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Ultracontractivity and Functional Inequalities on Infinite Graphs

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Abstract We prove the equivalence between some functional inequalities and the ultracontractivity property of the heat semigroup on infinite graphs. These functional inequalities include Sobolev inequalities, Nash inequalities, Faber–Krahn inequalities, and log-Sobolev inequalities. We also show that, under the assumptions of volume growth and CDE$(n, 0)$, which is regarded as the natural notion of curvature on graphs, these four functional inequalities and the ultracontractivity property of the heat semigroup are all true on graphs.

Keywords Ultracontractivity · Laplacian · Sobolev-type inequalities · CDE(n, K)

Mathematics Subject Classification 53C21 · 58J35

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1 Introduction

One can consider the heat equation associated with the Laplace operator Δ,

$$\partial_t u = \Delta u, \tag{1.1}$$

which leads in general to a smoothing effect under the form of ultracontractivity. This means that, if $u(t, x)$ satisfies (1.1), then there exists $\gamma(t) \to 0$ as $t \to \infty$ such that, for any $x \in V$ and $t > 0$,

$$\|u(t, x)\| \leq \gamma(t)\|u(0, x)\|.$$

One may reformulate this by saying that the semigroup $P_t = \mathrm{e}^{t\Delta}$ satisfies the estimate

$$\|P_t\|_{1\to\infty} \leq \gamma(t). \tag{1.2}$$

It turns out that there is a strong relationship between the geometry of the Laplacian Δ and the smoothing effect of the associated heat equation. The connection is made through functional inequalities, such as the families of Sobolev equalities, Faber–Krahn inequalities, Nash equalities, and log-Sobolev inequalities, in such a way that these inequalities are all equivalent. This has recently been studied extensively, e.g., in [13] on manifolds and [7] on metric spaces. The remarkable smoothing properties under the form of ultracontractivity, due to Varopoulos [20], may be established by different methods: Carlen, Kusuoka, and Stroock used Nash inequalities [4], while Davies and Simon [9] used log-Sobolev inequalities.

In the works of many authors such as Varopoulos, Grigor'yan, Bakry–Coulhon–Ledoux–Saloff–Coste, it is shown that the discrete forms of Sobolev inequalities, Nash equalities, Faber–Krahn inequalities, and discrete-time uniform bounds of the heat kernel are all equivalent on graphs (see [6]). However, on graphs, there are no related results involving log-Sobolev inequalities and continuous-time uniform upper bounds of the heat kernel. In this paper, in the setting of graphs, we give log-Sobolev inequalities in discrete forms, and show the equivalence with the above functional inequalities. This is proved by showing that log-Sobolev inequalities and Nash equalities are separately equivalent to the ultracontractivity property of the heat semigroup, i.e., the uniform upper bound of the heat kernel, for both continuous and discrete time.

Bakry and Émery [1] suggested a notion analogous to curvature that would work in the very general framework of a Markov semigroup. On graphs, curvature conditions have been extensively studied in literature (see, e.g., [3,15]) and have proved to be useful to estimate the heat kernel. Until now, however, no notion of curvature on graphs has been sufficient to imply these four functional inequalities. In this work, we consider non-negatively curved graphs, in the sense of $\mathrm{CDE}(n, 0)$, and derive all of the above functional inequalities under the assumption of volume condition. Furthermore, we derive a uniform upper bound for the heat kernel, in both continuous and discrete time, under the same assumptions. However, it seems to be impossible to prove a continuous-time uniform upper bound for the heat kernel on graphs. Indeed, in the papers of Davies [8] and Pang [19], they obtain upper bounds for the heat kernel on graphs, by using certain functions defined as the Legendre transform. From their results (see, e.g., [19, Cor. 2.7]), it is not difficult to see that, for small $t/d(x,y)$, the continuous-time uniform upper estimate is not true on graphs. Meanwhile, we prove this for any non-negatively curved graph under the assumption of polynomial volume of growth.

The remainder of this manuscript is organized as follows: In Sect. 2, we provide the basic setting and give the main results. In Sect. 3, we prove the equivalence of the log-Sobolev inequalities and the ultracontractivity of the heat semigroup. In Sect. 4, we prove the equivalence of the Nash equalities and the ultracontractivity of the heat semigroup.

2 Setting and Main Results

Let us now introduce the necessary definitions and notations to state the results. Let $G=(V,E)$ be an infinite connected graph. We allow the edges on the graph to be weighted by making use of a symmetric weight function $\omega\colon V\times V\to[0,\infty)$ so that the edge xy from x to y has weight $\omega_{xy}>0$, and we write $x\sim y$. The symmetry assumption means that $\omega_{xy}=\omega_{yx}$ for all vertices x and y. In this paper, we are interested in locally finite graphs, i.e., where the degree of each vertex is finite:

$$m(x):=\sum_{y\sim x}\omega_{xy}<\infty\quad\text{for any } x\in V.$$

We denote by $V^{\mathbb{R}}$ the space of real-valued functions on V and by $\ell^p=\big\{f\in V^{\mathbb{R}}:\sum_{x\in V}m(x)|f(x)|^p<\infty\big\}$, $1\le p<\infty$, the space of ℓ^p-integrable functions on V with respect to the degree m. If $p=2$, let the inner product be defined by $\langle f,g\rangle=\sum_{x\in V}m(x)f(x)g(x)$, so that the space ℓ^2 is a Hilbert space. For $p=\infty$, let $\ell^\infty=\{f\in V^{\mathbb{R}}:\sup_{x\in V}|f(x)|<\infty\}$ be the set of bounded functions. For all $1\le p\le\infty$, we endow these spaces with their standard norms.

We denote by $C_c(V)\subset\ell^2$ the set of functions $f\in V^{\mathbb{R}}$ with finite support. The graph is endowed with its natural graph metric $d(x,y)$, i.e., the smallest number of edges of a path between two vertices x and y. We define balls $B(x,r)=\{y\in V:d(x,y)\le r\}$, and the volume of a subset A of V, $V(A)=\sum_{x\in A}m(x)$. We will write $V(x,r)$ for $V(B(x,r))$.

For any function $f\in V^{\mathbb{R}}$ and any $x\in V$, let $\Delta\colon V^{\mathbb{R}}\to V^{\mathbb{R}}$ on G be the graph Laplacian. It is defined by

$$\Delta f(x)=\frac{1}{m(x)}\sum_{y\sim x}\omega_{xy}(f(y)-f(x)).\tag{2.1}$$

To the operator Δ is associated the semigroup $P_t\colon\ell^p\to\ell^p$ defined by

$$P_tf(x)=\sum_{y\in V}m(y)p(t,x,y)f(y),$$

where $p(t, x, y)$ is the heat kernel with continuous time on infinite graphs (see [16] and also [21]). $P_t f(x)$ is then a solution of the heat equation. We know that the operator P_t is contractive and self-adjoint on $C_c(V)$, and that the semigroup property holds on $C_c(V)$ as well. We shall keep considering the discrete-time heat kernel $p_k(x, y)$ on G because of its probabilistic significance, which is defined by

$$\begin{cases} p_0(x, y) = \delta_{xy}, \\ p_{k+1}(x, z) = \sum_{y \in V} p(x, y) p_k(y, z), \end{cases}$$

where $p(x, y) := \omega_{xy}/m(x)$ is the transition probability of the random walk on the graph, and $\delta_{xy} = 1$ when $x = y$, and 0 otherwise.

Now, following [3,15], we introduce the gradient forms associated to the Laplacian and the curvature dimension conditions on graphs.

Definition 2.1 The gradient form Γ and the iterated gradient form Γ_2 are defined by

$$\begin{aligned} 2\Gamma(f, g)(x) &= (\Delta(fg) - f\Delta(g) - g\Delta(f))(x) \\ &= \frac{1}{m(x)} \sum_{y \sim x} \omega_{xy}(f(y) - f(x))(g(y) - g(x)), \end{aligned}$$

$$2\Gamma_2(f, g)(x) = (\Delta\Gamma(f, g) - \Gamma(f, \Delta(g)) - \Gamma(g, \Delta(f)))(x).$$

We write $\Gamma(f) = \Gamma(f, f)$, $\Gamma_2(f) = \Gamma_2(f, f)$ for short.

Definition 2.2 We say that a graph G satisfies the curvature dimension inequality $\mathrm{CDE}(x, n, K)$ if, for any positive function $f \in V^{\mathbb{R}}$ such that $(\Delta f)(x) < 0$, we have

$$\widetilde{\Gamma_2}(f)(x) := \Gamma_2(f)(x) - \Gamma\left(f, \frac{\Gamma(f)}{f}\right)(x) \geq \frac{1}{n}(\Delta f(x))^2 + K\Gamma(f)(x). \qquad (2.2)$$

We say that $\mathrm{CDE}(n, K)$ is satisfied if $\mathrm{CDE}(x, n, K)$ is satisfied for all $x \in V$.

To simplify the notation, we denote $\langle f \rangle = \sum_{x \in V} m(x) f(x)$. In this paper we will consider the following inequalities on graphs:

Definition 2.3 Let $D > 2$. We shall consider the following properties on G:

- (LS) (log-Sobolev inequality) $\langle f^2 \log f \rangle - \|f\|_2^2 \log \|f\|_2 \leq \varepsilon \langle \Gamma(f) \rangle + \beta(\varepsilon) \|f\|_2^2$, where $\beta(\varepsilon)$ is a monotonically decreasing continuous function of ε, for all $\varepsilon > 0$, for any function f with finite support in G;
- (S) (Sobolev inequality) $\|f\|_{2D/(D-2)} \leq c\langle \Gamma(f) \rangle$, for any function f with finite support in G;
- (N) (Nash inequality) $\|f\|_2^{2+4/D} \leq c\langle \Gamma(f) \rangle \|f\|_1^{4/D}$, for any function f with finite support in G;
- (FK) (Faber–Krahn inequality) $\lambda_1(\Omega) \geq cV(\Omega)^{-2/D}$, for every finite subset Ω of G, where $\lambda_1(\Omega) = \inf\{\langle \Gamma(f) \rangle / \|f\|_2^2 ; \operatorname{supp}(f) \subset \Omega\}$.

We also study a similar upper estimate for the continuous-time heat kernel $p(t, x, y)$ and the discrete-time heat kernel $p_k(x, y)$ separately. In fact, $p(t, x, y)$ is not an exact analogue of $p_k(x, y)$.

Definition 2.4 We define two estimates of the heat kernel on G as follows:

(CUE) (Continuous-time uniform upper estimate) $\sup_{x,y\in V} p(t, x, y) \le Ct^{-D/2}$.

(DUE) (Discrete-time uniform upper estimate) $\sup_{x,y\in V} p_k(x, y)/m(x) \le Ck^{-D/2}$.

Study of the heat kernel upper bounds and the above inequalities has been the subject of great investigations for decades. Many authors (such as Varopoulos, Grigor'yan, Coulhon–Ledoux, etc.) have contributed to the development of this area in a very general setting (metric measure spaces) containing graphs as a particular case; for example, the paper [6] gathered some conclusions on graphs. (N) and (S) are equivalent from the Hölder inequality and the truncated functions technique (see [2]). Moreover, (N) implies (FK) by the Hölder inequality, the converse being mainly due to Grigor'yan [13]. In fact, the equivalence of (N) and (DUE), in a general setting, is covered in the paper [4]. Moreover, by proving a Nash-type inequality, the authors proved that (FK) implies the continuous-time uniform upper estimate of the Dirichlet heat kernel in an abstract setting (see [14, Lems. 5.4 and 5.5]).

As alluded to in the introduction, there are no related results involving (CUE) and (LS) on graphs. In this paper, we prove that (CUE) is equivalent to (LS) and (N) separately (see Theorems 3.5 and 4.1). We basically use Davies and Simon's method [7,9]. A new proof given by Patrick deserves to be mentioned here [17]. They used the Nash-type inequality as an intermediate step, instead of the ℓ^p version of (LS) and Stein's interpolation.

We summarize the above results and our conclusions as follows: The main contribution of this theorem is to provide the equivalence between (CUE), (LS) and (N) on graphs.

Theorem 2.5 *Let $D > 2$. The following properties are equivalent on graphs:*

1. *Sobolev inequality* (S).
2. *Nash inequality* (N).
3. *Faber–Krahn inequality* (FK).
4. *Discrete-time uniform upper estimate* (DUE).
5. *Continuous-time uniform upper estimate* (CUE).
6. log-*Sobolev inequality* (LS) *with* $\beta(\varepsilon) = c - \frac{D}{4}\log\varepsilon$.

Remark 2.6 Note that the requirement $D > 2$ is only necessary for the Sobolev inequality to ensure that $2D/(D-2) > 0$ in (S), but not for the rest of the results. Actually, the proof that the ultracontractive bounds (CUE) will transit through Nash inequalities (N) (see Theorem 4.1) and log-Sobolev inequality (LS) (see Theorem 3.5) can be extended to any $D > 0$.

Another main purpose of this paper is to reveal that non-negatively curved graphs with the assumption of polynomial volume growth can ensure the above properties.

In this paper, we say that the graph satisfies *polynomial volume growth* if, for all $x \in V$, $r \ge 0$, there exists $D > 0$ such that

$$V(x, r) \ge cr^D. \tag{V}$$

This condition is true on Abelian Cayley graphs which also satisfy CDE$(n, 0)$.

Theorem 2.7 *Let $D = D(n) > 2$, and assume a graph G satisfies* CDE$(n, 0)$ *and* (V)*, then all of the properties* (S), (N), (FK), (DUE), (CUE) *hold with appropriate constants, and also* (LS) *holds with $\beta(\varepsilon) = c(n) - (D(n)/4 \log \varepsilon)$.*

To prove Theorem 2.7, we need the global estimate of the heat kernel from [3] (see Theorem 7.6).

Lemma 2.8 *Suppose G satisfies* CDE$(n, 0)$*, then there exists a constant C', so that, for $t > 0$,*

$$p(t, x, y) \leq C' \frac{1}{V(x, \sqrt{t})}. \tag{2.3}$$

Proof of Theorem 2.7 Adding the condition (V), we immediately obtain (CUE) from Lemma 2.8, and combining with Theorem 2.5, we get Theorem 2.7.

Remark 2.9 Note that, if we do not consider (DUE), the above results can be extended to bounded and unbounded Laplacians on weighted graphs. For bounded Laplacians, the proof is basically the same as for the normalized Laplacian. For unbounded Laplacians, it is slightly more complicated. In another paper, Gong and Lin proved that (N) $\Leftrightarrow$ (CUE) $\Rightarrow$ (LS) for unbounded Laplacians on a complete graph with nondegenerate measure (see [11, Thms. 3.1 and 4.1]). Furthermore, from [12, Thm. 1.3], under the same assumptions, (2.3) holds if the graph satisfies CDE$'(n, 0)$. Therefore, for unbounded Laplacians, we can derive that, if a complete graph with nondegenerate measure satisfies CDE$'(n, 0)$ and (V), then (S), (N), (FK), (CUE) hold with appropriate constants, and also (LS) with $\beta(\varepsilon) = c(n) - (D(n)/4) \log \varepsilon$.

3 log-Sobolev Inequality and Ultracontractivity on Graphs

We say that the operator $P_t = \mathrm{e}^{t\Delta}$ is ultracontractive if P_t is bounded from ℓ^2 to ℓ^∞ for all $t \geq 0$. Let $\|A\|_{p \to q}$ be the norm of an operator A from ℓ^p to ℓ^q, that is $\|A\|_{p \to q} := \sup_{f \in \ell^p} \|Af\|_q / \|f\|_p$. We have, by duality, for all $t > 0$,

$$\|P_{t/2}\|_{2 \to \infty} = \|P_{t/2}\|_{1 \to 2} = \|P_t\|_{1 \to \infty}^{1/2}.$$

Indeed, the result follows from the semigroup property of the operator P_t, namely $P_{t/2} \circ P_{t/2} = P_t$, the symmetric property $P_t^* = P_t$, as well as the following well-known equality:

$$\|A^* A\|_{1 \to \infty} = \|A\|_{1 \to 2}^2.$$

Moreover,

$$\|P_t\|_{1 \to \infty} = \sup_{x, y \in V} p(t, x, y);$$

that is to say, obtaining the ultracontractivity property we mentioned before is the same as obtaining a uniform upper bound for the heat kernel $p(t,x,y)$.

Now, we introduce an analogous result on graphs from Davies' theorem [7].

Theorem 3.1 *For any $f \in \ell^2$, if the ultracontractivity property*

$$\|P_t f\|_\infty \le \mathrm{e}^{M(t)}\|f\|_2$$

holds, where $M(t)$ is a continuous and decreasing function of t, then for any $0 \le f \in C_\mathrm{c}(V)$, the logarithmic Sobolev inequality

$$\sum_{x\in V} m(x)f(x)^2\log f(x) \le \varepsilon\langle\Gamma(f)\rangle + \beta(\varepsilon)\|f\|_2^2 + \|f\|_2^2\log\|f\|_2$$

holds with $\beta(\varepsilon) = M(\varepsilon)$ for any $\varepsilon > 0$.

We assume that, if $f(x) = 0$, we take $f(x)^2\log f(x)$ to be zero in the above inequality. In fact, there is a similar result in [15] (see Lemma 7.2) where the authors restrict to functions in $\ell^\infty(V,\mu)$. However, the proof is basically the same. We simply reproduce it here for the sake of completeness.

Proof Let $p(s)$ be a bounded and continuous function of s such that $p(s) \ge 2$ and $p'(s)$ is bounded. For any $0 \le f \in C_\mathrm{c}(V)$, note that the functions $(P_s f)^{p(s)}\log P_s f$ and $\Delta P_s f(P_s f)^{p(s)-1}$ are in ℓ^1. Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\|P_s f\|_{p(s)}^{p(s)} = p'(s)\langle(P_s f)^{p(s)}\log P_s f\rangle + p(s)\langle\Delta P_s f(P_s f)^{p(s)-1}\rangle.$$

At $s = 0$, and specializing to $p(s) = 2t/(t-s)$, $0 \le s < t$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\|P_s f\|_{p(s)}^{p(s)}\big|_{s=0} = \frac{2}{t}\langle f^2\log f\rangle + 2\langle f\Delta f\rangle.$$

We assume $\|f\|_2 = 1$. Combining the ultracontractivity property and the fact that $\|P_t f\|_2 \le \|f\|_2$, and using the Stein interpolation theorem, we have

$$\|P_s f\|_{p(s)} \le \mathrm{e}^{M(t)s/t}.$$

From this point, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\|P_s f\|_{p(s)}^{p(s)}\big|_{s=0} \le \frac{2M(t)}{t},$$

by observing $\|P_s f\|_{p(s)}^{p(s)}\big|_{s=0} = 1$, $\mathrm{e}^{M(t)sp(s)/t}\big|_{s=0} = 1$, and

$$1 \ge \lim_{s\to 0^+}\frac{\|P_s f\|_{p(s)}^{p(s)} - 1}{\mathrm{e}^{M(t)sp(s)/t} - 1} = \frac{\mathrm{d}}{\mathrm{d}s}\|P_s f\|_{p(s)}^{p(s)}\big|_{s=0}\frac{t}{2M(t)}.$$

Using the fact that $-\langle f \Delta f\rangle = \langle \Gamma(f)\rangle$ from the symmetry of the weight of each edge, and combining with the above equality, we obtain

$$\langle f^2 \log f\rangle \le t\langle \Gamma(f)\rangle + M(t),\ t > 0.$$

If $\|f\|_2 \ne 1$, we put $f = g/\|g\|_2$ in the above inequality, and switch notation from t to ε, yielding the logarithmic Sobolev inequality as we desired. □

Now we turn to the converse of the above result. First, we introduce the following lemma, which is similar to a result of Varopoulos [20] on a measure space:

Lemma 3.2 *If there exists a monotonically decreasing continuous function $\beta(\varepsilon)$ such that, for any $\varepsilon > 0$ and $0 \le f \in C_c(V)$,*

$$\langle f^2 \log f\rangle \le \varepsilon\langle \Gamma(f)\rangle + \beta(\varepsilon)\|f\|_2^2 + \|f\|_2^2 \log \|f\|_2, \tag{3.1}$$

then, for all $2 < p < \infty$, we have

$$\langle f^p \log f\rangle \le \varepsilon\langle \Gamma(f^{p-1}, f)\rangle + \frac{2\beta(\varepsilon)}{p}\|f\|_p^p + \|f\|_p^p \log \|f\|_p.$$

Proof Putting $f = g^{p/2}$ $(2 < p < \infty)$ in (3.1), for all $0 \le g \in C_c(V)$, we obtain

$$\frac{p}{2}\langle g^2 \log g\rangle \le \varepsilon\langle \Gamma(g^{p/2})\rangle + \beta(\varepsilon)\|g\|_p^p + \frac{p}{2}\|g\|_p^p \log \|g\|_p.$$

We observe that the following inequality between $\Gamma(g^{p/2})$ and $\Gamma(g^{p-1}, g)$ holds:

$$\Gamma(g^{p/2}) \le \frac{p^2}{4(p-1)}\Gamma(g^{p-1}, g).$$

To prove this, we use the definition of Γ and the Schwartz inequality as follows:

$$\begin{aligned}(\alpha^{p/2} - \beta^{p/2})^2 &= \left(\int_\alpha^\beta \frac{p}{2} s^{p/2-1}\,\mathrm{d}s\right)^2 \le \frac{p^2}{4}(\alpha-\beta)\int_\alpha^\beta s^{p-2}\,\mathrm{d}s\\ &= \frac{p^2}{4(p-1)}(\alpha-\beta)(\alpha^{p-1} - \beta^{p-1}).\end{aligned}$$

Then

$$\frac{p}{2}\langle g^2 \log g\rangle \le \frac{\varepsilon p^2}{4(p-1)}\langle \Gamma(g^{p-1}, g)\rangle + \beta(\varepsilon)\|g\|_p^p + \frac{p}{2}\|g\|_p^p \log \|g\|_p.$$

Switching the notation from g to f yields the result. □

The following theorem is a discrete analogue of the result of Davies and Simon [9]:

Theorem 3.3 *Let $\varepsilon(p) > 0$ and $\delta(p)$ be two continuous functions defined for all $2 < p < \infty$ such that*

$$\langle f^p \log f\rangle \le \varepsilon(p)\langle\Gamma(f^{p-1}, f)\rangle + \delta(p)\|f\|_p^p + \|f\|_p^p \log\|f\|_p$$

for any $0 \le f \in C_{\mathrm{c}}(V)$. If

$$t = \int_2^\infty \frac{\varepsilon(p)}{p}\,\mathrm{d}p, \quad M = \int_2^\infty \frac{\delta(p)}{p}\,\mathrm{d}p$$

are both finite, then

$$\|P_t\|_{2\to\infty} \le \mathrm{e}^M.$$

Proof Define the function $p(s)$ for $0 \le s < t$ by

$$\frac{\mathrm{d}p}{\mathrm{d}s} = \frac{p}{\varepsilon(p)}, \quad p(0) = 2, \tag{3.2}$$

so that $p(s)$ is monotonically increasing and $p(s) \to \infty$ as $s \to t$. Define also the function $N(s)$ for $0 \le s < t$ by

$$\frac{\mathrm{d}N}{\mathrm{d}s} = \frac{\delta(p)}{\varepsilon(p)}, \quad N(0) = 0,$$

so that $N(s) \to M$ as $s \to t$. We consider the functional $\log\big(\mathrm{e}^{-N(s)}\|P_s f\|_{p(s)}\big)$, for any $0 < s < t$ and any $0 \le f \in C_{\mathrm{c}}(V)$. We obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\log\left(\mathrm{e}^{-N(s)}\|P_s f\|_{p(s)}\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\left(-N(s) + \frac{1}{p(s)}\log\|P_s f\|_{p(s)}^{p(s)}\right) = \frac{\delta(p)}{\varepsilon(p)} - \frac{1}{p^2}\frac{p}{\varepsilon(p)}\log\|P_s f\|_p^p\\
&\qquad + \frac{1}{p\|P_s f\|_p^p}\left(\frac{p}{\varepsilon(p)}\left\langle (P_s f)^p \log P_s f\right\rangle - p\langle\Gamma((P_s f)^{p-1}, P_s f)\rangle\right)\\
&= \frac{1}{\varepsilon(p)\|P_s f\|_p^p}\Big(\left\langle (P_s f)^p \log P_s f\right\rangle\\
&\qquad -\varepsilon(p)\langle\Gamma((P_s f)^{p-1}, P_s f)\rangle - \delta(p)\|P_s f\|_p^p - \|P_s f\|_p^p\log\|P_s f\|_p\Big)\\
&\le 0.
\end{aligned}$$

So, for all $0 \le s < t$,

$$\mathrm{e}^{-N(s)}\|P_s f\|_{p(s)} \le \|f\|_2.$$

We can derive that $\|P_t f\|_p^p$ is a decreasing function with respect to t as follows:

$$\begin{aligned}\partial_t \|P_t f\|_p^p &= \langle p(P_t f)^{p-1} \Delta P_t f \rangle = -p\langle \Gamma((P_t f)^{p-1}, P_t f)\rangle \\ &\le -p \cdot \frac{4(p-1)}{p^2} \langle \Gamma((P_t f)^{p/2})\rangle \le 0.\end{aligned}$$

Therefore, combining the above two inequalities, for all $0 \le s < t$, we have

$$\|P_t f\|_{p(s)} \le \|P_s f\|_{p(s)} \le \mathrm{e}^{N(s)} \|f\|_2.$$

Taking the limit $s \to t$, we obtain

$$\|P_t f\|_\infty \le \mathrm{e}^M \|f\|_2.$$

If $0 \le f \in \ell^2$, then there exists an increasing sequence of functions $\{f_n\}_{n\in\mathbb{N}}$, i.e., $0 \le f_n \le f_{n+1} \in C_c(V)$, such that $f_n(x) \le f(x)$ for any $x \in V$, and $\|f_n - f\|_2 \to 0$ $(n \to \infty)$; for example, take $f(x) = \sum_{n=1}^\infty [f_n(x) - f_{n-1}(x)]$ and $f_0 = 0$.

Since $\|P_t f_n - P_t f\|_2 \to 0$ $(n \to \infty)$ and from the above calculations, we have

$$\|P_t f_n\|_\infty \le \mathrm{e}^M \|f_n\|_2.$$

Therefore,

$$\|P_t f\|_\infty \le \mathrm{e}^M \|f\|_2.$$

For a general $f \in \ell^2$, we know $|P_t f| \le P_t|f|$ by the positivity of P_t, so

$$\|P_t f\|_\infty \le \|P_t|f|\|_\infty \le \mathrm{e}^M \|f\|_2.$$

This completes the proof. □

In the above theorem, we can choose

$$\varepsilon(p) = \frac{2t}{p}, \quad \delta(p) = \frac{2\beta(\varepsilon(p))}{p},$$

so that the solution of (3.2) is

$$p(s) = \frac{2t}{t-s},$$

and

$$M = \int_2^\infty \frac{\delta(p)}{p}\,\mathrm{d}p = \int_2^\infty \frac{2\beta(\varepsilon(p))}{p^2}\,\mathrm{d}p = \frac{1}{t}\int_0^t \beta(\varepsilon)\,\mathrm{d}\varepsilon = M(t).$$

Therefore, combining Lemma 3.2 with Theorem 3.3, we obtain the following result:

Corollary 3.4 *Let $\beta(\varepsilon)$ be a monotonically decreasing continuous function of ε such that, for all $\varepsilon > 0$ and $0 \leq f \in C_c(V)$,*

$$\langle f^2 \log f\rangle \leq \varepsilon\langle\Gamma(f)\rangle + \beta(\varepsilon)\|f\|_2^2 + \|f\|_2^2 \log\|f\|_2.$$

If

$$M(t) = \frac{1}{t}\int_0^t \beta(\varepsilon)\,\mathrm{d}\varepsilon$$

is finite for all $t > 0$, then P_t is ultracontractive, and for all $0 < t < \infty$,

$$\|P_t\|_{2\to\infty} \leq \mathrm{e}^{M(t)}.$$

Now, we give an example of the relationship between the bounds on $\|P_t\|_{2\to\infty}$ and the efficiency of log-Solobev inequality using Theorem 3.1 and Corollary 3.4. If there exist constants $c_1 > 0$ and $N > 0$ such that, for all $t > 0$,

$$\mathrm{e}^{M(t)} \leq c_1 t^{-N/4},$$

then there exists a constant $c_2 > 0$ such that, for all $\varepsilon > 0$,

$$\beta(\varepsilon) \leq c_2 - \frac{N}{4}\log\varepsilon.$$

Conversely, the above inequality implies that there exists a constant $c_3 > 0$ such that, for all $t > 0$,

$$\mathrm{e}^{M(t)} \leq c_3 t^{-N/4}.$$

From the relationship between the upper bound on $p(t,x,y)$ and $\|P_t\|_{2\to\infty}$ mentioned above, we have the following conclusion:

Theorem 3.5 *The following two inequalities are equivalent:*

- *For some constants $C > 0$, $N > 0$ and for all $t > 0$,*

$$\sup_{x,y\in V} p(t,x,y) \leq Ct^{-N/2}.$$

- *For some constant $C' > 0$ and for all $\varepsilon > 0$,*

$$\langle f^2 \log f\rangle \leq \varepsilon\langle\Gamma(f)\rangle + \left(C' - \frac{N}{4}\log\varepsilon\right)\|f\|_2^2 + \|f\|_2^2 \log\|f\|_2.$$

4 Nash Inequalities and Ultracontractivity on Graphs

The equivalence between Nash inequalities and the ultracontractivity property goes back to Nash [18] and was further studied by Fabes and Stroock [10] in the smooth setting, and [5] and [4] in a measure space.

Theorem 4.1 *Let* $\mu > 0$. *The following two bounds are equivalent:*

(1) *For some constant* $c_1 > 0$ *and all* $t > 0$, $f \in \ell^2$,

$$\|P_t f\|_\infty \le c_1 t^{-\mu/4} \|f\|_2.$$

(2) *For some constant* $c_2 > 0$ *and all* $0 \le f \in C_{\mathrm{c}}(V)$,

$$\|f\|_2^{2+4/\mu} \le c_2 \langle \Gamma(f) \rangle \|f\|_1^{4/\mu}.$$

Proof First, we introduce an equality similar to the one in [3] that we will use later. For any $f \in C_{\mathrm{c}}(V)$ and all $s > 0$, from the facts that P_t is self-adjoint and P_t commutes with Δ, and the semigroup property of P_t (that is, $P_{t/2} \circ P_{t/2} = P_t$), we obtain

$$\begin{aligned}
\langle f, f\rangle - \langle P_s f, f\rangle &= \langle f - P_s f, f\rangle = \sum_{x \in V} \mu(x) f(x)(P_0 f - P_s f)(x) \\
&= -\int_0^s \sum_{x \in V} \mu(x) f(x) \partial_t P_t f(x)\, \mathrm{d}t \\
&= -\int_0^s \sum_{x \in V} \mu(x) f(x) \Delta P_t f(x)\, \mathrm{d}t \\
&= -\int_0^s \sum_{x \in V} \mu(x) P_{t/2} f(x) \Delta P_{t/2} f(x)\, \mathrm{d}t = \int_0^s \langle \Gamma(P_{t/2} f)\rangle\, \mathrm{d}t.
\end{aligned}$$

Given (1), we have $\|P_t f\|_2 \le c_1 t^{-\mu/4} \|f\|_1$ by duality. Then, for all $f \in C_{\mathrm{c}}(V)$,

$$\begin{aligned}
c_1^2 t^{-\mu/2} \|f\|_1^2 &\ge \|P_t f\|_2^2 = \langle P_{2t} f, f\rangle \\
&= \langle f, f\rangle - \textstyle\int_0^{2t} \langle \Gamma(P_{s/2} f)\rangle\, \mathrm{d}s \ge \langle f, f\rangle - 2t \langle \Gamma(f)\rangle.
\end{aligned}$$

In the last step, we used that the function $\langle \Gamma(P_t f)\rangle$ is nonincreasing with respect to t, for any $t > 0$, which follows from

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \langle \Gamma(P_t f)\rangle &= \frac{\mathrm{d}}{\mathrm{d}t} \Big(\frac{1}{2} \sum_{x \in V} \sum_{y \sim x} \omega_{xy} (P_t f(y) - P_t f(x))^2 \Big) \\
&= \sum_{x \in V} \sum_{y \sim x} \omega_{xy} (P_t f(y) - P_t f(x))(\Delta P_t f(y) - \Delta P_t f(x)) \\
&= 2 \langle \Gamma(P_t f, \Delta P_t f)\rangle = -2 \langle \Delta P_t f, \Delta P_t f\rangle \le 0.
\end{aligned}$$

Therefore,

$$\|f\|_2^2 \le 2t\langle\Gamma(f)\rangle + c_1^2 t^{-\mu/2}\|f\|_1^2,$$

and (2) follows by putting

$$t = \langle\Gamma(f)\rangle \mathrm{e}^{-2/(\mu+2)}\|f\|_1^{4/(\mu+2)}.$$

Conversely, given (2), for all $0 \le f \in C_{\mathrm{c}}(V)$, we know that $\|f\|_1 \ge \|P_t f\|_1$, so we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}\|P_t f\|_2^2 = \langle\Gamma(P_t f)\rangle \ge \frac{\|P_t f\|_2^{2+4/\mu}}{c_2\|P_t f\|_1^{4/\mu}} \ge \frac{\|P_t f\|_2^{2+4/\mu}}{c_2\|f\|_1^{4/\mu}}.$$

Hence,

$$-\frac{\mathrm{d}}{\mathrm{d}t}(\|P_t f\|_2^{-4/\mu}) \ge \frac{2}{c_2\mu\|f\|_1^{4/\mu}},$$

and, integrating the above inequality from 0 to t, we obtain

$$\|P_t f\|_2^{-4/\mu} \ge \|P_t f\|_2^{-4/\mu} - \|f\|^{-2/\mu} \ge \frac{2t}{c_2\mu\|f\|_1^{4/\mu}}.$$

Thus,

$$\|P_t f\|_2 \le \left(\frac{c_2\mu}{2t}\right)^{\mu/4}\|f\|_1 = c_1 t^{-\mu/4}\|f\|_1.$$

Finally, (1) follows by duality.

As in the proof of Theorem 3.3, for general $f \in \ell^2$, we have the same conclusion.

Proof of Theorem 2.5. From Theorem 3.5 [(CUE) $\Leftrightarrow$ (LS) with $\beta(\varepsilon) = c - (D/4)\log\varepsilon$] and Theorem 4.1 [(CUE) $\Leftrightarrow$ (N)], and that we already know that (S) $\Leftrightarrow$ (N) $\Leftrightarrow$ (FK) $\Leftrightarrow$ (DUE) from the summary in Sect. 2, we finally obtain our desired result.

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