Rank n swapping algebra for the $\mathrm{PSL}(n, \mathbb{R})$ Hitchin component

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Abstract

F. Labourie [arXiv:1212.5015] characterized the Hitchin components for $\mathrm{PSL}(n, \mathbb{R})$ for any $n > 1$ by using the swapping algebra, where the swapping algebra should be understood as a ring equipped with a Poisson bracket. We introduce the rank n swapping algebra, which is the quotient of the swapping algebra by the $(n+1) \times (n+1)$ determinant relations. The main results are the well-definedness of the rank n swapping algebra and the "cross-ratio" in its fraction algebra. As a consequence, we use the sub fraction algebra of the rank n swapping algebra generated by these "cross-ratios" to characterize the $\mathrm{PSL}(n, \mathbb{R})$ Hitchin component for a fixed $n > 1$. We also show the relation between the rank 2 swapping algebra and the cluster $\mathcal{X}_{\mathrm{PGL}(2,\mathbb{R}),D_k}$-space.

1. Introduction

1.1 Background

Let S be a connected oriented closed surface of genus $g > 1$. When G is a reductive Lie group, the character variety is

$$R(S, G) := \{ \text{ homomorphisms } \rho : \pi_1(S) \to G \} // G,$$

where the group G acts on homomorphisms above by conjugation, and the quotient is taken in the sense of geometric invariant theory [MFK94]. When $G = \mathrm{PSL}(2, \mathbb{R})$, the character variety $R(S, \mathrm{PSL}(2, \mathbb{R}))$ has $4g - 3$ connected components [G88]. Two of these components correspond to all discrete faithful homomorphisms from $\pi_1(S)$ to $\mathrm{PSL}(2, \mathbb{R})$. By the uniformization theorem, any one of the two components is diffeomorphic to the Teichmüller space of complex structures on S up to isotopy. For $n \geq 2$, we define *n-Fuchsian representation* to be a representation ρ, which can be written as $\rho = i \circ \rho_0$, where ρ_0 is a discrete faithful representation of $\pi_1(S)$ with values in $\mathrm{PSL}(2, \mathbb{R})$ and i is the irreducible representation of $\mathrm{PSL}(2, \mathbb{R})$ in $\mathrm{PSL}(n, \mathbb{R})$. In [H92], N. Hitchin found one of the connected components of the character variety $R(S, \mathrm{PSL}(n, \mathbb{R}))$, which contains the n-Fuchsian representations, called *Hitchin component* and denoted by $H_n(S)$. By N. Hitchin [H92], the GIT quotient of the Hitchin component $H_n(S)$ coincides with its usual topological quotient. Furthermore, the Hitchin component $H_n(S)$ is diffeomorphic to a ball $\mathbb{R}^{(2g-2)(n^2-1)}$.

A decade later, F. Labourie [L06] and O. Guichard [Gu08] showed that every ρ in the Hitchin component $H_n(S)$ is one to one associated to a ρ-equivariant $(\xi_\rho(\gamma x) = \rho(\gamma)\xi_\rho(x))$ hyperconvex Frenet curve ξ_ρ from the boundary at infinity of $\pi_1(S)$—$\partial_\infty \pi_1(S)$ to $\mathbb{RP}^{n-1}$, where *hyperconvex*

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means that for any pairwise distinct points $(x_1,...,x_p)$ with $p \leq n$, the sum $\xi(x_1)+...+\xi(x_p)$ is direct. Let ξ^*_ρ be its associated ρ-equivariant osculating hyperplane curve from $\partial_\infty\pi_1(S)$ to $\mathbb{RP}^{n-1*}$. Let $\tilde{\xi_\rho}$ ($\tilde{\xi^*_\rho}$ resp.) be the lifts of ξ_ρ (ξ^*_ρ resp.) with values in $\mathbb{R}^n$ ($\mathbb{R}^{n*}$ resp.). F. Labourie defined the *weak cross ratio* $\mathbb{B}_\rho$ of four different points x,y,z,t in $\partial_\infty\pi_1(S)$:

$$\mathbb{B}_\rho(x,y,z,t) = \frac{\left\langle \tilde{\xi}(x) \middle| \tilde{\xi^*}(z) \right\rangle}{\left\langle \tilde{\xi}(x) \middle| \tilde{\xi^*}(t) \right\rangle} \cdot \frac{\left\langle \tilde{\xi}(y) \middle| \tilde{\xi^*}(t) \right\rangle}{\left\langle \tilde{\xi}(y) \middle| \tilde{\xi^*}(z) \right\rangle}. \tag{1}$$

Such cross ratios are the only cross ratios, called the rank n cross ratios, that satisfy some symmetry properties, normalisation properties, multiplicative cocycle identities, $\pi_1(S)$-invariant properties and $\mathbb{R}^n$-linear algebraic properties [L07]. Therefore, the space of the rank n cross ratios identifies with the Hitchin component $H_n(S)$.

Later on, F. Labourie [L12] defined the swapping algebra to characterize the union of the Hitchin components $\bigcup_{n=2}^\infty H_n(S)$. The swapping algebra is defined on the ordered pair of points of a subset $\mathcal{P} \subseteq S^1$. More precisely, we represent an ordered pair (x,y) of $\mathcal{P}$ by the expression xy, and we consider the ring $\mathcal{Z}(\mathcal{P}) := \mathbb{K}[\{xy\}_{\forall x,y\in\mathcal{P}}]/(\{xx|\forall x \in \mathcal{P}\})$ over a field $\mathbb{K}$ of characteristic zero. Then we equip $\mathcal{Z}(\mathcal{P})$ with a Poisson bracket $\{\cdot,\cdot\}$, called the swapping bracket, by extending the formula on generators for any $rx, sy \in \mathcal{P}$:

$$\{rx, sy\} = \mathcal{J}(rx,sy)\cdot ry \cdot sx, \tag{2}$$

to $\mathcal{Z}(\mathcal{P})$ by using Leibniz's rule. We will define the *linking number* $\mathcal{J}(rx,sy)$ in Section 2. Therefore, the *swapping algebra* of $\mathcal{P}$ is $(\mathcal{Z}(\mathcal{P}),\{\cdot,\cdot\})$. Let x,y,z,t belong to $\mathcal{P}$ so that $x\neq t$ and $y \neq z$. The *cross fraction* determined by (x,y,z,t) is the element:

$$[x,y,z,t] := \frac{xz}{xt}\cdot\frac{yt}{yz}.$$

Let $\mathcal{B}(\mathcal{P})$ be the sub fraction ring of $\mathcal{Z}(\mathcal{P})$ generated by all the cross fractions. Then, the *swapping multifraction algebra of* $\mathcal{P}$ is $(\mathcal{B}(\mathcal{P}),\{\cdot,\cdot\})$. Let $\mathcal{R}$ be the subset of $\partial_\infty\pi_1(S)$ given by the end points of periodic geodesics. F. Labourie consider a natural homomorphism I from $\mathcal{B}(\mathcal{R})$ to $C^\infty(H_n(S))$ by extending the following formula on generators to $\mathcal{B}(\mathcal{R})$:

$$I([x,y,z,t]) = \mathbb{B}_\rho(x,y,z,t). \tag{3}$$

Theorem 1.1 [F. Labourie [L12]] *Let S be a connected oriented closed surface of genus $g>1$. Let $\{\cdot,\cdot\}$ be the swapping bracket. For $n\geq 2$, let $\{\cdot,\cdot\}_S$ be the Atiyah-Bott-Goldman Poisson bracket [AB83][G84] of the Hitchin component $H_n(S)$. If $\Gamma_1,...,\Gamma_k,...$ is a vanishing sequence of finite index subgroups of $\pi_1(S)$. Let $S_k = \mathbb{H}^2/\Gamma_k$, vanishing means that any two primitive representatives of $\pi_1(S)$ in the sequence $S_1,...,S_k,...$ intersect simply at zero or one point at last. For any $b_0,b_1\in\mathcal{B}(\mathcal{R})$, we have*

$$\lim_{k\to\infty}\{I(b_0),I(b_1)\}_{S_k} = I\circ\{b_0,b_1\}.$$

The above theorem is true for any integer $n>1$, therefore the swapping multifraction algebra $(\mathcal{B}(\mathcal{R}),\{\cdot,\cdot\})$ asymptotically characterizes the union of Hitchin components $\bigcup_{n=2}^\infty H_n(S)$.

F. Labourie also showed that, for the space $\mathcal{L}_n$ of the Drinfeld-Sokolov reduction [DS85][Se91] on the space of $\mathrm{PSL}(n,\mathbb{R})$-Hitchin opers with trivial holonomy, the natural homomorphism i from the swapping multifraction algebra $\mathcal{B}(S^1)$ to the function space $C^\infty(\mathcal{L}_n)$ is Poisson with respect to the swapping bracket and the Poisson bracket corresponding to second Gelfand-Dickey symplectic structure.

Both the homomorphism I and the homomorphism i have large kernels arising from linear algebra of $\mathbb{R}^n$. Is the swapping algebra $(\mathcal{Z}(\mathcal{P}),\{\cdot,\cdot\})$ still well-defined after divided by these corresponding linear algebraic relations? Is the associated sub fraction algebra generated by all the cross fractions well-defined? These two questions are the main focus of this paper.

1.2 Rank n swapping algebra and the main results

For $n \geq 2$, let $R_n(\mathcal{P})$ be the ideal of $\mathcal{Z}(\mathcal{P})$ generated by

$$\left\{ D \in \mathcal{Z}(\mathcal{P}) \mid D = \det \begin{pmatrix} x_1y_1 & ... & x_1y_{n+1} \\ ... & ... & ... \\ x_{n+1}y_1 & ... & x_{n+1}y_{n+1} \end{pmatrix}, \forall x_1,...,x_{n+1},y_1,...,y_{n+1} \in \mathcal{P} \right\}.$$

Let $\mathcal{Z}_n(\mathcal{P})$ be the quotient ring $\mathcal{Z}(\mathcal{P})/R_n(\mathcal{P})$. The following two theorems are the main results of this paper, which will be proven in Section 3 and 4. By induction on corresponding positions of the points on the circle, we prove the following theorem.

Theorem 1.2 *For $n \geq 2$, $R_n(\mathcal{P})$ is a Poisson ideal with respect to the swapping bracket, thus $\mathcal{Z}_n(\mathcal{P})$ inherits a Poisson bracket from the swapping bracket.*

It then follows the Theorem 1.2 that the *rank n swapping algebra of $\mathcal{P}$* is the compatible pair $(\mathcal{Z}_n(\mathcal{P}),\{\cdot,\cdot\})$. For the well-definedness of the cross fractions of the ring $\mathcal{Z}_n(\mathcal{P})$, by using very classical geometric invariant theory [CP76][W39] and Lie group cohomology [CE48], we prove the following theorem.

Theorem 1.3 *For $n \geq 2$, the quotient ring $\mathcal{Z}_n(\mathcal{P})$ is an integral domain.*

Let $\mathcal{B}_n(\mathcal{P})$ be the sub fraction ring of $\mathcal{Z}_n(\mathcal{P})$ generated by all the cross fractions. Then, the *rank n swapping multifraction algebra of $\mathcal{P}$* is the pair $(\mathcal{B}_n(\mathcal{P}),\{\cdot,\cdot\})$. Thus, the homomorphism I naturally factors through $\mathcal{B}_n(\mathcal{P})$ because of the rank n cross ratio conditions [L07], which provides a homomorphism

$$I_n : \mathcal{B}_n(\mathcal{R}) \to C^\infty(H_n(S)).$$

Then we can replace I by I_n in Theorem 1.1. Therefore, for a fixed $n \geq 2$, the rank n swapping multifraction algebra $(\mathcal{B}_n(\mathcal{P}),\{\cdot,\cdot\})$ is the Poisson algebra which characterizes $H_n(S)$. But the homomorphism I_n is not injective, since the image of the cross fractions are $\pi_1(S)$ invariant. Still, the non-injectivity and the asymptotic behavior of I_n are two obstructions to characterize $(H_n(S),\omega_{ABG})$ exactly. We suggest that these two obstructions are worth of being investigated.

For the homomorphism i, we do not have these two obstructions. Similarly, we also have a homomorphism

$$i_n : \mathcal{B}_n(S^1) \to C^\infty(\mathcal{L}_n)$$

induced from the homomorphism i. The homomorphism i_n is Poisson by Theorem 10.7.2 in [L12] and Theorem 1.2, and injective by Theorem 4.6. As a conseqence, the rank n swapping multifraction algebra $(\mathcal{B}_n(S^1),\{\cdot,\cdot\})$ should be regarded as the dual of $\mathcal{W}_n$ algebra.

1.3 Rank 2 swapping algebra and the cluster $\mathcal{X}_{\mathrm{PGL}(2,\mathbb{R}),D_k}$-space

Let S be a connected oriented surface with non-empty boundary and a finite set P of special points on boundary, considered modulo isotopy. The rank n swapping algebra also relates to the Fock-Goncharov's cluster-$\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),S}$ space. V. Fock and A. Goncharov [FG06] introduced the positive structure in sense of [L94][L98] and the cluster algebraic structure for the moduli space

$\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),S}$ of framed local systems of the surface S. The positive part of the moduli space $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),S}$ is related to the Hitchin component $H_n(S)$. (For the surface S with boundary or punctures, we can still define $H_n(S)$, but the monodromy around a boundary component is conjugated to an upper or lower triangular totally positive matrix.) Moreover, they introduced a special coordinate system for the cluster $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),S}$-space in [FG06] Section 9, which generalizes Thurston's shear coordinates for Teichmüller space [T86]. (The Fock-Goncharov coordinates are also used in case of the closed surface S of genus $g>1$ [FD14].) This coordinate system is local, because it depends on the ideal triangulation $\mathcal{T}$. Moreover, the coordinate system for $\mathcal{T}$ gives us a split tori $\mathbb{T}_{\mathcal{T}}$ of $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),S}$. The space $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),S}$ is a variety glued by all these $\mathbb{T}_{\mathcal{T}}$, and the transition function from $\mathbb{T}_{\mathcal{T}}$ to another $\mathbb{T}_{\mathcal{T}'}$ is defined by a positive rational transformation corresponding to a composition of flips, where each flip is a composition of mutations in its cluster algebraic structure. The positive structure of $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),S}$ arises from the positivity of the rational transformations.

Let D_k be a disc with k special points on the boundary. In the last section, we will prove the following theorem.

Theorem 1.4 *Given an ideal triangulation $\mathcal{T}$ of D_k, there is an injective and Poisson homomorphims from the fraction algebra generated by the Fock-Goncharov coordinates for the cluster $\mathcal{X}_{\mathrm{PGL}(2,\mathbb{R}),D_k}$-space to the rank 2 swapping multifraction algebra $(\mathcal{B}_n(\mathcal{P}),\{\cdot,\cdot\})$, with respect to the natural Fock-Goncharov Poisson bracket and the swapping bracket.*

Then we will show that the cluster dynamic of the cluster $\mathcal{X}_{\mathrm{PGL}(2,\mathbb{R}),D_k}$-space can also be interpreted by the rank 2 swapping algebra. As a consequence, the natural Fock-Goncharov Poisson bracket does not depend on the triangulations. The above theorem is generalized for $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),D_k}$-space in the following papers. For $n=3$, in Chapter 3 of [Su14], we showed a complicated homomorphism, where k flags of $\mathbb{RP}^2$ correspond to the set $\mathcal{P}$ with k elements. For a general n, the homomorphism is discussed in [Su15], where the set $\mathcal{P}$ has $(n-1)\cdot k$ elements, each flag of $\mathbb{RP}^{n-1}$ corresponding to $n-1$ points near each other on the boundary S^1.

Therefore, the rank n swapping algebra provides the links among the Hitchin component $H_n(S)$, $\mathcal{W}_n$ algebra and the cluster $\mathcal{X}_{\mathrm{PGL}(2,\mathbb{R}),D_k}$-space.

1.4 Further discussions

In the upcoming paper [Su1511], we will define a quantized version of the rank n swapping algebra. The quantization of $\mathcal{X}_{D_k,\mathrm{PSL}(n,\mathbb{R})}$ by Fock-Goncharov [FG06] [FG09] is embedded into our quantization of the rank n swapping algebra. We will glue the rank n swapping algebras to characterize the cluster $\mathcal{X}_{S,\mathrm{PSL}(n,\mathbb{R})}$-space for the surface S in general. We expect to build a TQFT and some geometric invariants from the rank n swapping algebra.

In [Su1412], we relate the rank n swapping algebra to the discrete integrable system of the configuration space of N-twisted polygon in $\mathbb{RP}^{n-1}$ [FV93][SOT10][KS13]. When $n=2$, there is a bi-hamiltonian structure for the configuration space of N-twisted polygon in $\mathbb{RP}^{n-1}$. This was conjectured in [SOT10] for $n=3$. We expect that there exsits a bi-hamiltonian structure for n in general.

2. Swapping algebra revisited

In this section, we will recall some basic definitions about the swapping algebra introduced by F. Labourie in Section 2 of [L12]. The new part of this section is that we take care of the

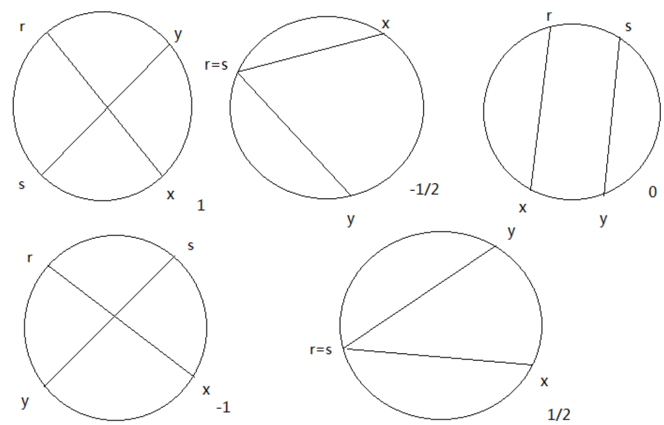

FIGURE 1. Linking number between rx and sy.

compatibilities of the rings related to $\mathcal{Z}(\mathcal{P})$ and the swapping bracket, particularly the sub fraction ring $\mathcal{B}(\mathcal{P})$ generated by "cross ratios".

2.1 Linking number

Definition 2.1 [LINKING NUMBER] *Let (r, x, s, y) be a quadruple of four points in S^1. The* linking number *between rx and sy is*

$$\mathcal{J}(rx, sy) = \frac{1}{2} \cdot (\sigma(r-x) \cdot \sigma(r-y) \cdot \sigma(y-x) - \sigma(r-x) \cdot \sigma(r-s) \cdot \sigma(s-x)), \tag{4}$$

such that for any $a \in \mathbb{R}$, we define $\sigma(a)$ as follows. Remove any point o different from r, x, s, y in S^1 in order to get an interval $]0,1[$. Then the points $r, x, s, y \in S^1$ correspond to the real numbers in $]0,1[$, $\sigma(a) = -1; 0; 1$ whenever $a < 0$; $a = 0$; $a > 0$ respectively.

In fact, the value of $\mathcal{J}(rx, sy)$ belongs to $\{0, \pm 1, \pm \frac{1}{2}\}$, depends on the corresponding positions of r, x, s, y and does not depend on the choice of the point o. In Figure 1, we describe five possible values of $\mathcal{J}(rx, sy)$.

2.2 Swapping algebra

Let $\mathcal{P}$ be a cyclic subset of S^1, we represent an ordered pair (r, x) of $\mathcal{P}$ by the expression rx. Then we consider the associative commutative ring

$$\mathcal{Z}(\mathcal{P}) := \mathbb{K}[\{xy\}_{\forall x,y \in \mathcal{P}}]/\{xx | \forall x \in \mathcal{P}\}$$

over a field $\mathbb{K}$ of characteristic 0, where $\{xy\}_{\forall x,y \in \mathcal{P}}$ are variables. Then we equip $\mathcal{Z}(\mathcal{P})$ with a swapping bracket.

Definition 2.2 [SWAPPING BRACKET [L12]] The swapping bracket *over $\mathcal{Z}(\mathcal{P})$ is defined by extending the following formula for any rx, sy in $\mathcal{P}$ to $\mathcal{Z}(\mathcal{P})$ by using* Leibniz's rule*:*

$$\{rx, sy\} = \mathcal{J}(rx, sy) \cdot ry \cdot sx. \tag{5}$$

By direct computations, F. Labourie proved the following theorem.

Theorem 2.3 [F. LABOURIE [L12]] *The swapping bracket is Poisson.*

Definition 2.4 [SWAPPING ALGEBRA] The swapping algebra of $\mathcal{P}$ *is the ring* $\mathcal{Z}(\mathcal{P})$ *equipped with the swapping bracket, denoted by* $(\mathcal{Z}(\mathcal{P}), \{\cdot,\cdot\})$.

2.3 Swapping multifraction algebra

In this subsection, we consider the rings related to $\mathcal{Z}(\mathcal{P})$ and their compatibilities with the swapping bracket.

Definition 2.5 [CLOSED UNDER SWAPPING BRACKET] *For a ring* R*, if* $\forall a, b \in R$*, we have* $\{a, b\} \in R$*, then we say that* R *is* closed under swapping bracket.

Since $\mathcal{Z}(\mathcal{P})$ is an integral domain, let $\mathcal{Q}(\mathcal{P})$ be the total fraction ring of $\mathcal{Z}(\mathcal{P})$. By Leibniz's rule, we have $\{a, \frac{1}{b}\} = -\frac{\{a,b\}}{b^2}$, thus the swapping bracket is well defined on $\mathcal{Q}(\mathcal{P})$. Therefore we have the following definition.

Definition 2.6 [SWAPPING FRACTION ALGEBRA OF $\mathcal{P}$] *The* swapping fraction algebra of $\mathcal{P}$ *is the ring* $\mathcal{Q}(\mathcal{P})$ *equipped with the induced swapping bracket, denoted by* $(\mathcal{Q}(\mathcal{P}), \{\cdot,\cdot\})$.

Definition 2.7 [CROSS FRACTION] *Let* x, y, z, t *belong to* $\mathcal{P}$ *so that* $x \neq t$ *and* $y \neq z$*. The* cross fraction *determined by* (x, y, z, t) *is the element of* $\mathcal{Q}(\mathcal{P})$*:*

$$[x, y, z, t] := \frac{xz}{xt} \cdot \frac{yt}{yz}. \tag{6}$$

Remark 2.8 *Notice that the cross fractions verify the following* cross-ratio conditions *[L07]:*

Symmetry: $[a, b, c, d] = [b, a, d, c]$,

Normalisation: $[a, b, c, d] = 0$ *if and only if* $a = c$ *or* $b = d$,

Normalisation: $[a, b, c, d] = 1$ *if and only if* $a = b$ *or* $c = d$,

Cocycle identity: $[a, b, c, d] \cdot [a, b, d, e] = [a, b, c, e]$,

Cocycle identity: $[a, b, d, e] \cdot [b, c, d, e] = [a, c, e, f]$.

Let $\mathcal{CR}(\mathcal{P}) = \{[x, y, z, t] \in \mathcal{Q}(\mathcal{P}) \mid \forall x, y, z, t \in \mathcal{P},\ x \neq t, y \neq z\}$ be the set of all the cross-fractions in $\mathcal{Q}(\mathcal{P})$. Let $\mathcal{B}(\mathcal{P})$ be the subring of $\mathcal{Q}(\mathcal{P})$ generated by $\mathcal{CR}(\mathcal{P})$.

Proposition 2.9 *The ring* $\mathcal{B}(\mathcal{P})$ *is closed under swapping bracket.*

Proof. By Leibniz's rule, $\forall c_1, ..., c_n, d_1, ..., d_m \in \mathcal{Z}(\mathcal{P})$

$$\frac{\{c_1 \cdots c_n, d_1 \cdots d_m\}}{c_1 \cdots c_n \cdot d_1 \cdots d_m} = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\{c_i, d_j\}}{c_i \cdot d_j}, \tag{7}$$

we only need to show that for any two elements $[x, y, z, t]$ and $[u, v, w, s]$ in $\mathcal{CR}(\mathcal{P})$, where $x \neq t$, $y \neq z$, $u \neq s$, $v \neq w$ in $\mathcal{P}$, then $\{\frac{xz}{xt} \cdot \frac{yt}{yz}, \frac{uw}{us} \cdot \frac{vs}{vw}\} \in \mathcal{B}(\mathcal{P})$. Let $e_1 = xz$, $e_2 = \frac{1}{xt}$, $e_3 = yt$, $e_4 = \frac{1}{yz}$, $h_1 = uw$, $h_2 = \frac{1}{us}$, $h_3 = vs$, $h_4 = \frac{1}{vw}$. By the definition of the swapping bracket, we have

$$\frac{\{e_1, h_1\}}{e_1 \cdot h_1} = \mathcal{J}(xz, uw) \cdot \frac{xw}{xz} \cdot \frac{uz}{uw} \in \mathcal{B}(\mathcal{P}).$$

Then by the Leibniz's rule, we deduce that for any $e, h \in \mathcal{Z}(\mathcal{P})$, we have

$$\frac{\{e, \frac{1}{h}\}}{e/h} = -\frac{\{e, h\}}{e \cdot h},\ \frac{\{\frac{1}{e}, h\}}{h/e} = -\frac{\{e, h\}}{e \cdot h},\ \frac{\{\frac{1}{e}, \frac{1}{h}\}}{1/(eh)} = \frac{\{e, h\}}{e \cdot h}.$$

So for any $i,j=1,2,3,4$, we have $\frac{\{e_i,h_j\}}{e_i\cdot h_j}\in\mathcal{B}(\mathcal{P})$. Since $e_1e_2e_3e_4$ and $h_1h_2h_3h_4$ are also in $\mathcal{B}(\mathcal{P})$, so

$$\{e_1e_2e_3e_4,h_1h_2h_3h_4\}=\sum_{i=1}^{4}\sum_{j=1}^{4}\frac{\{e_i,h_j\}}{e_i\cdot h_j}\cdot(e_1e_2e_3e_4h_1h_2h_3h_4)\in\mathcal{B}(\mathcal{P}).$$

Finally, we conclude that $\mathcal{B}(\mathcal{P})$ is closed under swapping bracket. □

Definition 2.10 [SWAPPING MULTIFRACTION ALGEBRA OF $\mathcal{P}$] The swapping multifraction algebra of $\mathcal{P}$ *is the ring $\mathcal{B}(\mathcal{P})$ equipped with the swapping bracket, denoted by $(\mathcal{B}(\mathcal{P}),\{\cdot,\cdot\})$.*

3. Rank n swapping algebra

Swapping algebra $(\mathcal{Z}(\mathcal{P}),\{\cdot,\cdot\})$ corresponds to $\bigcup_{n=2}^{\infty}H_n(S)$. In this section, we define the rank n swapping algebra $\mathcal{Z}_n(\mathcal{P})$, in order to restrict the correspondence for a fixed n. In theorem 3.4, we will prove that the ring $\mathcal{Z}_n(\mathcal{P})$ is compatible with the swapping bracket.

3.1 The rank n swapping ring $\mathcal{Z}_n(\mathcal{P})$

Notation 3.1 *Let*

$$\Delta((x_1,...,x_{n+1}),(y_1,...,y_{n+1}))=\det\begin{pmatrix} x_1y_1 & ... & x_1y_{n+1}\\ ... & ... & ...\\ x_{n+1}y_1 & ... & x_{n+1}y_{n+1}\end{pmatrix}.$$

Inspired by linear algebra for $\mathbb{R}^n$, and the space of the rank n cross-ratios identified with the Hitchin component $H_n(S)$ [L07], we define the rank n swapping ring as follows.

Definition 3.2 [THE RANK n SWAPPING RING $\mathcal{Z}_n(\mathcal{P})$] *For $n\geq 2$, let $R_n(\mathcal{P})$ be the ideal of $\mathcal{Z}(\mathcal{P})$ generated by $\{D\in\mathcal{Z}(\mathcal{P})\mid D=\Delta((x_1,...,x_{n+1}),(y_1,...,y_{n+1})),\forall x_1,...,x_{n+1},y_1,...,y_{n+1}\in\mathcal{P}\}$.*

The rank n swapping ring $\mathcal{Z}_n(\mathcal{P})$ *is the quotient ring $\mathcal{Z}(\mathcal{P})/R_n(\mathcal{P})$.*

Remark 3.3 *Decomposing the determinant D in the first row, we have by induction that*

$$R_2(\mathcal{P})\supseteq R_3(\mathcal{P})\supseteq ...\supseteq R_n(\mathcal{P}). \tag{8}$$

3.2 Swapping bracket over $\mathcal{Z}_n(\mathcal{P})$

We will prove by induction the first fundamental theorem of the rank n swapping algebra.

Theorem 3.4 [**First main result**] *For $n\geq 2$, the ideal $R_n(\mathcal{P})$ is a Poisson ideal with respect to the swapping bracket. Thus the ring $\mathcal{Z}_n(\mathcal{P})$ inherits a Poisson bracket from the swapping bracket.*

Proof. The above theorem is equivalent to say that for any $h\in R_n(\mathcal{P})$ and any $f\in\mathcal{Z}(\mathcal{P})$, we have $\{f,h\}\in R_n(\mathcal{P})$ where $n\geq 2$. By the Leibniz's rule of the swapping bracket, it suffices to prove the case where $f=ab\in\mathcal{Z}(\mathcal{P})$, $h=\Delta((x_1,...,x_{n+1}),(y_1,...,y_{n+1}))$. The points $x_1,...,x_{n+1}$ ($y_1,...,y_{n+1}$ *resp.*) should be different from each other in $\mathcal{P}$, otherwise $h=0$. Therefore the theorem follows from Lemma 3.5. □

Lemma 3.5 *Let $n\geq 2$. Let $x_1,...,x_{n+1}$ ($y_1,...,y_{n+1}$ resp.) in $\mathcal{P}$ be different from each other and ordered anticlockwise, a,b belong to $\mathcal{P}$ and $x_1,...,x_l,y_1,...,y_k$ are on the right side of $\overrightarrow{ab}$ (include*

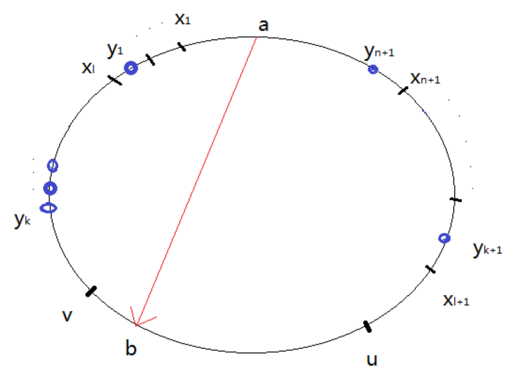

FIGURE 2. $\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}$.

coinciding with a or b) as illustrated in Figure 2. Let u (v resp.) be strictly on the left (right resp.) side of $\overrightarrow{ab}$. *Let*

$$\Delta^R(a,b) = \sum_{d=1}^{l} \mathcal{J}(ab, x_d u) \cdot x_d b \cdot \Delta((x_1, ..., x_{d-1}, a, x_{d+1}, ..., x_{n+1}), (y_1, ..., y_{n+1}))$$
$$+ \sum_{d=1}^{k} \mathcal{J}(ab, u y_d) \cdot a y_d \cdot \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{d-1}, b, y_{d+1}, ..., y_{n+1})), \tag{9}$$

We obtain that

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = \Delta^R(a,b). \tag{10}$$

Proof. The main idea of the proof is to consider the change of $\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}$ when ab moves topologically in the circle with special points $a, b, x_1, ..., x_{n+1}, y_1, ..., y_{n+1}$.

We will prove that

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = \Delta^R(a,b)$$

by induction on the number of elements of $\{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}\}$ on the right side of $\overrightarrow{ab}$ (includes coinciding with a or b), which is $m = l + k$. Let S_{n+1} be the permutation group of $\{1, ..., n+1\}$, the signature of $\sigma \in S_{n+1}$ denoted by $sgn(\sigma)$, is defined as 1 if σ is even and -1 if σ is odd. Then we have

$$\Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1})) = \sum_{\sigma \in S_{n+1}} sgn(\sigma) \prod_{i=1}^{n+1} x_i y_{\sigma(i)}. \tag{11}$$

By the Leibniz's rule, we obtain that

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = \sum_{\sigma \in S_{n+1}} sgn(\sigma) \prod_{i=1}^{n+1} x_i y_{\sigma(i)} \sum_{j=1}^{n+1} \frac{\{ab, x_j y_{\sigma(j)}\}}{x_j y_{\sigma(j)}}$$
$$= \sum_{\sigma \in S_{n+1}} sgn(\sigma) \prod_{i=1}^{n+1} x_i y_{\sigma(i)} \left(\sum_{j=1}^{n+1} \frac{\mathcal{J}(ab, x_j y_{\sigma(j)}) \cdot a y_{\sigma(j)} \cdot x_j b}{x_j y_{\sigma(j)}} \right). \tag{12}$$

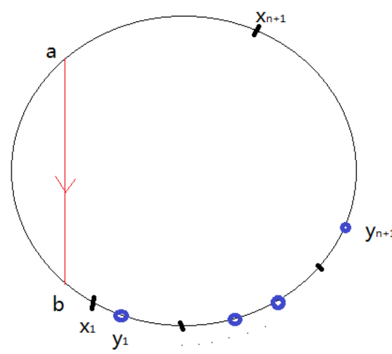

FIGURE 3. $m = 0$.

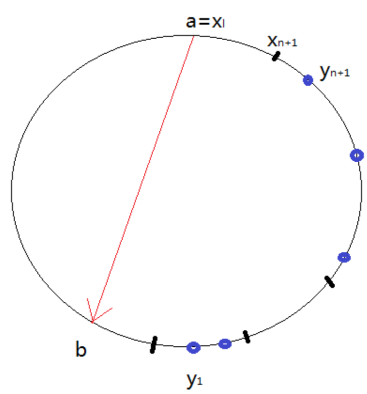

FIGURE 4. $m = q + 1$ case $a = x_l$.

When $m = 0$ as illustrated in Figure 3, since $\mathcal{J}(ab, x_j y_{\sigma(j)}) = 0$, we have $\{ab, x_j y_{\sigma(j)}\} = 0$ for any $j = 1, ..., n + 1$ and any $\sigma \in S_{n+1}$. By Equation 12, we have

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = 0 = \Delta^R(a, b)$$

in this case.

Suppose

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = \Delta^R(a, b)$$

for $m = q \geq 0$.

When $m = q + 1$, suppose that x_l is the first point of $\{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}\}$ on the right side of $\overrightarrow{ab}$ (include coinciding with a or b) with respect to the clockwise orientation.

(i) If x_l coincides with a as illustrated in Figure 4, then $m = 1$. So we have $\mathcal{J}(ab, x_l y_{\sigma(l)}) = \frac{1}{2}$ and $\mathcal{J}(ab, x_j y_{\sigma(j)}) = 0$ for $j \neq l$. By Equation 12, we have

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = \frac{1}{2} \cdot ab \cdot \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1})) = \Delta^R(a, b).$$

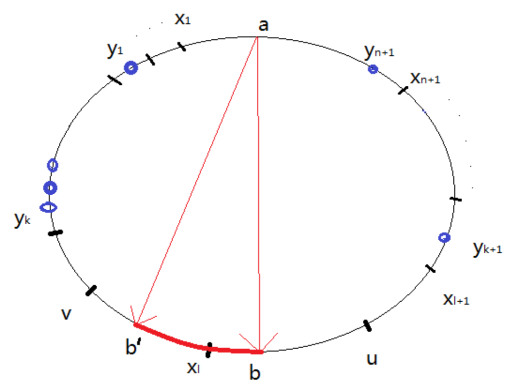

FIGURE 5. $m = q + 1$ case $a \neq x_l$.

(ii) If x_l does not coincide with a, we move b clockwise to the point b', such that $b' \neq x_l$ and the intersection between $\{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}\}$ and the arc bb' is x_l as illustrated in Figure 5.

Then $\{ab', \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}$ corresponds to the case $m = q$. Thus we have

$$\begin{aligned} &\{ab', \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} \\ &= \sum_{d=1}^{l-1} \mathcal{J}(ab', x_d u) \cdot x_d b \cdot \Delta((x_1, ..., x_{d-1}, a, x_{d+1}, ..., x_{n+1}), (y_1, ..., y_{n+1})) \\ &+ \sum_{d=1}^{k} \mathcal{J}(ab', u y_d) \cdot a y_d \cdot \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{d-1}, b, y_{d+1}, ..., y_{n+1})). \end{aligned} \tag{13}$$

On the other hand, by Equation 12,

$$\begin{aligned} &\{ab', \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} \\ &= \sum_{\sigma \in S_{n+1}} sgn(\sigma) \prod_{i=1}^{n+1} x_i y_{\sigma(i)} \left(\sum_{j=1}^{n+1} \frac{\mathcal{J}(ab', x_j y_{\sigma(j)}) \cdot a y_{\sigma(j)} \cdot x_j b'}{x_j y_{\sigma(j)}} \right) \end{aligned} \tag{14}$$

is a polynomial of $ab', x_1 b', ..., x_{n+1} b'$, denoted by $P(ab', x_1 b', ..., x_{n+1} b')$. Then

$$P(ab, x_1 b, ..., x_{n+1} b) = \sum_{\sigma \in S_{n+1}} sgn(\sigma) \prod_{i=1}^{n+1} x_i y_{\sigma(i)} \left(\sum_{j=1}^{n+1} \frac{\mathcal{J}(ab', x_j y_{\sigma(j)}) \cdot a y_{\sigma(j)} \cdot x_j b}{x_j y_{\sigma(j)}} \right).$$

By the cocycle identity [L12]: $\mathcal{J}(ab, xy) - \mathcal{J}(ab', xy) = \mathcal{J}(b'b, xy)$, we have

$$\begin{aligned} &\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} - P(ab, x_1 b, ..., x_{n+1} b) \\ &= \sum_{\sigma \in S_{n+1}} sgn(\sigma) \prod_{i=1}^{n+1} x_i y_{\sigma(i)} \left(\sum_{j=1}^{n+1} \frac{(\mathcal{J}(ab, x_j y_{\sigma(j)}) - \mathcal{J}(ab', x_j y_{\sigma(j)})) \cdot a y_{\sigma(j)} \cdot x_j b}{x_j y_{\sigma(j)}} \right) \\ &= \sum_{\sigma \in S_{n+1}} sgn(\sigma) \prod_{i=1}^{n+1} x_i y_{\sigma(i)} \left(\sum_{j=1}^{n+1} \frac{\mathcal{J}(b'b, x_j y_{\sigma(j)}) \cdot a y_{\sigma(j)} \cdot x_j b}{x_j y_{\sigma(j)}} \right) \end{aligned} \tag{15}$$

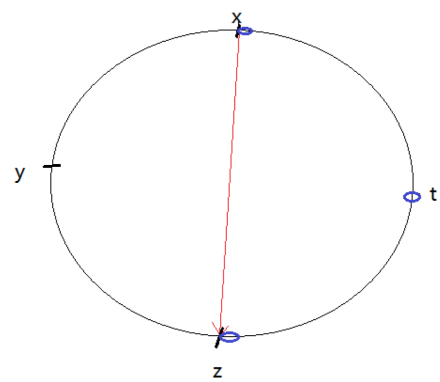

FIGURE 6. Example.

Since $\mathcal{J}(b'b, x_j y_{\sigma(j)}) = 0$ when $j \neq l$, $\mathcal{J}(b'b, x_l y_{\sigma(l)}) = \mathcal{J}(ab, x_l u)$. So the above sum equals to

$$\sum_{\sigma \in S_{n+1}} sgn(\sigma) \prod_{i=1}^{n+1} x_i y_{\sigma(i)} \cdot \frac{\mathcal{J}(b'b, x_l y_{\sigma(l)}) \cdot a y_{\sigma(l)} \cdot x_l b}{x_l y_{\sigma(l)}} \\ = \mathcal{J}(ab, x_l u) \cdot x_l b \cdot \Delta((x_1, ..., x_{l-1}, a, x_{l+1}, ..., x_{n+1}), (y_1, ..., y_{n+1})). \tag{16}$$

Since $\mathcal{J}(ab', x_d u) = \mathcal{J}(ab, x_d u)$ for $d = 1, ..., l-1$, $\mathcal{J}(ab', u y_d) = \mathcal{J}(ab, u y_d)$ for $d = 1, ..., k$, by Equations 13 15 16, we have

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = \Delta^R(a, b)$$

in this case.

When y_k is the first point of $\{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}\}$ on the right side of $\overrightarrow{ab}$ (include coinciding with a or b) with respect to clockwise orientation, the result follows the similar argument. By induction, we have

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = \Delta^R(a, b)$$

in general.

Finally, we conclude that

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = \Delta^R(a, b).$$

□

Remark 3.6 *We can also consider $\Delta^L(a,b)$ similar to $\Delta^R(a,b)$ with respect to the left side of $\overrightarrow{ab}$, The equation $\Delta^R(a,b) = \Delta^L(a,b)$ follows*

$$\sum_{i=1}^{n+1} a y_i \cdot \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{i-1}, b, y_{i+1}, ..., y_{n+1})) \\ = \sum_{i=1}^{n+1} x_i b \cdot \Delta((x_1, ..., x_{i-1}, a, x_{i+1}, ..., x_{n+1}), (y_1, ..., y_{n+1})). \tag{17}$$

Example 3.7 *As shown in Figure 6, we have*

$$\{xz, \Delta((x,z,y),(z,x,t))\} = -xt \cdot \Delta((x,z,y),(z,x,z)) = 0. \tag{18}$$

Therefore, the swapping bracket over $\mathcal{Z}_n(\mathcal{P})$ is well defined for $n \geq 2$.

Definition 3.8 [RANK n SWAPPING ALGEBRA OF $\mathcal{P}$] *For $n \geq 2$, the* rank n swapping algebra of $\mathcal{P}$ *is the rank n swapping ring $\mathcal{Z}_n(\mathcal{P})$ equipped with the swapping bracket, denoted by $(\mathcal{Z}_n(\mathcal{P}), \{\cdot,\cdot\})$.*

4. The ring $\mathcal{Z}_n(\mathcal{P})$ is an integral domain

In this section, we will show that the cross fractions are well-defined in the fraction ring of $\mathcal{Z}_n(\mathcal{P})$, by proving that the ring $\mathcal{Z}_n(\mathcal{P})$ is an integral domain. The strategy of the proof is the following. First, we introduce a geometric model studied by H. Weyl [W39] and C. D. Concini and C. Procesi [CP76] to characterize the ring $\mathcal{Z}_n(\mathcal{P})$ as a $\mathrm{GL}(n,\mathbb{K})$-module. Then, we transfer the integrality of the ring $\mathcal{Z}_n(\mathcal{P})$ to another ring $K_{n,p}^{\mathrm{GL}(n,\mathbb{K})}$ by an injective ring homomorphism. The homomorphism is recovered by a long exact sequence of Lie group cohomology with values in $\mathrm{GL}(n,\mathbb{K})$-modules by Proposition 4.9. In the end, we prove that the ring $K_{n,p}$ is integral, which will complete the proof of the theorem.

4.1 A geometric model for $\mathcal{Z}_n(\mathcal{P})$

Let us introduce a geometric model to characterize $\mathcal{Z}_n(\mathcal{P})$. Let $M_{n,p} = (\mathbb{K}^n \times \mathbb{K}^{n*})^p$ be the space of p vectors in $\mathbb{K}^n$ and p co-vectors in $\mathbb{K}^{n*}$.

Notation 4.1 *Let $a_i = (a_{i,1}, ..., a_{i,n})^T$, $b_i = \sum_{l=1}^n b_{i,l}\sigma_l$ where $a_{i,l}, b_{i,l} \in \mathbb{K}$, $\sigma_l \in \mathbb{K}^{n*}$ and $\sigma_l(a_i) = a_{i,l}$. We define the product between a vector a_i in $\mathbb{K}^n$ and a co-vector b_j in $\mathbb{K}^{n*}$ by*

$$\langle a_i | b_j \rangle := b_j(a_i) = \sum_{k=1}^n a_{i,l} \cdot b_{j,l}. \tag{19}$$

The group $\mathrm{GL}(n,\mathbb{K})$ acts naturally on the vectors and the covectors by

$$g \circ a_i := g \cdot (a_{i,1}, ..., a_{i,n})^T,$$

$$g \circ b_j := (b_{i,1}, ..., b_{i,n}) \cdot (g^{-1}) \cdot (\sigma_1, ..., \sigma_n)^T$$

where T is the transpose of the matrix. When we consider the action on their products, we write $b_j = (b_{i,1}, ..., b_{i,n})^T$ in column as a_i, then

$$g \circ b_j := (g^{-1})^T \cdot (b_{i,1}, ..., b_{i,n})^T.$$

For any $g \in \mathrm{GL}(n,\mathbb{K})$, $a, b \in \mathbb{K}[M_{n,p}]$,

$$g \circ (a \cdot b) := (g \circ a) \cdot (g \circ b).$$

It induces a $\mathrm{GL}(n,\mathbb{K})$ action on $\mathbb{K}[M_{n,p}]$ satisfying:

- *For any $g \in \mathrm{GL}(n,\mathbb{K})$, $a, b \in \mathbb{K}[M_{n,p}]$, we have*

$$g \circ (a+b) = g \circ a + g \circ b,$$

- *For any $g_1, g_2 \in \mathrm{GL}(n,\mathbb{K})$, $a \in \mathbb{K}[M_{n,p}]$, we have*

$$g_1 \circ (g_2 \circ a) = (g_1 \cdot g_2) \circ a.$$

Then the polynomial ring $\mathbb{K}[M_{n,p}]$ is a $\mathrm{GL}(n,\mathbb{K})$-module.

Let $B_{n\mathbb{K}}$ be the subring of $\mathbb{K}[M_{n,p}]$ generated by $\{\langle a_i|b_j\rangle\}_{i=1,j=1}^{p}$. We denote the $\mathrm{GL}(n,\mathbb{K})$ invariant ring of $\mathbb{K}[M_{n,p}]$ by $\mathbb{K}[M_{n,p}]^{\mathrm{GL}(n,\mathbb{K})}$. Since $\langle a_i|b_j\rangle \in \mathbb{K}[M_{n,p}]$ is invariant under $\mathrm{GL}(n,\mathbb{K})$ action, we have $B_{n\mathbb{K}} \subseteq \mathbb{K}[M_{n,p}]^{\mathrm{GL}(n,\mathbb{K})}$. Moreover, C. D. Concini and C. Procesi proved that

Theorem 4.2 [C. D. CONCINI AND C. PROCESI [CP76] [1]] $B_{n\mathbb{K}} = \mathbb{K}[M_{n,p}]^{\mathrm{GL}(n,\mathbb{K})}$.

Since $\mathbb{K}[M_{n,p}]$ is an integral domain, they obtained the following corollary.

Corollary 4.3 [C. D. CONCINI AND C. PROCESI [CP76]] *The subring $B_{n\mathbb{K}}$ is an integral domain.*

H. Weyl describe $B_{n\mathbb{K}}$ as a quotient ring.

Theorem 4.4 [H. WEYL [W39]] *All the relations in $B_{n\mathbb{K}}$ are generated by* $R = \{f \in B_{n\mathbb{K}} \mid f = \det\begin{pmatrix} \langle a_{i_1}|b_{j_1}\rangle & \dots & \langle a_{i_1}|b_{j_{n+1}}\rangle \\ \dots & \dots & \dots \\ \langle a_{i_{n+1}}|b_{j_1}\rangle & \dots & \langle a_{i_{n+1}}|a_{j_{n+1}}\rangle \end{pmatrix}, \forall i_k, j_l = 1,\dots,p\}$.

Remark 4.5 *In other words, let W be the polynomial ring $\mathbb{K}[\{x_{i,j}\}_{i,j=1}^{p}]$, $r = \{f \in W \mid f = \det\begin{pmatrix} x_{i_1,j_1} & \dots & x_{i_1,j_{n+1}} \\ \dots & \dots & \dots \\ x_{i_{n+1},j_1} & \dots & x_{i_{n+1},j_{n+1}} \end{pmatrix}, \forall i_k, j_l = 1,\dots,p\}$, let T be the ideal of W generated by r, then we have $B_{n\mathbb{K}} \cong W/T$.*

Let us recall that $\mathcal{Z}_n(\mathcal{P}) = \mathcal{Z}(\mathcal{P})/R_n(\mathcal{P})$ is the rank n swapping ring where $\mathcal{P} = \{x_1,\dots,x_p\}$. When we identify a_i with x_i on the left and b_i with x_i on the right of the pairs of points in $\mathcal{Z}_n(\mathcal{P})$, we obtain the main result of this subsection below.

Theorem 4.6 *Let $\mathcal{Z}_n(\mathcal{P})$ be the rank n swapping ring. Let $S_{n\mathbb{K}}$ be the ideal of $B_{n\mathbb{K}}$ generated by $\{\langle a_i|b_i\rangle\}_{i=1}^{p}$, then $B_{n\mathbb{K}}/S_{n\mathbb{K}} \cong \mathcal{Z}_n(\mathcal{P})$.*

4.2 Proof of the secound main result

Theorem 4.7 [**Second main result**] *For $n > 1$, $\mathcal{Z}_n(\mathcal{P})$ is an integral domain.*

Firstly, let us first consider the following $\mathrm{GL}(n,\mathbb{K})$-modules:

(i) Let L be the ideal of $\mathbb{K}[M_{n,p}]$ generated by $(\{\langle a_i|b_i\rangle\}_{i=1}^{p})$,

(ii) let $K_{n,p}$ be the quotient ring $\mathbb{K}[M_{n,p}]/L$,

(iii) let $S_{n\mathbb{K}}$ be the ideal of $B_{n\mathbb{K}}$ generated by $\{\langle a_i|b_i\rangle\}_{i=1}^{p}$.

Thus there is an exact sequence of $\mathrm{GL}(n,\mathbb{K})$-modules (the right arrows are not only module homomorphisms, but also ring homomorphisms):

$$0 \to L \to \mathbb{K}[M_{n,p}] \to K_{n,p} \to 0. \tag{20}$$

By Lie group cohomology [CE48], the exact sequence above induces the long exact sequence :

$$0 \to L^{\mathrm{GL}(n,\mathbb{K})} \to \mathbb{K}[M_{n,p}]^{\mathrm{GL}(n,\mathbb{K})} \to K_{n,p}^{\mathrm{GL}(n,\mathbb{K})} \to H^1(\mathrm{GL}(n,\mathbb{K}), L) \to \dots. \tag{21}$$

[1]Thanks for the reference provided by J. B. Bost.

Lemma 4.8 *Let S be the finite subset $\{\langle a_i|b_i\rangle\}_{i=1}^p$. Let $\mathbb{K}$ be a field of characteristic 0. Then*

$$(\mathbb{K}[M_{n,p}]\cdot S)^{\mathrm{GL}(n,\mathbb{K})} = \mathbb{K}[M_{n,p}]^{\mathrm{GL}(n,\mathbb{K})}\cdot S.$$

Proof. The proof follows from Weyl's unitary trick. Let

$$\mathrm{U}(n) = \{g\in \mathrm{GL}(n,\mathbb{K}) \mid g\cdot \bar{g}^T = I\}.$$

We want to prove that

$$(\mathbb{K}[M_p]\cdot S)^{\mathrm{U}(n)} = \mathbb{K}[M_p]^{\mathrm{U}(n)}\cdot S.$$

Notice first that one inclusion is obvious

$$(\mathbb{K}[M_p]\cdot S)^{\mathrm{U}(n)} \supseteq \mathbb{K}[M_p]^{\mathrm{U}(n)}\cdot S.$$

We next prove the other inclusion:

$$(\mathbb{K}[M_p]\cdot S)^{\mathrm{U}(n)} \subseteq \mathbb{K}[M_p]^{\mathrm{U}(n)}\cdot S.$$

For this, let $\mathrm{d}g$ be a Haar measure on $\mathrm{U}(n)$. Let x belongs to $(\mathbb{K}[M_p]\cdot S)^{\mathrm{U}(n)}$. We represent x by $\sum_{l=1}^k t_l\cdot s_l$, where $t_l\in\mathbb{K}[M_p]$ and $s_l\in S$. Since $S\subseteq \mathbb{K}[M_p]^{\mathrm{GL}(n,\mathbb{K})}\subseteq \mathbb{K}[M_p]^{\mathrm{U}(n)}$, for any $g\in\mathrm{U}(n)$, $g\circ s_l = s_l$. Thus we have

$$x = g\circ x = \sum_{l=1}^k (g\circ t_l)\cdot(g\circ s_l) = \sum_{l=1}^k (g\circ t_l)\cdot s_l.$$

By integrating over $\mathrm{U}(n)$:

$$g\circ x = \int_{\mathrm{U}(n)}\sum_{l=1}^k (g\circ t_l)\cdot s_l\ \mathrm{d}g = \sum_{l=1}^k\left(\int_{\mathrm{U}(n)} g\circ t_l\ \mathrm{d}g\right)\cdot s_l, \tag{22}$$

where

$$b_l = \int_{\mathrm{U}(n)} g\circ t_l\ \mathrm{d}g \in \mathbb{K}[M_p].$$

For any g_1 in $\mathrm{U}(n)$, we have

$$\begin{aligned} g_1\circ b_l &= \int_{\mathrm{U}(n)} g_1\circ(g\circ t_l)\ \mathrm{d}g = \int_{\mathrm{U}(n)} ((g_1\cdot g)\circ t_l)\ \mathrm{d}g \\ &= \int_{\mathrm{U}(n)} ((g_1\cdot g)\circ t_l)\ \mathrm{d}\,(g_1\cdot g) = b_l. \end{aligned} \tag{23}$$

Thus b_l belongs to $\mathbb{K}[M_p]^{\mathrm{U}(n)}$, x belongs to $\mathbb{K}[M_p]^{\mathrm{U}(n)}\cdot S$. Hence

$$(\mathbb{K}[M_p]\cdot S)^{\mathrm{U}(n)} \subseteq \mathbb{K}[M_p]^{\mathrm{U}(n)}\cdot S.$$

Therefore, we obtain

$$(\mathbb{K}[M_p]\cdot S)^{\mathrm{U}(n)} = \mathbb{K}[M_p]^{\mathrm{U}(n)}\cdot S.$$

By extending the ground field $\mathbb{K}$ of $\mathrm{U}(n)$, the property is extended to $\mathrm{GL}(n,\mathbb{K})$. Therefore, we conclude that

$$(\mathbb{K}[M_{n,p}]\cdot S)^{\mathrm{GL}(n,\mathbb{K})} = \mathbb{K}[M_{n,p}]^{\mathrm{GL}(n,\mathbb{K})}\cdot S.$$

This complete the proof of the lemma. □

Then, the integrality of $\mathcal{Z}_n(\mathcal{P})$ is transferred to another ring $K_{n,p}^{\mathrm{GL}(n,\mathbb{K})}$ by the following proposition.

Proposition 4.9 *There is a ring homomorphism $\theta : B_{n\mathbb{K}}/S_{n\mathbb{K}} \to K_{n,p}^{\mathrm{GL}(n,\mathbb{K})}$ induced from the long exact sequence:*

$$0 \to L^{\mathrm{GL}(n,\mathbb{K})} \to \mathbb{K}[M_{n,p}]^{\mathrm{GL}(n,\mathbb{K})} \to K_{n,p}^{\mathrm{GL}(n,\mathbb{K})} \to H^1(\mathrm{GL}(n,\mathbb{K}),L) \to ...., \tag{24}$$

which is injective.

Proof. By Theorem 4.2, we have $\mathbb{K}[M_{n,p}]^{\mathrm{GL}(n,\mathbb{K})} = B_{n\mathbb{K}}$. By Lemma 4.8, we have

$L^{\mathrm{GL}(n,\mathbb{K})} = (\mathbb{K}[M_{n,p}] \cdot (\{\langle a_i|b_i\rangle\}_{i=1}^p))^{\mathrm{GL}(n,\mathbb{K})} = \mathbb{K}[M_{n,p}]^{\mathrm{GL}(n,\mathbb{K})} \cdot (\{\langle a_i|b_i\rangle\}_{i=1}^p) = B_{n\mathbb{K}} \cdot (\{\langle a_i|b_i\rangle\}_{i=1}^p) = S_{n\mathbb{K}}$.

Hence Long exact sequence 21 becomes into:

$$0 \to S_{n\mathbb{K}} \to B_{n\mathbb{K}} \to K_{n,p}^{\mathrm{GL}(n,\mathbb{K})} \to H^1(\mathrm{GL}(n,\mathbb{K}),L) \to ... \tag{25}$$

Therefore, there is an injective module homomorphism θ from $B_{n\mathbb{K}}/S_{n\mathbb{K}}$ to $K_{n,p}^{\mathrm{GL}(n,\mathbb{K})}$. By definition of θ with respect to Exact sequence 20, for any $a, b \in B_{n\mathbb{K}}/S_{n\mathbb{K}}$, we have

$$\theta(a \cdot b) = \theta(a) \cdot \theta(b).$$

Thus the module homomorphism θ is also a ring homomorphism. As such, we conclude that the ring homomorphism θ is injective. □

Therefore, we only need to prove that $K_{n,p}$ is integral. This is the content of the following proposition.

Proposition 4.10 *For $n > 1$, $K_{n,p}$ is an integral domain.*

Proof. We prove the proposition by induction on the number of the vectors or covectors p. Let us start with $p = 1$. When $p = 1$, $K_{n,1} = \mathbb{K}[M_{n,1}]/(\{\sum_{k=1}^n a_{1,k} \cdot b_{1,k}\})$. Let us define the degree of a monomial in $\mathbb{K}[M_{n,1}]$ to be the sum of the degrees in all the variables. Let the degree of a polynomial f in $\mathbb{K}[M_{n,1}]$ be the maximal degree of the monomials in f, denoted by $\deg(f)$. Suppose that $\sum_{k=1}^n a_{1,k}.b_{1,k}$ is a reducible polynomial in $\mathbb{K}[M_{n,1}]$, we have

$$\sum_{k=1}^n a_{1,k}.b_{1,k} = g \cdot h,$$

where $g, h \in \mathbb{K}[M_{n,1}]$, $\deg(g) > 0$ and $\deg(h) > 0$. Since $\mathbb{K}[M_{n,1}]$ is an integral domain, $2 = \deg(gh) = \deg(g) + \deg(h)$, so we have $\deg(g) = \deg(h) = 1$. Suppose that

$$g = \lambda_0 + \lambda_1 \cdot c_1 + ... + \lambda_r \cdot c_r,$$

$$h = \mu_0 + \mu_1 \cdot d_1 + ... + \mu_s \cdot d_s,$$

where $\lambda_1, ..., \lambda_r, \mu_1, ..., \mu_s$ are non zero elements in $\mathbb{K}$, $c_1, ..., c_r$ ($d_1, ..., d_s$ resp.) are different elements in $\{a_{1,k}, b_{1,k}\}_{k=1}^n$. Since there is no square in $g \cdot h$, we have

$$\{c_1, ..., c_r\} \cap \{d_1, ..., d_s\} = \emptyset.$$

Because there are n monomials in gh, we obtain

$$r \cdot s = n.$$

Moreover, there are $2n$ variables in $g \cdot h$, we have

$$r + s = 2n,$$

thus

$$r \cdot s \geq 2n - 1.$$

Since $n>1$, we obtain that

$$r\cdot s\geq 2n-1>n=r\cdot s,$$

which is a contradiction. We therefore conclude that $\sum_{k=1}^n a_{1,k}.b_{1,k}$ is an irreducible polynomial in $\mathbb{K}[M_{n,p}]$. Since $\mathbb{K}[M_{n,1}]$ is an integral domain, we obtain that $K_{n,1}$ is an integral domain. Suppose that the proposition is true for $p=m\geq 1$. When $p=m+1$,

$$\begin{aligned} K_{n,m+1} &= \mathbb{K}\left[\{a_{i,k},b_{i,k}\}_{i,k=1}^{m+1,n}\right] / \left(\{\sum_{k=1}^{n} a_{i,k}.b_{i,k}\}_{i=1}^{m+1}\right) \\ &= K_{n,m}\left[\{a_{m+1,k},b_{m+1,k}\}_{k=1}^{n} / \left(\sum_{k=1}^{n} a_{m+1,k}\cdot b_{m+1,k}\right)\right., \end{aligned} \tag{26}$$

we have $K_{n,m}$ is an integral domain by induction, thus $K_{n,m}[\{a_{m+1,k},b_{m+1,k}\}_{k=1}^n]$ is an integral domain. By the above argument, the polynomial $\sum_{k=1}^n a_{m+1,k}\cdot b_{m+1,k}$ is an irreducible polynomial over $\mathbb{K}[\{a_{m+1,k},b_{m+1,k}\}_{k=1}^n]$. Moreover, $a_{m+1,k},b_{m+1,k}(k=1,...,n)$ are not variables that appear in $K_{n,m}$, so $\sum_{k=1}^n a_{m+1,k}\cdot b_{m+1,k}$ is an irreducible polynomial over $K_{n,m}[\{a_{m+1,k},b_{m+1,k}\}_{k=1}^n]$. Hence $K_{n,m+1}$ is an integral domain.

We therefore conclude that $K_{n,p}$ is an integral domain for any $p\geq 1$ and $n>1$. □

Proof of Theorem 4.7. By Proposition 4.10, the ring $K_{n,p}$ is an integral domain, we deduce that the invariant ring $K_{n,p}^{\mathrm{GL}(n,\mathbb{K})}$ is an integral domain. By Proposition 4.9, there is an injective ring homomorphism θ from $B_{n\mathbb{K}}/S_{n\mathbb{K}}$ to $K_{n,p}^{\mathrm{GL}(n,\mathbb{K})}$, so $B_{n\mathbb{K}}/S_{n\mathbb{K}}$ is an integral domain. Moreover, by Theorem 4.6 $\mathcal{Z}_n(\mathcal{P})\cong B_{n\mathbb{K}}/S_{n\mathbb{K}}$, we conclude that for $n>1$, the rank n swapping ring $\mathcal{Z}_n(\mathcal{P})$ is an integral domain. □

Remark 4.11 *The ring $\mathcal{Z}_1(\mathcal{P})$ is not an integral domain, since*

$$D=xy\cdot yz=\det\begin{pmatrix} xy & xz\\ yy & yz\end{pmatrix}$$

is zero in $\mathcal{Z}_1(\mathcal{P})$, but we have xy and yz are not zero in $\mathcal{Z}_1(\mathcal{P})$ whenever $x\neq y$, $y\neq z$.

4.3 Rank n swapping multifraction algebra of $\mathcal{P}$

After Theorem 4.7, we define rank n swapping multifraction algebra of $\mathcal{P}$ without any obstruction.

Definition 4.12 [RANK n SWAPPING FRACTION ALGEBRA OF $\mathcal{P}$] *Let $\mathcal{Q}_n(\mathcal{P})$ be the total fraction ring of $\mathcal{Z}_n(\mathcal{P})$. The* rank n swapping fraction algebra *of $\mathcal{P}$ is the total fraction ring $\mathcal{Q}_n(\mathcal{P})$ equipped with the swapping bracket, denoted by $(\mathcal{Q}_n(\mathcal{P}),\{\cdot,\cdot\})$.*

Let $\mathcal{CR}_n(\mathcal{P})=\left\{[x,y,z,t]=\frac{xz}{xt}\cdot\frac{yt}{yz}\in\mathcal{Q}_n(\mathcal{P})\;\middle|\;\forall x,y,z,t\in\mathcal{P},\ x\neq t,\ y\neq z\right\}$ be the set of all the cross fractions in $\mathcal{Q}_n(\mathcal{P})$. Let $\mathcal{B}_n(\mathcal{P})$ be the sub fraction ring of $\mathcal{Q}_n(\mathcal{P})$ generated by $\mathcal{CR}_n(\mathcal{P})$.

Similar to Proposition 2.9, we have the following.

Proposition 4.13 *The sub fraction ring $\mathcal{B}_n(\mathcal{P})$ is closed under swapping bracket.*

Definition 4.14 [RANK n SWAPPING MULTIFRACTION ALGEBRA OF $\mathcal{P}$] *Let $n\geq 2$, the* rank n swapping multifraction algebra of $\mathcal{P}$ *is the sub fraction ring $\mathcal{B}_n(\mathcal{P})$ equipped with the closed swapping bracket, denoted by $(\mathcal{B}_n(\mathcal{P}),\{\cdot,\cdot\})$.*

Then the ring homomorphism I from $\mathcal{B}(\mathcal{R})$ to $C^\infty(H_n(S))$ for any $n>1$, induces the homomorphism I_n from $\mathcal{B}_n(\mathcal{R})$ to $C^\infty(H_n(S))$ by extending the following formula on generators to $\mathcal{B}_n(\mathcal{R})$:

$$I_n([x,y,z,t]) = \mathbb{B}_\rho(x,y,z,t), \tag{27}$$

for any $[x,y,z,t] \in \mathcal{CR}_n(\mathcal{R})$. By the rank n cross ratio condition, the homomorphism I_n is well-defined. Then we rephrase Theorem 1.1 as follows.

Theorem 4.15 [F. LABOURIE [L12]] *With the same conditions as in Theorem 1.1, for any* $b_0, b_1 \in \mathcal{B}_n(\mathcal{R})$, *we have*

$$\lim_{n\to\infty} \{I_n(b_0), I_n(b_1)\}_{S_n} = I_n \circ \{b_0, b_1\}.$$

Hence the rank n swapping multifraction algebra $(\mathcal{B}_n(\mathcal{R}), \{\cdot,\cdot\})$ characterizes the Hitchin component $H_n(S)$ for a fixed $n>1$.

5. Cluster $\mathcal{X}_{\mathrm{PGL}(2,\mathbb{R}),D_k}$-space

Even though the rank n swapping multifraction algebra $(\mathcal{B}_n(\mathcal{P}), \{\cdot,\cdot\})$ characterizes the $\mathrm{PSL}(n,\mathbb{R})$ Hitchin component asymptotically, we still have the rank n swapping multifraction algebra $(\mathcal{B}_n(\mathcal{P}), \{\cdot,\cdot\})$ characterizes the related object—cluster $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),D_k}$-space without this asymptotic behavior where D_k is a disc with k special points on the boundary. We will show a simple case when $n=2$. We show that the cluster dynamic of $\mathcal{X}_{\mathrm{PGL}(2,\mathbb{R}),D_k}$ can be demonstrated in the rank 2 swapping algebra. As a byproduct, we reprove that the Fock-Goncharov Poisson bracket for $\mathcal{X}_{\mathrm{PGL}(2,\mathbb{R}),D_k}$ is independent of the ideal triangulation.

5.1 Cluster $\mathcal{X}_{\mathrm{PGL}(2,\mathbb{R}),D_k}$-space and rank 2 swapping algebra

Let S be an oriented surface with non-empty boundary and a finite set P of special points on boundary, considered modulo isotopy. In [FG06], Fock and Goncharov introduced the moduli space $\mathcal{X}_{G,S}$($\mathcal{A}_{G,S}$ resp.) which is a pair (∇, f), where ∇ is a flat connection on the principal G bundle on the surface S, f is a flat section of $\partial S\backslash P$ with values in the flag variety G/B (decorated flag variety G/U resp.). They found that the pair of two moduli spaces $(\mathcal{X}_{G,S}, \mathcal{A}_{G^L,S})$ is equipped with a cluster ensemble structure. Particularly, the moduli space $\mathcal{X}_{G,S}$ is called the *cluster* $\mathcal{X}_{G,S}$*-space*. Moreover, each one of the moduli spaces $\mathcal{X}_{G,S}, \mathcal{A}_{G,S}$ is equipped with a positive structure. When the set P is empty, the hole on the surface S should be regarded as the puncture, the positive part of $\mathcal{X}_{\mathrm{PGL}(2,\mathbb{R}),S}$ is related to the Teichmüller space of S, and the positive part of $\mathcal{A}_{\mathrm{SL}(2,\mathbb{R}),S}$ is related to Penner's decorated Teichmüller space [P87]. The fact that Penner's decorated Teichmüller space is related to a cluster algebra was independently observed by M. Gekhtman, M. Shapiro, and A. Vainshtein [GSV05].

When D_k is a disc with k special points on the boundary, the generic cluster $\mathcal{X}_{\mathrm{PGL}(2,\mathbb{R}),D_k}$-space corresponds to the generic configuration space $\mathrm{Conf}_{2,k}$ of k flags in $\mathbb{RP}^1$ up to projective transformations. Given a generic configuration of k flags $m^1, y^1, z^1, t^1, n^1, x^1, \ldots$ in $\mathbb{RP}^1$, let P_k be the associated convex k-gon with k vertices $m, y, z, t, n, x, \ldots$ as illustrated in Figure 7. The ideal triangulation of D_k corresponds to the triangulation of P_k. Given a triangulation $\mathcal{T}$ of the k-gon P_k, for any pair of triangles $(\Delta_{xyz}, \Delta_{xzt})$ of $\mathcal{T}$ where x,y,z,t are anticlockwise ordered, the Fock-Goncharov coordinate [FG06] corresponding to the inner edge xz is

$$X_{xz} = -\frac{\Omega\left(\hat{y}^1 \wedge \hat{z}^1\right)}{\Omega\left(\hat{t}^1 \wedge \hat{z}^1\right)} \cdot \frac{\Omega\left(\hat{t}^1 \wedge \hat{x}^1\right)}{\Omega\left(\hat{y}^1 \wedge \hat{x}^1\right)}.$$

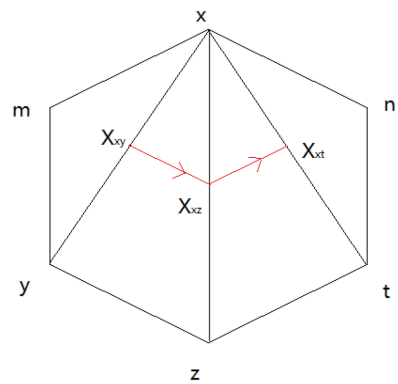

Figure 7. Fock-Goncharov coordinates for the triangulation $\mathcal{T}$.

By definition $X_{xz} = X_{zx}$, so there are $k-3$ different coordinates.

Definition 5.1 [Fock-Goncharov algebra] *Let $\mathcal{A}(\mathcal{T})$ be the fraction ring generated by $k-3$ Fock-Goncharov coordinates for the triangulation $\mathcal{T}$, the natural Fock-Goncharov Poisson bracket $\{\cdot,\cdot\}_2$ is defined on the fraction ring $\mathcal{A}(\mathcal{T})$ by extending the following map on the generators:*

$$\{X_{ab}, X_{cd}\}_2 = \varepsilon_{ab,cd} \cdot X_{ab} \cdot X_{cd}$$

for any inner edges ab, cd, where the value of $\varepsilon_{ab,cd}$ only depend on the anticlockwise orientation of P_k as illustrated in Figure 7. More precisely, $\varepsilon_{ab,cd} = 1$($\varepsilon_{ab,cd} = -1$ resp.) when $a = c$ and Δ_{abd} is a triangle of $\mathcal{T}$ such that a, b, d are ordered anticlockwise(clockwise resp.) in P_k; otherwise $\varepsilon_{ab,cd} = 0$.

The Fock-Goncharov algebra *of $\mathcal{T}$ is a pair $(\mathcal{A}(\mathcal{T}), \{\cdot,\cdot\}_2)$.*

Definition 5.2 *Let $\mathcal{P}$ be the vertices of the convex k-gon P_k. We define an injective ring homomorphism $\theta_{\mathcal{T}}$ from $\mathcal{A}(\mathcal{T})$ to $\mathcal{B}_2(\mathcal{P})$, by extending the following map on the generators:*

$$\theta_{\mathcal{T}}(X_{xz}) := -\frac{yz}{tz} \cdot \frac{tx}{yx} \tag{28}$$

for any inner edge of $\mathcal{T}$.

Theorem 5.3 *The injective ring homomorphism $\theta_{\mathcal{T}}$ is Poisson with respect to the Poisson bracket $\{\cdot,\cdot\}_2$ and the swapping bracket $\{\cdot,\cdot\}$.*

Proof. By direct calculations, for any inner edge ab of the triangulation $\mathcal{T}$, we have

$$\frac{\left\{ab, \frac{yz}{tz} \cdot \frac{tx}{yx}\right\}}{ab \cdot \frac{yz}{tz} \cdot \frac{tx}{yx}} = \begin{cases} 1 & \text{if } ab = zx; \\ -1 & \text{if } ab = xz; \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

The theorem follows from the above equation and the Leibniz's rule. □

5.2 Cluster dynamic in rank 2 swapping algebra

The cluster $\mathcal{X}$-space is introduced by Fock and Goncharov [FG06] by using the same set-up as the cluster algebra [FZ02]. We consider the case for the cluster $\mathcal{X}_{\mathrm{PGL}(2,\mathbb{R}),D_k}$-space.

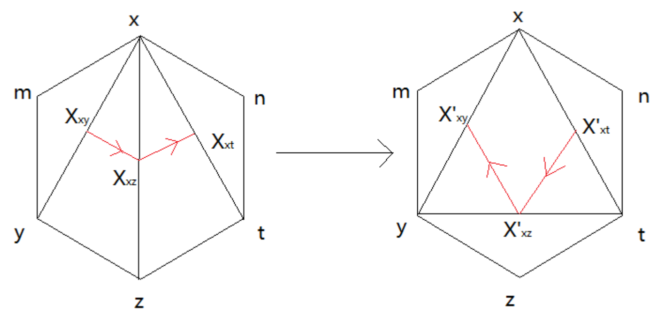

FIGURE 8. Cluster transformation at xz.

Definition 5.4 [CLUSTER $\mathcal{X}_{\mathrm{PGL}(2,\mathbb{R}),D_k}$-SPACE [FG04] [FG06]] *Let $I_{\mathcal{T}}$ be the set of $k-3$ inner edges of the triangulation $\mathcal{T}$ of D_k. The function ε from $I_{\mathcal{T}} \times I_{\mathcal{T}}$ to $\mathbb{Z}$ is defined as in Definition 5.1, a* seed *is $\mathbf{I}_{\mathcal{T}} = (I_{\mathcal{T}}, \varepsilon)$.*

A mutation *at the edge $e \in I_{\mathcal{T}}$ changes the seed $\mathbf{I}_{\mathcal{T}}$ to a new one $\mathbf{I}_{\mathcal{T}'} = (I_{\mathcal{T}'}, \varepsilon')$, where the edge e of the triangulation $\mathcal{T}$ is changed into the edge e' of $\mathcal{T}$ by a flip illustrated in Figure 8. We identify $I_{\mathcal{T}}$ with $I_{\mathcal{T}'}$ by identifying e with e', where*

$$\varepsilon'_{i,j} = \begin{cases} -\varepsilon_{i,j} \text{ if } e \in \{i,j\}; \\ \varepsilon_{i,j} + \varepsilon_{i,e} \max\{0, \varepsilon_{i,e}\varepsilon_{e,j}\} \text{ if } e \notin \{i,j\}. \end{cases}$$

A cluster transformation *is a composition of mutations and automorphisms of seeds.*

We assign to the seed $\mathbf{I}_{\mathcal{T}}$ ($\mathbf{I}_{\mathcal{T}'}$ resp.) the split tori $\mathbb{T}_{\mathcal{T}}$($\mathbb{T}_{\mathcal{T}'}$ resp.) associated to the the Fock-Goncharov coordinates $\{X_i\}_{i \in I_{\mathcal{T}}}$ ($\{X'_i\}_{i \in I_{\mathcal{T}'}}$ resp.). Then the transition function from $\mathbb{T}_{\mathcal{T}}$ to $\mathbb{T}_{\mathcal{T}'}$ is

$$\mu_e(X'_i) = g_e(X_i) = \begin{cases} X_i(1+X_e)^{-\varepsilon_{i,e}} & \text{if } e \neq i, \varepsilon_{i,e} \leq 0; \\ X_i(1+X_e^{-1})^{-\varepsilon_{i,e}} & \text{if } e \neq i, \varepsilon_{i,e} > 0; \\ X_e^{-1} & \text{if } i = e. \end{cases}$$

Any two triangulations are related by a composition of flips, therefore any two split tori are also related by a composition of the rational functions as above.

The cluster $\mathcal{X}_{\mathrm{PGL}(2,\mathbb{R}),D_k}$-space *is obtained by gluing all the possible algebraic tori $\mathbb{T}_{\mathcal{T}}$ according to the transition functions described as above.*

We show the cluster dynamic of $\mathcal{X}_{\mathrm{PGL}(2,\mathbb{R}),D_k}$ in the rank 2 swapping algebra as follows.

Lemma 5.5 *The triangulation $\mathcal{T}'$ is the flip of $\mathcal{T}$ at the edge e. Then*

$$\theta_{\mathcal{T}} \circ g_e(X_i) = \theta_{\mathcal{T}'}(X'_i).$$

Proof. Let us consider the case where $e = xz$ and the triangulations $\mathcal{T}, \mathcal{T}'$ is illustrated in Figure 8. For $i = xz$, we have

$$\theta_{\mathcal{T}'}(X'_{xz}) = -\frac{zt}{xt} \cdot \frac{xy}{zy},$$

$$\theta_{\mathcal{T}} \circ g_e(X_{xz}) = \theta_{\mathcal{T}}\left(\frac{1}{X_{xz}}\right) = -\frac{tz}{yz} \cdot \frac{yx}{tx}.$$

By the rank 2 swapping algebra relations:

$$yz \cdot zt \cdot ty + tz \cdot zy \cdot yt = 0,$$

$$tx \cdot xy \cdot yt + yx \cdot xt \cdot ty = 0,$$

we obtain

$$\theta_{\mathcal{T}} \circ g_e(X_{xz}) = \theta_{\mathcal{T}'}(X'_{xz}).$$

For $i = xy$, we have

$$\theta_{\mathcal{T}'}(X'_{xy}) = -\frac{my}{ty} \cdot \frac{tx}{mx},$$

$$\theta_{\mathcal{T}} \circ g_e(X_{xy}) = \theta_{\mathcal{T}}\left(X_{xy} \cdot \left(1 + X_{xz}^{-1}\right)^{-1}\right) = -\frac{my}{zy} \cdot \frac{zx}{mx}\left(1 - \frac{tz}{yz} \cdot \frac{yx}{tx}\right)^{-1}.$$

By the rank 2 swapping algebra relation:

$$yz \cdot ty \cdot zx + yx \cdot tz \cdot zy - yz \cdot zy \cdot tx = 0,$$

we obtain

$$\theta_{\mathcal{T}} \circ g_e(X_{xy}) = \theta_{\mathcal{T}'}(X'_{xy}).$$

We have same results for the other inner edges and the other cases different from the one illustrated in Figure 8, by the similar arguments. We therefore conclude that

$$\theta_{\mathcal{T}} \circ g_e(X_i) = \theta_{\mathcal{T}'}(X'_i).$$

□

Proposition 5.6 *The homomorphism μ_e preserves the Poisson bracket, so the Poisson bracket $\{\cdot,\cdot\}_2$ does not depend on the triangulation $\mathcal{T}$.*

Proof. We need to prove that

$$\{\mu_e(X'_i), \mu_e(X'_j)\}_2 = \mu_e\left(\{X'_i, X'_j\}_2\right),$$

which is equivalent to

$$\{g_e(X_i), g_e(X_j)\}_2 = \varepsilon'_{i,j} \cdot g_e(X_i) \cdot g_e(X_j).$$

Since $\theta_{\mathcal{T}'}$ is injective, we only need to prove that

$$\theta_{\mathcal{T}} \circ \{g_e(X_i), g_e(X_j)\}_2 = \theta_{\mathcal{T}}\left(\varepsilon'_{i,j} \cdot g_e(X_i) \cdot g_e(X_j)\right).$$

By Theorem 5.3 and Lemma 5.5, we have

$$\begin{aligned}
&\theta_{\mathcal{T}} \circ \{g_e(X_i), g_e(X_j)\}_2 = \{\theta_{\mathcal{T}} \circ g_e(X_i), \theta_{\mathcal{T}} \circ g_e(X_j)\} \\
&= \{\theta_{\mathcal{T}'}(X'_i), \theta_{\mathcal{T}'}(X'_j)\} = \theta_{\mathcal{T}'} \circ \{X'_i, X'_j\}_2 \\
&= \varepsilon'_{i,j} \cdot \theta_{\mathcal{T}'}(X_i) \cdot \theta_{\mathcal{T}'}(X_j) = \theta_{\mathcal{T}}\left(\varepsilon'_{i,j} \cdot g_e(X_i) \cdot g_e(X_j)\right).
\end{aligned} \tag{30}$$

We therefore conclude that the homomorphism μ_e preserves the Poisson bracket. □

Remark 5.7 *For n in general, the generalized injective ring homomorphism θ_{T_n} is shown in [Su15], where the set $\mathcal{P}$ has $(n-1)\cdot k$ elements, each flag of $\mathbb{RP}^{n-1}$ corresponds to $n-1$ points near each other on the boundary S^1. We expect to glue the rank n swapping algebras for the purpose of characterizing $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),S}$ for the surface case.*

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