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Calculus of variations on locally finite graphs

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Abstract
Let $G = (V, E)$ be a locally finite graph. Firstly, using calculus of variations, including a direct method of variation and the mountain-pass theory, we get sequences of solutions to several local equations on G (the Schrödinger equation, the mean field equation, and the Yamabe equation). Secondly, we derive uniform estimates for those local solution sequences. Finally, we obtain global solutions by extracting convergent sequence of solutions. Our method can be described as a variational method from local to global.

Keywords Analysis on graph · Variational method on graph · Sobolev embedding theorem

Mathematics Subject Classification 35R02 · 34B45

1 Introduction

Partial differential equations on Euclidean space or manifolds are important topics in mathematical physics and differential geometry. As their discrete versions, it is important to study the difference equations on graph, particularly the existence problem for such equations.

About five years ago, joined with Grigor'yan, we systematically raised and studied Kazdan–Warner equations, Yamabe equations and Schördinger equations on graphs in [6–8]. We first established the Sobolev spaces and the functional framework. Then the problem of solving the equations is transformed into finding critical points of various functionals. As a consequence, variational methods are applied to these problems. If

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the graph is finite, then all the Sobolev spaces have finite dimensions, and whence they are pre-compact. For this reason, the variational problems for finite graph are comparatively simple [6,7]. Since the graph has no concept of dimension, if it includes infinite vertices, the Sobolev embedding theorems becomes unusual. An easy-to-understand one was observed by us [8] under the assumption that the graph is locally finite and its measure has positive lower bound (see next section for details). Any other Sobolev embedding theorem for infinite graph would be extremely interesting.

In recent years, the research in this field has aroused great interest. Motivated by [8,15], Zhang–Zhao [17] obtained nontrivial solutions to certain nonlinear Schrödinger equation. Similar equations on infinite metric graphs were studied by Akduman–Pankov [2]. The Kazdan–Warner equation was extended by Keller–Schwarz [11] to canonically compactifiable graphs, and by Ge–Jiang [5] to certain infinite graph. For other related works, we refer the reader to [9,10,12–14,16] and the references therein.

In this paper, we study various equations on locally finite graphs, say Schrördinger equation, Mean field equation and Yamabe equation. Assuming that the weights of the graph have a positive lower bound and the distance function of the graph belongs to L^p, we derive a Sobolev embedding theorem, which is crucial in our analysis. In addition to the Sobolev embedding theorem, we also employ calculus of variations, including a direct method of variation and the mountain-pass theorem. It is remarkable that we show how to get solutions from local to global by using variational method.

2 Notations and main results

Let $G = (V, E)$ be a connected graph, where V denotes the vertex set and E denotes the edge set. For any edge $xy \in E$, we assume that its weight $w_{xy} > 0$ and that $w_{xy} = w_{yx}$. Let $\mu : V \to \mathbb{R}^+$ be a finite measure. For any function $u : V \to \mathbb{R}$, the Laplacian of u is defined as

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x)), \tag{1}$$

where $y \sim x$ means $xy \in E$ or y is adjacent to x. The gradient form is written by

$$\Gamma(u, v)(x) = \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x))(v(y) - v(x)). \tag{2}$$

Denote $\Gamma(u) = \Gamma(u, u)$ and $\nabla u \nabla v = \Gamma(u, v)$. The length of the gradient of u is represented by

$$|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x))^2 \right)^{1/2}. \tag{3}$$

The integral of a function f on V is given as

$$\int_V f d\mu = \sum_{x \in V} \mu(x) f(x). \tag{4}$$

For any $q > 0$, we let $L^q(V)$ be a linear space of functions $f : V \to \mathbb{R}$ with the norm

$$\|f\|_{L^q(V)} = \left(\int_V |f|^q d\mu \right)^{1/q}. \tag{5}$$

While $L^\infty(V)$ includes all functions $f : V \to \mathbb{R}$ satisfying

$$\|f\|_{L^\infty(V)} = \sup_{x \in V} |f(x)| < \infty.$$

If $x, y \in V$ and y is adjacent to x, then the distance between x and y is defined as 1. While if y is not adjacent to x, then there exists a shortest path γ connecting y and x, and thus the distance between x and y is defined as the number of edges belonging to γ. Given any $O \in V$. Denote the distance between x and O by

$$\rho(x) = \rho(x, O). \tag{6}$$

For any integer $k \geq 1$, we denote a ball centered at O with radius k by

$$B_k = B_k(O) = \{x \in V : \rho(x) < k\}. \tag{7}$$

The boundary of B_k is written as

$$\partial B_k = \{x \in V : \rho(x) = k\}. \tag{8}$$

According to [7], $W_0^{1,2}(B_k)$ stands for a Sobolev space including all functions $u : B_k \to \mathbb{R}$ with $u = 0$ on the boundary ∂B_k given as in (8). For any fixed k, it is pre-compact. Precisely, if (u_j) is a bounded sequence in $W_0^{1,2}(B_k)$, i.e.

$$\|u_k\|_{W_0^{1,2}(B_k)} = \left(\int_{B_k} |\nabla u_k|^2 d\mu \right)^{1/2} \leq C, \tag{9}$$

where the notations in (2), (3) and (4) are used, then there exists a subsequence of (u_j) converging to some function u under the norm in (9).

Recall another important Sobolev space $W^{1,2}(V)$ including all functions $u : V \to \mathbb{R}$ with

$$\|u\|_{W^{1,2}(V)} = \left(\int_V (|\nabla u|^2 + u^2) d\mu \right)^{1/2} < +\infty. \tag{10}$$

Let $C_c(V)$ be a set of all functions with finite support, and $W_0^{1,2}(V)$ be a completion of $C_c(V)$ under the norm as in (10). Both of $W^{1,2}(V)$ and $W_0^{1,2}(V)$ are Hilbert spaces with the same inner product $\langle u, v\rangle = \int_V(\nabla u \nabla v + uv)d\mu$.

A connected graph is said to be locally finite if for any fixed $O \in V$, B_k is a finite subgraph. In [8], we made a key observation under the assumption that G is locally finite, and there exists a constant $\mu_0 > 0$ satisfying

$$\mu(x) \geq \mu_0 \quad \text{for all} \quad x \in V. \tag{11}$$

Namely, a Sobolev embedding theorem holds.

Theorem 1 [8]. *Let $G = (V, E)$ be a connected locally finite graph. If (11) is satisfied, then for any $u \in W^{1,2}(V)$ and any $2 \leq q \leq \infty$, there exists a positive constant C depending only on q and μ_0 satisfying $\|u\|_{L^q(V)} \leq C\|u\|_{W^{1,2}(V)}$. In particular,*

$$\|u\|_{L^\infty(V)} \leq \frac{1}{\sqrt{\mu_0}}\|u\|_{W^{1,2}(V)}.$$

If instead of (11), there exists some constant $w_0 > 0$ such that

$$w_{xy} \geq w_0 \quad \text{for all} \quad y \sim x,\ \ x, y \in V, \tag{12}$$

and the distance function $\rho(x)$ defined as in (6) belongs to $L^p(V)$, we shall prove a Sobolev embedding as follows.

Theorem 2 *Let $G = (V, E)$ be a connected locally finite graph. If the weights w_{xy} satisfy (12), and the distance function $\rho(x) = \rho(x, O) \in L^p(V)$ for some $p > 0$ and some $O \in V$, then there exists some constant C depending only on w_0, $\mu(O)$ and p such that*

$$\|u\|_{L^p(V)} \leq C(\|\rho\|_{L^p(V)} + 1)\|u\|_{W^{1,2}(V)}.$$

If a function $h : V \to \mathbb{R}$ has a positive lower bound on V, then we define a subspace of $W_0^{1,2}(V)$, which is also a Hilbert space, namely

$$\mathscr{H} = \left\{u \in W_0^{1,2}(V) : \int_V (|\nabla u|^2 + hu^2)d\mu < \infty\right\} \tag{13}$$

with an inner product

$$\langle u, v\rangle_{\mathscr{H}} = \int_V (\nabla u \nabla v + huv)d\mu. \tag{14}$$

The first equation we concern is the following linear Schrödinger equation

$$\begin{cases} -\Delta u + hu = f\ d\text{in } V \\ u \in \mathscr{H}, \end{cases} \tag{15}$$

where Δ is the Laplacian operator defined as in (1), and $\mathscr{H}$ is defined as in (13). We now state the following existence result.

Theorem 3 *Let $G = (V, E)$ be a connected locally finite graph. Assume there is some constant $a_0 > 0$ such that $h(x) \geq a_0$ for all $x \in V$. If one of the following three hypotheses is satisfied:*

(i) $f \in L^2(V)$;
(ii) $\mu(x) \geq \mu_0 > 0$ *for all* $x \in V$, $f \in L^1(V)$;
(iii) *the weights of the graph satisfies* (12), *the distance function* $\rho(x) = \rho(x, O) \in L^p(V)$ *for some* $p > 1$, $O \in V$, *and* $f \in L^{p/(p-1)}(V)$,
then the Eq. (15) *has a unique solution. If in addition* $f \geq 0$ *and* $f \not\equiv 0$ *on* V, *then* $u(x) > 0$ *for all* $x \in V$.

The second equation we concern is the mean field equation, which is also known as the Kazdan–Warnar equation, namely

$$\Delta u = f - ge^u \quad \text{in} \quad V. \tag{16}$$

Theorem 4 *Let $G = (V, E)$ be a locally finite graph. Suppose that $g \leq f < 0$ on V and $g \in L^1(V)$. Then the Eq.* (16) *has a solution.*

We remak that using a method of the heat equation, Ge-Jiang [5] obtained similar result as that of Theorem 4 under different assumptions on f and g. In the case $g > 0$, it is not likely to find a nontrivial solution as in Theorem 4 in general. The main difficulty is that $\int_V |\nabla u|^2 d\mu$ does not control $\|u\|_{W^{1,2}(V)}$ if V is an infinite graph. However, it is natural to consider the following mean field equation

$$\begin{cases} -\Delta u + hu = \frac{ge^u}{\int_V ge^u d\mu} - f \text{ in } \quad V \\ u \in \mathscr{H} \cap L^\infty(V), \end{cases} \tag{17}$$

where h has a positive lower bound, and $\mathscr{H}$ is defined as in (13). To seek solutions of (17), we need certain Trudinger–Moser embedding. It suffices to assume (11) for the graph in order to get that kind of embedding. Precisely we have the following:

Theorem 5 *Let $G = (V, E)$ be a connected locally finite graph. Suppose* (11) *is satisfied, there exists some constant $a_0 > 0$ such that $h(x) \geq a_0$ for all $x \in V$, $g \geq 0$ and $g \not\equiv 0$ on V, $g \in L^1(V)$, and $f \in L^q(V)$ for some $q \in [1, 2]$. Then the Eq.* (17) *has a solution.*

Note that in Theorem 5, the function f allows the form $\sum_{i=1}^{\ell} c_i \delta_{x_i}$ for some constants $c_1, \cdots, c_\ell$, where δ_{x_i} stands for the Dirac function satisfying

$$\delta_{x_i}(x) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{if } x \neq x_i. \end{cases}$$

As a consequence, it makes sense to consider Chern–Simons–Higgs model in locally finite graph. Such a model in finite graph was recently studied by Huang–Lin–Yau [10].

The third equation we are interested in is the Yamabe equation

$$\begin{cases} -\Delta u + hu = |u|^{q-2}u \quad \text{in} \quad V \\ u \in \mathscr{H}, \end{cases} \tag{18}$$

where h has a positive lower bound, $\mathscr{H}$ is defined as in (13), and $q > 2$. In order to find a solution to the Eq. (18), we seek the Sobolev embedding theorem, say Theorem 1 or Theorem 2. Inspired by [1,4,15], we have solved this problem in [8] by employing Theorem 1. For application of Theorem 2, we state the following:

Theorem 6 *Let $G = (V, E)$ be a connected locally finite graph. Let O be a fixed point of V, the distance function $\rho(x) = \rho(x, O) \in L^p(V)$ for some $p > 2$. Suppose $h(x) \geq a_0 > 0$ for some constant a_0 and all $x \in V$. If further $1/h \in L^1(V)$ or $h(x) \to +\infty$ as $\rho(x) \to +\infty$, then for any q with $2 < q < p$, the Eq. (18) has a nontrivial solution.*

The remaining part of this paper is organized as follows: In Sect. 3, a Sobolev embedding theorem (Theorem 2) is proved; In Sect. 4, we study the linear Schrödinger equation, and prove Theorem 3; In Sect. 5, the mean field equations are discussed, and Theorems 4 and 5 are proved; In Sect. 6, we consider the Yamabe equation and prove Theorem 6. Throughout this paper, we do not distinguish sequence and subsequence, and denote various constants by the same C.

3 A Sobolev embedding

In this section, using definitions of $W^{1,2}(V)$ and $L^p(V)$, we prove Theorem 2.

Proof of Theorem 2 Let O be a fixed point in V. For any $x \in V$, we denote the distance between x and O by $\rho(x) = \rho(x, O)$. Choose a shortest path $\gamma = \{x_1, \cdots, x_{k+1}\}$ connecting x and O. In particular $x_1 = x, \cdots, x_{k+1} = O$, x_i is adjacent to x_{i+1} for all $1 \leq i \leq k$, and $k = \rho(x)$. For any $u \in W^{1,2}(V)$, we get

$$|u(x)| \leq |u(x_1) - u(x_2)| + \cdots + |u(x_k) - u(x_{k+1})| + |u(O)|. \tag{19}$$

Noting that (10) implies

$$\|u\|_{W^{1,2}(V)} = \left(\sum_{z \in V,\, y \sim z} w_{zy}(u(y) - u(z))^2 + \sum_{z \in V} \mu(z)u^2(z) \right)^{1/2}, \tag{20}$$

and that $\mu(z) > 0$ for all $z \in V$, we have

$$|u(O)| \leq \frac{1}{\sqrt{\mu(O)}} \|u\|_{W^{1,2}(V)}; \tag{21}$$

since $w_{zy} \geq w_0 > 0$ for all z adjacent to y, in view of (20),

$$\begin{aligned}\sum_{i=1}^{k} |u(x_i) - u(x_{i+1})| &\leq k \max_{1 \leq i \leq k} |u(x_i) - u(x_{i+1})| \\ &\leq \frac{k}{\sqrt{w_0}} \max_{1 \leq i \leq k} \sqrt{w_{x_i x_{i+1}}} |u(x_i) - u(x_{i+1})| \\ &\leq \frac{1}{\sqrt{w_0}} \rho(x) \|u\|_{W^{1,2}(V)}.\end{aligned} \tag{22}$$

Combining (19), (21) and (22), we obtain

$$|u(x)| \leq \left(\frac{1}{\sqrt{w_0}} \rho(x) + \frac{1}{\sqrt{\mu(O)}} \right) \|u\|_{W^{1,2}(V)}. \tag{23}$$

Since $\rho \in L^p(V)$ for some $p > 0$ and $\rho(x, y) \geq 1$ for all $x \neq y$, in view of (5), there holds

$$\begin{aligned}\|1\|_{L^p(V)} = \left(\sum_{z \in V} \mu(z) \right)^{1/p} &\leq \left(\sum_{z \in V} \mu(z) \rho^p(z) + \mu(O) \right)^{1/p} \\ &\leq 2^{1/p} \max \left\{ \left(\sum_{z \in V} \mu(z) \rho^p(z) \right)^{1/p}, \ \mu(O)^{1/p} \right\} \\ &= 2^{1/p} \max \left\{ \|\rho\|_{L^p(V)}, \ \mu(O)^{1/p} \right\}.\end{aligned}$$

This together with (23) leads to

$$\begin{aligned}\|u\|_{L^p(V)} &\leq C \left(\frac{1}{\sqrt{w_0}} \|\rho\|_{L^p(V)} + \frac{1}{\sqrt{\mu(O)}} \|1\|_{L^p(V)} \right) \|u\|_{W^{12}(V)} \\ &\leq C(\|\rho\|_{L^p(V)} + 1) \|u\|_{W^{1,2}(V)}\end{aligned}$$

for some constant C depending only on w_0, $\mu(O)$ and p, as we expected. □

4 Schrödinger equation

In this section, we prove Theorem 3 by using a direct method of variation from local to global.

Proof of Theorem 3 Fix some point $O \in V$. Denote the distance between x and O by $\rho(x) = \rho(x, O)$. For any positive integer k, we write $B_k = \{x \in V : \rho(x) < k\}$. Note that $h(x) \geq a_0 > 0$ for all $x \in V$. Let $W_0^{1,2}(B_k)$ be the Sobolev space including all functions $u : B_k \to \mathbb{R}$, $u = 0$ on ∂B_k, with the norm

$$\|u\|_{W_0^{1,2}(B_k)} = \left(\int_{B_k} (|\nabla u|^2 + hu^2) d\mu \right)^{1/2}. \tag{24}$$

For any fixed k, the norm in (24) is equivalent to that in (9), due to the Poincaré inequality

$$\int_{B_k} u^2 d\mu \leq C_k \int_{B_k} |\nabla u|^2 d\mu, \quad \forall u \in W_0^{1,2}(B_k),$$

where C_k is a constant depending on k. In general, C_k tends to infinity as $k \to \infty$. It is convenient for us to use (24) as the norm in $W_0^{1,2}(B_k)$. Define a functional $J_k : W_0^{1,2}(B_k) \to \mathbb{R}$ by

$$J_k(u) = \frac{1}{2} \int_{B_k} (|\nabla u|^2 + hu^2) d\mu - \int_{B_k} fud\mu. \tag{25}$$

Set $\Lambda_k = \inf_{u \in W_0^{1,2}(B_k)} J_k(u)$. Obviously

$$\Lambda_k \leq J_k(0) = 0. \tag{26}$$

Case (i). $f \in L^2(V)$.

By the Hölder inequality and the Young inequality, we have

$$\begin{aligned} \int_{B_k} fud\mu &\leq \frac{1}{\sqrt{a_0}} \left(\int_V f^2 d\mu \right)^{1/2} \left(\int_{B_k} hu^2 d\mu \right)^{1/2} \\ &\leq \frac{1}{\sqrt{a_0}} \|f\|_{L^2(V)} \|u\|_{W_0^{1,2}(B_k)} \\ &\leq \frac{1}{4} \|u\|^2_{W_0^{1,2}(B_k)} + \frac{1}{a_0} \|f\|^2_{L^2(V)}, \end{aligned} \tag{27}$$

where $\|u\|_{W_0^{1,2}(B_k)}$ is defined as in (24). It follows from (25) and (27) that

$$J_k(u) \geq \frac{1}{4} \|u\|^2_{W_0^{1,2}(B_k)} - \frac{1}{a_0} \|f\|^2_{L^2(V)}. \tag{28}$$

Hence

$$\Lambda_k = \inf_{u \in W_0^{1,2}(B_k)} J_k(u) \geq -\frac{1}{a_0} \|f\|^2_{L^2(V)}. \tag{29}$$

Combining (26) and (29), we know that (Λ_k) is a bounded sequence of numbers. Now we fix a positive integer k and take a sequence of functions $(\widetilde{u}_j) \subset W_0^{1,2}(B_k)$

satisfying

$$J_k(\widetilde{u}_j) \to \Lambda_k \quad \text{as} \quad j \to \infty. \tag{30}$$

It follows from (28) that $(\widetilde{u}_j)$ is bounded in $W_0^{1,2}(B_k)$. By the Sobolev embedding theorem for finite graph [7], there exists a $u_k \in W_0^{1,2}(B_k)$ such that up to a subsequence, $\widetilde{u}_j$ converges to u_k under the norm (24). Clearly $J_k(u_k) = \Lambda_k$, and u_k satisfies the Euler–Lagrange equation

$$\begin{cases} -\Delta u_k + h u_k = f & \text{in } B_k \\ u_k = 0 & \text{on } \partial B_k. \end{cases} \tag{31}$$

Noting that (Λ_k) is bounded, in view of (28) and (30), we obtain

$$\|u_k\|^2_{W_0^{1,2}(B_k)} = \int_{B_k} (|\nabla u_k|^2 + h u_k^2) d\mu \leq C \tag{32}$$

for some constant C independent of k. For any finite set $K \subset V$, there holds $K \subset B_k$ for sufficiently large k. The power of (32) is evident. It ensures that

$$\|u_k\|_{L^\infty(K)} \leq \frac{1}{\sqrt{a_0} \min_{x \in K} \mu(x)} \|u_k\|^2_{W_0^{1,2}(B_k)} \leq C.$$

Note that (u_k) is naturally viewed as a sequence of functions defined on V, say $u_k \equiv 0$ on $V \setminus B_k$. There would exist a subsequence of (u_k) (which is still denoted by (u_k)) and a function u^* such that (u_k) converges to u^* locally uniformly in V, i.e. for any fixed positive integer ℓ,

$$\lim_{k \to \infty} u_k(x) = u^*(x) \quad \text{for all} \quad x \in B_\ell.$$

Now we show that

$$u^* \in \mathscr{H}. \tag{33}$$

Since u_k is viewed as a function on the whole V, $u_k = 0$ on $V \setminus B_k$, and the weights of the graph is symmetric, i.e. $w_{xy} = w_{yx}$ for all y adjacent to x, we have the following estimate

$$\begin{aligned} \|u_k\|^2_{\mathscr{H}} &= \sum_{y \sim x} w_{xy}(u_k(y) - u_k(x))^2 + \sum_{x \in V} \mu(x) h(x) u_k^2(x) \\ &= \sum_{y \sim x,\, x \in B_k} w_{xy}(u_k(y) - u_k(x))^2 + \sum_{x \in B_k} \mu(x) h(x) u_k^2(x) \\ &\quad + \sum_{y \sim x,\, x \in \partial B_k} w_{xy}(u_k(y) - u_k(x))^2 \end{aligned}$$

$$
\begin{aligned}
&\leq 2 \sum_{y \sim x,\, x \in B_k} w_{xy}(u_k(y) - u_k(x))^2 + \sum_{x \in B_k} \mu(x)h(x)u_k^2(x) \\
&\leq 2\|u_k\|^2_{W_0^{1,2}(B_k)}.
\end{aligned} \tag{34}
$$

Up to a subsequence, we assume (u_k) converges to u^* locally uniformly in V. In view of (32) and (34), we know that (u_k) is bounded in $\mathscr{H}$. Since every Hilbert space is weakly compact, it follows that up to a subsequence, (u_k) converges to some function u_1^* weakly in $\mathscr{H}$. This in particular implies

$$
\int_V u_k \phi d\mu \to \int_V u_1^* \phi d\mu, \quad \forall \phi \in C_c(V).
$$

Let $z \in V$ be any fixed point. In the above estimate, we take ϕ satisfying $\phi(x) = 1$ at $x = z$ and $\phi(x) = 0$ at $x \neq z$. Then $u_k(z) \to u_1^*(z)$. Hence by the uniqueness of the limit, $u_1^*(z) \equiv u^*(z)$ for all $z \in V$, and (33) follows immediately.

It then follows from (31) that for any fixed $x \in V$, there holds

$$
-\Delta u^*(x) + h(x)u^*(x) = f(x).
$$

Therefore u^* is a solution of (15). To prove that u^* is a unique solution of (15), it suffices to show the homogenueous equation

$$
\begin{cases} -\Delta u + hu = 0 \\ u \in \mathscr{H} \end{cases} \tag{35}
$$

has only one solution $u \equiv 0$. Since $u \in \mathscr{H}$, there exists a sequence $(\varphi_k) \subset C_c(V)$ such that $\varphi_k \to u$ in $\mathscr{H}$. Testing (35) by φ_k, we have by integration by parts

$$
\langle \varphi_k, u \rangle_{\mathscr{H}} = \int_V (\nabla u \nabla \varphi_k + hu\varphi_k) d\mu = 0,
$$

where $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ is the inner product in $\mathscr{H}$ defined as in (14). Passing to the limit $k \to \infty$, we conclude $\langle u, u \rangle_{\mathscr{H}} = 0$, and thus $u \equiv 0$. This confirms the uniqueness of u^*.

If $f(x) \geq 0$ for all $x \in V$, then applying the maximum principle to (31), we obtain $u_k(x) \geq 0$ for all $x \in B_k$. Indeed, suppose there exists some $x_0 \in B_k$ satisfying $\min_{B_k} u_k = u_k(x_0) < 0$, we have by (31) that

$$
-\Delta u_k(x_0) = f(x_0) - h(x_0)u_k(x_0) > 0.
$$

This is impossible, and leads to $u_k \geq 0$ on B_k. As a consequence, $u^*(x) \geq 0$ for all $x \in V$. Since $f \not\equiv 0$, one has $u^* \not\equiv 0$. We now prove $u^*(x) > 0$ for all $x \in V$. Suppose not, there would be a point $x^* \in V$ such that $u^*(x^*) = 0 = \min_V u^*$ and $\Delta u^*(x^*) > 0$. It follows that

$$
0 > -\Delta u^*(x^*) = f(x^*) \geq 0,
$$

which is a contradiction, and implies $u^*(x) > 0$ for all $x \in V$.

Case (ii). $\mu(x) \geq \mu_0 > 0$ *for all* $x \in V$.

By the Sobolev embedding theorem (Theorem 1), we have for all $u \in W_0^{1,2}(B_k)$,

$$\|u\|_{L^\infty(B_k)} \leq \frac{1}{\sqrt{\mu_0}}\|u\|_{W_0^{1,2}(B_k)}.$$

Similar to (27), there holds

$$\begin{aligned}\int_{B_k} fud\mu &\leq \|u\|_{L^\infty(B_k)}\|f\|_{L^1(B_k)} \\ &\leq \frac{1}{\sqrt{\mu_0}}\|u\|_{W_0^{1,2}(B_k)}\|f\|_{L^1(V)} \\ &\leq \frac{1}{4}\|u\|^2_{W_0^{1,2}(B_k)} + \frac{1}{\mu_0}\|f\|^2_{L^1(V)}.\end{aligned}$$

In the same way, for any $u \in W_0^{1,2}(B_k)$, we obtain analogs of (28) and (29), namely

$$J_k(u) \geq \frac{1}{4}\|u\|^2_{W_0^{1,2}(B_k)} - \frac{1}{\mu_0}\|f\|^2_{L^1(V)}$$

and

$$\Lambda_k = \inf_{u \in W_0^{1,2}(B_k)} J_k(u) \geq -\frac{1}{\mu_0}\|f\|^2_{L^1(V)}.$$

The remaining part of the proof is completely analogous to that of the case (i), and is omitted.

Case (iii). $w_{xy} \geq w_0 > 0$ *for all* y *adjacent to* x, $\rho \in L^p(V)$ *and* $f \in L^{p/(p-1)}(V)$ *for some* $p \in [1, \infty]$, *in particular* $f \in L^\infty(V)$ *if* $p = 1$.

It follows from the Sobolev embedding (Theorem 2) that there exists some constant C depending only on w_0, $\mu(O)$, $\|\rho\|_{L^p(V)}$ and p satisfying

$$\|u\|_{L^p(B_k)} \leq C\|u\|_{W_0^{1,2}(B_k)}, \quad \forall u \in W_0^{1,2}(B_k).$$

Similar to (27), we have

$$\begin{aligned}\int_{B_k} fud\mu &\leq \|u\|_{L^p(B_k)}\|f\|_{L^{\frac{p}{p-1}}(V)} \\ &\leq C\|u\|_{W_0^{1,2}(B_k)}\|f\|_{L^{\frac{p}{p-1}}(V)} \\ &\leq \frac{1}{4}\|u\|^2_{W_0^{1,2}(B_k)} + C^2\|f\|^2_{L^{\frac{p}{p-1}}(V)}.\end{aligned}$$

As a consequence, we obtain analogs of (28) and (29) as follows:

$$J_k(u) \geq \frac{1}{4}\|u\|^2_{W_0^{1,2}(B_k)} - C^2\|f\|^2_{L^{\frac{p}{p-1}}(V)},$$

and

$$\Lambda_k = \inf_{u\in W_0^{1,2}(B_k)} J_k(u) \geq -C^2\|f\|^2_{L^{\frac{p}{p-1}}(V)}.$$

Again the remaining part of the proof in this case is completely analogous to that of Case (i), and thus is omitted. □

5 Mean field equation

In this section, we consider mean field equations. Precisely we prove Theorems 4 and 5 by variational method from local to global.

5.1 The case $g \leq f < 0$

Proof of Theorem 4 Fix some point $O \in V$. For any $x \in V$, $\rho(x) = \rho(x, O)$ denotes the distance between x and O. For any positive integer k, we let $B_k = \{x \in V : \rho(x) < k\}$, and define a functional $J_k : W_0^{1,2}(B_k) \to \mathbb{R}$ by

$$J_k(u) = \frac{1}{2}\int_{B_k} |\nabla u|^2 d\mu + \int_{B_k} f u d\mu - \int_{B_k} g e^u d\mu.$$

Step 1. *For any positive integer k, J_k has a lower bound on $W_0^{1,2}(B_k)$.*

Since $g \leq f < 0$ and $g \in L^1(V)$, we have also $f \in L^1(V)$. An elementary inequality $e^t \geq 1 + t$ for all $t \in \mathbb{R}$ implies that for all $u \in W_0^{1,2}(B_k)$,

$$\begin{aligned} J_k(u) &\geq \int_{B_k} f u d\mu - \int_{B_k} g e^u d\mu \\ &\geq \int_{B_k} f(u - e^u) d\mu \\ &\geq \int_{B_k} (-f) d\mu \\ &= \int_V (-f) d\mu + o_k(1), \end{aligned} \tag{36}$$

where $o_k(1) \to 0$ as $k \to \infty$. Denoting $c_k = \int_{B_k}(-f)d\mu$, we obtain $J_k(u) \geq c_k$ for all $u \in W_0^{1,2}(B_k)$.

Step 2. *For any positive integer k, there exists a function $u_k \in W_0^{1,2}(B_k)$ such that*

$$J_k(u_k) = \Lambda_k = \inf_{u \in W_0^{1,2}(B_k)} J_k(u). \tag{37}$$

Moreover u_k satisfies the Euler–Lagrange equation

$$\begin{cases} \Delta u_k = f - g e^{u_k} & \text{in } B_k \\ u_k = 0 & \text{on } \partial B_k. \end{cases} \tag{38}$$

Obviously there holds

$$\Lambda_k \le J_k(0) = \int_{B_k} (-g) d\mu \le \int_V (-g) d\mu.$$

This together with (36) gives

$$\|f\|_{L^1(V)} + o_k(1) \le \Lambda_k \le \|g\|_{L^1(V)}. \tag{39}$$

Take a minimizing sequence $(\widetilde{u}_j) \subset W_0^{1,2}(B_k)$ satisfying

$$J_k(\widetilde{u}_j) \to \Lambda_k = \inf_{u \in W_0^{1,2}(B_k)} J_k(u) \quad \text{as} \quad j \to \infty. \tag{40}$$

For any function $v : V \to \mathbb{R}$, we write

$$v^+(x) = \begin{cases} v(x) & \text{if } v(x) > 0 \\ 0 & \text{if } v(x) \le 0; \end{cases} \quad v^-(x) = \begin{cases} v(x) & \text{if } v(x) < 0 \\ 0 & \text{if } v(x) \ge 0. \end{cases}$$

To see a lower bound of $J_k(\widetilde{u}_j)$, we calculate

$$\begin{aligned} J_k(\widetilde{u}_j) = & \frac{1}{2} \int_{B_k} |\nabla \widetilde{u}_j|^2 d\mu + \int_{B_k} (f \widetilde{u}_j^+ - g e^{\widetilde{u}_j^+}) d\mu \\ & - \int_{B_k} g e^{\widetilde{u}_j^-} d\mu + \int_{B_k} f \widetilde{u}_j^- d\mu + \int_{B_k} g d\mu \\ \ge & \frac{1}{2} \int_{B_k} |\nabla \widetilde{u}_j|^2 d\mu + \int_{B_k} (f \widetilde{u}_j^+ - g e^{\widetilde{u}_j^+}) d\mu + \int_{B_k} f \widetilde{u}_j^- d\mu + \int_{B_k} g d\mu. \end{aligned} \tag{41}$$

Combining (40) and (41), and noting that $g(x) \le f(x) < 0, \widetilde{u}_j^-(x) \le 0$ for all $x \in B_k$, we conclude that $(\widetilde{u}_j^-)$ is bounded in B_k with respect to j, or equivalently there exists some constant C depending on k such that

$$|\widetilde{u}_j^-(x)| \le C \quad \text{for all} \quad x \in B_k. \tag{42}$$

Note also that

$$f\widetilde{u}_j^+ - ge^{\widetilde{u}_j^+} \geq f\widetilde{u}_j^+ - fe^{\widetilde{u}_j^+} \geq -\frac{f}{2}(\widetilde{u}_j^+)^2,$$

which together with (40) and (41) leads to

$$\widetilde{u}_j^+(x) \leq C \quad \text{for all} \quad x \in B_k, \tag{43}$$

where C is some constant depending on k. It follows from (42) and (43) that (u_j) is uniformly bounded in B_k with respect to j. Hence there exist a subsequence of $(\widetilde{u}_j)$, which is still denoted by $(\widetilde{u}_j)$, and a function $u_k \in W_0^{1,2}(B_k)$ such that $\widetilde{u}_j$ converges to u_k uniformly in B_k as $j \to \infty$. This together with (40) immediately leads to (37). By a straightforward calculation, u_k satisfies the Euler–Lagrange equation (38).

Step 3. *For any finite set $A \subset V$, (u_k) is uniformly bounded in A.*

Let A be a finite subset of V. An obvious analog of (41) reads

$$\begin{aligned} J_k(u_k) &\geq \frac{1}{2}\int_{B_k} |\nabla u_k|^2 d\mu + \int_{B_k} (fu_k^+ - ge^{u_k^+})d\mu + \int_{B_k} fu_k^- d\mu + \int_{B_k} gd\mu \\ &\geq \int_A (fu_k^+ - ge^{u_k^+})d\mu + \int_A fu_k^- d\mu + \int_{B_k} gd\mu, \end{aligned}$$

provided that k is sufficiently large. As a consequence, one derives

$$\max_{x\in A} |u_k^-(x)| \leq \frac{J_k(u_k) - \int_{B_k} gd\mu}{\min_{x\in A} \mu(x)|f(x)|}; \quad \max_{x\in A} u_k^+(x) \leq \sqrt{\frac{2J_k(u_k) - 2\int_{B_k} gd\mu}{\min_{x\in A} \mu(x)|f(x)|}}. \tag{44}$$

Combining (37), (39) and (44), we conclude that there exists some constant C depending only on h, g, μ and A such that

$$\max_{x\in A} |u_k(x)| \leq C.$$

Step 4. *There exists a subsequence of (u_k), which is still denoted by (u_k), and a function $u^* : V \to \mathbb{R}$ such that (u_k) converges to u^* locally uniformly in V. Moreover, u^* is a solution of the equation (16).*

By Step 3, (u_k) is uniformly bounded in B_1. Hence there exists a subsequence of (u_k), which is written as $(u_{1,k})$, and a function u_1^* such that $u_{1,k}$ converges to u_1^* in B_1. By Step 3 again, $(u_{1,k})$ is uniformly bounded in B_2. Then there would exist a subsequence of $(u_{1,k})$, which is written as $(u_{2,k})$, and a function u_2^* such that $u_{2,k}$ convergence to u_2^* uniformly in B_2. Obviously $u_2^* = u_1^*$ on B_1. Repeating this process, one finds a diagonal subsequence $(u_{k,k})$, which is still denoted by (u_k), and a function $u^* : V \to \mathbb{R}$ such that for any finite set $A \subset V$, (u_k) converges to u^* uniformly in A. For any fixed $x \in V$, passing to the limit $k \to \infty$ in (38), we obtain

$$\Delta u^*(x) = f(x) - g(x)e^{u^*(x)}.$$

This ends the final step and completes the proof of the theorem. □

5.2 The case $g > 0$

Proof of Theorem 5 Fix some point $O \in V$. For any $x \in V$, $\rho(x) = \rho(x, O)$ denotes the distance between x and O. Let $B_k = \{x \in V : \rho(x) < k\}$, $W_0^{1,2}(B_k)$ be the Sobolev space including all functions u satisfying $u = 0$ on ∂B_k, with the norm

$$\|u\|_{W_0^{1,2}(B_k)} = \left(\int_{B_k} (|\nabla u|^2 + hu^2) d\mu\right)^{1/2},$$

where $h(x) \geq a_0 > 0$, $\mu(x) \geq \mu_0 > 0$ for all $x \in V$. Define a functional $J_k : W_0^{1,2}(B_k) \to \mathbb{R}$ by

$$J_k(u) = \frac{1}{2}\int_{B_k} (|\nabla u|^2 + hu^2) d\mu + \int_{B_k} fud\mu - \log\int_{B_k} ge^u d\mu. \tag{45}$$

Since $f \in L^q(V)$ for some q with $1 \leq q \leq 2$, we have by the Sobolev embedding (Theorem 1),

$$\left|\int_{B_k} fud\mu\right| \leq \|f\|_{L^q(V)}\|u\|_{L^p(B_k)} \leq C\|f\|_{L^q(V)}\|u\|_{W_0^{1,2}(B_k)} \tag{46}$$

for some constant C depending only on μ_0, a_0 and q, where $1/p + 1/q = 1$. Since

$$\|v\|_{L^\infty(B_k)} \leq \frac{1}{\sqrt{\mu_0 a_0}}\|v\|_{W_0^{1,2}(B_k)}, \quad \forall v \in W_0^{1,2}(B_k),$$

there holds for any $\epsilon > 0$,

$$e^u \leq e^{\frac{u^2}{4\epsilon\|u\|^2_{W_0^{1,2}(B_k)}} + \epsilon\|u\|^2_{W_0^{1,2}(B_k)}} \leq e^{\frac{1}{4\epsilon\mu_0 a_0} + \epsilon\|u\|^2_{W_0^{1,2}(B_k)}}.$$

It then follows that

$$\log\int_{B_k} ge^u d\mu \leq \log\|g\|_{L^1(V)} + \frac{1}{4\epsilon\mu_0 a_0} + \epsilon\|u\|^2_{W_0^{1,2}(B_k)}. \tag{47}$$

Note that $\|g\|_{L^1(V)} > 0$, since $g \geq 0$ but $g \not\equiv 0$. Inserting (46) and (47) into (45), we obtain

$$\begin{aligned} J_k(u) \geq \left(\frac{1}{2} - \epsilon\right)\|u\|^2_{W_0^{1,2}(B_k)} - \frac{1}{4}\|u\|^2_{W_0^{1,2}(B_k)} \\ -C^2\|f\|^2_{L^q(V)} - \log\|g\|_{L^1(V)} - \frac{1}{4\epsilon\mu_0 a_0}. \end{aligned}$$

Choosing $\epsilon = 1/8$, we immediately have for any $u \in W_0^{1,2}(B_k)$,

$$J_k(u) \geq \frac{1}{8}\|u\|^2_{W_0^{1,2}(B_k)} - C^2\|f\|^2_{L^q(V)} - \log\|g\|_{L^1(V)} - \frac{2}{\mu_0 a_0}. \tag{48}$$

Hence J_k has a lower bound in $W_0^{1,2}(B_k)$. Take a minimizing sequence $(\widetilde{u}_j) \subset W_0^{1,2}(B_k)$ such that

$$J_k(\widetilde{u}_j) \to \Lambda_k = \inf_{u \in W_0^{1,2}(B_k)} J_k(u) \quad \text{as} \quad j \to \infty. \tag{49}$$

Since $g \geq 0$ and there exists some $x_0 \in V$ such that $g(x_0) > 0$, there holds

$$\mu(x_0)g(x_0) \leq \int_{B_k} g d\mu,$$

and thus

$$\Lambda_k \leq J_k(0) = -\log \int_{B_k} g d\mu \leq -\log(\mu(x_0)g(x_0)). \tag{50}$$

Combining (48), (49) and (50), we have

$$\|\widetilde{u}_j\|_{W_0^{1,2}(B_k)} \leq C$$

for some constant C independent of k. Hence there exists a subsequence of $(\widetilde{u}_j)$, which is still denoted by $(\widetilde{u}_j)$, and a function $u_k \in W_0^{1,2}(B_k)$ such that $(\widetilde{u}_j)$ converges to u_k uniformly in B_k as $j \to \infty$. It is easy to see that u_k is a minimizer of J_k, or equivalently

$$J_k(u_k) = \Lambda_k = \inf_{u \in W_0^{1,2}(B_k)} J_k(u).$$

Moreover u_k satisfies the Euler–Lagrange equation

$$\begin{cases} -\Delta u_k + h u_k = \frac{1}{\gamma_k} g e^{u_k} - f \text{ in } B_k \\ u_k \in W_0^{1,2}(B_k), \ \gamma_k = \int_{B_k} g e^{u_k} d\mu. \end{cases} \tag{51}$$

Since (Λ_k) is bounded due to (48) and (50), we conclude that

$$\|u_k\|_{W_0^{1,2}(B_k)} \leq C \tag{52}$$

for some constant C independent of k. Using the same argument as Step 4 of the proof of Theorem 4, one easily extracts a subsequence of u_k, which is still denoted by u_k, and finds some function u^* such that (u_k) converges to u^* locally uniformly in V. In view of (52), the Sobolev embedding theorem (Theorem 1) implies

$$\|u_k\|_{L^\infty(B_k)} \leq \frac{1}{\sqrt{\mu_0}}\|u_k\|_{W_0^{1,2}(B_k)} \leq C. \tag{53}$$

This immediately leads to

$$e^{-C}\|g\|_{L^1(B_k)} \leq \gamma_k \leq e^C\|g\|_{L^1(B_k)},$$

where γ_k is given as in (51). Then up to a subsequence, γ_k converges to some number γ^* with

$$e^{-C}\|g\|_{L^1(V)} \leq \gamma^* \leq e^C\|g\|_{L^1(V)}. \tag{54}$$

It follows from (51) and (54) that

$$-\Delta u^* + hu^* = \frac{1}{\gamma^*}ge^{u^*} - f \quad \text{in} \quad V. \tag{55}$$

We now prove

$$\gamma^* = \int_V ge^{u^*}d\mu. \tag{56}$$

On one hand, for any fixed $\ell > 1$, there holds

$$\int_{B_\ell} ge^{u^*}d\mu = \lim_{k\to\infty}\int_{B_\ell} ge^{u_k}d\mu \leq \lim_{k\to\infty}\int_{B_k} ge^{u_k}d\mu = \gamma^*,$$

which leads to

$$\int_V ge^{u^*}d\mu \leq \gamma^*. \tag{57}$$

On the other hand, in view of (53) and the assumption $g \in L^1(V)$, for any $\eta > 0$, there would exist a sufficiently large $\ell_0 > 1$ such that if $\ell \geq \ell_0$, then

$$\int_{B_k} ge^{u_k}d\mu \leq \eta + \int_{B_\ell} ge^{u_k}d\mu. \tag{58}$$

Indeed, (53) and $g \in L^1(V)$ lead to

$$\int_{B_k\setminus B_\ell} ge^{u_k}d\mu \leq e^C\int_{V\setminus B_\ell} gd\mu = o_\ell(1),$$

where $o_\ell(1) \to 0$ as $\ell \to \infty$. Thus (58) is satisfied. Passing to the limit $k \to \infty$ first, and then $\ell \to \infty$ in (58), we obtain

$$\gamma^* \leq \eta + \int_V g e^{u^*} d\mu.$$

Since $\eta > 0$ is arbitrary, there must hold

$$\gamma^* \leq \int_V g e^{u^*} d\mu. \tag{59}$$

Hence (56) follows from (57) and (59) immediately. Combining (56) and (55), we conclude that u^* is a solution of

$$\begin{cases} -\Delta u^* + h u^* = \frac{1}{\gamma^*} g e^{u^*} - f \ \text{ in } \ V \\ \gamma^* = \int_V g e^{u^*} d\mu. \end{cases}$$

Since u_k is naturally viewed as a function on V, using the same argument as the proof of (33), we conclude from (52) and (34) that $u^* \in \mathscr{H}$. This completes the proof of the theorem. □

6 Yamabe equation

In this section, using the mountain-pass theorem due to Ambrosetti–Rabinowitz [3], we prove the existence of nontrivial solutions to the Yamabe equation (18). The key estimate is the Sobolev embedding theorem. In [8], we have used Theorem 1 under the assumption (11). Here we shall apply Theorem 2 to the mountain-pass theory. Our assumptions on the locally finite graph are $w_{xy} \geq w_0 > 0$ for all y adjacent to x, and

$$\int_V \rho^p d\mu = \sum_{x \in V} \mu(x) \rho^p(x) < +\infty$$

for some $p > 2$, where $\rho(x) = \rho(x, O)$ denotes the distance between x and O. It seems that Theorem 2 has a lot of room for improvement.

To begin with, we have the following compactness embedding for $\mathscr{H}$, where $\mathscr{H}$ is a Hilbert space defined as in (13).

Lemma 7 *If $h \geq a_0 > 0$ and $1/h \in L^1(V)$, then $\mathscr{H}$ is embedded in $L^q(V)$ compactly for all $1 \leq q < p$; If $h \geq a_0 > 0$ and $h(x) \to +\infty$ as $\rho(x) \to +\infty$, then $\mathscr{H}$ is embedded in $L^q(V)$ compactly for all $2 \leq q < p$.*

Proof Suppose (u_k) is a bounded sequence in $\mathscr{H}$, namely

$$\|u_k\|_{\mathscr{H}}^2 = \int_V (|\nabla u_k|^2 + h u_k^2) d\mu \leq C. \tag{60}$$

Since the Hilbert space $\mathscr{H}$ is reflexive, there exists some function $u \in \mathscr{H}$ such that up to a subsequence, (u_k) converges to u weakly in $\mathscr{H}$, locally uniformly in V. If $1/h \in L^1(V)$, then for any $\epsilon > 0$, there exists some $\ell > 1$ such that

$$\int_{V\setminus B_\ell} \frac{1}{h} d\mu < \epsilon^2.$$

Moreover, there holds

$$\begin{aligned}\int_V |u_k - u| d\mu &\leq \int_{B_\ell} |u_k - u| d\mu + \left(\int_{V\setminus B_\ell} \frac{1}{h} d\mu\right)^{1/2} \left(\int_{V\setminus B_\ell} h|u_k - u|^2 d\mu\right)^{1/2} \\ &\leq C\epsilon + o_k(1).\end{aligned}$$

This immediately implies

$$\lim_{k\to\infty} \|u_k - u\|_{L^1(V)} = 0. \tag{61}$$

For any $q \in (1, p)$, there exists a unique $\lambda \in (0, 1)$ such that $q = \lambda + (1 - \lambda)p$. By the Hölder inequality, (60) and Theorem 2,

$$\begin{aligned}\int_V |u_k - u|^q d\mu &\leq \left(\int_V |u_k - u| d\mu\right)^{\lambda} \left(\int_V |u_k - u|^p d\mu\right)^{1-\lambda} \\ &\leq C \left(\int_V |u_k - u| d\mu\right)^{\lambda},\end{aligned}$$

which together with (61) leads to

$$\lim_{k\to\infty} \|u_k - u\|_{L^q(V)} = 0. \tag{62}$$

If $h(x) \to +\infty$ as $\rho(x) \to \infty$, then for any $\epsilon > 0$, there exists some $\ell_1 > 1$ such that

$$h(x) \geq \frac{C}{\epsilon} \quad \text{for all} \quad x \in V \setminus B_{\ell_1}.$$

As a consequence

$$\begin{aligned}\int_V |u_k - u|^2 d\mu &= \int_{B_{\ell_1}} |u_k - u|^2 d\mu + \int_{V\setminus B_{\ell_1}} |u_k - u|^2 d\mu \\ &\leq \frac{\epsilon}{C} \int_{V\setminus B_{\ell_1}} h|u_k - u|^2 d\mu + o_k(1).\end{aligned}$$

This implies that

$$\lim_{k\to\infty} \|u_k - u\|_{L^2(V)} = 0. \tag{63}$$

Using the same argument as in the proof of (62), we obtain from (63) that

$$\lim_{k\to\infty} \|u_k - u\|_{L^q(V)} = 0 \quad \text{for all} \quad 2 < q < p.$$

This ends the proof of the lemma. □

Let f be a function of one variable defined by

$$f(s) = |s|^{q-2}s, \quad s \in \mathbb{R} \tag{64}$$

and F be its primitive function, namely

$$F(s) = \int_0^s f(t)dt = \frac{1}{q}|s|^q, \quad s \in \mathbb{R}. \tag{65}$$

Obviously $sf(s) = qF(s)$ for all $s \in \mathbb{R}$. Define a functional $J : \mathscr{H} \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_V (|\nabla u|^2 + hu^2)d\mu - \int_V F(u)d\mu. \tag{66}$$

Lemma 8 *Assume $q \in (2, p)$, f, F and J are defined as in (64), (65) and (66) respectively. Then for any $c \in \mathbb{R}$, J satisfies the $(PS)_c$ condition. Precisely, if for any sequence $(u_k) \subset \mathscr{H}$ with $J(u_k) \to c$ and $J'(u_k) \to 0$, then up to a subsequence, $(u_k) \to u$ in $\mathscr{H}$ for some function $u \in \mathscr{H}$.*

Proof Since $(u_k) \subset \mathscr{H}$, $J(u_k) \to c$ and $J'(u_k) \to 0$, we have

$$\frac{1}{2}\|u_k\|^2_{\mathscr{H}} - \int_V F(u_k)d\mu = c + o_k(1) \tag{67}$$

$$\langle u_k, \phi\rangle_{\mathscr{H}} - \int_V f(u_k)\phi d\mu = o_k(1)\|\phi\|_{\mathscr{H}}, \quad \forall \phi \in \mathscr{H}. \tag{68}$$

Taking $\phi = u_k$ in (68) and noting that $u_k(x)f(u_k(x)) = qF(u_k(x))$ for all $x \in V$, we obtain

$$\frac{q}{2}\|u_k\|^2_{\mathscr{H}} - qc + o_k(1) = \|u_k\|^2_{\mathscr{H}} + o_k(1)\|u_k\|_{\mathscr{H}}. \tag{69}$$

Since $2 < q < p$, (69) implies that (u_k) is bounded in $\mathscr{H}$. By Lemma 7, there exist a subsequence of (u_k), which is still denoted by (u_k), and some function $u \in \mathscr{H}$ such that

$$\lim_{k\to\infty} \int_V |u_k - u|^q d\mu = 0. \tag{70}$$

One calculates

$$\begin{aligned}\int_V |F(u_k) - F(u)| d\mu &= \int_V |f(\xi_k)||u_k - u| d\mu \\ &\leq \int_V (|u_k|^{q-1} + |u|^{q-1})|u_k - u| d\mu \\ &\leq C\|u_k - u\|_{L^q(V)}, \end{aligned} \tag{71}$$

where Theorem 2 is used, C is a constant independent of k, and ξ_k lies between u_k and u. Combining (70) and (71), we obtain

$$\lim_{k\to\infty} \int_V F(u_k) d\mu = \int_V F(u) d\mu. \tag{72}$$

In the same way,

$$\begin{aligned}\left| \int_V f(u_k)(u_k - u) d\mu \right| &\leq \left(\int_V |f(u_k)|^{\frac{q}{q-1}} d\mu \right)^{1-1/q} \left(\int_V |u_k - u|^q d\mu \right)^{1/q} \\ &\leq \|u_k\|_{L^q(V)}^{q-1} \|u_k - u\|_{L^q(V)} \\ &\leq C\|u_k - u\|_{L^q(V)}. \end{aligned}$$

As a consequence

$$\lim_{k\to\infty} \int_V f(u_k)(u_k - u) d\mu = 0. \tag{73}$$

Taking $\phi = u_k - u$ in (68) and noting (73), we obtain

$$\langle u_k, u_k - u \rangle_{\mathscr{H}} = o_k(1). \tag{74}$$

Since up to a subsequence, $u_k \rightharpoonup u$ weakly in $\mathscr{H}$, it follows that

$$\langle u, u_k - u \rangle_{\mathscr{H}} = o_k(1). \tag{75}$$

Combining (74) and (75), we conclude that (u_k) converges to u in $\mathscr{H}$. In view of (67), (68), (72) and (73), we have

$$J(u) = c, \quad J'(u) = 0.$$

This ends the proof of the lemma. □

Proof of Theorem 6 Let $J \in C^1(\mathscr{H}, \mathbb{R})$ be the functional defined as in (66). We claim that J satisfies (H_1) $J(0) = 0$; (H_2) for some $\delta > 0$, $\inf_{\|u\|_{\mathscr{H}} = \delta} J(u) > 0$; (H_3)

$J(v) < 0$ for some $v \in \mathscr{H}$ with $\|v\|_{\mathscr{H}} > \delta$. Firstly, (H_1) is obvious. Secondly, to see (H_2), we have by Lemma 7,

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|u\|_{\mathscr{H}}^2 - \frac{1}{q}\int_V |u|^q d\mu \\ &\geq \frac{1}{2}\|u\|_{\mathscr{H}}^2 - C\|u\|_{\mathscr{H}}^q \end{aligned}$$

for some constant C depending on q. Hence, if $\|u\|_{\mathscr{H}} = \delta$ for sufficiently small $\delta > 0$, there holds $J(u) \geq C > 0$ for some constant C depending on q and δ. This confirms (H_2). Finally, to see (H_3), we take a function

$$u_0(x) = \begin{cases} 1, & x = O \\ 0, & x \neq O \end{cases}$$

for some fixed point $O \in V$. It then follows that

$$\begin{aligned} J(tu_0) &= \frac{t^2}{2}\|u_0\|_{\mathscr{H}}^2 - \frac{t^q}{q}\int_V u_0^q d\mu \\ &\to -\infty \quad \text{as} \quad t \to +\infty, \end{aligned}$$

since $2 < q < p$. If we choose $v = tu_0$ for sufficiently large $t > 0$, then $J(v) < 0$ and (H_3) holds.

Let

$$c = \min_{\gamma \in \Gamma} \max_{u \in \gamma} J(u),$$

where $\Gamma = \{\gamma | \gamma : [0,1] \to \mathscr{H}$ is a C^1 curve with $\gamma(0) = 0, \gamma(1) = v\}$. Clearly $0 < c < +\infty$. In view of Lemma 8, applying the mountain-pass theorem due to Ambrosetti–Rabinowitz [3], we conclude that c is a critical value of J. In particular, there exists some $u \in \mathscr{H}$ such that $J(u) = c$, $J'(u) = 0$. Clearly $u \not\equiv 0$, and u satisfies the Euler–Lagrange equation (18). □

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