UNIVERSAL HIERARCHICAL STRUCTURE OF QUASIPERIODIC EIGENFUNCTIONS

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Abstract. We determine exact exponential asymptotics of eigenfunctions and of corresponding transfer matrices of the almost Mathieu operators for all frequencies in the localization regime. This uncovers a universal structure in their behavior, governed by the continued fraction expansion of the frequency, explaining some predictions in physics literature. In addition it proves the arithmetic version of the frequency transition conjecture. Finally, it leads to an explicit description of several non-regularity phenomena in the corresponding non-uniformly hyperbolic cocycles, which is also of interest as both the first natural example of some of those phenomena and, more generally, the first non-artificial model where non-regularity can be explicitly studied.

1. Introduction

A very captivating question and a longstanding theoretical challenge in solid state physics is to determine/understand the hierarchical structure of spectral features of operators describing 2D Bloch electrons in perpendicular magnetic fields, as related to the continued fraction expansion of the magnetic flux. Such structure was first predicted in the work of Azbel in 1964 [11]. It was numerically confirmed through the famous butterfly plot and further argued for by Hofstadter in [29], for the spectrum of the almost Mathieu operator. This was even before this model was linked to the integer quantum Hall effect [48] and other important phenomena. Mathematically, it is known that the spectrum is a Cantor set for all irrational fluxes [5], and moreover, even all gaps predicted by the gap labeling are open in the non-critical case [6, 8]. Both were very important challenges in themselves, even though these results, while strongly indicate, do not describe or explain the hierarchical structure, and the problem of its description/explanation remains open, even in physics. As for the understanding the hierarchical behavior of the eigenfunctions, despite significant numerical studies and even a discovery of Bethe Ansatz solutions [1, 49] it has also remained an important open challenge even at the physics level. Certain results indicating the hierarchical structure in the corresponding semi-classical/perturbative regimes were previously obtained in the works of Sinai, Helffer-Sjostrand, and Buslaev-Fedotov (see [22, 28, 47], and also [52] for a different model).

In this paper we address the latter problem by describing the universal self-similar exponential structure of eigenfunctions throughout the entire localization regime. We determine explicit universal functions $f(k)$ and $g(k)$, depending only on the Lyapunov exponent and the position of k in the hierarchy defined by the denominators q_n of the continued fraction approximants of the flux α, that completely define the exponential behavior of, correspondingly, eigenfunctions and norms of the transfer matrices of the almost Mathieu operators, for all eigenvalues corresponding to a.e. phase , see Theorem 2.1. [1] Our result holds for *all* frequency

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and coupling pairs in the localization regime. Since the behavior is fully determined by the frequency and does not depend on the phase, it is the same, eventually, around any starting point, so is also seen unfolding at different scales when magnified around local eigenfunction maxima, thus describing the exponential universality in the hierarchical structure, see, for example, Theorems 2.2,2.4.

While the almost Mathieu family is precisely the one of main interest in physics literature, it also presents the simplest case of analytic quasiperiodic operator, so a natural question is which features discovered for the almost Mathieu would hold for this more general class. Not all do, in particular, the ones that exploit the self-dual nature of the family $H_{\lambda,\alpha,\theta}$ often cannot be expected to hold in general. In case of Theorems 2.1 and 2.2, we conjecture that they should in fact hold for general analytic (or even more general) potentials, for a.e. phase and with $\ln|\lambda|$ replaced by the Lyapunov exponent $L(E)$ (see Footnote 4), but with otherwise the same or very similar statements. The hierarchical structure theorems 2.2 and 2.4 are also expected to hold universally for most (albeit not all, as in the present paper) appropriate local maxima. Some of our qualitative corollaries may hold in even higher generality. Establishing this fully would require certain new ideas since so far even an arithmetic version of localization for the Diophantine case has not been established for the general analytic family, the current state-of-the-art result by Bourgain-Goldstein [18] being measure theoretic in α. However, some ideas of our method can already be transferred to general trigonometric polynomials [35]. Moreover, our method was used recently in [27] to show that the same f and g govern the asymptotics of eigenfunctions and universality around the local maxima throughout the a.e. localization regime in another popular object, the Maryland model.

Since we are interested in exponential growth/decay, the behavior of f and g becomes most interesting in case of frequencies with exponential rate of approximation by the rationals. In general, localization for quasiperiodic operators is a classical case of a small denominator problem, and has been traditionally approached in a perturbative way: through KAM-type schemes for large couplings [21, 24, 47] (which, being KAM-type schemes, all required Diophantine conditions on frequencies) or through perturbation of periodic operators (Liouville frequency). Unlike the random case, where, in dimension one, localization holds for all couplings, a distinctive feature of quasiperiodic operators is the presence of metal-insulator transitions as couplings increase. Even when non-perturbative methods, for the almost Mathieu and then for general analytic potentials, were developed in the 90s [18, 31], allowing to obtain localization for a.e. frequency throughout the regime of positive Lyapunov exponents, they still required Diophantine conditions, and exponentially approximated frequencies that are neither far from nor close enough to rationals remained a challenge, as for them there was nothing left to perturb about or to remove. Moreover, it has become clear that for all frequencies, the true localization threshold should be arithmetically determined and happen precisely where the exponential growth provided by the Lyapunov exponent beats the exponential strength of the small denominators. Thus the most interesting regime - the neighborhood of the transition - required dealing with the exponential frequencies not amenable to perturbations/parameter removals, adding a strong number theoretic flavor to the problem. The precise second transition conjecture was stated for the almost Mathieu operator [30]. Our analysis provides also a (constructive) solution to the full arithmetic version of the transition in frequency and explains the role of frequency resonances in the phenomenon of localization, in a sharp way.

The almost Mathieu operator (AMO) is the (discrete) quasiperiodic Schrödinger operator on $\ell^2(\mathbb{Z})$:

$$(H_{\lambda,\alpha,\theta}u)(n) = u(n+1) + u(n-1) + 2\lambda\cos 2\pi(\theta + n\alpha)u(n),$$

where λ is the coupling, α is the frequency, and θ is the phase.

It is the central quasiperiodic model due to coming from physics and attracting continued interest there. First appearing in Peierls [44], it arises as related, in two different ways, to a two-dimensional electron subject to a perpendicular magnetic field and plays a central role in the Thouless et al theory of the integer quantum Hall effect. For further background, history, and surveys of results see [20, 32, 39, 43] and references therein.

Almost Mathieu operator has a transition from zero to positive Lyapunov exponents on the spectrum at $|\lambda| = 1$ (the critical coupling) leading to the conjecture, dating back to [2], that it induces a transition from absolutely continuous to pure point spectrum. For Diophantine α this was proved in [31]. The result was extended to all α, θ for $|\lambda| < 1$ (the subcritical regime) in [4], solving one of the Simon's problems [46]. For the supercritical regime ($|\lambda| > 1$) it is known however that the nature of the spectrum should depend on the arithmetic properties of α [10, 26].

Set

$$\beta = \beta(\alpha) = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}, \tag{1}$$

where $\frac{p_n}{q_n}$ are the continued fraction approximants of α.

For any irrational number α, we say that phase $\theta \in (0, 1)$ is Diophantine with respect to α, if there exist $\kappa > 0$ and $\nu > 0$ such that

$$||2\theta + k\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^\nu}, \tag{2}$$

for any $k \in \mathbb{Z}\backslash\{0\}$, where $||x||_{\mathbb{R}/\mathbb{Z}} = \text{dist}(x, \mathbb{Z})$. Clearly, for any irrational number α, the set of phases which are Diophantine with respect to α is of full Lebesgue measure. The conjecture in [30] states that for α-Diophantine (thus almost every) θ, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization (i.e., has only pure point spectrum with exponentially decaying eigenfunctions) if $|\lambda| > e^\beta$, and has, for all θ, purely singular continuous spectrum for $1 < |\lambda| < e^\beta$.[2]

For $\beta = 0$ this follows from [31]. A progress towards the localization side of the above conjecture was made in [5] (localization for $|\lambda| > e^{\frac{16}{9}\beta}$, as a step in solving the Ten Martini problem) and in [50] (in a limited sense, for $|\lambda| > Ce^\beta$). The method developed in [5] that allowed to approach exponentially small denominators on the localization side was brought to its technical limits in [41], where the result for $|\lambda| > e^{\frac{3}{2}\beta}$ was obtained.

Lately, with the development of Avila's global theory and the proof of the almost reducibility conjecture [3], it has become possible to obtain non-perturbative reducibility directly, allowing to potentially argue localization for the dual model by duality, as was first done, in a perturbative regime in [14], avoiding the localization method completely. This was done recently by Avila-You-Zhou [9] who proved the full singular continuous part of the conjecture and a measure-theoretic (i.e. almost all θ) version of the pure point part (see also [33] where a simple alternative way to argue completeness in the duality argument was presented). The measure-theoretic (in phase) nature of the pure point result of [9] is, in fact, inherent in the duality argument. In contrast, our analysis provides a direct constructive proof for an arithmetically defined set of α-Diophantine θ, thus proving the full arithmetic version of the conjecture.

Our method can be used to also obtain *precise* asymptotics of *arbitrary* solutions of $H_{\lambda,\alpha,\theta}\varphi = E\varphi$ where E is an eigenvalue. Combined with the arguments of Last-Simon [40], this allows

[2] The original conjecture is slightly stronger in that it allows for not just polynomial, but any subexponential approximation of 2θ by $k\alpha$. The same goes for our proof, with obvious modifications. We choose to present the result, and thus also present the conjecture, for a slightly stronger Diophantine case in order to slightly simplify the argument.

us to find precise asymptotics of the norms of the transfer-matrices, providing the first example of this sort for non-uniformly hyperbolic dynamics. Since those norms sometimes differ significantly from the reciprocals of the eigenfunctions, this leads to further interesting and unusual consequencies, for example exponential tangencies between contracted and expanded directions at the resonant sites.

From this point of view, our analysis also provides, as far as we know, the first study of the dynamics of Lyapunov-Perron non-regular points, in a natural setting. An artificial example of irregular dynamics can be found in [12], p.23, however it is not even a cocycle over an ergodic transformation, and we are not aware of other such, even artificial, ergodic examples where the dynamics has been studied. Loosely, for a cocycle A over a transformation f acting on a space X (Lyapunov-Perron) non-regular points $x \in X$ are the ones at which Oseledets multiplicative ergodic theorem does not hold coherently in both directions. They therefore form a measure zero set with respect to any invariant measure on X.[3] Yet, it is precisely the non-regular points that are of interest in the study of Schrödinger cocycles in the non-uniformly hyperbolic (positive Lyapunov exponent) regime, since spectral measures, for every fixed phase, are always supported on energies where there exists a solution polynomially bounded in both directions, so the (hyperbolic) cocycle defined at such energies is always non-regular at precisely the relevant phases. Thus the non-regular points capture the entire action from the point of view of spectral theory, so become the most important ones to study. One can also discuss stronger non-regularity notions: absence of forward regularity and, even stronger, non-exactness of the Lyapunov exponent [12]. While it is not difficult to see that energies in the support of singular continuous spectral measure in the non-uniformly hyperbolic regime always provide examples of non-exactness, our analysis gives the first non-trivial example of non-exactness with non-zero upper limit (Corollary 2.13). Finally, as we understand, this work provides also the first natural example of an even stronger manifestation of the lack of regularity, the exponential tangencies (Corollary 2.14). Tangencies between contracted and expanded directions are a characteristic feature of nonuniform hyperbolicity (and, in particular, always happen at the maxima of the eigenfunctions). They complicate proofs of positivity of the Lyapunov exponents and are viewed as a difficulty to avoid through e.g. the parameter exclusion [15, 17, 51]. However, when the tangencies are only subexponentially deep they do not in themselves lead to non-exactness. Here we observe the first natural example of *exponentially* strong tangencies (with the rate determined by the arithmetics of α and the positions precisely along the sequence of resonances.)

The localization-for-the-exponential-regime method of [5] consists of different arguments for non-resonant (meaning sufficiently far from jq_n on the corresponding scale) sites and for the resonant ones (the rest). It is the resonant sites that lead to dealing with the smallest denominators and that necessitate the $|\lambda| > e^{\frac{16}{9}\beta}$ requirement in [5]. Here we start with the same basic setup, and only technically modify the non-resonant statement of [5]. However we develop a completely new bootstrap technique to handle the resonant sites, allowing us to get to the transition and obtain the fine estimates. The estimates from below (that coincide with our estimates from above) are also new. In general, the statements that are technically similar to the ones in the existing literature are collected in the Appendices, while all the results/proofs in the body of the paper are, in their pivotal parts, not like anything that has appeared before.

The key elements of the technique developed in this paper are robust and have made it possible to approach other scenarios. As such, in the upcoming work we prove the sharp phase

[3] Although in the uniformly hyperbolic situations this set can be of full Hausdorff dimension [13].

transition for Diophantine α and all θ and establish sharp exponential asymptotics of eigenfunctions and transfer matrices in the corresponding pure point regime [36]. Moreover, our analysis reveals there a universal *reflective-hierarchical* structure in the entire regime of phase-induced resonances, a phenomenon not even previously discovered in physics literature. Thus while in this paper we develop a complete understanding of frequency induced resonances, in [36] we develop new methods motivated by the ideas of this manuscript to obtain a complete understanding of phase induced resonances. In other follow-up works we determine the exact exponent of the exponential decay rate in expectation for the Diophantine case [34] and study delicate properties of the singular continuous regime, obtaining upper bounds on fractal dimensions of the spectral measure and quantum dynamics for the almost Mathieu operator [37], as well as potentials defined by general trigonometric polynomials [35].

Except for a few standard, general (e.g. uniform upper semicontinuity) or very simple to verify statements, this paper is entirely self contained. The only technically involved fact that we use without proving it in the paper is Lemma 3.1 [19] but this is not even necessary if we replace $\ln|\lambda|$ by the Lyapunov exponent $L(E)$ throughout the manuscript. [4]

2. Main results

Let

$$A_k(\theta) = \prod_{j=k-1}^{0} A(\theta + j\alpha) = A(\theta+(k-1)\alpha)A(\theta+(k-2)\alpha)\cdots A(\theta) \tag{3}$$

and

$$A_{-k}(\theta) = A_k^{-1}(\theta - k\alpha) \tag{4}$$

for $k \geq 1$, where $A(\theta) = \begin{pmatrix} E - 2\lambda\cos 2\pi\theta & -1 \\ 1 & 0 \end{pmatrix}$. A_k is called the (k-step) transfer matrix. As is clear from the definition, it also depends on θ and E but since those parameters will be usually fixed, we omit this from the notation.

Given $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ we define functions $f, g : \mathbb{Z}^+ \to \mathbb{R}^+$ in the following way. Let $\frac{p_n}{q_n}$ be the continued fraction approximants to α. For any $\frac{q_n}{2} \leq k < \frac{q_{n+1}}{2}$, define $f(k), g(k)$ as follows:

Case 1: $q_{n+1}^{\frac{8}{9}} \geq \frac{q_n}{2}$ or $k \geq q_n$.

If $\ell q_n \leq k < (\ell+1)q_n$ with $\ell \geq 1$, set

$$f(k) = e^{-|k-\ell q_n|\ln|\lambda|}\bar{r}_\ell^n + e^{-|k-(\ell+1)q_n|\ln|\lambda|}\bar{r}_{\ell+1}^n, \tag{5}$$

and

$$g(k) = e^{-|k-\ell q_n|\ln|\lambda|}\frac{q_{n+1}}{\bar{r}_\ell^n} + e^{-|k-(\ell+1)q_n|\ln|\lambda|}\frac{q_{n+1}}{\bar{r}_{\ell+1}^n}, \tag{6}$$

where for $\ell \geq 1$,

$$\bar{r}_\ell^n = e^{-(\ln|\lambda| - \frac{\ln q_{n+1}}{q_n} + \frac{\ln \ell}{q_n})\ell q_n}.$$

Set also $\bar{r}_0^n = 1$ for convenience.

If $\frac{q_n}{2} \leq k < q_n$, set

$$f(k) = e^{-k\ln|\lambda|} + e^{-|k-q_n|\ln|\lambda|}\bar{r}_1^n, \tag{7}$$

and

$$g(k) = e^{k\ln|\lambda|}. \tag{8}$$

[4] In fact, $\ln|\lambda|$ is being used in this paper as a shortcut for $L(E)$.

Case 2: $q_{n+1}^{\frac{8}{9}} < \frac{q_n}{2}$ and $\frac{q_n}{2} \leq k \leq \min\{q_n, \frac{q_{n+1}}{2}\}$.
Set

$$f(k) = e^{-k\ln|\lambda|}, \tag{9}$$

and

$$g(k) = e^{k\ln|\lambda|}. \tag{10}$$

Notice that f, g only depend on α and λ but not on θ or E. $f(k)$ decays and $g(k)$ grows exponentially, globally, at varying rates that depend on the position of k in the hierarchy defined by the continued fraction expansion of α, see Fig.1 and Fig.2.

We say that ϕ is a generalized eigenfunction of H with generalized eigenvalue E, if

$$H\phi = E\phi, \text{ and } |\phi(k)| \leq \hat{C}(1+|k|). \tag{11}$$

Our first main result is that in the entire regime $|\lambda| > e^{\beta}$, the exponential asymptotics of the generalized eigenfunctions and norms of transfer matrices at the generalized eigenvalues are completely determined by $f(k), g(k)$.

Theorem 2.1. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be such that $|\lambda| > e^{\beta(\alpha)}$. Suppose θ is Diophantine with respect to α, E is a generalized eigenvalue of $H_{\lambda,\alpha,\theta}$ and ϕ is the generalized eigenfunction. Let $U(k) = \begin{pmatrix} \phi(k) \\ \phi(k-1) \end{pmatrix}$. Then for any $\varepsilon > 0$, there exists K (depending on $\lambda, \alpha, \hat{C}, \varepsilon$ and Diophantine constants κ, ν) such that for any $|k| \geq K$, $U(k)$ and A_k satisfy*

$$f(|k|)e^{-\varepsilon|k|} \leq ||U(k)|| \leq f(|k|)e^{\varepsilon|k|}, \tag{12}$$

and

$$g(|k|)e^{-\varepsilon|k|} \leq ||A_k|| \leq g(|k|)e^{\varepsilon|k|}. \tag{13}$$

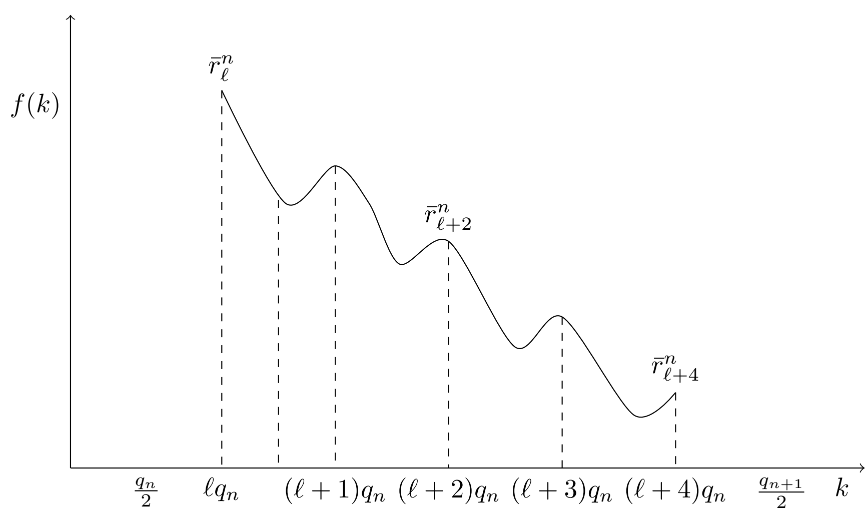

Fig.1

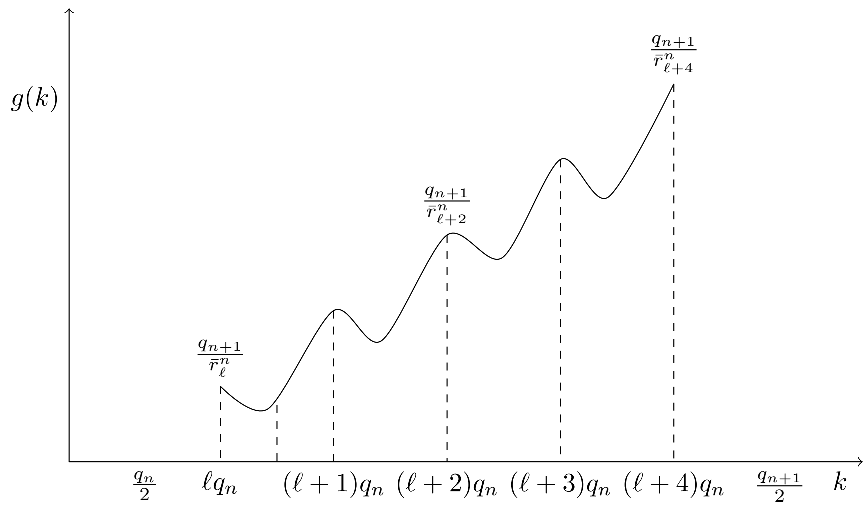

Fig.2

Certainly, there is nothing special about $k = 0$, so the behavior described in Theorem 2.1 happens around arbitrary point $k = k_0$. This implies the self-similar nature of the eigenfunctions): $U(k)$ behave as described at scale q_n but when looked at in windows of size $q_k, q_k < q_{n-1}$ will demonstrate the same universal behavior around appropriate local maxima/minima.

To make the above precise, let ϕ be an eigenfunction, and $U(k) = \begin{pmatrix} \phi(k) \\ \phi(k-1) \end{pmatrix}$. Let $I^j_{\varsigma_1,\varsigma_2} = [-\varsigma_1 q_j, \varsigma_2 q_j]$, for some $0 < \varsigma_1, \varsigma_2 \leq 1$. We will say k_0 is a local j-maximum of ϕ if $||U(k_0)|| \geq ||U(k)||$ for $k - k_0 \in I^j_{\varsigma_1,\varsigma_2}$. Occasionally, we will also use terminology (j, ς)-maximum for a local j-maximum on an interval $I^j_{\varsigma,\varsigma}$.

We will say a local j-maximum k_0 is *nonresonant* if

$$||2\theta + (2k_0 + k)\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{{q_{j-1}}^{\nu}},$$

for all $|k| \leq 2q_{j-1}$ and

$$||2\theta + (2k_0 + k)\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^{\nu}}, \tag{14}$$

for all $2q_{j-1} < |k| \leq 2q_j$.

We will say a local j-maximum is *strongly nonresonant* if

$$||2\theta + (2k_0 + k)\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^{\nu}}, \tag{15}$$

for all $0 < |k| \leq 2q_j$.

An immediate corollary of Theorem 2.1 is the universality of behavior at all (strongly) nonresonant local maxima.

Theorem 2.2. *Given $\varepsilon > 0$, there exists $j(\varepsilon) < \infty$ such that if k_0 is a local j-maximum for $j > j(\epsilon)$, then the following two statements hold:*

If k_0 is nonresonant, then

$$f(|s|)e^{-\varepsilon|s|} \leq \frac{||U(k_0 + s)||}{||U(k_0)||} \leq f(|s|)e^{\varepsilon|s|}, \tag{16}$$

for all $2s \in I^j_{\varsigma_1,\varsigma_2}$, $|s| > \frac{q_{j-1}}{2}$.

If k_0 *is strongly nonresonant, then*

$$f(|s|)e^{-\varepsilon|s|} \leq \frac{||U(k_0+s)||}{||U(k_0)||} \leq f(|s|)e^{\varepsilon|s|}, \tag{17}$$

for all $2s \in I^j_{\varsigma_1,\varsigma_2}$.

Remark 2.3. (1) For the neighborhood of a local j-maximum described in the Theorem 2.2 only the behavior of $f(s)$ for $q_{j-1}/2 < |s| \leq q_j/2$ is relevant. Thus f implicitly depends on j but through the scale-independent mechanism described in (5),(7) and (9).

(2) Actually, a modification in our proof allows to formulate (16) in Theorem 2.2 with non-resonant condition (14) only required for $2q_{j-1} < |k| \leq q_j$ rather than for $2q_{j-1} < |k| \leq 2q_j$.

In case $\beta(\alpha) > 0$, Theorem 2.1 also guarantees an abundance (and a hierarchical structure) of local maxima of each eigenfunction. Let k_0 be a global maximum[5] .

Universal hierarchical structure of an eigenfunction

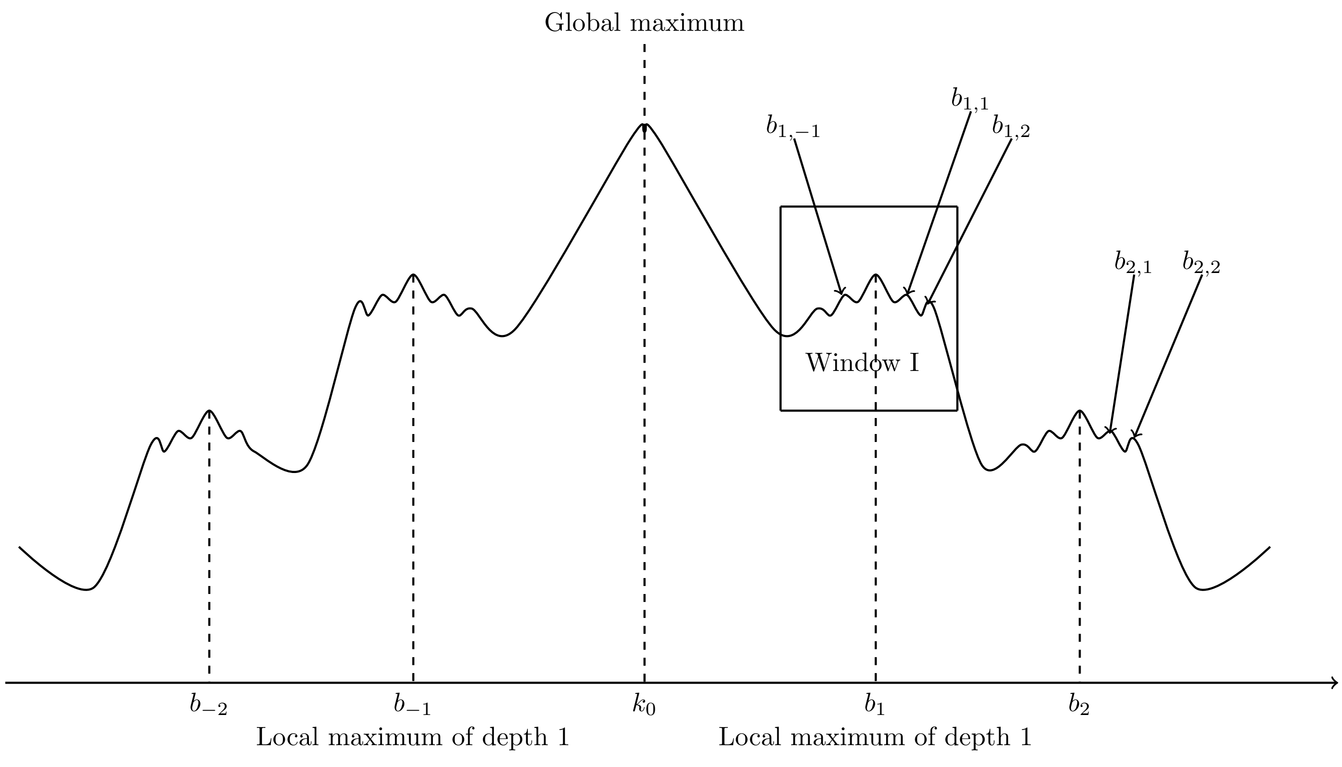

Fig.3

[5]If there are several, what follows is true for each.

Window I

Local maximum of depth 1

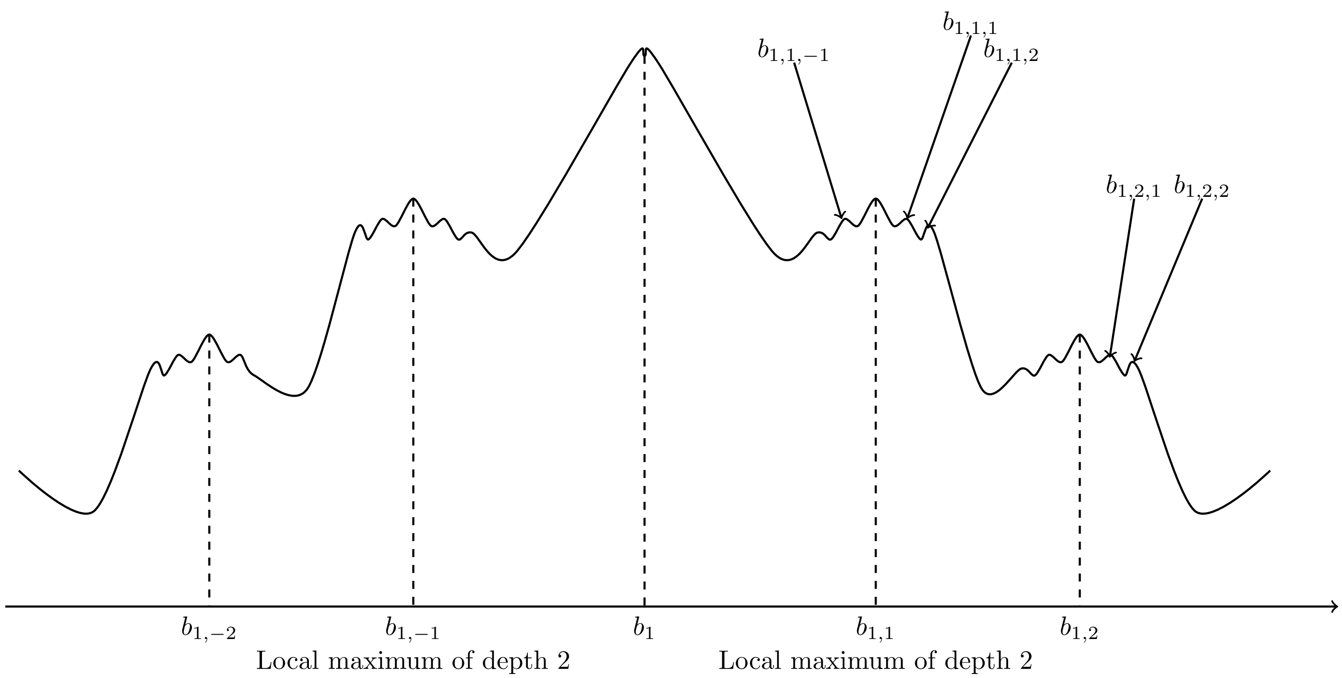

Fig.4

We first describe the hierarchical structure of local maxima informally. We will say that a scale n_{j_0} is exponential if $\ln q_{n_{j_0}+1} > cq_{n_{j_0}}$. Then there is a *constant* scale $\hat{n}_0$ thus a constant $C := q_{\hat{n}_0+1}$, such that for any exponential scale n_j and any eigenfunction there are local n_j-maxima within distance C of $k_0 + sq_{n_{j_0}}$ for each $0 < |s| < e^{cq_{n_{j_0}}}$. Moreover, these are all the local n_{j_0}-maxima in $[k_0 - e^{cq_{n_{j_0}}}, k_0 + e^{cq_{n_{j_0}}}]$. The exponential behavior of the eigenfunction in the local neighborhood (of size $\sim q_{n_{j_0}}$) of each such local maximum, normalized by the value at the local maximum is given by f. Note that only exponential behavior at the corresponding scale is determined by f and fluctuations of much smaller size are invisible. Now, let $n_{j_1} < n_{j_0}$ be another exponential scale. Denoting "depth 1" local maximum located near $k_0 + a_{n_{j_0}} q_{n_{j_0}}$ by $b_{a_{n_{j_0}}}$ we then have a similar picture around $b_{a_{n_{j_0}}}$: there are local n_{j_1}-maxima in the vicinity of $b_{a_{n_{j_0}}} + sq_{n_{j_1}}$ for each $0 < |s| < e^{cq_{n_{j_1}}}$. Again, this describes all the local $q_{n_{j_1}}$-maxima within an exponentially large interval. And again, the exponential (for the n_{j_1} scale) behavior in the local neighborhood (of size $\sim q_{n_{j_1}}$) of each such local maximum, normalized by the value at the local maximum is given by f. Denoting those "depth 2" local maxima located near $b_{a_{n_{j_0}}} + a_{n_{j_1}} q_{n_{j_1}}$, by $b_{a_{n_{j_0}}, a_{n_{j_1}}}$ we then get the same picture taking the magnifying glass another level deeper and so on. At the end we obtain a complete hierarchical structure of local maxima that we denote by $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ with each "depth $s+1$" local maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ being in the corresponding vicinity of the "depth s" local maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_{s-1}}}}$ and with universal behavior at the corresponding scale around each. The quality of the approximation of the position of the next maximum gets lower with each level of depth, yet the depth of the hierarchy that can be so achieved is at least $j/2-C$, see Corollary 2.7. Fig. 3 schematically illustrates the structure of local maxima of depth one and two, and Fig. 4 illustrates that the the neighborhood of a local maximum appropriately magnified looks like a picture of the global maximum.

We now describe the hierarchical structure precisely. Suppose

$$||2(\theta+k_0\alpha)+k\alpha||_{\mathbb{R}/\mathbb{Z}}>\frac{\kappa}{|k|^{\nu}}, \tag{18}$$

for any $k\in\mathbb{Z}\backslash\{0\}$. Fix $0<\varsigma,\epsilon$ with $\varsigma+2\epsilon<1$. Let $n_j\to\infty$ be such that $\ln q_{n_j+1}\geq(\varsigma+2\epsilon)\ln|\lambda|q_{n_j}$. Let $\mathfrak{c}_j=(\ln q_{n_j+1}-\ln|a_{n_j}|)/\ln|\lambda|q_{n_j}-\epsilon$. We have $\mathfrak{c}_j>\epsilon$ for $0<a_{n_j}<e^{\varsigma\ln|\lambda|q_{n_j}}$. Then we have

Theorem 2.4. *There exists $\hat{n}_0(\alpha,\lambda,\kappa,\nu,\epsilon)<\infty$ such that for any $j_0>j_1>\cdots>j_k$, $n_{j_k}\geq\hat{n}_0+k$, and $0<a_{n_{j_i}}<e^{\varsigma\ln|\lambda|q_{n_{j_i}}}, i=0,1,\ldots,k$, for all $0\leq s\leq k$ there exists a local n_{j_s}-maximum $b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}}$ on the interval $b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}}+I^{n_{j_s}}_{\mathfrak{c}_{j_s},1}$ for all $0\leq s\leq k$ such that the following holds:*

- **I:** $|b_{a_{n_{j_0}}}-(k_0+a_{n_{j_0}}q_{n_{j_0}})|\leq q_{\hat{n}_0+1}$,
- **II:** *For any* $1\leq s\leq k$, $|b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}}-(b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_{s-1}}}}+a_{n_{j_s}}q_{n_{j_s}})|\leq q_{\hat{n}_0+s+1}$.
- **III:** *if* $2(x-b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_k}}})\in I^{n_{j_k}}_{\mathfrak{c}_{j_k},1}$ *and* $|x-b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_k}}}|\geq q_{\hat{n}_0+k}$, *then for each* $s=0,1,...,k$,

$$f(x_s)e^{-\varepsilon|x_s|}\leq\frac{||U(x)||}{||U(b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}})||}\leq f(x_s)e^{\varepsilon|x_s|}, \tag{19}$$

where $x_s=|x-b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}}|$ *is large enough.*

Moreover, every local n_{j_s}-maximum on the interval $b_{a_{n_j},a_{n_{j_1}},\ldots,a_{n_{j_{s-1}}}}+[-e^{\epsilon\ln\lambda q_{n_{j_s}}},e^{\epsilon\ln\lambda q_{n_{j_s}}}]$ is of the form $b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}}$ for some $a_{n_{j_s}}$.

Remark 2.5. By I of Theorem 2.4, the local maximum can be determined up to a constant $K_0=q_{\hat{n}_0+1}$. Actually, if k_0 is only a local n_j+1-maximum, we can still make sure that I, II and III of Theorem 2.4 hold. This is the local version of Theorem 2.4, see Theorem 7.3.

Remark 2.6. $q_{\hat{n}_0+1}$ is the scale at which *phase* resonances of $\theta+k_0\alpha$ still can appear. Notably, it determines the precision of pinpointing local n_{j_0}-maxima in a (exponentially large in $q_{n_{j_0}}$) neighborhood of k_0, for any j_0. When we go down the hierarchy, the precision decreases, but note that except for the very last scale it stays at least iterated logarithmically [6] small in the corresponding scale $q_{n_{j_s}}$

Thus for $x\in b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}}+[-\frac{c_{j_s}}{2}q_{n_{j_s}},\frac{1}{2}q_{n_{j_s}}]$, the behavior of $\phi(x)$ is described by the same universal f in each $q_{n_{j_s}}$-window around the corresponding local maximum $b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}},s=0,1,...,k$. We call such a structure *hierarchical*, and we will say that a local j-maximum is k-hierarchical if the complete hierarchy goes down at least k levels (for a precise definition see Section 7). We then have an immediate corollary

Corollary 2.7. *There exists $C=C(\alpha,\lambda,\kappa,\nu,\epsilon)$ such that every local n_j-maximum in $[k_0-e^{\varsigma\ln|\lambda|q_{n_j}},k_0+e^{\varsigma\ln|\lambda|q_{n_j}}]$ is at least $(j/2-C)$-hierarchical.*

[6] for most scales even much less

Remark 2.8. The estimate on the depth of the hierarchy in the corollary assumes the worst case scenario when all scales after $\hat{n}_0$ are Liouville. Otherwise the hierarchical structure will go even much deeper. Note that a local n_j-maximum that is not an n_{j+1}-maximum cannot be k-hierarchical for $k > j$.

Another interesting corollary of Theorem 2.1 is

Theorem 2.9. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be such that $|\lambda| > e^{\beta(\alpha)}$ and θ is Diophantine with respect to α. Then $H_{\lambda,\alpha,\theta}$ has Anderson localization, with eigenfunctions decaying at the rate $\ln|\lambda| - \beta$.*

This solves the arithmetic version of the second transition conjecture in that it establishes localization throughout the entire regime of (α, λ) where localization may hold for any θ (see the discussion in the introduction), for an arithmetically defined full measure set of θ.

We note that Theorem 2.9 cannot be upgraded to *all* θ in the regime $|\lambda| > e^{\beta}$ [38] so exclusion of a certain arithmetically defined set where the spectrum must be singular continuous is necessary. There is a conjecture of where in this regime the transition in θ happens [30] but we do not explore it in this work. The sharp transition in θ for Diophantine α will be established in the follow-up work [36]. Also, it could be added that, for all θ, $H_{\lambda,\alpha,\theta}$ has no localization (i.e., no exponentially decaying eigenfunctions) if $|\lambda| = e^{\beta}$ (see Appendix A.1).

Remark 2.10. Theorems 2.1, 2.9 cover the optimal range of (α, λ) for a.e. θ. For Theorem 2.9, even though some θ have to be excluded [38], we do not claim the Diophantine condition on θ is optimal. At the same time, exponentially strong θ-resonances (exponentially small lower bound in (2) instead of a polynomial) will make Theorem 2.1 false as stated, no matter how small the exponent, and would require differently defined f and g. In [36] we obtain f' and g' that govern the exponential behavior of eigenfunctions and transfermatrices for *all* θ throughout the entire pure point regime corresponding to Diophantine α.

Let $\psi(k)$ denote any solution to $H_{\lambda,\alpha,\theta}\psi = E\psi$ that is linearly independent with respect to $\phi(k)$. Let $\tilde{U}(k) = \begin{pmatrix} \psi(k) \\ \psi(k-1) \end{pmatrix}$. An immediate counterpart of (13) is the following

Corollary 2.11. *Under the conditions of Theorem 2.1 for large k vectors $\tilde{U}(k)$ satisfy*

$$g(|k|)e^{-\varepsilon|k|} \leq ||\tilde{U}(k)|| \leq g(|k|)e^{\varepsilon|k|}. \tag{20}$$

Thus every solution is expanding at the rate $g(k)$ except for one that is exponentially decaying at the rate $f(k)$.

It is well known that for E in the spectrum the dynamics of the transfer-matrix cocycle A_k is nonuniformly hyperbolic. Moreover, E being a generalized eigenvalue of $H_{\lambda,\alpha,\theta}$ already implies that the behavior of A_k is non-regular. Theorem 2.1 provides precise information on how the non-regular behavior unfolds in this case. Previously, a study of some features of the non-regular behavior for the almost Mathieu operator was made in [23]. We are not aware though of other non-artificially constructed examples of non-uniformly hyperbolic systems where non-regular behavior can be described with such precision as in the present work.

The information provided by Theorem 2.1 leads to many interesting corollaries which will be explored elsewhere. Here we only want to list a few immediate sharp consequences.

Corollary 2.12. *Under the condition of Theorem 2.1, we have*

i)

$$\limsup_{k\to\infty} \frac{\ln||A_k||}{k} = \limsup_{k\to\infty} \frac{\ln||\tilde{U}(k)||}{k} = \ln|\lambda|,$$

ii)

$$\liminf_{k\to\infty}\frac{\ln||A_k||}{k}=\liminf_{k\to\infty}\frac{\ln||\tilde{U}(k)||}{k}=\ln|\lambda|-\beta.$$

iii) *Outside an explicit sequence of lower density zero,* [7]

$$\lim_{k\to\infty}\frac{\ln||A_k||}{k}=\lim_{k\to\infty}\frac{\ln||\tilde{U}(k)||}{k}=\ln|\lambda|.$$

Therefore the Lyapunov behavior for the norm fails to hold only along a sequence of density zero. It is interesting that the situation is different for the eigenfunctions. While, just like the overall growth of $\|A_k\|$ is $\ln|\lambda|-\beta$, the overall rate of decay of the eigenfunctions is also $\ln|\lambda|-\beta$, they however decay at the Lyapunov rate only outside a sequence of positive upper density. That is

Corollary 2.13. *Under the condition of Theorem 2.1, we have*

i)

$$\limsup_{k\to\infty}\frac{-\ln||U(k)||}{k}=\ln|\lambda|,$$

ii)

$$\liminf_{k\to\infty}\frac{-\ln||U(k)||}{k}=\ln|\lambda|-\beta.$$

iii) *There is an explicit sequence of upper density* $1-\frac{1}{2}\frac{\beta}{\ln|\lambda|}$, [8], *along which*

$$\lim_{k\to\infty}\frac{-\ln||U(k)||}{k}=\ln|\lambda|.$$

iv) *There is an explicit sequence of upper density* $\frac{1}{2}\frac{\beta}{\ln|\lambda|}$, [9] *along which*

$$\limsup_{k\to\infty}\frac{-\ln||U(k)||}{k}<\ln|\lambda|.$$

The fact that g is not always the reciprocal of f leads also to another interesting phenomenon.

Let $0\leq\delta_k\leq\frac{\pi}{2}$ be the angle between vectors $U(k)$ and $\tilde{U}(k)$.

Corollary 2.14. *We have*

$$\limsup_{k\to\infty}\frac{\ln\delta_k}{k}=0, \tag{21}$$

and

$$\liminf_{k\to\infty}\frac{\ln\delta_k}{k}=-\beta. \tag{22}$$

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[7] It will be clear from the proof that the sequence with convergence to the Lyapunov exponent contains $q_n, n=1,\cdots$.

[8] It will be clear from the proof that the sequence contains $\lfloor\frac{q_n}{2}\rfloor, n=1,\cdots$.

[9] As will be clear from the proof, this sequence can have lower density ranging from 0 to $\frac{1}{2}\frac{\beta}{\ln|\lambda|}$ depending on finer continued fraction properties of α.

As becomes clear from the proof, neighborhoods of resonances q_n are the places of exponential tangencies between contracted and expanded directions, with the rate approaching $-\beta$ along a subsequence.[10] Exponential tangencies also happen around points of the form jq_n but at lower strength. This means, in particular, that A_k with $k \sim q_n$ is exponentially close to a matrix with the trace $e^{(\ln|\lambda|-\beta)k}$.

The rest of this paper is organized in the following way. We list the definitions and standard preliminaries in Section 3. We also include there the non-resonant regularity statement. While similar to the corresponding statements in [5, 41, 42], it differs in enough technical details that a proof is needed for completeness. We present this proof in Appendix B. Section 4 is devoted to the bootstrap localization argument, establishing sharp upper bounds for the resonant case. Section 5 is devoted to the lower bounds. In Section 6 we prove the statements about eigenfunctions: (12) of Theorem 2.1, Theorems 2.2 and 2.9. In Section 7, we will prove the hierarchical structure Theorem 2.4 and Corollary 2.7. In Section 8, we study the growth of transfer matrices and prove (13) of Theorem 2.1. The remaining Corollaries are proved in Section 9.

3. Preliminaries

Fix $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ such that $\beta(\alpha) < \infty$. Unless stated otherwise, we always assume $\lambda > e^{\beta}$ (for $\lambda < -e^{\beta}$, notice that $H_{\lambda,\alpha,\theta} = H_{-\lambda,\alpha,\theta+\frac{1}{2}}$), θ is Diophantine with respect to α and E is a generalized eigenvalue. We also assume ϕ is the corresponding generalized eigenfunction of $H_{\lambda,\alpha,\theta}$. Without loss of generality assume $|\phi(0)|^2 + |\phi(-1)|^2 = 1$. Let ψ be any solution to $H_{\lambda,\alpha,\theta}\psi = E\psi$ linear independent with respect to ϕ, i.e., $|\psi(0)|^2 + |\psi(-1)|^2 = 1$ and

$$\phi(-1)\psi(0) - \phi(0)\psi(-1) = c, \tag{23}$$

where $c \neq 0$.

Then by the constancy of the Wronskian, one has

$$\phi(k+1)\psi(k) - \phi(k)\psi(k+1) = c. \tag{24}$$

We also will denote by φ an *arbitrary* solution, so either ψ or ϕ. Thus for any k, m, one has

$$\begin{pmatrix} \varphi(k+m) \\ \varphi(k+m-1) \end{pmatrix} = A_k(\theta + m\alpha) \begin{pmatrix} \varphi(m) \\ \varphi(m-1) \end{pmatrix}. \tag{25}$$

The Lyapunov exponent is given by

$$L(E) = \lim_{k\to\infty} \frac{1}{k} \int_{\mathbb{R}/\mathbb{Z}} \ln \|A_k(\theta)\| d\theta. \tag{26}$$

The Lyapunov exponent can be computed precisely for E in the spectrum of $H_{\lambda,\alpha,\theta}$. We denote the spectrum by $\Sigma_{\lambda,\alpha}$ (it does not depend on θ).

Lemma 3.1. *[19] For $E \in \Sigma_{\lambda,\alpha}$ and $\lambda > 1$, we have $L(E) = \ln\lambda$.*

Recall that we always assume $E \in \Sigma_{\lambda,\alpha}$ so by upper semicontinuity and unique ergodicity (e.g. [25]) one has

$$\ln\lambda = \lim_{k\to\infty} \sup_{\theta\in\mathbb{R}/\mathbb{Z}} \frac{1}{k} \ln \|A_k(\theta)\|, \tag{27}$$

that is, the convergence in (27) is uniform with respect to $\theta \in \mathbb{R}$. Precisely, $\forall \varepsilon > 0$,

$$\|A_k(\theta)\| \leq e^{(\ln\lambda+\varepsilon)k}, \text{for } k \text{ large enough.} \tag{28}$$

[10]In fact the rate is close to $-\frac{\ln q_{n+1}}{q_n}$ for any large n.

We start with the basic setup going back to [31]. Let us denote

$$P_k(\theta) = \det(R_{[0,k-1]}(H_{\lambda,\alpha,\theta} - E)R_{[0,k-1]}).$$

It is easy to check that

$$A_k(\theta) = \begin{pmatrix} P_k(\theta) & -P_{k-1}(\theta+\alpha) \\ P_{k-1}(\theta) & -P_{k-2}(\theta+\alpha) \end{pmatrix}. \tag{29}$$

By Cramer's rule for given x_1 and $x_2 = x_1 + k - 1$, with $y \in I = [x_1, x_2] \subset \mathbb{Z}$, one has

$$|G_I(x_1, y)| = \left| \frac{P_{x_2-y}(\theta + (y+1)\alpha)}{P_k(\theta + x_1\alpha)} \right|, \tag{30}$$

$$|G_I(y, x_2)| = \left| \frac{P_{y-x_1}(\theta + x_1\alpha)}{P_k(\theta + x_1\alpha)} \right|. \tag{31}$$

By (28) and (29), the numerators in (30) and (31) can be bounded uniformly with respect to θ. Namely, for any $\varepsilon > 0$,

$$|P_k(\theta)| \leq e^{(\ln\lambda + \varepsilon)k} \tag{32}$$

for k large enough.

Definition 3.2. Fix $\tau > 0$, $0 < \delta < 1/2$. A point $y \in \mathbb{Z}$ will be called (τ, k) regular with δ if there exists an interval $[x_1, x_2]$ containing y, where $x_2 = x_1 + k - 1$, such that

$$|G_{[x_1,x_2]}(y, x_i)| < e^{-\tau|y-x_i|} \text{ and } |y - x_i| \geq \delta k \text{ for } i = 1, 2.$$

It is easy to check that

$$\varphi(x) = -G_{[x_1,x_2]}(x_1, x)\varphi(x_1 - 1) - G_{[x_1,x_2]}(x, x_2)\varphi(x_2 + 1), \tag{33}$$

where $x \in I = [x_1, x_2] \subset \mathbb{Z}$.

Definition 3.3. We say that the set $\{\theta_1, \cdots, \theta_{k+1}\}$ is ϵ-uniform if

$$\max_{x\in[-1,1]} \max_{i=1,\cdots,k+1} \prod_{j=1, j\neq i}^{k+1} \frac{|x - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} < e^{k\epsilon}. \tag{34}$$

Let $A_{k,r} = \{\theta \in \mathbb{R} \mid P_k(\cos 2\pi(\theta - \frac{1}{2}(k-1)\alpha))| \leq e^{(k+1)r}\}$ with $k \in \mathbb{N}$ and $r > 0$. We have the following Lemma.

Lemma 3.4. *(Lemma 9.3 ,[5]) Suppose $\{\theta_1, \cdots, \theta_{k+1}\}$ is ϵ_1-uniform. Then there exists some θ_i in set $\{\theta_1, \cdots, \theta_{k+1}\}$ such that $\theta_i \notin A_{k,\ln\lambda-\epsilon}$ if $\epsilon > \epsilon_1$ and k is sufficiently large.*

Proof. Straightforward calculation. □

We say θ is *n-Diophantine* with respect to α, if for some $\kappa > 0, \nu > 1$ the following hold

$$||2\theta + k\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{{q_n}^{\nu}}, \tag{35}$$

for all $|k| \leq 2q_n$ and

$$||2\theta + k\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^{\nu}}, \tag{36}$$

for all $2q_n < |k| \leq 2q_{n+1}$.

Define $b_n = q_n^t$ with $\frac{8}{9} \leq t < 1$ (t will be defined later). For any $k > 0$, we will distinguish two cases with respect to n:

(i) $|k - \ell q_n| \leq b_n$ for some $\ell \geq 1$, called *$n-$resonance*.

(ii) $|k - \ell q_n| > b_n$ for all $\ell \geq 0$, called *$n-$nonresonance*.

For the $n-$nonresonant y, let n_0 be the least positive integer such that $4q_{n-n_0} \le dist(y, q_n\mathbb{Z})$. Let s be the largest positive integer such that $4sq_{n-n_0} \le dist(y, q_n\mathbb{Z})$. Notice that $n_0 \le C(\alpha)$.

The following theorem is similar to a statement appearing in[5] with modifications in [41, 42]. We present a proof in Appendix B.

Theorem 3.5. *Assume $\lambda > e^{\beta(\alpha)}$. Suppose either*

i) $b_n \le |y| < Cb_{n+1}$, where $C > 1$ is a fixed constant, and θ is n-Diophantine with respect to α

or

ii) $0 \le |y| < q_n$ and θ satisfies (35)

Then for any $\varepsilon > 0$ and n large enough, if y is $n-$nonresonant, we have y is $(\ln\lambda + 8\ln(sq_{n-n_0}/q_{n-n_0+1})/q_{n-n_0} - \varepsilon, 4sq_{n-n_0} - 1)$ regular with $\delta = \frac{1}{4}$.

Remark 3.6. If θ is $n-1$-Diophantine with respect to α, then (35) holds.

Remark 3.7. In the nonresonant case, for any $\varepsilon > 0$, $\frac{8}{9} \le t < 1$, one has $\ln\lambda + 8\ln(sq_{n-n_0}/q_{n-n_0+1})/q_{n-n_0} \ge \ln\lambda - 8(1-t)\beta - \varepsilon > 0$. In addition, we have $\ln\lambda + 8\ln(sq_{n-n_0}/q_{n-n_0+1})/q_{n-n_0} \ge \ln\lambda - 2\varepsilon$ if t is close to 1.

Remark 3.8. In the present paper, we only use Theorem 3.5 with $C = 50C_\star$, where $C_\star$ is given by (37) (see the next section).

4. Bootstrap resonant localization

In this section we assume θ is $n-$Diophantine with respect to α. Clearly, it is enough to consider $k > 0$. In this section we study the resonant case. Suppose there exists some $k \in [b_n, b_{n+1}]$ such that k is $n-$resonant. Then we have $b_{n+1} \ge \frac{q_n}{2}$. For any $\varepsilon > 0$, choose $\eta = \frac{\varepsilon}{C}$, where C is a large constant (depending on λ, α).

Let

$$C_* = 2(1 + \lfloor \frac{\ln\lambda}{\ln\lambda - \beta} \rfloor), \tag{37}$$

where $\lfloor m \rfloor$ denotes the smallest integer not exceeding m.

For an arbitrary solution φ satisfying $H\varphi = E\varphi$, let

$$r_j^{n,\varphi} = \sup_{|r| \le 10\eta} |\varphi(jq_n + rq_n)|,$$

where $|j| \le 50C_* \frac{b_{n+1}}{q_n}$.

Fix ψ satisfying (23) and denote by

$$R_j^n = r_j^{n,\psi},$$

and

$$r_j^n = r_j^{n,\phi}.$$

Since we keep n fixed in this section we omit the dependence on n from the notation and write r_j^φ, R_j, and r_j.

Note that below we always assume n is large enough.[11]In the next Lemma and its variant, Lemma 4.2, we establish exponential decay of the eigenfunctions at non-resonant points, at the nearly Lyapunov rate, with respect to the distance to the resonances.

[11] The required largeness of n will depend on $\alpha, \theta, \hat{C}$ in (11) and ε whenever ε is (implicitly) present in the statement.

Lemma 4.1. *Let $k \in [jq_n, (j+1)q_n]$ with $dist(k, q_n\mathbb{Z}) \geq 10\eta q_n$. Suppose either*
i) $|j| \leq 48C_* \frac{b_{n+1}}{q_n}$ and $b_{n+1} \geq \frac{q_n}{2}$,
or
ii) $j = 0$,
then for sufficiently large n,

$$|\varphi(k)| \leq \max\{r_j^{\varphi} \exp\{-(\ln\lambda - 2\eta)(d_j - 3\eta q_n)\}, r_{j+1}^{\varphi} \exp\{-(\ln\lambda - 2\eta)(d_{j+1} - 3\eta q_n)\}\}, \tag{38}$$

where $d_j = |k - jq_n|$ and $d_{j+1} = |k - jq_n - q_n|$.

Proof. The proof builds on the ideas used in the proof of Lemma 9.11 in [5] and Lemma 3.2 in [41]. However it requires a more careful approach.

We first prove the case i).

For any $y \in [jq_n + \eta q_n, (j+1)q_n - \eta q_n]$, apply i) of Theorem 3.5 with $C = 50C_\star$. Notice that in this case, we have

$$\ln\lambda + 8\ln(sq_{n-n_0}/q_{n-n_0+1})/q_{n-n_0} - \eta \geq \ln\lambda - 2\eta.$$

Thus y is regular with $\tau = \ln\lambda - 2\eta$. Therefore there exists an interval $I(y) = [x_1, x_2] \subset [jq_n, (j+1)q_n]$ such that $y \in I(y)$ and

$$\operatorname{dist}(y, \partial I(y)) \geq \frac{1}{4}|I(y)| \geq q_{n-n_0} \tag{39}$$

and

$$|G_{I(y)}(y, x_i)| \leq e^{-(\ln\lambda - 2\eta)|y - x_i|},\ i = 1, 2, \tag{40}$$

where $\partial I(y)$ is the boundary of the interval $I(y)$, i.e.,$\{x_1, x_2\}$, and $|I(y)|$ is the size of $I(y)\cap\mathbb{Z}$, i.e., $|I(y)| = x_2 - x_1 + 1$. For $z \in \partial I(y)$, let z' be the neighbor of z, (i.e., $|z - z'| = 1$) not belonging to $I(y)$.

If $x_2 + 1 \leq (j+1)q_n - \eta q_n$ or $x_1 - 1 \geq jq_n + \eta q_n$, we can expand $\varphi(x_2 + 1)$ or $\varphi(x_1 - 1)$ using (33). We can continue this process until we arrive to z such that $z + 1 > (j+1)q_n - \eta q_n$ or $z - 1 < jq_n + \eta q_n$, or the iterating number reaches $\lfloor \frac{2q_n}{q_{n-n_0}} \rfloor$. Thus, by (33)

$$\varphi(k) = \sum_{s; z_{i+1} \in \partial I(z_i')} G_{I(k)}(k, z_1) G_{I(z_1')}(z_1', z_2) \cdots G_{I(z_s')}(z_s', z_{s+1}) \varphi(z_{s+1}'), \tag{41}$$

where in each term of the summation one has $jq_n + \eta q_n + 1 \leq z_i \leq (j+1)q_n - \eta q_n - 1$, $i = 1, \cdots, s$, and either $z_{s+1} \notin [jq_n + \eta q_n + 1, (j+1)q_n - \eta q_n - 1]$, $s + 1 < \lfloor \frac{2q_n}{q_{n-n_0}} \rfloor$; or $s + 1 = \lfloor \frac{2q_n}{q_{n-n_0}} \rfloor$. We should mention that $z_{s+1} \in [jq_n, (j+1)q_n]$.

If $z_{s+1} \in [jq_n, jq_n + \eta q_n]$, $s + 1 < \lfloor \frac{2q_n}{q_{n-n_0}} \rfloor$, this implies

$$|\varphi(z_{s+1}')| \leq r_j^{\varphi}.$$

By (40), we have

$$\begin{aligned} |G_{I(k)}(k, z_1) G_{I(z_1')}(z_1', z_2) \cdots G_{I(z_s')}(z_s', z_{s+1}) \varphi(z_{s+1}')| \\ \leq\ & r_j^{\varphi} e^{-(\ln\lambda - 2\eta)(|k - z_1| + \sum_{i=1}^{s} |z_i' - z_{i+1}|)} \\ \leq\ & r_j^{\varphi} e^{-(\ln\lambda - 2\eta)(|k - z_{s+1}| - (s+1))} \\ \leq\ & r_j^{\varphi} e^{-(\ln\lambda - 2\eta)(d_j - 2\eta q_n - 4 - \frac{2q_n}{q_{n-n_0}})}. \end{aligned} \tag{42}$$

If $z_{s+1}\in[(j+1)q_n-\eta q_n,(j+1)q_n]$, $s+1<\lfloor\frac{2q_n}{q_{n-n_0}}\rfloor$, by the same arguments, we have

(43)
$$|G_{I(k)}(k,z_1)G_{I(z_1')}(z_1',z_2)\cdots G_{I(z_s')}(z_s',z_{s+1})\varphi(z_{s+1}')|\leq r_{j+1}^{\varphi}e^{-(\ln\lambda-2\eta)(d_{j+1}-2\eta q_n-4-\frac{2q_n}{q_{n-n_0}})}.$$

If $s+1=\lfloor\frac{2q_n}{q_{n-n_0}}\rfloor$, using (39) and (40), we obtain

(44)
$$|G_{I(k)}(k,z_1)G_{I(z_1')}(z_1',z_2)\cdots G_{I(z_s')}(z_s',z_{s+1})\varphi(z_{s+1}')|\leq e^{-(\ln\lambda-2\eta)q_{n-n_0}\lfloor\frac{2q_n}{q_{n-n_0}}\rfloor}|\varphi(z_{s+1}')|.$$

Notice that the total number of terms in (41) is at most $2^{\lfloor\frac{2q_n}{q_{n-n_0}}\rfloor}$ and $d_j,d_{j+1}\geq 10\eta q_n$. By (42), (43) and (44), we have

(45)
$$|\varphi(k)|\leq\max\{r_j^{\varphi}e^{-(\ln\lambda-2\eta)(d_j-3\eta q_n)},r_{j+1}^{\varphi}e^{-(\ln\lambda-2\eta)(d_{j+1}-3\eta q_n)},\max_{p\in[jq_n,(j+1)q_n]}\{e^{-(\ln\lambda-2\eta)q_n}|\varphi(p)|\}\}.$$

Now we will show that for any $p\in[jq_n,(j+1)q_n]$, one has $|\varphi(p)|\leq\max\{r_j^{\varphi},r_{j+1}^{\varphi}\}$. Then (45) implies case i) of Lemma 4.1. Otherwise, by the definition of r_j^{φ}, if $|\varphi(p')|$ is the largest one of $|\varphi(z)|, z\in[jq_n+10\eta q_n+1,(j+1)q_n-10\eta q_n-1]$, then $|\varphi(p')|>\max\{r_j^{\varphi},r_{j+1}^{\varphi}\}$. Applying (45) to $\varphi(p')$ and noticing that $dist(p',q_n\mathbb{Z})\geq 10\eta q_n$, we get

$$|\varphi(p')|\leq e^{-7(\ln\lambda-2\eta)\eta q_n}\max\{r_j^{\varphi},r_{j+1}^{\varphi},|\varphi(p')|\}.$$

This is impossible because $|\varphi(p')|>\max\{r_j^{\varphi},r_{j+1}^{\varphi}\}$.

Now we turn to the proof of case ii). Notice that in proving case i) of Lemma 4.1, we only used case i) of Theorem 3.5. Using case ii) of Theorem 3.5 instead we can prove case ii) of Lemma 4.1 by the same reasoning. In order to avoid repetition, we omit the details. □

Lemma 4.1 is sufficient for our current purposes, but for the purposes of Section 7 we will need a similar statement that allows for shifts and reflections. For $B\in\mathbb{Z}$, let $r_{j,\pm}^{n,\varphi}(B)=\sup_{|r|\leq 10\eta}|\varphi(B\pm(jq_n+rq_n))|$. For $y\in[B\pm jq_n\pm\eta q_n,B\pm(j+1)q_n\mp\eta q_n]$, let n_0 be the least positive integer such that $4q_{n-n_0}\leq dist(y-B,q_n\mathbb{Z})$ and s be the largest positive integer such that $4sq_{n-n_0}\leq dist(y-B,q_n\mathbb{Z})$. Since we only used the appropriate regularity of the non-resonant y, the proof of Lemma 4.1 also establishes the following Lemma

Lemma 4.2. *Suppose for any $y\in[B\pm jq_n\pm\eta q_n,B\pm(j+1)q_n\mp\eta q_n]$, y is $(\ln\lambda+8\ln(sq_{n-n_0}/q_{n-n_0+1})/q_{n-n_0}-\varepsilon,4sq_{n-n_0}-1)$ regular with $\delta=\frac{1}{4}$. Let $k-B\in\pm[jq_n,(j+1)q_n]$ with $dist(k-B,q_n\mathbb{Z})\geq 10\eta q_n$. Suppose either*

i) $|j|\leq 48C_*\frac{b_{n+1}}{q_n}$ and $b_{n+1}\geq\frac{q_n}{2}$,

or

ii) $j=0$,

then for sufficiently large n,

Then we have

(46)
$$|\varphi(k)|\leq\max\{r_{j,\pm}^{\varphi}(B)\exp\{-(\ln\lambda-2\eta)(d_j-3\eta q_n)\},r_{j+1,\pm}^{\varphi}(B)\exp\{-(\ln\lambda-2\eta)(d_{j+1}-3\eta q_n)\}\}.$$

where $d_j=|k-B\mp jq_n|$ and $d_{j+1}=|k-B\mp(j+1)q_n|$.

By Theorem 3.5 , Lemma 4.1 is a particular case of Lemma 4.2, when $B=0$ and the sign is a $+$. Going back to this case, we will prove

Lemma 4.3. *For $1\leq j\leq 46C_\star\frac{b_{n+1}}{q_n}$ with $b_{n+1}\geq\frac{q_n}{2}$, the following holds*

$$r_j^{\varphi}\leq\max\{r_{j\pm1}^{\varphi}\frac{q_{n+1}}{j}\exp\{-(\ln\lambda-C\eta)q_n\}\}. \tag{47}$$

Proof. Fix j with $1 \leq j \leq 46C_* \frac{b_{n+1}}{q_n}$ and $|r| \leq 10\eta q_n$. Set $I_1, I_2 \subset \mathbb{Z}$ as follows

$$\begin{aligned} I_1 &= [-\lfloor \frac{1}{2} q_n \rfloor, q_n - \lfloor \frac{1}{2} q_n \rfloor - 1], \\ I_2 &= [jq_n - \lfloor \frac{1}{2} q_n \rfloor, (j+1)q_n - \lfloor \frac{1}{2} q_n \rfloor - 1]. \end{aligned}$$

Let $\theta_m = \theta + m\alpha$ for $m \in I_1 \cup I_2$. The set $\{\theta_m\}_{m \in I_1 \cup I_2}$ consists of $2q_n$ elements.

By arguments similar to those in Lemma 9.13 in [5] or Theorem 3.1 in [41], one has $\{\theta_m\}$ is $\frac{\ln q_{n+1} - \ln j}{2q_n} + \varepsilon$ uniform for any $\varepsilon > 0$. Since our case is slightly different we prove it as Theorem B.5 in Appendix B. Combining with Lemma 3.4, there exists some j_0 with $j_0 \in I_1 \cup I_2$ such that $\theta_{j_0} \notin A_{2q_n - 1, \ln\lambda - \frac{\ln q_{n+1} - \ln j}{2q_n} - \eta}$.

First, we assume $j_0 \in I_2$.

Set $I = [j_0 - q_n + 1, j_0 + q_n - 1] = [x_1, x_2]$. In (32), let $\varepsilon = \eta$. Combining with (30) and (31), it is easy to verify

$$|G_I(jq_n + r, x_i)| \leq e^{(\ln\lambda + \eta)(2q_n - 1 - |jq_n + r - x_i|) - (2q_n - 1)(\ln\lambda - \frac{\ln q_{n+1} - \ln j}{2q_n} - \eta)}.$$

Using (33), we obtain

$$|\varphi(jq_n + r)| \leq \sum_{i=1,2} \frac{q_{n+1}}{j} e^{5\eta q_n} |\varphi(x_i')| e^{-|jq_n + r - x_i| \ln\lambda}, \tag{48}$$

where $x_1' = x_1 - 1$ and $x_2' = x_2 + 1$.

Let $d_j^i = |x_i - jq_n|$, $i = 1, 2$. It is easy to check that

$$|jq_n + r - x_i| + d_j^i, |jq_n + r - x_i| + d_{j\pm1}^i \geq q_n - |r|, \tag{49}$$

and

$$|jq_n + r - x_i| + d_{j\pm2}^i \geq 2q_n - |r|. \tag{50}$$

If $dist(x_i, q_n\mathbb{Z}) \geq 10\eta q_n$, then we bound $\varphi(x_i)$ in (48) using (38). If $dist(x_i, q_n\mathbb{Z}) \leq 10\eta q_n$, then we bound $\varphi(x_i)$ in (48) by some proper r_j. Combining with (49), (50), we have

$$r_j^\varphi \leq \max\{r_{j\pm1}^\varphi \frac{q_{n+1}}{j} \exp\{-(\ln\lambda - C\eta)q_n\}, r_j^\varphi \frac{q_{n+1}}{j} \exp\{-(\ln\lambda - C\eta)q_n\}, r_{j\pm2}^\varphi \frac{q_{n+1}}{j} \exp\{-2(\ln\lambda - C\eta)q_n\}\}.$$

However

$$\begin{aligned} r_j^\varphi &\leq r_j^\varphi \frac{q_{n+1}}{j} \exp\{-(\ln\lambda - C\eta)q_n\} \\ &\leq r_j^\varphi \exp\{-(\ln\lambda - \beta - C\eta)q_n\} \end{aligned}$$

cannot happen, so we must have

$$r_j^\varphi \leq \max\{r_{j\pm1}^\varphi \frac{q_{n+1}}{j} \exp\{-(\ln\lambda - C\eta)q_n\}, r_{j\pm2}^\varphi \frac{q_{n+1}}{j} \exp\{-2(\ln\lambda - C\eta)q_n\}\}. \tag{51}$$

In particular,

$$r_j^\varphi \leq \exp\{-(\ln\lambda - \beta - C\eta)q_n\} \max\{r_{j\pm1}^\varphi\, r_{j\pm2}^\varphi\}. \tag{52}$$

If $j_0 \in I_1$, then (52) holds for $j = 0$. Let $\varphi = \phi$ in (52). We get

$$|\phi(0)|, |\phi(-1)| \leq \exp\{-(\ln\lambda - \beta - C\eta)q_n\},$$

this is in contradiction with $|\phi(0)|^2 + |\phi(-1)|^2 = 1$. Therefore $j_0 \in I_2$, so (51) holds for any φ.

By (25) and (28), we have

$$|| \begin{pmatrix} \varphi(k_1) \\ \varphi(k_1 - 1) \end{pmatrix} || \geq Ce^{-(\ln\lambda + \varepsilon)|k_1 - k_2|} || \begin{pmatrix} \varphi(k_2) \\ \varphi(k_2 - 1) \end{pmatrix} ||. \tag{53}$$

This implies

$$r^{\varphi}_{j\pm 2} \leq r^{\varphi}_{j\pm 1} \exp\{(\ln\lambda + C\eta)q_n\},$$

thus (51) becomes

$$r^{\varphi}_{j} \leq \max\{r^{\varphi}_{j\pm 1} \frac{q_{n+1}}{j} \exp\{-(\ln\lambda - C\eta)q_n\}\}, \tag{54}$$

for any $1 \leq j \leq 46C_* \frac{b_{n+1}}{q_n}$. □

For solution ϕ and ψ we can also get a more subtle estimate.

Theorem 4.4. *For $1 \leq j \leq 10\frac{b_{n+1}}{q_n}$ with $b_{n+1} \geq \frac{q_n}{2}$, the following holds*

$$r_j \leq r_{j-1} \exp\{-(\ln\lambda - C\eta)q_n\}\frac{q_{n+1}}{j}. \tag{55}$$

Proof. Let $\varphi = \phi$ in Lemma 4.3. We must have

$$r_j \leq \max\{r_{j\pm 1} \frac{q_{n+1}}{j} \exp\{-(\ln\lambda - C\eta)q_n\}\}, \tag{56}$$

for any $1 \leq j \leq 46C_* \frac{b_{n+1}}{q_n}$.

Suppose for some $1 \leq j \leq 10\frac{b_{n+1}}{q_n}$, the following holds,

$$r_j \leq r_{j+1} \frac{q_{n+1}}{j} \exp\{-(\ln\lambda - C\eta)q_n\} \leq r_{j+1} \exp\{-(\ln\lambda - \beta - C\eta)q_n\}. \tag{57}$$

Applying (56) to $j+1$, we obtain

$$r_{j+1} \leq \max\{r_j, r_{j+2}\} \frac{q_{n+1}}{j+1} \exp\{-(\ln\lambda - C\eta)q_n\}. \tag{58}$$

Combining with (57), we must have

$$r_{j+1} \leq r_{j+2} \exp\{-(\ln\lambda - \beta - C\eta)q_n\}. \tag{59}$$

Generally, for any $0 < p \leq (C_* + 1)j - 1$, we obtain

$$r_{j+p} \leq r_{j+p+1} \exp\{-(\ln\lambda - \beta - C\eta)q_n\}. \tag{60}$$

Thus

$$r_{(C_*+1)j} \geq r_j \exp\{(\ln\lambda - \beta - C\eta)C_* j q_n\}. \tag{61}$$

Clearly, by (53), one has

$$r_j \geq \exp\{-(\ln\lambda + C\eta) j q_n\}.$$

Then

$$r_{(C_*+1)j} \geq \exp\{((C_* - 1)\ln\lambda - C_*\beta - C\eta) j q_n\}. \tag{62}$$

By the definition of C_*, one has

$$(C_* - 1)\ln\lambda - C_*\beta > 0.$$

Thus (62) is in contradiction with the fact that $|\phi(k)| \leq 1 + |k|$.

Now that (57) can not happen, from (56), we must have

$$r_j \leq r_{j-1} \frac{q_{n+1}}{j} \exp\{-(\ln\lambda - C\eta)q_n\}. \tag{63}$$

□

Theorem 4.5. *For $0 \leq j \leq 8\frac{b_{n+1}}{q_n}$ with $b_{n+1} \geq \frac{q_n}{2}$, the following holds*

$$R_j \leq R_{j+1} \exp\{-(\ln\lambda - C\eta)q_n\}\frac{q_{n+1}}{j+1}. \tag{64}$$

Proof. If $j = 0$, (64) holds directly by (55) (applying it with $j = 1$) and (24). Now we consider $j \geq 1$. Let $\varphi = \psi$ in Lemma 4.3. Then (54) also holds for R_j with $j \geq 1$, that is

$$R_j \leq \max\{R_{j\pm1}\frac{q_{n+1}}{j+1}\exp\{-(\ln\lambda - C\eta)q_n\}\}. \tag{65}$$

Suppose for some $j \geq 1$

$$R_j \leq R_{j-1}\frac{q_{n+1}}{j+1}\exp\{-(\ln\lambda - C\eta)q_n\}. \tag{66}$$

Applying (65) to $j-1$ and taking into account (66), one has

$$R_{j-1} \leq R_{j-2}\frac{q_{n+1}}{j}\exp\{-(\ln\lambda - C\eta)q_n\}. \tag{67}$$

Iterating j times, we must have

$$R_j \leq R_0\frac{q_{n+1}^j}{(j+1)!}\exp\{-(\ln\lambda - C\eta)jq_n\} \leq R_0\exp\{-(\ln\lambda - \beta - C\eta)jq_n\}. \tag{68}$$

Similarly, iterating (55) j times, we have

$$r_j \leq r_0\exp\{-(\ln\lambda - \beta - C\eta)jq_n\}. \tag{69}$$

(68) and (69) contradict (24). This implies (66) can not happen, thus we must have (64). □

5. Lower bounds on decaying solution in the resonant case

In this section we assume θ is $n-$Diophantine with respect to α. We will study the lower bound on ϕ for the resonant sites. Recall that $b_{n+1} \geq \frac{q_n}{2}$ in this case.

Theorem 5.1. *Let $\tilde{r}_j = ||\begin{pmatrix}\phi(jq_n)\\ \phi(jq_n-1)\end{pmatrix}||$. Suppose $1 \leq j \leq 8\frac{b_{n+1}}{q_n}$ with $b_{n+1} \geq \frac{q_n}{2}$, then we must have*

$$\tilde{r}_j \geq \frac{q_{n+1}}{j}e^{-(\ln\lambda+\varepsilon)q_n}\tilde{r}_{j-1}. \tag{70}$$

We first list two standard facts.

Lemma 5.2. *([45]) Let $A^1, A^2, \cdots, A^n$ and $B^1, B^2, \cdots, B^n$ be 2×2 matrices with $||\prod_{m=0}^{\ell-1}A^{j+m}|| \leq Ce^{d\ell}$ for some constant C and d. Then*

$$||(A^n+B^n)\cdots(A^1+B^1) - A^n\cdots A^1|| \leq Ce^{dn}(\prod_{j=1}^n(1+Ce^{-d}||B^j||)-1).$$

Lemma 5.3. *For any $\varepsilon > 0$ and large n the following hold,*

$$||A_{q_n}(\theta+q_n\alpha) - A_{q_n}(\theta)|| \leq \frac{1}{q_{n+1}}e^{(\ln\lambda+\varepsilon)q_n}, \tag{71}$$

and

$$||A_{q_n}^{-1}(\theta+q_n\alpha) - A_{q_n}^{-1}(\theta)|| \leq \frac{1}{q_{n+1}}e^{(\ln\lambda+\varepsilon)q_n}. \tag{72}$$

Proof. We only prove (71) for simplicity. By the DC approximation(or see (150) in Appendix), we have

$$||q_n\alpha||_{\mathbb{R}/\mathbb{Z}} \le \frac{1}{q_{n+1}}.$$

This implies

$$||A(\theta+q_n\alpha)-A(\theta)|| \le \frac{C}{q_{n+1}}.$$

Applying Lemma 5.2 and (28), one has

$$||A_{q_n}(\theta+q_n\alpha)-A_{q_n}(\theta)|| \le e^{(\ln\lambda+\varepsilon)q_n}((1+\frac{C}{q_{n+1}})^{q_n}-1). \tag{73}$$

Using the fact $|e^y-1|\le ye^y$ for $y>0$, we obtain

$$\begin{aligned}(1+\frac{C}{q_{n+1}})^{q_n}-1 &\le q_n(1+\frac{C}{q_{n+1}})^{q_n}\ln(1+\frac{C}{q_{n+1}})\\ &\le C\frac{q_n}{q_{n+1}}.\end{aligned}$$

Combining this with (73) completes the proof. □

Lemma 5.4. *For any* $0\le j\le 8\frac{b_{n+1}}{q_n}-1$, *one of the following two estimates must hold,*

$$\tilde r_{j+1} \ge \frac{q_{n+1}}{j+1}e^{-(\ln\lambda+\varepsilon)q_n}\tilde r_j, \tag{74}$$

or

$$\tilde r_{j+1}\tilde r_{j-1} \ge (1-\frac{1}{10(j+1)})^2(1-\frac{1}{10(j+1)^2})\tilde r_j^2. \tag{75}$$

Proof. Suppose

$$\tilde r_{j+1} \le \frac{q_{n+1}}{j+1}e^{-(\ln\lambda+\varepsilon)q_n}\tilde r_j. \tag{76}$$

Let $U_j=\begin{pmatrix}\phi(jq_n)\\ \phi(jq_n-1)\end{pmatrix}$, then for $n>0$, one has

$$U_j = A_{q_n}(\theta+(j-1)q_n\alpha)U_{j-1}.$$

Denote $B=A_{q_n}(\theta+jq_n\alpha)$. Notice that $\det B=1$. We have

$$B^2+(\mathrm{Tr}B)B+I=0. \tag{77}$$

Case 1: $\mathrm{Tr}B\le\frac{\tilde r_j}{\gamma\tilde r_{j+1}}$, where

$$1-\frac{1}{\gamma}=\frac{1}{10(j+1)}.$$

Applying (77) to U_j, one has

$$B^2U_j+(\mathrm{Tr}B)BU_j+U_j=0. \tag{78}$$

Notice that $U_{j+1}=BU_j$, thus

$$||(\mathrm{Tr}B)BU_j|| \le \frac{1}{\gamma}\tilde r_j.$$

Thus we have

$$||B^2U_j|| \ge (1-\frac{1}{\gamma})\tilde r_j = \frac{1}{10(j+1)}\tilde r_j. \tag{79}$$

This is impossible. Indeed, from the following estimate

$$\begin{aligned} ||U_{j+2}-B^2U_j|| &\leq ||A_{q_n}(\theta+(j+1)q_n\alpha)-A_{q_n}(\theta+jq_n\alpha)||\ \ ||U_{j+1}|| \\ &\leq e^{(\ln\lambda+\frac{1}{2}\varepsilon)q_n}\frac{1}{q_{n+1}}\tilde{r}_{j+1} \\ &\leq \frac{1}{100(j+1)}\tilde{r}_j, \end{aligned}$$

where the second inequality holds by (71) and the third inequality holds by assumption (76), combining with (79), one has

$$||U_{j+2}||=\tilde{r}_{j+2}\geq \frac{9}{100(j+1)}\tilde{r}_j. \tag{80}$$

However, by (55) and (53),

$$\tilde{r}_{j+2}\leq \frac{q_{n+1}^2}{(j+1)(j+2)}e^{-2(\ln\lambda-C\eta)q_n}\tilde{r}_j.$$

This is in contradiction with (80).

Case 2: It remains to consider

$$\mathrm{Tr}B\geq \frac{\tilde{r}_j}{\gamma\tilde{r}_{j+1}}. \tag{81}$$

From (77),

$$BU_j+(\mathrm{Tr}B)U_j+B^{-1}U_j=0. \tag{82}$$

First by assumption (76), one has

$$\tilde{r}_{j+1}\leq \frac{1}{10(j+1)}\tilde{r}_j,$$

then

$$\begin{aligned} ||BU_j||=\tilde{r}_{j+1} &\leq \frac{\tilde{r}_j}{\gamma\tilde{r}_{j+1}}\frac{\tilde{r}_j}{10(j+1)^2} \\ &\leq ||(\mathrm{Tr}B)U_j||\frac{1}{10(j+1)^2}. \end{aligned}$$

Thus by (82), we have

$$\begin{aligned} ||B^{-1}U_j|| &\geq (1-\frac{1}{10(j+1)^2})||(\mathrm{Tr}B)U_j|| \\ &\geq (1-\frac{1}{10(j+1)^2})\frac{\tilde{r}_j^2}{\gamma\tilde{r}_{j+1}} \qquad (83) \\ &\geq (1-\frac{1}{10(j+1)^2})\frac{1}{\gamma}\frac{j+1}{q_{n+1}}e^{(\ln\lambda+\varepsilon)q_n}\tilde{r}_j, \qquad (84) \end{aligned}$$

where the second inequality holds by (81) and the third inequality hold by (76).

By (72), the following holds

$$\begin{aligned} ||U_{j-1}-B^{-1}U_j|| &\leq ||A_{q_n}^{-1}(\theta+(j-1)q_n\alpha)-A_{q_n}^{-1}(\theta+jq_n\alpha)||\ \ ||U_j|| \\ &\leq e^{(\ln\lambda+\frac{1}{2}\varepsilon)q_n}\frac{1}{q_{n+1}}\tilde{r}_j \\ &\leq \frac{1}{10(j+1)}||B^{-1}U_j||, \qquad (85) \end{aligned}$$

where the third inequality holds by (84).

Putting (83) and (85) together, we have

$$\begin{aligned}\tilde r_{j-1} = ||U_{j-1}|| &\geq (1-\frac{1}{10(j+1)})||B^{-1}U_j|| \\ &\geq (1-\frac{1}{10(j+1)})^2(1-\frac{1}{10(j+1)^2})\frac{\tilde r_j^2}{\tilde r_{j+1}}.\end{aligned}$$

This implies (75). □

Proof of Theorem 5.1.

Proof. We can proceed by induction.

Set $j=0$ in Lemma 5.4. By (55), the second case (75) can not happen, thus Theorem 5.1 holds for $j=1$.

Suppose (70) holds for $p=j-1$, that is

$$\tilde r_{j-1} \geq \frac{q_{n+1}}{j-1}e^{-(\ln\lambda+\varepsilon)q_n}\tilde r_{j-2}. \tag{86}$$

We will show (70) holds for $p=j$. Let us apply Lemma 5.4 to $p=j-1$. If (74) holds for $p=j-1$, the result follows. Otherwise by (75), we have

$$\begin{aligned}\tilde r_j &\geq (1-\frac{1}{10j})^2(1-\frac{1}{10j^2})\tilde r_{j-1}\frac{\tilde r_{j-1}}{\tilde r_{j-2}} \\ &\geq (1-\frac{1}{10j})^2(1-\frac{1}{10j^2})\tilde r_{j-1}\frac{q_{n+1}}{j-1}e^{-(\ln\lambda+\varepsilon)q_n} \\ &\geq \tilde r_{j-1}\frac{q_{n+1}}{j}e^{-(\ln\lambda+\varepsilon)q_n},\end{aligned}$$

where the second inequality holds by (86).

□

6. Decaying solutions. Proof of (12), Theorems 2.2, 2.4 and 2.9

In this section the dependence on n will play a role, so we go back to the $r_j^n,\tilde r_j^n$ notation. We first give a series of auxiliary facts. Recall footnote 11.

Theorem 6.1. *Assume θ is $n-$Diophantine with respect to α. For any $1\leq j\leq 10\frac{b_{n+1}}{q_n}$ with $b_{n+1}\geq\frac{q_n}{2}$, we have*

$$\bar r_j^n e^{-\varepsilon jq_n}\leq r_j^n\leq \bar r_j^n e^{\varepsilon jq_n}$$

and

$$\bar r_j^n e^{-\varepsilon jq_n}\leq \tilde r_j^n\leq \bar r_j^n e^{\varepsilon jq_n}.$$

Proof. For any $\varepsilon>0$, we choose η small enough.Using (55) j times, we have

$$r_j^n\leq\frac{q_{n+1}^j}{j!}\exp\{-(\ln\lambda-\varepsilon)jq_n\}.$$

Similarly, using (70) j times, we have

$$\tilde r_j^n\geq\frac{q_{n+1}^j}{j!}\exp\{-(\ln\lambda+\varepsilon)jq_n\}.$$

By Stirling formula and (53), we obtain the theorem. □

Theorem 6.2. *Assume θ is $n-$Diophantine with respect to α. Assume $jq_n \leq k < (j+1)q_n$ with $0 \leq j \leq 8\frac{b_{n+1}}{q_n}$, $b_{n+1} \geq \frac{q_n}{2}$ and $k \geq \frac{q_n}{4}$. We have*

$$||U(k)|| \leq \max\{e^{-|k-jq_n|\ln\lambda}\tilde{r}_j^n, e^{-|k-(j+1)q_n|\ln\lambda}\tilde{r}_{j+1}^n\}e^{\varepsilon q_n}, \tag{87}$$

$$||U(k)|| \geq \max\{e^{-|k-jq_n|\ln\lambda}\tilde{r}_j^n, e^{-|k-(j+1)q_n|\ln\lambda}\tilde{r}_{j+1}^n\}e^{-\varepsilon q_n}. \tag{88}$$

In particular, we have

$$||U(k)|| \leq \max\{e^{-|k-jq_n|\ln\lambda}\bar{r}_j^n, e^{-|k-(j+1)q_n|\ln\lambda}\bar{r}_{j+1}^n\}e^{\varepsilon|k|}, \tag{89}$$

$$||U(k)|| \geq \max\{e^{-|k-jq_n|\ln\lambda}\bar{r}_j^n, e^{-|k-(j+1)q_n|\ln\lambda}\bar{r}_{j+1}^n\}e^{-\varepsilon|k|}. \tag{90}$$

Proof. (89) and (90) just follows from (87), (88) and Theorem 6.1. Thus it suffices to prove (87), (88). Clearly, by (53), one has

$$||U(k)|| \geq \max\{e^{-|k-jq_n|\ln\lambda}\tilde{r}_j^n, e^{-|k-(j+1)q_n|\ln\lambda}\tilde{r}_{j+1}^n\}e^{-\varepsilon q_n}.$$

This implies (90) by Theorem 6.1.

We now turn to (89). If $|k-jq_n| \leq 10\eta q_n$ or $|k-(j+1)q_n| \leq 10\eta q_n$, the result follows from Theorem 6.1 and (53). If $|k-jq_n| \geq 10\eta q_n$ and $|k-(j+1)q_n| \geq 10\eta q_n$, it follows from Lemma 4.1, Theorem 6.1 and (53). □

Theorem 6.3. *For $q_n^{\frac{8}{9}} \leq k \leq \frac{q_n}{2}$, let n_0 be the smallest positive integer such that $q_{n-n_0} \leq k < q_{n-n_0+1}$. Suppose $jq_{n-n_0} \leq k < (j+1)q_{n-n_0}$ with $j \geq 1$. If θ is $k-$Diophantine with respect to α for $k = n-n_0$ and $k = n-1$,then*

$$||U(k)|| \leq \max\{e^{-|k-jq_{n-n_0}|\ln\lambda}\bar{r}_j^{n-n_0}, e^{-|k-(j+1)q_{n-n_0}|\ln\lambda}\bar{r}_{j+1}^{n-n_0}\}e^{\varepsilon k}, \tag{91}$$

and

$$||U(k)|| \geq \max\{e^{-|k-jq_{n-n_0}|\ln\lambda}\bar{r}_j^{n-n_0}, e^{-|k-(j+1)q_{n-n_0}|\ln\lambda}\bar{r}_{j+1}^{n-n_0}\}e^{-\varepsilon k}. \tag{92}$$

Proof. Set $t_0 = 1 - \frac{\varepsilon}{8\beta}$. Let $t = t_0$ in the definition of resonance, i.e. $b_n = q_n^{t_0}$.

Case 1: $k \leq q_{n-n_0+1}^{t_0}$. In this case, one has

$$q_{n-n_0} \leq k \leq q_{n-n_0+1}^{t_0}.$$

The result holds by Theorem 6.2.

Case 2: $k \geq q_{n-n_0+1}^{t_0}$. Then

$$\begin{aligned}
\bar{r}_j^{n-n_0} &\leq \exp\{-(\ln\lambda - \frac{\ln q_{n-n_0+1}}{q_{n-n_0}} + \frac{\ln q_{n-n_0+1}^{t_0}}{q_{n-n_0}} - \varepsilon)jq_{n-n_0}\} \\
&\leq \exp\{-(\ln\lambda - (1-t_0)\beta - \varepsilon)jq_{n-n_0}\} \\
&\leq \exp\{-(\ln\lambda - 2\varepsilon)jq_{n-n_0}\},
\end{aligned}$$

where the third inequality holds by the definition of t_0. Noting that $k \leq q_{n-n_0+1}$, one has

$$\bar{r}_j^{n-n_0} \geq \exp\{-jq_{n-n_0}(\ln\lambda + \varepsilon)\}.$$

Similarly,

$$\exp\{-(j+1)q_{n-n_0}(\ln\lambda+\varepsilon)\} \leq \bar{r}_{j+1}^{n-n_0} \leq \exp\{-(j+1)q_{n-n_0}(\ln\lambda - \varepsilon)\}.$$

Thus in order to prove case 2, it suffices to show

$$e^{-(\ln\lambda+\varepsilon)k} \leq ||U(k)|| \leq e^{-(\ln\lambda-\varepsilon)k}. \tag{93}$$

The left inequality holds by (53).

We start to prove the right inequality. For any $y \in [\varepsilon k, k]$ or $y \in [q_n - k, q_n - \varepsilon k]$, let n_0' be the least positive integer such that $4q_{n-n_0'} \le dist(y, q_n\mathbb{Z})$, thus $n_0' \ge n_0$. Let s be the largest positive integer such that $4sq_{n-n_0'} \le dist(y, q_n\mathbb{Z})$.

Set $I_1, I_2 \subset \mathbb{Z}$ as follows

$$\begin{aligned} I_1 &= [-sq_{n-n_0'}, sq_{n-n_0'} - 1], \\ I_2 &= [y - sq_{n-n_0'}, y + sq_{n-n_0'} - 1], \end{aligned}$$

and let $\theta_j = \theta + j\alpha$ for $j \in I_1 \cup I_2$. The set $\{\theta_j\}_{j \in I_1 \cup I_2}$ consists of $4sq_{n-n_0'}$ elements. By case ii) of Theorem 3.5, y is $(\ln\lambda + 8\ln(sq_{n-n_0'}/q_{n-n_0'+1})/q_{n-n_0'} - \varepsilon, 4sq_{n-n_0'} - 1)$ regular with $\delta = \frac{1}{4}$. Notice that

$$(s+1)q_{n-n_0'} \ge \varepsilon q_{n-n_0+1}^{t_0} \ge \varepsilon q_{n-n_0'+1}^{t_0},$$

thus we have

$$\begin{aligned} \ln\lambda + 8\ln(sq_{n-n_0'}/q_{n-n_0'+1})/q_{n-n_0'} &\ge \ln\lambda - 8(1-t_0)\beta - \varepsilon \\ &\ge \ln\lambda - 2\varepsilon. \end{aligned}$$

This implies for any $y \in [\varepsilon k, k]$ or $y \in [q_n - k, q_n - \varepsilon k]$, there exists an interval $I(y) = [x_1, x_2] \subset [0, q_n]$ with $y \in I(y)$ such that

$$dist(y, \partial I(y)) \ge \frac{1}{2} q_{n-n_0'} \tag{94}$$

and

$$|G_{I(y)}(y, x_i)| \le e^{-(\ln\lambda - \varepsilon)|y - x_i|}, \ i = 1, 2. \tag{95}$$

For any $y \in (k, q_n - k)$, let s be the largest positive integer such that $sq_{n-n_0} \le dist(y, q_n\mathbb{Z})$ and set $I_1, I_2 \subset \mathbb{Z}$ as follows:

$$\begin{aligned} I_1 &= [-\lfloor \frac{sq_{n-n_0}}{2} \rfloor, sq_{n-n_0} - \lfloor \frac{sq_{n-n_0}}{2} \rfloor - 1], \\ I_2 &= [y - \lfloor \frac{sq_{n-n_0}}{2} \rfloor, y + sq_{n-n_0} - \lfloor \frac{sq_{n-n_0}}{2} \rfloor - 1]. \end{aligned}$$

By the same reason, (94) and (95) also hold for $n_0' = n_0$.

Arguing exactly as in the proof of Lemma 4.1, with (39) replaced with (94) and (40) with (95), we obtain

$$||U(k)|| \le \max\{\hat{r}_0 \exp\{-(\ln\lambda - 2\varepsilon)(k - 3\varepsilon k)\}, \hat{r}_1^n \exp\{-(\ln\lambda - 2\varepsilon)(q_n - k - 3\varepsilon k)\}\}, \tag{96}$$

where $\hat{r}_j = \max_{|r| \le 10\varepsilon} ||U(jq_n + rk)||$ with $j = 0, 1$. Using that $k \le \frac{q_n}{2}$, one has

$$||U(k)|| \le e^{-(\ln\lambda - \varepsilon)k},$$

this implies (93) and thus the theorem.

□

Remark 6.4. The assumption that θ is $n - n_0$ Diophantine with respect to α is sufficient for the proof of case 1. The assumption that θ satisfies (35) is sufficient for the proof of case ii) of Theorem 3.5, and therefore for the proof of case 2. Then by Remark 3.6, the assumption that θ is $n - 1$ Diophantine with respect to α is sufficient for the proof of case 2.

Remark 6.5. Suppose we only consider $q_n^{\frac{8}{9}} \le k \le cq_n$ with $c \le \frac{1}{2}$ in Theorem 6.3. Theorem 6.3 still holds if we only have (11) for function $\phi(k)$ on $[-\mathfrak{c}q_n, 2cq_n]$ for some $\mathfrak{c} > 0$.

In order to prove (12), it suffices to prove the following Theorem, which is a stronger local version of (12).

Theorem 6.6. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be such that $|\lambda| > e^{\beta(\alpha)}$. Suppose E is a generalized eigenvalue of $H_{\lambda,\alpha,\theta}$ and ϕ is the generalized eigenfunction. Let $U(k) = \begin{pmatrix} \phi(k) \\ \phi(k-1) \end{pmatrix}$. Then for any $\varepsilon > 0, \kappa > 0, \nu > 1$, there exists $\hat{n}_0$ (depending on $\alpha, E, \kappa, \nu, \varepsilon$[12])such that, if θ is n-Diophantine with respect to α with Diophantine constants κ, ν for some $n \geq \hat{n}_0$, we have $U(k)$ satisfy*

$$f(|k|)e^{-\varepsilon|k|} \leq ||U(k)|| \leq f(|k|)e^{\varepsilon|k|}, \tag{97}$$

for $\frac{q_n}{2} \leq |k| \leq \frac{q_{n+1}}{2}$.

Proof. It remains to collect several already proved statements that cover different scenarios.

Case i: $\frac{q_n}{2} \leq q_{n+1}^{\frac{8}{9}}$.

For $\frac{q_n}{2} \leq k \leq 4q_{n+1}^{\frac{8}{9}}$ the result follows from Theorem 6.2.

For $4q_{n+1}^{\frac{8}{9}} \leq k \leq \frac{q_{n+1}}{2}$, (97) follows from Theorem 6.3 (notice that now $k \geq 2q_n$, so $n_0 = 1$).

Case ii: $q_{n+1}^{\frac{8}{9}} \leq \frac{q_n}{2}$.

Case ii.1: $\frac{q_n}{2} \leq k \leq \min\{q_n, \frac{q_{n+1}}{2}\}$.

If $q_n = q_{n-1} + q_{n-2}$, then $q_{n-1} \geq \frac{q_n}{2}$. By the proof of case 2 in Theorem 6.3 (by Remark 6.4, the assumption that θ is n-Diophantine is enough), one has for any $q_{n-1} \leq k \leq \min\{q_n, \frac{q_{n+1}}{2}\}$

$$|\phi(k)| \leq e^{-(\ln\lambda - \varepsilon)k}.$$

This leads to

$$|\phi(k)| \leq e^{-(\ln\lambda - \varepsilon)k}.$$

for $\frac{q_n}{2} \leq k \leq \min\{q_n, \frac{q_{n+1}}{2}\}$. This also implies (12).

If $q_n = jq_{n-1} + q_{n-2}$ with $j \geq 2$, we have $\frac{q_n}{2} \geq q_{n-1}$. By case 2 in Theorem 6.3 (by Remark 6.4, the assumption that θ is n-Diophantine is enough) again (with $n + 1 - n_0 = n - 1$), we obtain (97).

Case ii.2: $q_n \leq k \leq \frac{q_{n+1}}{2}$

In this case (97) follows directly from Theorem 6.3 (with $n + 1 - n_0 = n$), because $n_0 = 1$ so that the fact θ is n Diophantine can guarantee both cases 1 and 2 of Theorem 6.3. □

Proof of Theorem 2.2

Proof. The proof follows that of (97) by shifting by k_0 units, and Remark 6.5. □

Proof of Theorem 2.9.

Proof. Assume θ is Diophantine with respect to α. First by the definition of $\beta(\alpha)$, one has for any large n and any ℓ

$$\bar{r}_\ell^n \leq e^{-(\ln\lambda - \beta - \varepsilon)\ell q_n}.$$

Combining with the definition of $f(k)$ and (12), we have

$$|\phi(k)| \leq e^{-(\ln\lambda - \beta - \varepsilon)k}. \tag{98}$$

We therefore established that every generalized eigenfunction decays exponentially, which by Schnol's Theorem [16] implies the localization statement.

By the definition of $\beta(\alpha)$ again, there exists a subsequence q_{n_k} of q_n such that

$$q_{n_k+1} \geq e^{(\beta-\varepsilon)q_{n_k}}. \tag{99}$$

[12] The dependence on E is through the constant $\hat{C}$ in (11).

By Theorem 2.1 (or 6.1) and the definition of $\bar{r}_j^n$ we have for any $k > 0$

$$||U(q_{n_k})|| \geq e^{-(\ln\lambda-\beta+\varepsilon)q_{n_k}}. \tag{100}$$

Thus (98) and (100) imply that the decay rate is just $\ln\lambda - \beta$. □

7. Hierarchical structure

As we have already established Theorem 2.9 we know that each generalized eigenfunction decays exponentially so has a global maximum. Assume its global maximum (see Footnote 2) is at 0 and ϕ is normalized by $\|\phi\|_\infty = 1$. Note that then $\hat{C}$ in (11) is equal to 1 so all dependence of the largeness of n on E (see Footnote 12) disappears. Theorem 2.1 provides, for sufficiently large n plenty of local n-maxima in the vicinity of aq_n, but determined only with εq_n precision. In the next theorem we show that this precision can be improved all the way to an n-independent constant. We have

Theorem 7.1. *Fix κ, ν, ϵ. Then for sufficiently small ε there exists $\hat{n}_0(\kappa, \nu, \lambda, \alpha, \epsilon, \varepsilon)$ such that if θ is k-Diophantine for all $\hat{n}_0 \leq k \leq n$ with Diophantine constants κ, ν and $\frac{\ln q_{n+1} - \ln j}{q_n} > \epsilon \ln\lambda$ with $\epsilon > 0$, then*

$$\sup_{k\in[jq_n-\epsilon q_n+\varepsilon q_n, jq_n]} ||U(k)|| = \sup_{k\in[jq_n-K_0, jq_n]} ||U(k)||, \tag{101}$$

and

$$\sup_{k\in[jq_n, jq_n+\epsilon q_n-\varepsilon q_n]} ||U(k)|| = \sup_{k\in[jq_n, jq_n+K_0]} ||U(k)||, \tag{102}$$

where $K_0 = q_{\hat{n}_0+1}$.

Proof. We first give the proof of (101).

Let $k_0 \in [jq_n - \varepsilon q_n, jq_n]$ be such that

$$||U(k_0)|| = \sup_{k\in[jq_n-\varepsilon q_n, jq_n]} ||U(k)||.$$

By (55), (70),(87) and (88), one has

$$||U(k_0)|| = \sup_{k\in[jq_n-\epsilon q_n+\varepsilon q_n, jq_n]} ||U(k)||.$$

Suppose (101) does not hold, i.e., $k_0 \in [jq_n - \varepsilon q_n, jq_n - K_0]$.

Now we will reflect the elements in $[jq_n - \varepsilon q_n, jq_n]$ at $\frac{j}{2}q_n$. That is for any element $k \in [jq_n - \varepsilon q_n, jq_n]$, let $k' = jq_n - k$. Then $k' \in [0, \varepsilon q_n]$.

Choose n' such that $b_{n'} \leq k_0' < b_{n'+1}$ (where $k_0' = jq_n - k_0$). Then $n' \geq \hat{n}_0$.

Case 1. k_0' is n'-nonresonant, i.e., $\mathrm{dist}(k_0', q_{n'}\mathbb{Z}) \geq b_{n'}$.

Let n_0' be the least positive integer such that $4q_{n'-n_0'} \leq dist(k_0', q_{n'}\mathbb{Z})$. Let s be the largest positive integer such that $4sq_{n'-n_0'} \leq dist(k_0', q_{n'}\mathbb{Z})$. Set $I_1, I_2, I_2' \subset \mathbb{Z}$ as follows

$$\begin{aligned} I_1 &= [-sq_{n'-n_0'}, sq_{n'-n_0'} - 1], \\ I_2 &= [k_0 - sq_{n'-n_0'} + 1, k_0 + sq_{n'-n_0'}], \\ I_2' &= [k_0' - sq_{n'-n_0'}, k_0' + sq_{n'-n_0'} - 1], \end{aligned}$$

Notice that I_2' and I_2 are reflections of each other about $\frac{j}{2}q_n$.

By the Diophantine condition on θ with respect to α, for any $k_2 \in I_2 (k_2' \in I_2')$ and $k_1 \in I_1$, we have

$$\begin{aligned} ||2\theta + (k_1+k_2)\alpha||_{\mathbb{R}/\mathbb{Z}} &= ||2\theta - k_2'\alpha + jq_n\alpha + k_1\alpha||_{\mathbb{R}/\mathbb{Z}} \\ &\geq ||2\theta + (k_1 - k_2')\alpha||_{\mathbb{R}/\mathbb{Z}} - ||jq_n\alpha||_{\mathbb{R}/\mathbb{Z}} \\ &\geq ||2\theta + (k_1 - k_2')\alpha||_{\mathbb{R}/\mathbb{Z}} - \frac{j}{q_{n+1}} \\ &\geq ||2\theta + (k_1 - k_2')\alpha||_{\mathbb{R}/\mathbb{Z}} - e^{-\epsilon \ln \lambda q_n} \\ &\geq \frac{1}{2}||2\theta + (k_1 - k_2')\alpha||_{\mathbb{R}/\mathbb{Z}} \geq \frac{C}{q_{n'}^C}, \end{aligned} \tag{103}$$

where the last inequality holds by the fact $|k_1|, |k_2'| \leq C_\star b_{n'+1}$ so that we can apply Lemma B.2.

For any $k_2 \in I_2$ and $k_1 \in I_1$ ($k_1 + k_2' \neq 0$ by the construction of I_1, I_2), we also have

$$\begin{aligned} ||(k_2 - k_1)\alpha||_{\mathbb{R}/\mathbb{Z}} &= || - k_2'\alpha + jq_n\alpha - k_1||_{\mathbb{R}/\mathbb{Z}} \\ &\geq ||(-k_1 - k_2')\alpha||_{\mathbb{R}/\mathbb{Z}} - ||jq_n\alpha||_{\mathbb{R}/\mathbb{Z}} \\ &\geq ||(k_1 + k_2')\alpha||_{\mathbb{R}/\mathbb{Z}} - e^{-\epsilon \ln \lambda q_n} \\ &\geq \frac{1}{2}||(k_1 + k_2')\alpha||_{\mathbb{R}/\mathbb{Z}} \geq \frac{C}{q_{n'}^C}, \end{aligned} \tag{104}$$

where the last inequality holds by the fact $|k_1|, |k_2'| \leq C_\star b_{n'+1}$ and $k_1 - k_2' \neq q_{n'}\mathbb{Z}$ so that we can apply Lemma B.3.

By Theorem B.4, (103) and (104), we have k_0 is $(\hat{k}_0, \ln \lambda - \beta - \varepsilon)$ regular, where $\hat{k}_0 = 4sq_{n'-n_0'} - 1$. Let $I_2 = [x_1, x_2] \subset [jq_n - 2\varepsilon q_n, jq_n]$.

By (33), we have

$$|\phi(k_0)| \leq e^{-(\ln\lambda - \beta - \varepsilon)\frac{\hat{k}_0}{10}}(|\phi(x_1)| + |\phi(x_0)|) \leq e^{-(\ln\lambda - \beta - \varepsilon)\frac{\hat{k}_0}{10}}||U(k_0)||.$$

Similarly,

$$|\phi(k_0 - 1)| \leq e^{-(\ln\lambda - \beta - \varepsilon)\frac{\hat{k}_0}{10}}||U(k_0)||.$$

The last two inequalities imply that

$$||U(k_0)|| \leq e^{-(\ln\lambda - \beta - \varepsilon)\frac{\hat{k}_0}{10}}||U(k_0)||.$$

This is impossible.

Case 2. k_0' is n-resonant, i.e., $|k_0' - \ell q_{n'}| \leq b_{n'}$ for some ℓ.

From (103) and (104), we know that the small divisor condition does not change under reflection at $\frac{j}{2}q_n$. Following the proof of (54) and replacing Lemma 4.1 with a combination of Lemma 4.2 and Theorem B.4, we have

$$\mathtt{r}_\ell^{n',\phi} \leq \exp\{-(\ln\lambda - \beta - \varepsilon)q_{n'}\} \max\{\mathtt{r}_{\ell\pm1}^{n',\phi}\},$$

where

$$\mathtt{r}_\ell^{n',\phi} = \sup_{|r| \leq 10\varepsilon} |\phi(jq_n - (\ell q_{n'} + rq_{n'}))|.$$

This is contradicted to the fact that k_0 is the maximal point because $|k_0' - \ell q_{n'}| \leq b_{n'}$.

This completes the proof of (101).

Now we turn to the proof of (102). Let $k_0^r \in [jq_n, jq_n + \varepsilon q_n]$ be such that

$$||U(k_0^r)|| = \sup_{k \in [jq_n, jq_n + \varepsilon q_n]} ||U(k)||.$$

Suppose the Theorem does not hold, i.e., $k_0^r \in [jq_n + K_0, jq_n + \varepsilon q_n]$.

In this case we shift the elements in $[jq_n, jq_n + \varepsilon q_n]$ by $-jq_n$. That is for any element $k \in [jq_n, jq_n + \varepsilon q_n]$, let $k^{r,\prime} = k - jq_n$. Then $k^{r,\prime} \in [0, \varepsilon q_n]$. Then (102) holds by the same proof, only replacing all k' with $k^{r,\prime}$.

□

We restate the result of Theorem 7.1 as a more convenient

Theorem 7.2. *Fix κ, ν, ϵ. Then for sufficiently small ε there exists $\hat{n}_0(\kappa, \nu, \lambda, \alpha, \epsilon, \varepsilon)$ such that if k_0 is a local $(n+1)$-maximum, θ is k-Diophantine for all $\hat{n}_0 \le k \le n$ with Diophantine constants κ, ν, and*

$$\frac{\ln q_{n+1} - \ln j}{q_n} > \epsilon \ln \lambda.$$

Then

$$\sup_{k \in [k_0 + jq_n - \epsilon q_n + \varepsilon q_n, k_0 + (j+1)q_n]} ||U(k)|| = \sup_{k \in [k_0 + jq_n - K_0, k_0 + jq_n + K_0]} ||U(k)||, \tag{105}$$

where $K_0 = q_{\hat{n}_0 + 1}$.

Proof. By shifting the operator by k_0 units, we can assume $k_0 = 0$. Theorem 7.1 still holds if 0 is a local $(n+1)$-maximum by Remark 6.5.

□

We will now formulate a local version of the hierarchical structure Theorem 2.4.

Fix $0 < \varsigma, \epsilon$ with $\varsigma + 2\epsilon < 1$. Let $n_j \to \infty$ be such that $\ln q_{n_j+1} \ge (\varsigma + 2\epsilon) \ln |\lambda| q_{n_j}$. Let $\mathfrak{c}_j = (\ln q_{n_j+1} - \ln |a_{n_j}|)/\ln |\lambda| q_{n_j} - \epsilon$. $\mathfrak{c}_j > \epsilon$ for $0 < a_{n_j} < e^{\varsigma \ln |\lambda| q_{n_j}}$.

Theorem 7.3. *Suppose k_0 is a local $(n_{j_0}+1)$-maximum. Suppose $\theta + k_0\alpha$ is Diophantine with respect to α (with Diophantine constants κ, ν). Then there exists $\hat{n}_0(\alpha, \lambda, \kappa, \nu, \epsilon) < \infty$ such that for any $j_0 > j_1 > \cdots > j_k$, $n_{j_k} \ge \hat{n}_0 + k$, and $0 < a_{n_{j_i}} < e^{\varsigma \ln |\lambda| q_{n_{j_i}}}$, $i = 0, 1, \ldots, k$ for all $0 \le s \le k$ there exists a local n_{j_s}-maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ on the interval $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}} + I^{n_{j_s}}_{\mathfrak{c}_{j_s}, 1}$ for all $0 \le s \le k$ such that the following holds:*

I: *$|b_{a_{n_{j_0}}} - (k_0 + a_{n_{j_0}} q_{n_{j_0}})| \le q_{\hat{n}_0 + 1}$,*

II: *For any $1 \le s \le k$, $|b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}} - (b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_{s-1}}}} + a_{n_{j_s}} q_{n_{j_s}})| \le q_{\hat{n}_0 + s + 1}$.*

III: *if $2(x - b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_k}}}) \in I^{n_{j_k}}_{\mathfrak{c}_{j_k}, 1}$, then for each $s = 0, 1, \ldots, k$,*

$$f(x_s) e^{-\varepsilon |x_s|} \le \frac{||U(x)||}{||U(b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}})||} \le f(x_s) e^{\varepsilon |x_s|}, \tag{106}$$

where $x_s = |x - b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}|$ is large enough.

Moreover, every local n_{j_s}-maximum on the interval

$$b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_{s-1}}}} + [-e^{\epsilon \ln \lambda q_{n_{j_s}}}, e^{\epsilon \ln \lambda q_{n_{j_s}}}]$$

is of the form $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ for some $a_{n_{j_s}}$.

Proof. Let $\hat{n}_0 = \hat{n}_0(\kappa, 3\nu, \lambda, \alpha, \epsilon, \epsilon/10)$ be given by Theorem 7.2 .[13]

As long as

$$(\ln q_{n+1} - \ln |a_n|)/q_n \ge 2\epsilon \ln |\lambda| \tag{107}$$

---

[13] 3ν here can be easily relaxed to $(1 + \epsilon)\nu$.

with $0<\varsigma,\epsilon<1$, where $0<|a_n|\leq \frac{q_{n+1}}{2q_n}$, Theorem 7.2 (upon shifting by k_0 units) implies that there exists a local n-maximum b_{a_n} on interval $b_{a_n}+I^n_{\epsilon,1}$ such that

$$|b_{a_n}-(a_nq_n+k_0)|\leq K_0=q_{\hat{n}_0+1}. \tag{108}$$

Now let n_i be such that $\ln q_{n_i+1}\geq(\varsigma+2\epsilon)\ln|\lambda|q_{n_i}, i=t_0,t_0+1,\cdots,j$ for some $0<\varsigma,\epsilon<1$.

By (108), one has that there exists a local n_{j_0} maximum $b_{a_{n_{j_0}}}=a_{n_{j_0}}q_{n_{j_0}}+k_0+\hat{K}_{n_{j_0}}$ with $|\hat{K}_{n_{j_0}}|\leq K_0$.

Now we will prove that for $0\leq s\leq k$, there exists

$$|b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}}-a_{n_{j_s}}q_{n_{j_s}}-b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_{s-1}}}}|\leq K_s=q_{\hat{n}_0+s+1}.$$

Notice that $\sum_{i=0}^k K_i\leq 4K_k$. We will now prove Theorem 2.4 by induction on s.

By the assumption, one has

$$||2\theta+2k_0+k\alpha||_{\mathbb{R}/\mathbb{Z}}>\frac{\kappa}{|k|^{\nu}}, \tag{109}$$

for any $k\in\mathbb{Z}\backslash\{0\}$.

First we prove the case $s=1$. By the Diophantine condition on θ (109), we have for $|\ell|\leq q_{n_{j_1}+1}$, the following holds

$$\begin{aligned}
||2\theta+(2b_{a_{n_{j_0}}}+\ell)\alpha||_{\mathbb{R}/\mathbb{Z}} &\geq ||2\theta+(2k_0+\ell+2K_{n_{j_0}})\alpha||_{\mathbb{R}/\mathbb{Z}}-||2a_{n_{j_0}}q_{n_{j_0}}\alpha||_{\mathbb{R}/\mathbb{Z}}\\
&\geq \frac{\kappa}{(2K_0+|\ell|)^{\nu}}-\frac{2a_{n_{j_0}}}{q_{n_{j_0}+1}}\\
&\geq \frac{\kappa}{|\max\{K_0,\ell\}|^{2\nu}}-e^{-\epsilon\ln\lambda q_{n_{j_0}}}\\
&\geq \frac{\kappa}{|\max\{K_0,\ell\}|^{3\nu}}.
\end{aligned} \tag{110}$$

Therefore $\theta+b_{a_{n_{j_0}}}\alpha$ is $\hat{n}_0+1$-Diophantine with respect to α with parameters $3\nu,\kappa$, and by Theorem 7.2 again, there exists a local n_{j_1}-maximum such that $b_{a_{n_{j_0}},a_{n_{j_1}}}=a_{n_{j_1}}q_{n_{j_1}}+b_{a_{n_{j_0}}}+\hat{K}_{n_{j_1}}$ with $|\hat{K}_{n_{j_1}}|\leq K_1=q_{\hat{n}_0+2}$. This completes the first step.

Assume Theorem holds for $s=k-1$. It suffices to show it holds for $s=k$. By the Diophantine condition on θ (109) again, we have for $|\ell|\leq q_{n_{j_k}+1}$, the following holds,

$$\begin{aligned}
||2\theta+(2b_{a_{n_{j_0}},a_{n_{j_1}},\cdots,a_{n_{j_{k-1}}}}+\ell)\alpha||_{\mathbb{R}/\mathbb{Z}} &\geq ||2\theta+(2k_0+\ell+2\sum_{s=0}^{k-1}K_s)\alpha||_{\mathbb{R}/\mathbb{Z}}-\sum_{s=0}^{k-1}||2a_{n_{j_s}}q_{n_{j_s}}\alpha||_{\mathbb{R}/\mathbb{Z}}\\
&\geq \frac{\kappa}{(8K_{k-1}+|\ell|)^{\nu}}-\sum_{s=0}^{k-1}||2a_{n_{j_s}}q_{n_{j_s}}\alpha||_{\mathbb{R}/\mathbb{Z}}\\
&\geq \frac{\kappa}{|\max\{K_{k-1},\ell\}|^{2\nu}}-\sum_{s=0}^{k-1}e^{-\epsilon q_{n_{j_s}}}\\
&\geq \frac{\kappa}{|\max\{K_{k-1},\ell\}|^{3\nu}}.
\end{aligned}$$

Thus $\theta+b_{a_{n_{j_0}},a_{n_{j_1}},\cdots,a_{n_{j_{k-1}}}}\alpha$ is $\hat{n}_0+k$-Diophantine with respect to α with parameters $3\nu,\kappa$, and by Theorem 7.2 again, there exists a local n_{j-k}-maximum such that $b_{a_{n_{j_0}},a_{n_{j_1}},\cdots,a_{n_{j_k}}}=a_{n_{j_k}}q_{n_{j_k}}+b_{a_{n_{j_0}},a_{n_{j_1}},\cdots,a_{n_{j_{k-1}}}}+\hat{K}_{n_{j_k}}$ with $|\hat{K}_{n_{j_k}}|\leq K_k=q_{\hat{n}_0+k+1}$. This implies II holds for $s=k$. Thus we complete the proof of I and II.

III, as well as the moreover part, follow from Theorem 2.2 directly. □

Proof of Theorem 2.4

Proof. Since k_0 is a local $n_{j_0}+1$-maximum for every j, Theorem 2.4 follows from Theorem 7.3 directly. □

Theorem 7.3 describes a hierarchical structure around every local $(n_{j_0}+1)$-maximum.

We will say that a local n_{j_0}-maximum is *k-hierarchical* if there exists $\epsilon>0$, $j_0>j_1>\cdots>j_k$ with $n_{j_{i+1}}>e^{\epsilon n_{j_i}}$ and, for each $s=0,1,\ldots,k$, a collection of local n_{j_s}-maxima, $\{b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}}\}$ such that

I: All local (n_{j_s},ϵ)-maxima in $[b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}}-e^{\epsilon q_{n_{j_s}}},b_{a_{n_{j_0}},a_{n_{j_1}},\cdots,a_{n_{j_{s-1}}}}+e^{\epsilon q_{n_{j_s}}}]$ are given by $\{b_{a_{n_{j_0}},a_{n_{j_1}},\cdots,a_{n_{j_{s-1}}},a_{n_{j_s}}}\}$ with all possible choices of $a_{n_{j_s}}$.

II: if $2(x-b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_k}}})\in I_{\epsilon,\epsilon}^{n_{j_k}}$, then for each $s=0,1,...,k$,

$$f(x_s)e^{-\varepsilon|x_s|}\leq\frac{||U(x)||}{||U(b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}})||}\leq f(x_s)e^{\varepsilon|x_s|},\tag{111}$$

where $x_s=|x-b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}}|$ is large enough.

Proof of Corollary 2.7. Clearly, $b_{a_{n_1},\ldots,a_{n_s}}$ of Theorem 7.3 form the collection required for the definition of k-hierarchy, so it remains to estimate the number of levels of the hierarchy, that is find k such that $n_{j_k}\geq\hat{n}_0+k$. Clearly, $k=j/2-\lfloor\hat{n}_0/2\rfloor$ works. □

8. Growth of transfer matrices. Proof of (13)

Assume θ is Diophantine with respect to α in this and the following section.

Theorem 8.1. *Let $A(j)=||A_{jq_n}||$. Assume $jq_n\leq k<(j+1)q_n$ with $0\leq j\leq 48C_\star\frac{b_{n+1}}{q_n}$, $b_{n+1}\geq\frac{q_n}{2}$ and $k\geq\frac{q_n}{4}$. We have*

$$||A_k||\leq\max\{e^{-|k-jq_n|\ln\lambda}A(j),e^{-|k-(j+1)q_n|\ln\lambda}A(j+1)\}e^{\varepsilon k},\tag{112}$$

$$||A_k||\geq\max\{e^{-|k-jq_n|\ln\lambda}A(j),e^{-|k-(j+1)q_n|\ln\lambda}A(j+1)\}e^{-\varepsilon k}.\tag{113}$$

Proof. Let $\tilde{U}(k)=\begin{pmatrix}\psi(k)\\ \psi(k-1)\end{pmatrix}$. By Last-Simon's arguments ((8.6) in [40]), one has

$$||A_k||\geq||A_k\tilde{U}(0)||\geq c||A_k||.\tag{114}$$

Then (112) holds by (114),(53) and (38).

(113) holds directly by (28). □

Theorem 8.2. *Assume $1\leq j\leq 8\frac{b_{n+1}}{q_n}$ and $b_{n+1}\geq\frac{q_n}{2}$. Then*

$$\frac{q_{n+1}}{\bar{r}_j^n}e^{-\varepsilon jq_n}\leq A(j)\leq\frac{q_{n+1}}{\bar{r}_j^n}e^{\varepsilon jq_n}.\tag{115}$$

Proof. We first show the left inequality. Clearly

$$||A_k||\geq||U(k)||^{-1},\tag{116}$$

thus by (89) and (112), we must have for any $jq_n\leq k<(j+1)q_n$ with $j\geq 0$ and $k\geq\frac{q_n}{4}$,

(117)
$$\max\{e^{-|k-jq_n|\ln\lambda}A(j),e^{-|k-(j+1)q_n|\ln\lambda}A(j+1)\}e^{\varepsilon k}\geq(\max\{e^{-|k-jq_n|\ln\lambda}\bar{r}_j^n,e^{-|k-(j+1)q_n|\ln\lambda}\bar{r}_{j+1}^n\})^{-1}e^{-\varepsilon k}.$$

Let

$$k_0 = (j+1)q_n - \frac{\ln q_{n+1} - \ln(j+1)}{2\ln\lambda}.$$

One has $k_0 \geq \frac{q_n}{4}$, thus

$$\max\{e^{-|k_0 - jq_n|\ln\lambda}\bar{r}_j^n, e^{-|k_0-(j+1)q_n|\ln\lambda}\bar{r}_{j+1}^n\} \leq \bar{r}_{j+1}^n(\frac{j+1}{q_{n+1}})^{\frac{1}{2}}e^{\varepsilon k_0}.$$

Combining with (117), we have

$$\max\{e^{-|k_0-jq_n|\ln\lambda}A(j), e^{-|k_0-(j+1)q_n|\ln\lambda}A(j+1)\} \geq \frac{q_{n+1}^{\frac{1}{2}}}{(j+1)^{\frac{1}{2}}}(\bar{r}_{j+1}^n)^{-1}e^{-\varepsilon k_0}. \tag{118}$$

This implies that either

$$A(j) \geq e^{\ln\lambda q_n}(\bar{r}_{j+1}^n)^{-1}e^{-\varepsilon k_0}, \tag{119}$$

or

$$A(j+1) \geq \frac{q_{n+1}}{j+1}(\bar{r}_{j+1}^n)^{-1}e^{-\varepsilon k_0}. \tag{120}$$

Notice that by (64) and (114), we have

$$A(j+1) \geq A(j)e^{(\ln\lambda-\varepsilon)q_n}\frac{j+1}{q_{n+1}}. \tag{121}$$

By (119),(120) and (121), we obtain the left inequality of (115).

Now we turn to the proof of the right inequality of (115). By (8.5) and (8.7) in [40] we have

$$||A_kU(0)||^2 \leq ||A_k||^2m(k)^2 + ||A_k||^{-2}, \tag{122}$$

where

$$m(k) \leq C\sum_{p=k}^{\infty}\frac{1}{||A_p||^2}. \tag{123}$$

If $k \geq C_\star jq_n$ with $j \geq 1$, by (98) we have

$$\begin{aligned} ||A_k|| &\geq ||U(k)||^{-1} \\ &\geq e^{(\ln\lambda-\beta-\varepsilon)k} \end{aligned}$$

and by (28) we have

$$A(j) \leq e^{(\ln\lambda+\varepsilon)jq_n}.$$

This implies

$$||A_k|| \geq A(j)e^{\frac{\ln\lambda-\beta}{2}k}. \tag{124}$$

If $jq_n \leq k \leq C_\star jq_n$ with $j \geq 1$, let $j_0q_n \leq k < (j_0+1)q_n$ with $j \leq j_0 \leq C_\star j$. By (113) and (121), we have

$$\begin{aligned} ||A_k|| &\geq A(j_0)\max\{e^{-|k-j_0q_n|\ln\lambda}, e^{-|k-(j_0+1)q_n|\ln\lambda}e^{q_n\ln\lambda}\frac{j_0+1}{q_{n+1}}\}e^{-\varepsilon j_0q_n} \\ &\geq (\frac{j_0+1}{q_{n+1}})^{\frac{1}{2}}A(j_0)e^{-\varepsilon j_0q_n} \\ &\geq (\frac{j+1}{q_{n+1}})^{\frac{1}{2}}A(j)e^{-\varepsilon jq_n}. \end{aligned} \tag{125}$$

Thus by (124) and (125), we have

$$m(jq_n) \leq \frac{q_{n+1}}{jA(j)^2} e^{\varepsilon j q_n}. \tag{126}$$

Let $k = jq_n$ in (122). One has

$$\tilde{r}_j^2 \leq \frac{q_{n+1}^2}{j^2 A(j)^2} e^{\varepsilon j q_n}.$$

Thus by (6.1), we obtain

$$A(j) \leq \frac{q_{n+1}}{j\bar{r}_j^n} e^{\varepsilon j q_n}. \tag{127}$$

This implies the right inequality of (115). □

Theorems 8.1 and 8.2 imply the following theorem directly.

Theorem 8.3. *Assume* $jq_n \leq k < (j+1)q_n$ *with* $0 \leq j \leq 6\frac{b_{n+1}}{q_n}$, $b_{n+1} \geq \frac{q_n}{2}$. *We have, for* $k \geq q_n$,

$$||A_k|| \leq \max\{e^{-|k-jq_n|\ln\lambda} \frac{q_{n+1}}{\bar{r}_j^n}, e^{-|k-(j+1)q_n|\ln\lambda} \frac{q_{n+1}}{\bar{r}_{j+1}^n}\} e^{\varepsilon|k|}, \tag{128}$$

and

$$||A_k|| \geq \max\{e^{-|k-jq_n|\ln\lambda} \frac{q_{n+1}}{j\bar{r}_j^n}, e^{-|k-(j+1)q_n|\ln\lambda} \frac{q_{n+1}}{(j+1)\bar{r}_{j+1}^n}\} e^{-\varepsilon|k|}. \tag{129}$$

and for $\frac{q_n}{4} \leq k < q_n$,

$$||A_k|| \leq \max\{e^{-|k|\ln\lambda}, e^{-|k-q_n|\ln\lambda} \frac{q_{n+1}}{\bar{r}_1^n}\} e^{\varepsilon|k|}, \tag{130}$$

and

$$||A_k|| \geq \max\{e^{-|k|\ln\lambda}, e^{-|k-q_n|\ln\lambda} \frac{q_{n+1}}{\bar{r}_1^n}\} e^{-\varepsilon|k|}. \tag{131}$$

Theorem 8.4. *For any* $q_n^{\frac{8}{9}} \leq k \leq \frac{q_n}{2}$, *let* n_0 *be the smallest positive integer such that* $q_{n-n_0} \leq k < q_{n-n_0+1}$. *Suppose* $jq_{n-n_0} \leq k < (j+1)q_{n-n_0+1}$ *with* $j \geq 1$, *then the following holds,*

$$||A_k|| \leq \max\{e^{-|k-jq_{n-n_0}|\ln\lambda} \frac{q_{n-n_0+1}}{\bar{r}_j^{n-n_0}}, e^{-|k-(j+1)q_{n-n_0}|\ln\lambda} \frac{q_{n-n_0+1}}{\bar{r}_{j+1}^{n-n_0}}\} e^{\varepsilon|k|}, \tag{132}$$

and

$$||A_k|| \geq \max\{e^{-|k-jq_{n-n_0}|\ln\lambda} \frac{q_{n-n_0+1}}{\bar{r}_j^{n-n_0}}, e^{-|k-(j+1)q_{n-n_0}|\ln\lambda} \frac{q_{n-n_0+1}}{\bar{r}_{j+1}^{n-n_0}}\} e^{-\varepsilon|k|}. \tag{133}$$

Proof. As in the proof of Theorem 6.3, we split into the same two cases: 1 and 2. Case 1 can be done directly by Theorem 8.3. For case 2, as in the proof of case 2 of Theorem 6.3, it suffices to show

$$e^{(\ln\lambda-\varepsilon)k} \leq ||A_k|| \leq e^{(\ln\lambda+\varepsilon)k}, \tag{134}$$

which follows directly from (93),(116) and (28). □

Proof of (13)

Proof. The arguments are similar to the proof of (12) and consist of collecting the already proved facts, with the same cases.

Case i: $\frac{q_n}{2} \leq q_{n+1}^{\frac{8}{9}}$.

For $\frac{q_n}{2} \leq k \leq 4q_{n+1}^{\frac{8}{9}}$, the result follows from Theorem 8.3.

For $4q_{n+1}^{\frac{8}{9}} \leq k \leq \frac{q_{n+1}}{2}$, (13) follows from Theorem 8.4 (notice that now $k \geq 2q_n$, thus $n_0 = 1$).

Case ii: $q_{n+1}^{\frac{8}{9}} \leq \frac{q_n}{2}$.

Case ii.1: $\frac{q_n}{2} \leq k \leq \min\{q_n, \frac{q_{n+1}}{2}\}$.

If $q_n = q_{n-1} + q_{n-2}$, then $q_{n-1} \geq \frac{q_n}{2}$. This is the case 2 of Theorem 8.4. By (134), one has for any $q_{n-1} \leq k \leq \min\{q_n, \frac{q_{n+1}}{2}\}$

$$||A_k|| \geq e^{(\ln \lambda - \varepsilon)k}.$$

This leads to

$$||A_k|| \geq e^{(\ln \lambda - \varepsilon)k}.$$

for $\frac{q_n}{2} \leq k \leq \min\{q_n, \frac{q_{n+1}}{2}\}$. This also implies (13).

If $q_n = jq_{n-1} + q_{n-2}$ with $j \geq 2$, then $\frac{q_n}{2} \geq q_{n-1}$. (13) follows directly from Theorem 8.4 (notice that now $n + 1 - n_0 = n - 1$).

Case ii.2 $q_n \leq k \leq \frac{q_{n+1}}{2}$

In this case (13) follows directly from Theorem 8.4 (notice that now $n + 1 - n_0 = n$). □

9. Proof of the corollaries

Proof of Corollary 2.12

Proof. Due to (28), i) follows from iii). By Theorem 2.1 to prove iii), it is enough to show that for any $\varepsilon > 0$, sufficiently large n and $\varepsilon q_n < k < q_n$, we have

$$e^{(\ln \lambda - C\varepsilon)k} \leq g(k) \leq e^{(\ln \lambda + C\varepsilon)k}. \tag{135}$$

Let $m \leq n+1$ be such that $q_m/2 \leq k < q_{m+1}/2$. If $q_m \leq k$, we are in Case 1 of the definition of g with $m \leq n-1$. Notice that $\ell q_m \geq \varepsilon q_n \geq \varepsilon q_{m+1}$, which leads to $\frac{\ln \frac{q_{m+1}}{\ell}}{q_m}$ being small. Then (135) follows from (6). If $q_m/2 < k < q_m$ (135) is automatic by the definition of g.

It remains to establish ii).

First by (98), we must have

$$\liminf_{k \to \infty} \frac{\ln ||A_k||}{k} \geq \ln \lambda - \beta. \tag{136}$$

Let $j_k = \lfloor q_{n_k+1}^{\varepsilon} \rfloor$, where sequence q_{n_k} is given by (99). Then

$$\begin{aligned} g(j_k q_{n_k}) &= \frac{q_{n_k+1}}{\bar{r}_{j_k}^{n_k}} \\ &= e^{(\ln \lambda - \frac{\ln q_{n_k+1}}{q_{n_k}} + \frac{\ln j_k}{q_{n_k}}) j_k q_{n_k}} q_{n_k+1} \\ &\leq e^{(\ln \lambda - \beta + C\varepsilon) j_k q_{n_k}}. \end{aligned}$$

Combining with Theorem 2.1, we must have

$$\liminf_{k \to \infty} \frac{\ln ||A_k||}{k} \leq \ln \lambda - \beta. \tag{137}$$

ii) holds by (136) and (137). □

Proof of Corollary 2.13

Proof. i) follows from iii) and ii) follows from (98) and (100). To establish iii), we only need to show that for any $\varepsilon q_n \le k \le q_n - \frac{\beta}{2\ln\lambda} q_n - \varepsilon q_n$,

$$e^{-(\ln\lambda + C\varepsilon)k} \le ||U(k)|| \le e^{-(\ln\lambda - C\varepsilon)k}.$$

Let $m \le n+1$ be such that $q_m/2 \le k < q_{m+1}/2$.

Case 1: $m \le n-1$.

Then by Theorem 2.1, the statement is not immediate only in case 1, but then it follows from (5) and (7) since in that case $\bar{r}_\ell^m \le e^{-(\ln\lambda - \varepsilon)\ell q_m}$.

Case 2: $m = n$ or $m = n+1$.

In this case we have $\frac{q_m}{2} \le k \le q_m - \frac{\beta}{2\ln\lambda} q_m - \varepsilon q_m$ and $\frac{q_m}{2} \le k < \frac{q_{m+1}}{2}$. Then the statement holds by Theorem 2.1 and, in case 1, (7).

To prove iv) it suffices to show that for any $q_{n_j} - \frac{\beta}{2\ln\lambda} q_{n_j} + \varepsilon q_{n_j} \le k \le q_{n_j} - \varepsilon q_{n_j}$,we have

$$||U(k)|| \ge e^{-(\ln\lambda - c\varepsilon)k}$$

where q_{n_j} is a subsequence satisfying (99).Indeed, under this assumption we are in Case 1 of the definition of f and the second addend dominates in (7) leading to the statement. □

Remarks

- If we take for q_{n_k} a subsequence with *any* bounded away from zero exponential growth, we still get non-Lyapunov behavior on intervals of the form $[q_{n_k} - cq_{n_k} + \varepsilon q_{n_k}, q_{n_k} - \varepsilon q_{n_k}]$ for some $c < \frac{\beta}{2\ln\lambda}$.
- In fact, in all the arguments β can be replaced with $\ln q_{n+1}/q_n$.

Proof of Corollary 2.14

Proof. First by (24), one has

$$||U(k)||||\tilde{U}(k)|| \sin\delta_k = \frac{1}{2}. \tag{138}$$

Combining with (114), we have

$$\frac{1}{2||U(k)||||A_k||} \le \sin\delta_k \le \frac{1}{||U(k)||||A_k||}. \tag{139}$$

We first prove (21). Clearly, it suffices to show

$$\limsup_{k\to\infty} \frac{\ln\delta_k}{k} \ge 0. \tag{140}$$

Let $k_j = \lfloor \frac{1}{4} q_{n_j+1} \rfloor$ where sequence q_{n_j} is given by (99). By Theorem 2.1, we must have

$$||U(k_j)|| \le e^{-(\ln\lambda - \varepsilon)k_j},$$

and

$$||A_{k_j}|| \le e^{(\ln\lambda + \varepsilon)k_j}.$$

Combining with (139), we must have

$$\delta_{k_n} \ge e^{-\varepsilon k_n}. \tag{141}$$

This implies (140) and also implies (21).

Now we verify (22). By the definition of $f(k), g(k)$, for any large k, we have

$$f(k)g(k) \le e^{(\beta+\varepsilon)k}. \tag{142}$$

Then by (139) and Theorem 2.1 again, one has

$$\liminf_{k\to\infty}\frac{\ln\delta_k}{k}\geq-\beta. \tag{143}$$

Let $k_j=q_{n_j}$. One has

$$f(k_j)g(k_j)=q_{n_j+1}. \tag{144}$$

Combining with (139) and Theorem 2.1 again, we get

$$\lim_{j\to\infty}\frac{\ln\delta_{k_j}}{k_j}=-\beta. \tag{145}$$

(22) follows directly from (143) and (145). ☐

Proof of Corollary 2.11

Proof. This Corollary follows directly from (13) and (114). ☐

Appendix A. Gordon arguments for $\lambda\leq e^\beta$

Proposition A.1. *The almost Mathieu operator*

$$(H_{\lambda,\alpha,\theta}u)(n)=u(n+1)+u(n-1)+2\lambda\cos 2\pi(\theta+n\alpha)u(n),$$

has no localized eigenfunctions if $|\lambda|\leq e^\beta$.[14]

Proof. Otherwise, there exists a solution $\{u(n)\}_{n\in\mathbb{Z}}$ of $H_{\lambda,\alpha,\theta}u=Eu$ such that

$$|u(n)|\leq Ce^{-5c|n|}, \tag{146}$$

where $c>0$. Without loss of generality, assume the vector $\begin{pmatrix}u(0)\\u(-1)\end{pmatrix}$ is unit.

Let $\varphi(n)=\begin{pmatrix}u(n)\\u(n-1)\end{pmatrix}$. For simplicity, denote $\varphi=\varphi(0)$. By the definition of $\beta(\alpha)$, there exists a subsequence $\tilde{q}_k$ of q_n such that

$$||\tilde{q}_k\alpha||_{\mathbb{R}/\mathbb{Z}}\leq e^{-(\beta-\frac{c}{4})\tilde{q}_k}. \tag{147}$$

Denote $B=A_{\tilde{q}_k}(\theta)$. Then we have

$$B^2+(\mathrm{Tr}B)B+I=0. \tag{148}$$

Case 1: if $\mathrm{Tr}B\leq e^{2c\tilde{q}_k}$, one has either $||B^2\varphi||\geq\frac{1}{2}$ or $||B\varphi||\geq\frac{1}{2}e^{-2c\tilde{q}_k}$. By (146), we must have $||B^2\varphi||\geq\frac{1}{2}$. This is impossible. Indeed, from the following estimate

$$\begin{aligned}||\varphi(2\tilde{q}_k)-B^2\varphi|| &= ||A_{\tilde{q}_k}(\theta+\tilde{q}_k\alpha)-A_{\tilde{q}_k}(\theta)||\ \ ||\varphi(\tilde{q}_k)||\\ &\leq Ce^{\frac{3}{2}c\tilde{q}_k}e^{-5c\tilde{q}_k}\\ &\leq e^{-3c\tilde{q}_k},\end{aligned}$$

where the first inequality holds by (71). Then

$$||\varphi(2\tilde{q}_k)||\geq\frac{1}{4},$$

contradicting $||\varphi(2\tilde{q}_k)||\leq Ce^{-10c\tilde{q}_k}$.

[14] Localized here means exponentially decaying. One can exclude any decaying solutions for $|\lambda|<e^\beta$[9] but not for $|\lambda|=e^\beta$[7].

Case 2: if $\mathrm{Tr}B \geq e^{2c\tilde{q}_k}$, from (148), it is easy to see that either $||B\varphi|| \geq \frac{1}{2}e^{2c\tilde{q}_k}$ or $||B^{-1}\varphi|| \geq \frac{1}{2}e^{2c\tilde{q}_k}$ holds. By (146) again, we must have $||B^{-1}\varphi|| \geq \frac{1}{2}e^{2c\tilde{q}_k}$. By (72), the following holds

$$\begin{aligned} ||\varphi(-\tilde{q}_k) - B^{-1}\varphi|| &= ||A_{\tilde{q}_k}^{-1}(\theta - \tilde{q}_k\alpha) - A_{\tilde{q}_k}^{-1}(\theta)|| \ ||\varphi|| \\ &\leq e^{\frac{3}{2}c\tilde{q}_k}. \end{aligned}$$

Thus

$$||\varphi(-\tilde{q}_k)|| \geq \frac{1}{4}e^{2c\tilde{q}_k}.$$

This is also impossible.

□

Appendix B. Uniformity

We start with some basic facts.

Let $\frac{p_n}{q_n}$ be the continued fraction approximants to α. Then

$$\forall 1 \leq k < q_{n+1}, \mathrm{dist}(k\alpha, \mathbb{Z}) \geq |q_n\alpha - p_n|, \tag{149}$$

and

$$\frac{1}{2q_{n+1}} \leq \Delta_n := |q_n\alpha - p_n| \leq \frac{1}{q_{n+1}}. \tag{150}$$

Lemma B.1. *(Lemma 9.7, [5]) Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $x \in \mathbb{R}$ and $0 \leq \ell_0 \leq q_n - 1$ be such that $|\sin\pi(x+\ell_0\alpha)| = \inf_{0\leq\ell\leq q_n-1}|\sin\pi(x+\ell\alpha)|$, then for some absolute constant $C > 0$,*

$$-C\ln q_n \leq \sum_{\ell=0,\ell\neq\ell_0}^{q_n-1} \ln|\sin\pi(x+\ell\alpha)| + (q_n - 1)\ln 2 \leq C\ln q_n. \tag{151}$$

We now prove

Lemma B.2. *For any $|i|, |j| \leq 50C_*b_{n+1}$, if θ is n-Diophantine with respect to α, then the following estimate holds,*

$$\ln|\sin\pi(2\theta + (j+i)\alpha)| \geq -C\ln q_n. \tag{152}$$

Proof. By the Diophatine condition on θ, (2), one has that there exist $\kappa > 0$ and $\nu > 0$ such that

$$\min_{j,i\in[-q_n,q_n]} |\sin\pi(2\theta + (j+i)\alpha)| \geq \frac{\kappa}{q_n^\nu}. \tag{153}$$

Let $\ell_i, \ell_j \in \mathbb{Z}$ be such that $dist(i, q_n\mathbb{Z}) = |i - \ell_i q_n|$ and $dist(j, q_n\mathbb{Z}) = |j - \ell_j q_n|$. Then $|\ell_i|, |\ell_j| \leq 50C_*\frac{b_{n+1}}{q_n} + 1$. Let $i' = i - \ell_i q_n$ and $j' = j - \ell_j q_n$, then $i', j' \in [-q_n, q_n]$.

If $q_{n+1}^{1-t} > \frac{100C_*}{\kappa}q_n^{\nu+2}$, it is easy to verify that $|\ell_k\Delta_n| < \frac{\kappa}{q_n^{\nu+1}}$. Combining with (153), we have for any $|i|, |j| \leq 50C_*b_{n+1}$

$$\begin{aligned} &|\sin\pi(2\theta + (j+i)\alpha)| \\ &\quad = |\sin\pi(2\theta + (j'+i')\alpha)\cos\pi(\ell_i+\ell_j)\Delta_n \pm \cos\pi(2\theta + (j'+i')\alpha)\sin\pi(\ell_i+\ell_j)\Delta_n| \\ &\quad \geq \frac{\kappa}{100q_n^\nu} \end{aligned}$$

(the choice of $\pm$ depends on the sign of $q_n\alpha - p_n$).

If $q_{n+1}^{1-t} \leq \frac{100C_*}{\kappa} q_n^{\nu+2}$, we also have for any $|i|, |j| \leq 50C_* b_{n+1}$

$$|\sin \pi(2\theta + (j+i)\alpha)| \geq \frac{\kappa^{1+\frac{t\nu}{1-t}}}{(100C_*)^{\frac{\nu}{1-t}} q_n^{\frac{\nu t(\nu+2)}{1-t}}}.$$

Thus in both cases, we have

$$\min_{|i|,|j| \leq 50C_* b_{n+1}} \ln|\sin \pi(2\theta + (j+i)\alpha)| \geq -C \ln q_n. \tag{154}$$

□

Lemma B.3. *Assume* $|i|, |j| \leq 50C_* b_{n+1}$*, and* $i - j \neq q_n\mathbb{Z}$*. Then*

$$\ln|\sin \pi(j-i)\alpha| \geq -C \ln q_n. \tag{155}$$

Proof. By assumption, $|j - i| = \ell q_n + r$ with $0 \leq \ell \leq 100C_* \frac{b_{n+1}}{q_n}$ and $0 < r < q_n$. Then by (149) and (150) again, we also have

$$\begin{aligned} ||(j-i)\alpha||_{\mathbb{R}/\mathbb{Z}} &\geq ||r\alpha||_{\mathbb{R}/\mathbb{Z}} - |\ell|||q_n\alpha||_{\mathbb{R}/\mathbb{Z}} \\ &\geq \frac{1}{2q_n} - \frac{|\ell|}{q_{n+1}} \\ &\geq \frac{1}{2q_n} - \frac{100C_*}{q_{n+1}^{1-t}} \frac{1}{q_n} \\ &\geq \frac{1}{4q_n}. \end{aligned}$$

This implies (155). □

We are now ready to study the behavior at non-resonant points. For an n-nonresonant y, let, as before, n_0 be the least positive integer such that $4q_{n-n_0} \leq dist(y, q_n\mathbb{Z})$. Let s be the largest positive integer such that $4sq_{n-n_0} \leq dist(y, q_n\mathbb{Z})$. Recall that, automatically, $n_0 \leq C(\alpha)$. Set $I_1, I_2 \subset \mathbb{Z}$ as follows

$$\begin{aligned} I_1 &= [-sq_{n-n_0}, sq_{n-n_0} - 1], \\ I_2 &= [y - sq_{n-n_0}, y + sq_{n-n_0} - 1], \end{aligned}$$

We have

Theorem B.4. *For an* $n-$*nonresonant* y*, assume that*

$$\min_{j,i \in I_1 \cup I_2} \ln|\sin \pi(2\theta + (j+i)\alpha)| \geq -C \ln q_n. \tag{156}$$

and

$$\min_{i \neq j; i,j \in I_1 \cup I_2} \ln|\sin \pi(j-i)\alpha)| \geq -C \ln q_n. \tag{157}$$

Then for any $\varepsilon > 0$ *and* n *large enough, we have* y *is* $(\ln \lambda + 8 \ln(sq_{n-n_0}/q_{n-n_0+1})/q_{n-n_0} - \varepsilon, 4sq_{n-n_0} - 1)$ *regular with* $\delta = \frac{1}{4}$.

Proof. Without loss of generality assume $y > 0$. By the definition of s and n_0, we have $4sq_{n-n_0} \leq dist(y, q_n\mathbb{Z})$ and $4q_{n-n_0+1} > dist(y, q_n\mathbb{Z})$. This leads to $sq_{n-n_0} \leq q_{n-n_0+1}$. Let $\theta_j = \theta + j\alpha$ for $j \in I_1 \cup I_2$. The set $\{\theta_j\}_{j \in I_1 \cup I_2}$ consists of $4sq_{n-n_0}$ elements.

In (34), let $x = \cos 2\pi a$, $k = 4sq_{n-n_0} - 1$ and take the logarithm, then

$$\ln \prod_{j \in I_1 \cup I_2, j \neq i} \frac{|\cos 2\pi a - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|}$$

$$= \sum_{j \in I_1 \cup I_2, j \neq i} \ln|\cos 2\pi a - \cos 2\pi\theta_j| - \sum_{j \in I_1 \cup I_2, j \neq i} \ln|\cos 2\pi\theta_i - \cos 2\pi\theta_j|.$$

First, we estimate $\sum_{j \in I_1 \cup I_2, j \neq i} \ln|\cos 2\pi a - \cos 2\pi\theta_j|$. Obviously,

$$\begin{aligned} &\sum_{j \in I_1 \cup I_2, j \neq i} \ln|\cos 2\pi a - \cos 2\pi\theta_j| \\ &= \sum_{j \in I_1 \cup I_2, j \neq i} \ln|\sin \pi(a + \theta_j)| + \sum_{j \in I_1 \cup I_2, j \neq i} \ln|\sin \pi(a - \theta_j)| + (4sq_{n-n_0} - 1)\ln 2 \\ &= \Sigma_+ + \Sigma_- + (4sq_{n-n_0} - 1)\ln 2. \end{aligned}$$

Both Σ_+ and Σ_- consist of $4s$ terms of the form of (151), plus 4s terms of the form

$$\ln \min_{j=0,1,\cdots,q_{n-n_0}} |\sin \pi(x + j\alpha)|,$$

minus $\ln|\sin \pi(a \pm \theta_i)|$. Thus, using (151) 4s times for Σ_+ and Σ_- respectively, one has

$$\sum_{j \in I_1 \cup I_2, j \neq i} \ln|\cos 2\pi a - \cos 2\pi\theta_j| \leq -4sq_{n-n_0}\ln 2 + Cs \ln q_{n-n_0}. \tag{158}$$

If $a = \theta_i$, we obtain

$$\begin{aligned} &\sum_{j \in I_1 \cup I_2, j \neq i} \ln|\cos 2\pi\theta_i - \cos 2\pi\theta_j| \\ &= \sum_{j \in I_1 \cup I_2, j \neq i} \ln|\sin \pi(\theta_i + \theta_j)| + \sum_{j \in I_1 \cup I_2, j \neq i} \ln|\sin \pi(\theta_i - \theta_j)| + (4sq_{n-n_0} - 1)\ln 2 \\ &= \Sigma_+ + \Sigma_- + (4sq_{n-n_0} - 1)\ln 2, \end{aligned} \tag{159}$$

where

$$\Sigma_+ = \sum_{j \in I_1 \cup I_2, j \neq i} \ln|\sin \pi(2\theta + (i + j)\alpha)|,$$

and

$$\Sigma_- = \sum_{j \in I_1 \cup I_2, j \neq i} \ln|\sin \pi(i - j)\alpha|.$$

We will estimate Σ_+. Set $J_1 = [-s, s-1]$ and $J_2 = [s, 3s-1]$, which are two adjacent disjoint intervals of length $2s$. Then $I_1 \cup I_2$ can be represented as a disjoint union of segments B_j, $j \in J_1 \cup J_2$, each of length q_{n-n_0}. Applying (151) to each B_j, we obtain

$$\Sigma_+ \geq -4sq_{n-n_0}\ln 2 + \sum_{j \in J_1 \cup J_2} \ln|\sin \pi\hat{\theta}_j| - Cs \ln q_{n-n_0} - \ln|\sin 2\pi(\theta + i\alpha)|, \tag{160}$$

where

$$|\sin \pi\hat{\theta}_j| = \min_{\ell \in B_j} |\sin \pi(2\theta + (\ell + i)\alpha)|. \tag{161}$$

Next we estimate $\sum_{j \in J_1} \ln|\sin \pi\hat{\theta}_j|$. Assume that $\hat{\theta}_{j+1} = \hat{\theta}_j + q_{n-n_0}\alpha$ for every $j, j+1 \in J_1$. In this case, for any $i, j \in J_1$ and $i \neq j$, we have

$$||\hat{\theta}_i - \hat{\theta}_j||_{\mathbb{R}/\mathbb{Z}} \geq ||q_{n-n_0}\alpha||_{\mathbb{R}/\mathbb{Z}}. \tag{162}$$

Applying the Stirling formula, (156) and (162), one has

$$\sum_{j\in J_1}\ln|\sin 2\pi\hat{\theta}_j| > 2\sum_{j=1}^{s}\ln(j\Delta_{n-n_0})-C\ln q_n$$
$$> 2s\ln\frac{s}{q_{n-n_0+1}}-C\ln q_n-Cs. \tag{163}$$

In the other cases, decompose J_1 in maximal intervals T_κ such that for $j,j+1\in T_\kappa$ we have $\hat{\theta}_{j+1}=\hat{\theta}_j+q_{n-n_0}\alpha$. Notice that the boundary points of an interval T_κ are either boundary points of J_1 or satisfy $\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}}+\Delta_{n-n_0}\geq\frac{\Delta_{n-n_0-1}}{2}$. This follows from the fact that if $0<|z|<q_{n-n_0}$, then $\|\hat{\theta}_j+q_{n-n_0}\alpha\|_{\mathbb{R}/\mathbb{Z}}\leq\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}}+\Delta_{n-n_0}$, and $\|\hat{\theta}_j+(z+q_{n-n_0})\alpha\|_{\mathbb{R}/\mathbb{Z}}\geq\|z\alpha\|_{\mathbb{R}/\mathbb{Z}}-\|\hat{\theta}_j+q_{n-n_0}\alpha\|_{\mathbb{R}/\mathbb{Z}}\geq\Delta_{n-n_0-1}-\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}}-\Delta_{n-n_0}$. Assuming $T_\kappa\neq J_1$, then there exists $j\in T_\kappa$ such that $\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}}\geq\frac{\Delta_{n-n_0-1}}{2}-\Delta_{n-n_0}$.

If T_κ contains some j with $\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}}<\frac{\Delta_{n-n_0-1}}{10}$, then

$$|T_\kappa| \geq \frac{\frac{\Delta_{n-n_0-1}}{2}-\Delta_{n-n_0}-\frac{\Delta_{n-n_0-1}}{10}}{\Delta_{n-n_0}}$$
$$\geq \frac{1}{4}\frac{\Delta_{n-n_0-1}}{\Delta_{n-n_0}}-1\geq\frac{s}{8}-1, \tag{164}$$

since $sq_{n-n_0}\leq q_{n-n_0+1}$, where $|T_\kappa|=b-a+1$ if $T_\kappa=[a,b]$. For such T_κ, a similar estimate to (163) gives

$$\sum_{j\in T_\kappa}\ln|\sin\pi\hat{\theta}_j| \geq |T_\kappa|\ln\frac{|T_\kappa|}{q_{n-n_0+1}}-Cs-C\ln q_n$$
$$\geq |T_\kappa|\ln\frac{s}{q_{n-n_0+1}}-Cs-C\ln q_n. \tag{165}$$

If T_κ does not contain any j with $\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}}<\frac{\Delta_{n-n_0-1}}{10}$, then by (150)

$$\sum_{j\in T_\kappa}\ln|\sin\pi\hat{\theta}_j| \geq -|T_\kappa|\ln q_{n-n_0}-C|T_\kappa|$$
$$\geq |T_\kappa|\ln\frac{s}{q_{n-n_0+1}}-C|T_\kappa|. \tag{166}$$

By (165) and (166), one has

$$\sum_{j\in J_1}\ln|\sin\pi\hat{\theta}_j|\geq 2s\ln\frac{s}{q_{n-n_0+1}}-Cs-C\ln q_n. \tag{167}$$

Similarly,

$$\sum_{j\in J_2}\ln|\sin\pi\hat{\theta}_j|\geq 2s\ln\frac{s}{q_{n-n_0+1}}-Cs-C\ln q_n. \tag{168}$$

Putting (160), (167) and (168) together, we have

$$\Sigma_+>-4sq_{n-n_0}\ln 2+4s\ln\frac{s}{q_{n-n_0+1}}-Cs\ln q_{n-n_0}-C\ln q_n. \tag{169}$$

Now we start to estimate Σ_-. Replacing (156) with (157), and following the proof of (169), we obtain,

$$\Sigma_->-4sq_{n-n_0}\ln 2+4s\ln\frac{s}{q_{n-n_0+1}}-Cs\ln q_{n-n_0}-C\ln q_n. \tag{170}$$

From (159), (169) and (170), one has

$$\sum_{j\in I_1\cup I_2, j\neq i}\ln|\cos 2\pi\theta_i-\cos 2\pi\theta_j|$$

$$\geq -4sq_{n-n_0}\ln 2+8s\ln\frac{s}{q_{n-n_0+1}}-Cs\ln q_{n-n_0}-C\ln q_n. \tag{171}$$

By (158) and (171), we have

$$\max_{i\in I_1\cup I_2}\prod_{j\in I_1\cup I_2, j\neq i}\frac{|x-\cos 2\pi\theta_j|}{|\cos 2\pi\theta_i-\cos 2\pi\theta_j|}<e^{4sq_{n-n_0}(-2\ln(s/q_{n-n_0+1})/q_{n-n_0}+\varepsilon)}.$$

Combining with Lemma 3.4, there exists some j_0 with $j_0\in I_1\cup I_2$ such that

$$\theta_{j_0}\notin A_{4sq_{n-n_0}-1,\ln\lambda+2\ln(s/q_{n-n_0+1})/q_{n-n_0}-\varepsilon}.$$

First, we assume $j_0\in I_2$.

Set $I=[j_0-2sq_{n-n_0}+1, j_0+2sq_{n-n_0}-1]=[x_1,x_2]$. By (30), (31) and (32), it is easy to verify

$$|G_I(y,x_i)|\leq\exp\{(\ln\lambda+\varepsilon)(4sq_{n-n_0}-1-|y-x_i|)-4sq_{n-n_0}(\ln\lambda+2\ln(s/q_{n-n_0+1})/q_{n-n_0}-\varepsilon)\}.$$

Notice that $|y-x_i|\geq sq_{n-n_0}$, so we obtain

$$|G_I(y,x_i)|\leq\exp\{-(\ln\lambda+8\ln(s/q_{n-n_0+1})/q_{n-n_0}-2\varepsilon)|y-x_i|\}. \tag{172}$$

If $j_0\in I_1$, we may let $y=0$ or $y=1$ in (172). Combining with (33), we get

$$|\phi(0)|,|\phi(-1)|\leq 6sq_{n-n_0}\exp\{-(\ln\lambda+8\ln(s/q_{n-n_0+1})/q_{n-n_0}-2\varepsilon)sq_{n-n_0}\}.$$

This is in contradiction with $|\phi(0)|^2+|\phi(-1)|^2=1$. Thus $j_0\in I_2$, and the theorem follows from (172). □

Proof of Theorem 3.5.

In case i), (156) and (157) are obtained correspondingly from Lemmas B.2 and B.3, thus Theorem 3.5 follows from Theorem B.4. In case ii) it is easy to see that (156) and (157) also hold, so Theorem B.4 applies as well. □

Assume $b_{n+1}\geq\frac{q_n}{2}$. For any $1\leq j\leq 48C_*\frac{b_{n+1}}{q_n}$, we construct $I_1, I_2\subset\mathbb{Z}$ as follows

$$\begin{aligned} I_1 &= [-\lfloor\frac{1}{2}q_n\rfloor, q_n-\lfloor\frac{1}{2}q_n\rfloor-1],\\ I_2 &= [jq_n-\lfloor\frac{1}{2}q_n\rfloor,(j+1)q_n-\lfloor\frac{1}{2}q_n\rfloor-1],\end{aligned}$$

Let $\theta_m=\theta+m\alpha$ for $m\in I_1\cup I_2$. Then

Theorem B.5. *Suppose θ is n-Diophantine with respect to α. Then for any $\varepsilon>0$, the set $\{\theta_m\}_{m\in I_1\cup I_2}$ is $\frac{\ln q_{n+1}-\ln j}{2q_n}+\varepsilon$-uniform for sufficiently large n.*

Proof. In (34), let $x=\cos 2\pi a$, $k=2q_n-1$ and take the logarithm. Thus in order to prove the theorem, it suffices to show that

$$\begin{aligned}&\ln\prod_{m\in I_1\cup I_2,m\neq i}\frac{|\cos 2\pi a-\cos 2\pi\theta_m|}{|\cos 2\pi\theta_i-\cos 2\pi\theta_m|}\\ &=\sum_{m\in I_1\cup I_2,m\neq i}\ln|\cos 2\pi a-\cos 2\pi\theta_m|-\sum_{m\in I_1\cup I_2,m\neq i}\ln|\cos 2\pi\theta_i-\cos 2\pi\theta_m|\\ &\leq(2q_n-1)(\frac{\ln q_{n+1}-\ln j}{2q_n}+\varepsilon).\end{aligned}$$

First, we estimate $\sum_{m\in I_1\cup I_2, m\neq i}\ln|\cos 2\pi a-\cos 2\pi\theta_m|$. Obviously,

$$\begin{aligned}
&\sum_{m\in I_1\cup I_2, m\neq i}\ln|\cos 2\pi a-\cos 2\pi\theta_m| \\
&= \sum_{m\in I_1\cup I_2, m\neq i}\ln|\sin\pi(a+\theta_m)| + \sum_{m\in I_1\cup I_2, m\neq i}\ln|\sin\pi(a-\theta_m)| + (2q_n-1)\ln 2 \\
&= \Sigma_+ + \Sigma_- + (2q_n-1)\ln 2.
\end{aligned}$$

Both Σ_+ and Σ_- consist of 2 terms of the form of (151), plus two terms of the form

$$\min_{k=1,\cdots,q_n}\ln|\sin\pi(x+k\alpha)|,$$

minus $\ln|\sin\pi(a\pm\theta_i)$. Thus one has

$$\sum_{m\in I_1\cup I_2, m\neq i}\ln|\cos 2\pi a-\cos 2\pi\theta_m| \leq -2q_n\ln 2 + C\ln q_n.$$

Setting $a=\theta_i$ and using the first inequality of (151) two times, we obtain

$$\begin{aligned}
\sum_{m\in I_1\cup I_2, m\neq i}\ln|\cos 2\pi\theta_i-\cos 2\pi\theta_m| \quad \geq \quad & -2q_n\ln 2 - C\ln q_n + 2\min_{m,i\in I_1\cup I_2}\ln|\sin\pi(2\theta+(m+i)\alpha)| \\
& + \min_{m\in I_1\cup I_2, m\neq i}\ln|\sin\pi(m-i)\alpha|. \qquad (173)
\end{aligned}$$

By Lemma B.2, we also have

$$\min_{m,i\in I_1\cup I_2}\ln|\sin\pi(2\theta+(m+i)\alpha)| \geq -C\ln q_n. \qquad (174)$$

By (149) and (150), the corresponding minimum term of $\min_{m\in I_1\cup I_2, m\neq i}\ln|\sin\pi((m-i)\alpha)|$ is achieved at jq_n. It is easy to check that

$$\min\{\ln|\sin\pi jq_n\alpha|\} > -\ln\frac{q_{n+1}}{j} - C, \qquad (175)$$

since $\Delta_n \geq \frac{1}{2q_{n+1}}$.

Putting (173), (174) and (175) together, we obtain

$$\max_{x\in[-1,1]}\max_{i=1,\cdots,k+1}\prod_{m=1,m\neq i}^{k+1}\frac{|x-\cos 2\pi\theta_m|}{|\cos 2\pi\theta_i-\cos 2\pi\theta_m|} \leq e^{(2q_n-1)(\frac{\ln q_{n+1}-\ln j}{2q_n}+\varepsilon)}.$$

□

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