Omega) \le \lambda_2(\Omega) \le \cdots \le \lambda_N(\Omega)$, where $N = \#\Omega$. It is well-known that $\lambda_1(\Omega)$ is positive and its corresponding eigenfunctions $\phi_1(x)$ can be chosen as $\phi_1(x) > 0$ for all $x \in \Omega^\circ$ (see [3, 4, 8]). Furthermore, $\phi_1(x)$ can be normalized as $\sum_{x\in\Omega^\circ} \mu(x)\phi_1(x) = 1$.

2.2 The volume growth of graphs

A connected graph can be endowed with its graph distance $d(x, y)$, that is, the smallest number of edges of a path between two vertices x and y. For any $r \geq 0$, we define balls $B(x, r) = \{y \in V : \mathrm{d}(x, y) \leq r\}$. The volume of a subset U of V is denoted by $V(U)$, where $V(U) = \sum\limits_{x \in U} \mu(x)$. For convenience, we usually abbreviate $V\big(B(x, r)\big)$ as $V(x, r)$. In addition, we say that a graph has polynomial volume growth of degree $m > 0$, if there is a constant $c_0 > 0$, such that for all $x \in V$, $r \geq 0$,

$$V(x, r) \leq c_0 r^m.$$

2.3 The heat kernel on graphs

We say that a function $p : (0, +\infty) \times V \times V \to \mathbb{R}$ is a fundamental solution of the heat equation $u_t = \Delta u$ on G, if for any bounded initial condition $u_0 : V \to \mathbb{R}$, the function

$$u(t, x) = \sum_{y \in V} \mu(y) p(t, x, y) u_0(y) \quad (t > 0,\ x \in V)$$

is differentiable in t, satisfies the heat equation, and for any $x \in V$, $\lim\limits_{t \to 0^+} u(t, x) = u_0(x)$ holds.

For completeness, we recall some important properties of the heat kernel $p(t, x, y)$ on G as follows:

Proposition 2.1 (see [18, 19]) For $t, s > 0$ and any $x, y \in V$, we have

(i) $p(t, x, y) = p(t, y, x)$,

(ii) $p(t, x, y) > 0$,

(iii) $\sum\limits_{y \in V} \mu(y) p(t, x, y) \leq 1$,

(iv) $\partial_t p(t, x, y) = \Delta_x p(t, x, y) = \Delta_y p(t, x, y)$,

(v) $\sum\limits_{z \in V} \mu(z) p(t, x, z) p(s, z, y) = p(t + s, x, y)$.

Lin et al [14] utilized polynomial volume growth condition to obtain an on-diagonal lower estimate of heat kernels on graphs for large time. We recall it bellow.

Proposition 2.2 (see [14]) Assume that G satisfies polynomial volume growth, then for large enough t,

$$p(t, x, x) \geq \frac{1}{4V(x, C_0 t \log t)},$$

where $C_0 > 2e(\sqrt{D_\mu} \vee 1)$.

3 Proof of Theorem 1.1

As a preparation for the proof of Theorem 1.1, we first introduce two lemmas.

Lemma 3.1 Let G be a locally finite connected graph. For any $T > 0$, if g is bounded in V, then for any $x \in V$, $\sum\limits_{y \in V} \mu(y) p(t, x, y) g(y)$ converges uniformly in $(0, T]$.

Proof We begin with recalling a previous result which was obtained in [18]. If Δ is a bounded operator, then we have

$$P_t g(x) = e^{t\Delta} g(x) = \sum_{k=0}^{+\infty} \frac{t^k \Delta^k}{k!} g(x) = \sum_{y \in V} \mu(y) p(t, x, y) g(y). \tag{3.1}$$

Because g is bounded in V, we can assume that $|g(x)| \leq A$ in V, then

$$|\Delta g(x)| = \left| \frac{1}{\mu(x)} \sum_{y\sim x} \omega_{xy}\big(g(y) - g(x)\big) \right| \leq 2D_\mu A.$$

By iteration, we obtain, for any $k \in \mathbb{N}$ and $x \in V$,

$$\left|\Delta^k g(x)\right| \leq 2^k D_\mu^k A.$$

Thus, for any $t \in (0, T]$ and $x \in V$,

$$\left| \frac{t^k \Delta^k}{k!} g(x) \right| \leq \left| \frac{T^k \Delta^k}{k!} g(x) \right| \leq \frac{T^k}{k!} 2^k D_\mu^k A.$$

In view of

$$\sum_{k=0}^{+\infty} \frac{T^k}{k!} 2^k D_\mu^k A = Ae^{2D_\mu T} < \infty,$$

we deduce that $\sum\limits_{k=0}^{+\infty} \frac{t^k \Delta^k}{k!} g(x) = \sum\limits_{y\in V} \mu(y)p(t,x,y)g(y)$ converges uniformly in $(0, T]$.

This completes the proof of Lemma 3.1. □

Lemma 3.2 Let G be a locally finite connected graph. For any $t > 0$ and $x \in V$, if g is bounded in V, then we have

$$\sum_{y\in V} \mu(y)\Delta p(t,x,y)g(y) = \sum_{y\in V} \mu(y)p(t,x,y)\Delta g(y). \tag{3.2}$$

Proof As before, because g is bounded in V, we assume that $|g(x)| \leq A$ in V.

A direct computation yields

$$\begin{aligned}\sum_{y\in V} \mu(y)\Delta p(t,x,y)g(y) &= \sum_{y\in V}\sum_{z\in V} \omega_{yz}\big(p(t,x,z)g(y) - p(t,x,y)g(y)\big)\\ &= \sum_{y\in V}\sum_{z\in V} \omega_{yz}p(t,x,z)g(y) - \sum_{y\in V}\sum_{z\in V} \omega_{yz}p(t,x,y)g(y)\\ &= \sum_{z\in V}\sum_{y\in V} \omega_{yz}p(t,x,y)g(z) - \sum_{y\in V}\sum_{z\in V} \omega_{yz}p(t,x,y)g(y)\\ &= \sum_{y\in V}\sum_{z\in V} \omega_{yz}p(t,x,y)g(z) - \sum_{y\in V}\sum_{z\in V} \omega_{yz}p(t,x,y)g(y)\\ &= \sum_{y\in V} \mu(y)p(t,x,y)\Delta g(y).\end{aligned}$$

Note that the above summations can be exchanged, on account of

$$\sum_{y\in V}\sum_{z\in V} \left|\omega_{yz}p(t,x,y)g(z)\right| \leq \sum_{y\in V} \mu(y)p(t,x,y) \left(\sum_{z\in V} \frac{\omega_{yz}}{\mu(y)} |g(z)| \right) \leq D_\mu A.$$

Thus, Lemma 3.2 is proved. □

Proof of Theorem 1.1 Suppose that there exists a non-negative solution $u = u(t,x)$ of (1.1) in $[0, +\infty) \times V$. Because $a(x)$ is non-negative and not trivial in V, we can assume that $a(\nu) > 0$ with $\nu \in V$.

Taking an arbitrary $T > 0$, we put

$$J_T(s) = \sum_{x\in V} \mu(x)p(T-s,\nu,x)u(s,x) \quad (0 \leq s < T). \tag{3.3}$$

Obviously, J_T is continuous with respect to s. Because u is bounded, according to Lemma 3.1, we know that J_T exists even though G is locally finite.

Firstly, we show that J_T is positive for all $s \in [0, T)$.

Because $u(s, \nu)$ is non-negative in $[0, T)$ and f is non-negative in $[0, +\infty)$, it follows that for all $0 \le s < T$,

$$\frac{\partial u}{\partial s}(s, \nu) - \Delta u(s, \nu) \ge 0. \tag{3.4}$$

Note that

$$\Delta u(s, \nu) = \frac{1}{\mu(\nu)} \sum_{y \sim \nu} \omega_{\nu y} \big(u(s, y) - u(s, \nu)\big) \ge -\frac{1}{\mu(\nu)} \sum_{y \sim \nu} \omega_{\nu y} u(s, \nu) \ge -D_\mu u(s, \nu),$$

then inequality (3.4) gives

$$\frac{\partial u}{\partial s}(s, \nu) \ge -D_\mu u(s, \nu),$$

which, together with $a(\nu) = u(0, \nu) > 0$, yields

$$u(s, \nu) \ge u(0, \nu) \exp(-D_\mu s) > 0, \quad s \in [0, T).$$

Hence, for all $0 \le s < T$, we have

$$\sum_{x \in V} u(s, x) > 0.$$

In view of the fact that $p(T - s, \nu, x)$ is positive, then $J_T(s)$ is positive in $[0, T)$.

Secondly, we shall prove that J_T is differentiable with respect to s and satisfies the following equation

$$\frac{\mathrm{d}}{\mathrm{d}s} J_T(s) = \sum_{x \in V} \mu(x) p(T - s, \nu, x) f(u(s, x)). \tag{3.5}$$

Because u is bounded, by Lemma 3.1, we know that J_T is uniformly convergent. Note also that, f is continuous in $[0, +\infty)$, which and the boundedness of u imply that $f(u)$ is bounded. Utilizing the above arguments and Lemma 3.1, then $\frac{\mathrm{d}}{\mathrm{d}s} J_T$ is also uniformly convergent. Hence, we can exchange the order of summation and derivation, and then we get

$$\frac{\mathrm{d}}{\mathrm{d}s} J_T(s) = \sum_{x \in V} \left(\mu(x) \frac{\partial}{\partial s} p(T - s, \nu, x) u(s, x) + \mu(x) p(T - s, \nu, x) \frac{\partial}{\partial s} u(s, x) \right).$$

From the property of heat kernel and Lemmas 3.1 and 3.2, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} J_T(s) &= \sum_{x \in V} \Big(-\mu(x) \Delta p(T - s, \nu, x) u(s, x) + \mu(x) p(T - s, \nu, x) \Big(\Delta u(s, x) + f(u(s, x)) \Big) \Big) \\ &= -\sum_{x \in V} \mu(x) \Delta p(T - s, \nu, x) u(s, x) + \sum_{x \in V} \mu(x) p(T - s, \nu, x) \Delta u(s, x) \\ &\quad + \sum_{x \in V} \mu(x) p(T - s, \nu, x) f(u(s, x)) \\ &= \sum_{x \in V} \mu(x) p(T - s, \nu, x) f(u(s, x)). \end{aligned}$$

Thirdly, we need to show that

$$\frac{\mathrm{d}}{\mathrm{d}s} J_T \ge f(J_T). \tag{3.6}$$

Suppose that $0 < \sum\limits_{i=1}^{n} k_i \leq 1$ and $k_i \in [0,1)$, then there exists $k^* \in [0,1)$ such that $\sum\limits_{i=1}^{n} k_i + k^* = 1$. Because f is convex in $[0,+\infty)$, using Jensen's inequality, for any $x_1, x_2, \cdots, x_n \in [0,+\infty)$, we have

$$k_1 f(x_1) + k_2 f(x_2) + \cdots + k_n f(x_n) + k^* f(0) \geq f(k_1 x_1 + k_2 x_2 + \cdots + k_n x_n + k^* \cdot 0).$$

In view of $f(0)=0$, we have

$$k_1 f(x_1) + k_2 f(x_2) + \cdots + k_n f(x_n) \geq f(k_1 x_1 + k_2 x_2 + \cdots + k_n x_n).$$

Note that f is continuous, thus, if $\sum\limits_{i=1}^{\infty} k_i f(x_i)$ and $\sum\limits_{i=1}^{\infty} k_i x_i$ converge, we have

$$\sum_{i=1}^{\infty} k_i f(x_i) \geq f\left(\sum_{i=1}^{\infty} k_i x_i\right), \tag{3.7}$$

where $0 < \sum\limits_{i=1}^{\infty} k_i \leq 1 \quad (0 \leq k_i < 1)$.

We have shown that J_T and $\frac{\mathrm{d}}{\mathrm{d}s} J_T$ both are convergent, which together with (3.7) and

$$0 < \sum_{x \in V} \mu(x) p(T-s, \nu, x) \leq 1$$

yields

$$\sum_{x \in V} \mu(x) p(T-s, \nu, x) f(u(s,x)) \geq f\left(\sum_{x \in V} \mu(x) p(T-s, \nu, x) u(s,x)\right).$$

This is the desired inequality (3.6).

Next, we consider the following function:

$$Q(s) = F(J_T(0)) - F(J_T(s)) = \int_{J_T(0)}^{J_T(s)} \frac{1}{f(\tau)} \mathrm{d}\tau \qquad (0 \leq s < T).$$

It is not difficult to find that this function is well-defined, because $J_T(s) > 0$ for any $s \in [0,T)$ and $f(\tau) > 0$ for all $\tau > 0$.

We observe from (3.6) that for any $s \in [0,T)$,

$$Q'(s) = J_T'(s) \frac{1}{f(J_T(s))} \geq 1.$$

Owing to $Q(0)=0$ and using the Mean-value theorem, for any $0 < \varepsilon < T$, we get

$$Q(T-\varepsilon) \geq T - \varepsilon. \tag{3.8}$$

Because $f(\tau) > 0$ for all $\tau > 0$ and $J_T(s) > 0$ for any $s \in [0,T)$, we conclude that $F(J_T(s))$ is positive for all $s \in [0,T)$. Hence, we deduce from inequality (3.8) that

$$F(J_T(0)) > F(J_T(0)) - F(J_T(T-\varepsilon)) = Q(T-\varepsilon) \geq T - \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain

$$F\left(\sum_{x \in V} \mu(x) p(T, \nu, x) a(x)\right) \geq T. \tag{3.9}$$

From the given condition $V(x,r) \leq c_0 r^m$ $(c_0 > 0, r \geq 0, m > 0)$ and Proposition 2.2, we have, for large enough T,

$$p(T, \nu, \nu) \geq \frac{1}{4 c_0 C_0^m} \left(T \log T\right)^{-m}. \tag{3.10}$$

Hence, for sufficiently large T, we have

$$\sum_{x\in V}\mu(x)p(T,\nu,x)a(x)\geq\mu(\nu)p(T,\nu,\nu)a(\nu)\geq\widetilde{C}(T\log T)^{-m},\tag{3.11}$$

where $\widetilde{C}=\frac{\mu(\nu)a(\nu)}{4c_0C_0^m}>0$.

Letting us come back to inequality (3.9), then together with (3.11) and the fact that F is non-increasing, one obtains

$$F\left(\widetilde{C}(T\log T)^{-m}\right)\geq T\tag{3.12}$$

for large enough T.

In contrast, it is easy to observe from the limit

$$\lim_{T\to+\infty}\frac{\widetilde{C}^{-\frac{1}{m}}\log T}{T^{\delta}}=0\qquad(\delta>0)$$

that

$$\widetilde{C}^{-\frac{1}{m}}\log T<T^{\delta}\qquad(\delta>0)$$

for sufficiently large T. So, we can choose a real number $T_1>0$ such that

$$\widetilde{C}^{-\frac{1}{m}}\log T<T^{\frac{1-\theta}{\theta}}\qquad(0<\theta<1)$$

for all $T>T_1$. By $f(\tau)>0$ for $\tau>0$, we find that F is strictly decreasing in $(0,+\infty)$. Thus, we have

$$F\left(\widetilde{C}(T\log T)^{-m}\right)<F\left(T^{-\frac{m}{\theta}}\right)\tag{3.13}$$

for all $T>T_1$.

The given condition (1.3) in Theorem 1.1 shows that there exist real numbers $T_2\in(0,+\infty)$ and $\theta\in(0,1)$ such that for $T>T_2$,

$$F\left(T^{-\frac{m}{\theta}}\right)\leq T.\tag{3.14}$$

Combining (3.13) and (3.14), we obtain

$$F\left(\widetilde{C}(T\log T)^{-m}\right)<T$$

for all $T>\max\{T_1,T_2\}$. However, this contradicts with (3.12).

The proof of Theorem 1.1 is completed. □

4 Proof of Theorem 1.2

To prove Theorem 1.2, we need the following two lemmas.

Lemma 4.1 (Strong maximum principle) Let $G=(V,E)$ be a locally finite connected graph and $\Omega\subset V$ be finite. For any $T>0$, we assume that $v(t,x)$ is bounded and continuous with respect to t in $[0,T]\times\Omega$, which satisfies

$$\begin{cases}\dfrac{\partial}{\partial t}v(t,x)-\Delta_\Omega v(t,x)-k(t,x)v(t,x)\geq0, & (t,x)\in(0,T]\times\Omega^\circ,\\ v(0,x)\geq0, & x\in\Omega^\circ,\\ v(t,x)\geq0, & (t,x)\in[0,T]\times\partial\Omega,\end{cases}\tag{4.1}$$

where $k(t,x)$ is bounded in $(0,T]\times\Omega^\circ$. Then, $v(t,x)\geq0$ in $[0,T]\times\Omega$.

Proof Set $\mathcal{L}v(t,x) = v_t(t,x) - \Delta_\Omega v(t,x) - k(t,x)v(t,x)$. Then, we have $\mathcal{L}v \geq 0$ in $(0,T]\times\Omega^\circ$. Because $k(t,x)$ is bounded in $(0,T]\times\Omega^\circ$, we can assume that there exists a positive number l such that $l > k(t,x)$ for all $(t,x)\in(0,T]\times\Omega^\circ$. Using a transformation $\psi = ve^{-lt}$, we get

$$\begin{aligned}\mathcal{L}v(t,x) &= v_t(t,x) - \Delta_\Omega v(t,x) - k(t,x)v(t,x)\\ &= e^{lt}\left(\psi_t(t,x) - \Delta_\Omega\psi(t,x) - (k(t,x)-l)\psi(t,x)\right)\\ &= e^{lt}\widetilde{\mathcal{L}}\psi(t,x),\end{aligned}$$

where $\widetilde{\mathcal{L}}\psi(t,x) = \psi_t(t,x) - \Delta_\Omega\psi(t,x) - (k(t,x)-l)\psi(t,x)$. Noting that $\mathcal{L}v \geq 0$ in $(0,T]\times\Omega^\circ$, so we have $\widetilde{\mathcal{L}}\psi \geq 0$ in $(0,T]\times\Omega^\circ$.

It is easy to observe that

$$|\psi(t,x)| = |e^{-lt}v(t,x)| \leq |v(t,x)|$$

for any $(t,x)\in[0,T]\times\Omega^\circ$.

Because $v(t,x)$ is bounded and continuous with respect to t, we conclude that $\psi(t,x)$ is also bounded and continuous with respect to t in $[0,T]\times\Omega^\circ$, which implies that $\psi(t,x)$ exists a minimum value in $[0,T]\times\Omega^\circ$.

Let (t_0,x_0) be a minimum point of function ψ in $[0,T]\times\Omega^\circ$. To prove $v(t,x)\geq 0$ in $[0,T]\times\Omega^\circ$, we need to verify that $\psi(t,x)\geq 0$ in $[0,T]\times\Omega^\circ$; it is sufficient to prove $\psi(t_0,x_0)\geq 0$.

In the following, we will prove $\psi(t_0,x_0)\geq 0$ by contradiction. Assume that $\psi(t_0,x_0) < 0$.

Case 1 If $t_0 = 0$, then it follows from (4.1) that $\psi(0,x_0) = v(0,x_0)\geq 0$. This contradicts with $\psi(t_0,x_0) < 0$.

Case 2 If $t_0\in(0,T]$, then we deduce from $k(t_0,x_0) < l$ and $\psi(t_0,x_0) < 0$ that

$$\widetilde{\mathcal{L}}\psi(t_0,x_0) < \psi_t(t_0,x_0) - \Delta_\Omega\psi(t_0,x_0).$$

Because the function $t\mapsto\psi(t,x_0)$ in $(0,T]$ attains its minimum at $t = t_0$, we obtain

$$\psi_t(t_0,x_0)\leq 0 \tag{4.2}$$

(if $t_0 < T$, then $\psi_t(t_0,x_0) = 0$). Hence, we conclude that

$$\widetilde{\mathcal{L}}\psi(t_0,x_0) < -\Delta_\Omega\psi(t_0,x_0). \tag{4.3}$$

On the other hand, the function $x\mapsto\psi(t_0,x)$ in Ω° attains the minimum at $x = x_0$, thus,

$$\Delta_\Omega\psi(t_0,x_0) = \frac{1}{\mu(x_0)}\sum_{y\sim x_0}\omega_{x_0y}\big(\psi(t_0,y)-\psi(t_0,x_0)\big)\geq 0. \tag{4.4}$$

Applying (4.4) to (4.3), we obtain $\widetilde{\mathcal{L}}\psi(t_0,x_0) < 0$, which is a contradiction with $\widetilde{\mathcal{L}}\psi\geq 0$ in $(0,T]\times\Omega^\circ$.

Hence, we have $\psi(t_0,x_0)\geq 0$, which leads to $\psi(t,x)\geq 0$ in $[0,T]\times\Omega^\circ$.

In view of

$$v = e^{lt}\psi, \tag{4.5}$$

we deduce that $v(t,x)\geq 0$ in $[0,T]\times\Omega^\circ$.

In addition, from (4.1) we have $v(t,x)\geq 0$ in $[0,T]\times\partial\Omega$. Thus, we conclude that $v(t,x)\geq 0$ in $[0,T]\times\Omega$.

The proof of Lemma 4.1 is completed. □

Lemma 4.2 (Comparison principle) Let $G=(V,E)$ be a locally finite connected graph and $\Omega\subset V$ be finite. For any $T>0$, we assume that $u(t,x)$ and $\underline{u}(t,x)$ are bounded and continuous with respect to t in $[0,T]\times\Omega$, which satisfy

$$\begin{cases}\dfrac{\partial}{\partial t}u-\Delta_\Omega u-g(u)\geq\dfrac{\partial}{\partial t}\underline{u}-\Delta_\Omega\underline{u}-g(\underline{u}), & (t,x)\in(0,T]\times\Omega^\circ,\\ u(0,x)\geq\underline{u}(0,x), & x\in\Omega^\circ,\\ u(t,x)\geq\underline{u}(t,x), & (t,x)\in[0,T]\times\partial\Omega,\end{cases}\tag{4.6}$$

where $g\in C^1(\mathbb{R})$. Then, $u(t,x)\geq\underline{u}(t,x)$ in $[0,T]\times\Omega$.

Proof Set $v(t,x)=u(t,x)-\underline{u}(t,x)$. Then,

$$(4.6)\Longleftrightarrow\begin{cases}\dfrac{\partial}{\partial t}v(t,x)-\Delta_\Omega v(t,x)-(g(u(t,x))-g(\underline{u}(t,x)))\geq 0, & (t,x)\in(0,T]\times\Omega^\circ,\\ v(0,x)\geq 0, & x\in\Omega^\circ,\\ v(t,x)\geq 0, & (t,x)\in[0,T]\times\partial\Omega.\end{cases}$$

Define a function

$$k(t,x)=\begin{cases}\dfrac{g(u(t,x))-g(\underline{u}(t,x))}{u(t,x)-\underline{u}(t,x)} & \text{for } (t,x) \text{ such that } u(t,x)\neq\underline{u}(t,x),\\ 0 & \text{for } (t,x) \text{ such that } u(t,x)=\underline{u}(t,x).\end{cases}$$

Then,

$$\frac{\partial}{\partial t}v(t,x)-\Delta_\Omega v(t,x)-(g(u(t,x))-g(\underline{u}(t,x)))=\frac{\partial}{\partial t}v(t,x)-\Delta_\Omega v(t,x)-k(t,x)v(t,x).$$

Next, we shall show that $k(t,x)$ is bounded in $(0,T]\times\Omega^\circ$.

Because $u(t,x)$ and $\underline{u}(t,x)$ are bounded in $[0,T]\times\Omega$, there exists a constant $M>0$ such that for any $(t,x)\in[0,T]\times\Omega$,

$$|u(t,x)|\leq M\quad\text{and}\quad|\underline{u}(t,x)|\leq M.$$

Using the Mean-value theorem, we have

$$|k(t,x)|\leq|g'(\zeta)|,\quad\zeta\in(-M,M).$$

In view of $g\in C^1(\mathbb{R})$, we deduce from the above inequality that $k(t,x)$ is bounded in $[0,T]\times\Omega$, which implies that $k(t,x)$ is bounded in $(0,T]\times\Omega^\circ$.

Hence, it follows from Lemma 4.1 that

$$v(t,x)\geq 0 \text{ in } [0,T]\times\Omega\Longleftrightarrow u(t,x)\geq\underline{u}(t,x) \text{ in } [0,T]\times\Omega.$$

Thus, Lemma 4.2 is proved. □

Proof of Theorem 1.2 Suppose that there exists a solution $u=u(t,x)$ of (1.2) in $[0,+\infty)\times V$.

We consider the function

$$J(t)=\sum_{x\in\Omega^\circ}\mu(x)u(t,x)\phi_1(x)\qquad(t\geq 0),\tag{4.7}$$

where ϕ_1 is a eigenfunction corresponding to the smallest eigenvalue $\lambda_1(\Omega)$ (see Section 2). It is clear that $J(0)\equiv\kappa$ and $J(t)$ is continuous with respect to t.

Firstly, we show that $J(t)$ is positive for all $t\in[0,+\infty)$.

For an arbitrary $(t,x)\in[0,+\infty)\times\Omega$, there exists a positive number T such that $(t,x)\in[0,T]\times\Omega$. Consider the special case of Lemma 4.2 when $\underline{u}(t,x)\equiv 0$ and $g=f$, and then the conditions (4.6) stated in Lemma 4.2 become

$$\begin{cases}\dfrac{\partial}{\partial t}u(t,x)-\Delta_\Omega u(t,x)-f(u(t,x))\geq 0, & (t,x)\in(0,T]\times\Omega^\circ,\\ u(0,x)\geq 0, & x\in\Omega^\circ,\\ u(t,x)\geq 0, & (t,x)\in[0,T]\times\partial\Omega.\end{cases}\tag{4.8}$$

By comparing equations (1.2) provided by Theorem 1.2 with (4.8) above, we deduce from Lemma 4.2 that $u(t,x)\geq 0$ in $[0,T]\times\Omega$, which implies that $u(t,x)\geq 0$ for all $(t,x)\in[0,+\infty)\times\Omega$ in view of the arbitrariness of $(t,x)\in[0,+\infty)\times\Omega$.

Set $\Omega_1:=\{x\in\Omega^\circ: a(x)>0\}$. Because $a(x)$ is not trivial in Ω°, we get $\Omega_1\neq\varnothing$. For any $z\in\Omega_1$, because of the fact that f is non-negative in $[0,+\infty)$, we have

$$\frac{\partial u}{\partial t}(t,z)-\Delta_\Omega u(t,z)\geq 0\qquad(t\geq 0).\tag{4.9}$$

A simple calculation shows that

$$\Delta_\Omega u(t,z)\geq-\frac{1}{\mu(z)}\sum_{y\sim z}\omega_{zy}u(t,z)\geq -D_\mu u(t,z).\tag{4.10}$$

Combining (4.9) and (4.10), we verify that

$$\frac{\partial u}{\partial t}(t,z)\geq -D_\mu u(t,z),$$

which implies that

$$u(t,z)\geq a(z)\exp(-D_\mu t)>0\qquad(t\geq 0).$$

Hence, for any $t\geq 0$, we have

$$\sum_{x\in\Omega^\circ}u(t,x)>0.$$

Because $\mu(x)$ and $\phi_1(x)$ are positive in Ω°, we conclude that $J(t)>0$ for all $t\in[0,+\infty)$.

Secondly, we prove that

$$J'(t)\geq-\lambda_1 J(t)+f(J(t)).$$

By using the fact that Δ_Ω is self-adjoint, we have

$$\begin{aligned}J'(t)&=\sum_{x\in\Omega^\circ}\mu(x)u_t(t,x)\phi_1(x)\\&=\sum_{x\in\Omega^\circ}\mu(x)\phi_1(x)\Delta_\Omega u(t,x)+\sum_{x\in\Omega^\circ}\mu(x)\phi_1(x)f(u(t,x))\\&=\sum_{x\in\Omega^\circ}\mu(x)\Delta_\Omega\phi_1(x)u(t,x)+\sum_{x\in\Omega^\circ}\mu(x)\phi_1(x)f(u(t,x))\\&=-\lambda_1\sum_{x\in\Omega^\circ}\mu(x)\phi_1(x)u(t,x)+\sum_{x\in\Omega^\circ}\mu(x)\phi_1(x)f(u(t,x)).\end{aligned}\tag{4.11}$$

Because $\sum\limits_{x\in\Omega^\circ}\mu(x)\phi_1(x)=1$, by Jensen's inequality, we obtain

$$\sum_{x\in\Omega^\circ}\mu(x)\phi_1(x)f(u(t,x))\geq f\left(\sum_{x\in\Omega^\circ}\mu(x)\phi_1(x)u(t,x)\right)=f(J(t)).\tag{4.12}$$

Combining (4.11) with (4.12) gives

$$J'(t) \geq -\lambda_1 J(t) + f(J(t)). \tag{4.13}$$

Thirdly, we will claim that there exists a positive constant $K > \kappa$ such that $f(\tau) > 2\lambda_1\tau$ for any $\tau \geq K$.

We prove this assertion by contradiction. Assume that for any $K > \kappa$, there exists a $\tau^* \geq K$ such that

$$f(\tau^*) \leq 2\lambda_1\tau^*.$$

On the basis of the assumption above, given a fixed $K_0 > \kappa$, there exists a $\tau_0^* \geq K_0$ such that

$$f(\tau_0^*) \leq 2\lambda_1\tau_0^*.$$

In addition, for any $K > \tau_0^* > \kappa$, there exists a $\tau_k^* \geq K$ such that

$$f(\tau_k^*) \leq 2\lambda_1\tau_k^*.$$

Because $f(\tau)$ is convex in $[0, +\infty)$, for any $\tau \in (\tau_0^*, \tau_k^*)$, we have

$$f(\tau) \leq \frac{\tau_k^* - \tau}{\tau_k^* - \tau_0^*} f(\tau_0^*) + \frac{\tau - \tau_0^*}{\tau_k^* - \tau_0^*} f(\tau_k^*) \leq 2\lambda_1\tau_0^* \cdot \frac{\tau_k^* - \tau}{\tau_k^* - \tau_0^*} + 2\lambda_1\tau_k^* \cdot \frac{\tau - \tau_0^*}{\tau_k^* - \tau_0^*} = 2\lambda_1\tau.$$

In view of $\tau_0^* < K \leq \tau_k^*$, we get

$$f(\tau) \leq 2\lambda_1\tau, \quad \tau \in (\tau_0^*, K).$$

Note that K is an arbitrary number that is greater than τ_0^*, we obtain, for all $\tau \in (\tau_0^*, +\infty)$,

$$f(\tau) \leq 2\lambda_1\tau. \tag{4.14}$$

Because f is positive in $(\tau_0^*, +\infty)$, inequality (4.14) implies that

$$\frac{1}{f(\tau)} \geq \frac{1}{2\lambda_1\tau}$$

for all $\tau \in (\tau_0^*, +\infty)$, which contradicts with the fact that $\frac{1}{f}$ is integrable at $\tau \to \infty$. Hence, there exists $K > \kappa$ such that $f(\tau) > 2\lambda_1\tau$ for any $\tau \geq K$.

Now, we are in a position to prove the assertion of Theorem 1.2.

We observe that $f(\tau)$ and $f(\tau) - \lambda_1\tau$ are positive in $[\kappa, +\infty)$, thus,

$$\frac{1}{f(\tau) - \lambda_1\tau} < \frac{2}{f(\tau)} \qquad (\tau \geq K). \tag{4.15}$$

Because $\int_K^{+\infty} \frac{\mathrm{d}\tau}{f(\tau)} < +\infty$, inequality (4.15) implies that

$$\int_K^{+\infty} \frac{\mathrm{d}\tau}{f(\tau) - \lambda_1\tau} < +\infty.$$

This yields

$$\int_{J(0)}^{+\infty} \frac{\mathrm{d}\tau}{f(\tau) - \lambda_1\tau} = \int_{J(0)}^{K} \frac{\mathrm{d}\tau}{f(\tau) - \lambda_1\tau} + \int_K^{+\infty} \frac{\mathrm{d}\tau}{f(\tau) - \lambda_1\tau} < +\infty. \tag{4.16}$$

It is clear that there is no singularity in the above integral, because $J(0) \equiv \kappa > 0$ and $f(\tau)$, $f(\tau) - \lambda_1\tau$ are positive in $[\kappa, +\infty)$.

On the other hand, it follows from (4.13) that

$$\int_{J(0)}^{J(t)} \frac{\mathrm{d}J}{f(J)-\lambda_1 J} \geq t. \tag{4.17}$$

Letting $t \to +\infty$ in (4.17) leads to a contradiction with (4.16), which together with the property of continuation on the positive classical solution yields $J(t) \to +\infty$ as $t \to T^-$.

This completes the proof of Theorem 1.2. □

References

[1] Bauer F, Hua B, Jost J. The dual cheeger constant and spectra of infinite graphs. Adv Math, 2014, **251**(2): 147–194

[2] Bebernes J, Eberly D. Mathematical Problems from Combustion Theory. New York: Springer, 1989

[3] Chung F R K. Spectral Graph Theory, CBMS Reg Conf Ser Math. Providence, RI: Amer Math Soc, 1997

[4] Chung Y S, Lee Y S, Chung S Y. Extinction and positivity of the solutions of the heat equations with absorption on networks. J Math Anal Appl, 2011, **380**(2): 642–652

[5] Dodziuk J, Kendall W S. Combinatorial Laplacians and the isoperimetric inequality//Ellworthy K D. From Local Times to Global Geometry. Control and Physics. Pitman Research Notes in Mathematics Series 150. Essex: Longman Scientific and Technical, 1986: 68–74

[6] Fujita H. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. J Fac Sci Univ Tokyo Sect A Math, 1966, **13**(2): 109–124

[7] Fujita H. On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations. Symp Pure Mathematics, 1969, **18**: 105–113

[8] Grigor'yan A. Analysis on Graphs. University Bielefeld, 2009

[9] Grigor'yan A, Lin Y, Yang Y. Yamabe type equations on graphs. J Differential Equations, 2016, **261**(9): 4924–4943

[10] Grigor'yan A, Lin Y, Yang Y. Kazdan-Warner equation on graph. Calc Var Partial Differential Equations, 2016, **55**(4): 92 (13 pages)

[11] Grigor'yan A, Lin Y, Yang Y. Existence of positive solutions to some nonlinear equations on locally finite graphs. Sci China Math, 2017, **60**(7): 1311–1324

[12] Haeseler S, Keller M, Lenz D, Wojciechowski R. Laplacians on infinite graphs: Dirichlet and Neumann boundary conditions. J Spectr Theory, 2012, **2**(4): 397–432

[13] Kaplan S. On the growth of solutions of quasilinear parabolic equations. Comm Pure Appl Math, 1963, **16**: 305–333

[14] Lin Y, Wu Y. On-diagonal lower estimate of heat kernels on graphs. J Math Anal Appl, 2017, **456**: 1040–1048

[15] Lin Y, Wu Y. The existence and nonexistence of global solutions for a semilinear heat equation on graphs. Calc Var Partial Differential Equations, 2017, **56**(4): 102 (22 pages)

[16] Liu W, Chen K, Yu J. Extinction and asymptotic behavior of solutions for the ω-heat equation on graphs with source and interior absorption. J Math Anal Appl, 2016, **435**(1): 112–132

[17] Osgood W F. Beweis der Existenz einer Lösung der Differentialgleichung $dy/dx = f(x,y)$ ohne Hinzunahme der Cauchy-Lipschitzschen Bedingung. Monatshefte der Mathemratik unid Physik, 1898, **9**(1): 331–345

[18] Weber A. Analysis of the physical Laplacian and the heat flow on a locally finite graph. J Math Anal Appl, 2012, **370**(1): 146–158

[19] Wojciechowski R. Heat kernel and essential spectrum of infinite graphs. Indiana Univ Math J, 2008, **58**(3): 1419–1442

[20] Xin Q, Xu L, Mu C. Blow-up for the ω-heat equation with Dirichelet boundary conditions and a reaction term on graphs. Appl Anal, 2014, **93**(8): 1691–1701