# ATIYAH CLASSES OF STRONGLY HOMOTOPY LIE PAIRS

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ABSTRACT. The subject of this paper is strongly homotopy (SH) Lie algebras, also known as $L_\infty$-algebras. We extract an intrinsic character, the Atiyah class, which measures the nontriviality of an SH Lie algebra $A$ when it is extended to $L$. In fact, given such an SH Lie pair $(L, A)$, and any $A$-module $E$, there associates a canonical cohomology class, the Atiyah class $[\alpha^E]$, which generalizes earlier known Atiyah classes out of Lie algebra pairs. We show that the Atiyah class $[\alpha^{L/A}]$ induces a graded Lie algebra structure on $\mathrm{H}^\bullet_{\mathrm{CE}}(A, L/A[-2])$, and the Atiyah class $[\alpha^E]$ of any $A$-module $E$ induces a Lie algebra module structure on $\mathrm{H}^\bullet_{\mathrm{CE}}(A, E[-2])$. Moreover, Atiyah classes are invariant under gauge equivalent $A$-compatible infinitesimal deformations of $L$.

*Keywords*: Homotopical algebra, $L_\infty$-algebra, Atiyah class.
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## CONTENTS



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## INTRODUCTION

This work is motivated by two sources: Atiyah classes and strongly homotopy Lie algebras. Originally, the Atiyah class [3] of a holomorphic vector bundle $U$ over a complex manifold constitutes the obstruction to the existence of a holomorphic connection on $U$. Molino [26, 27] defined the Atiyah-Molino class of a foliation of a manifold to capture the existence of a locally projectable connection. The Atiyah class of a Lie algebra pair was studied by Wang [36], Nguyen-van [28] and Bordemann [6], to characterize the existence of invariant connections on a homogeneous space. Atiyah classes have enjoyed renewed vigor due to Kontsevich's seminal work on deformation quantization [16, 17]. They are also related to the Rozansky-Witten theory [15, 30]. Calaque and Van den Bergh [10] considered the Atiyah class of a DG module over a DG-algebra. They also inferred that, given a Lie algebra pair $(\mathfrak{d}, \mathfrak{g})$, the Atiyah class of the quotient $\mathfrak{d}/\mathfrak{g}$ coincides with the class capturing the obstruction to the "PBW problem" studied earlier by Calaque–Căldăraru–Tu [9] (see also [8, 14]).

The notion of strongly homotopy (SH) Lie algebras, also called $L_\infty[1]$-algebras (see Definition 1.5), was introduced by Lada and Stasheff [19, 20]. The investigation of SH Lie algebras from various perspectives started a while ago. Attention on this subject in the past ten years is largely due to its role in mathematical physics and supergeometry. For example, Kontsevich and Soibelman [18] approached this notion via the language of formal geometry. Meanwhile, Bashkirov and Voronov [5] used the Batalin-Vilkovisky formalism to treat an SH Lie algebra as a special pointed $\mathrm{BV}_\infty$-manifold. The interested reader is referred to a recent talk by Stasheff [31] in which many related topics are reviewed.

Motivated by the various constructions of Atiyah classes, we will study SH Lie algebra pairs, show that analogous Atiyah classes exist, and how they play the role of sending homotopical objects to Lie objects.

It is usually nontrivial to construct examples of $L_\infty[1]$-algebras. One "trivial" way is the semi-direct product of an $L_\infty[1]$-algebra $A$ with its module $B$ (see Proposition 1.28 (4)). We are particularly interested in grasping the information when a smaller $L_\infty[1]$-algebra, say $A$, "non-trivially" extends to a bigger one, say $L$. Hence we introduce the SH Lie pair $(L, A)$, where $L$ is an $L_\infty[1]$-algebra and $A \subset L$ is a sub-algebra. What we discovered, is the so-called Atiyah class $[\alpha^{L/A}]$ of $(L, A)$, which generalizes previous constructions of Lie algebra pairs. It measures how "nontrivial" it is when the sub-algebra $A$ is extended to $L$, while the $A$-module structure on $L/A$ is maintained. Moreover, it refines the homotopical data of $(L, A)$, to a canonical graded Lie algebra $\mathrm{H}^\bullet_{\mathrm{CE}}(A, L/A[-2])$ (see Theorem 3.5). One can even involve an external object — an $A$-module $E$, and use the Atiyah class $[\alpha^E]$ of $E$ to test the nontrivial information of $A$ being extended to $L$. Moreover, $[\alpha^E]$ gives rise to a Lie algebra module structure on $\mathrm{H}^\bullet_{\mathrm{CE}}(A, E[-2])$, over the aforesaid Lie algebra object (see Theorem 3.5).

The following is a summary of this paper:

After a review of $\mathbb{Z}$-graded linear algebra and SH Lie algebras in Section 1, we focus on the construction of Atiyah classes in Section 2. Given an SH Lie pair $(L, A)$ and an $A$-module $E$, one is able to extend the $A$-module structure on $E$ to an $L$-connection $\nabla$ on $E$. The curvature $R^\nabla$ measures the failure of $E$ being an $L$-module. From $R^\nabla$, we extract a particular element

$$\alpha^E_\nabla := (J \otimes 1)(R^\nabla) \in \mathcal{O}(A) \otimes A^\perp \otimes \mathrm{End}(E),$$

which is a degree 2 cocycle. Here $\mathcal{O}(A)$ is the graded algebra of formal power series on $A$. We call $\alpha^E_\nabla$ the Atiyah cocycle of the SH Lie pair $(L, A)$ with respect to the $A$-module $E$ and the $L$-connection $\nabla$ extending $(A, E)$.

**Theorem** (A). *The cohomology class, called Atiyah class,* $[\alpha^E_\nabla] \in \mathrm{H}^2_{\mathrm{CE}}(A, A^\perp \otimes \mathrm{End}(E))$ *is canonical, i.e., independent of the choice of* $\nabla$. *In particular, for the canonical* $A$*-module* $L/A$*, there associates a canonical Atiyah class* $[\alpha^{L/A}] \in \mathrm{H}^2_{\mathrm{CE}}(A, \mathrm{Hom}(L/A \otimes L/A, L/A))$.

In Section 3, we introduce the Atiyah operator and functor, which manifest the nature of Atiyah classes from different perspectives. The Atiyah operator is an $\mathcal{O}(A)$-linear map which arises from the construction of Atiyah cocycles:

$$\begin{aligned} \boldsymbol{\alpha}^E : &\qquad \mathcal{O}(A) \otimes E \longrightarrow \mathcal{O}(A) \otimes A^{\perp} \otimes E; \\ \text{Or,} \quad \boldsymbol{\alpha}^E : &\qquad (\mathcal{O}(A) \otimes L/A) \times (\mathcal{O}(A) \otimes E) \longrightarrow \mathcal{O}(A) \otimes E. \end{aligned}$$

We prove

**Theorem** (B). *The graded vector space $\mathrm{H}^{\bullet}_{\mathrm{CE}}(A, (L/A)[-2])$ with the binary operation induced by the Atiyah operator $\boldsymbol{\alpha}^{L/A}$ is a Lie algebra. Furthermore, if $E$ is an $A$-module, then $\mathrm{H}^{\bullet}_{\mathrm{CE}}(A, E[-2])$ is a Lie algebra module over $\mathrm{H}^{\bullet}_{\mathrm{CE}}(A, (L/A)[-2])$, with the action induced by the Atiyah operator $\boldsymbol{\alpha}^E$.*

This certainly generalizes previous results in [7, 12, 15]. An alternative point of view is that the process of taking Atiyah classes defines a functor, called Atiyah functor, from the category of $A$-modules to the category of $\mathrm{H}^{\bullet}_{\mathrm{CE}}(A, (L/A)[-2])$-modules.

In Section 4, we study a special kind of deformations of the given SH Lie algebra pair $(L, A)$, namely $A$-compatible infinitesimal deformations of $L$. Roughly speaking, they are $L_\infty[1]$-algebra structures on

$$L[\hbar] = L \oplus \hbar L,$$

where $\hbar$ is a formal parameter with $\hbar^2 = 0$, such that the subspace $A[\hbar]$ is trivially extended from $A$, and the $A[\hbar]$-module structure on $L[\hbar]/A[\hbar] = (L/A)[\hbar]$ is trivially extended from the $A$-module $L/A$. A small perturbation of $L[\hbar]$ is an isomorphism $\sigma : \mathcal{O}(L)[\hbar] \to \mathcal{O}(L)[\hbar]$ of graded algebras.

We prove the following invariance property of Atiyah classes under gauge equivalences (See Definition 4.3):

**Theorem** (C). *If two $A$-compatible infinitesimal deformations of $L$ are gauge equivalent, then the two associated Atiyah classes are the same.*

It is our hope that these results may lead to new insights in homotopical algebras and DG-manifolds. We would also like to point out other works that are related to the present paper: Chen, Stiénon and Xu [12] proposed a notion of the Atiyah class of a Lie algebroid pair $(L, A)$, which encompasses both the original Atiyah class of holomorphic vector bundles and the Atiyah-Molino class of a foliation as special cases. Shortly after that, an $L_\infty$-algebra structure on the space $\Gamma(\wedge^\bullet A^\vee \otimes L/A)$ was constructed in [21, 22], where the Atiyah class determines the 2-bracket $l_2$. A similar theory for Lie groupoid pairs is available in [23]. We also mention that Shoikhet [32] studied the Atiyah class of a DG-manifold; Costello [13] defined the Atiyah class of a DG-vector bundle in his geometric approach to Witten genus; Mehta, Stiénon and Xu [25] studied the Atiyah class of a DG-Lie algebroid with respect to a DG-vector bundle.

## 1. PRELIMINARIES

1.1. **Graded linear algebra.** Throughout this paper, we fix a base field $\mathbb{K}$ of characteristic zero. A $\mathbb{Z}$-graded vector space is a $\mathbb{K}$-vector space $V = \oplus_{n \in \mathbb{Z}} V^n$, where each $V^n = \{v \in V \mid |v| = n\}$ is an ordinary $\mathbb{K}$-vector space consisting of elements of homogeneous degree $n$. Henceforth, we will simply call $V$ a graded vector space. And $\mathbb{K}$ is considered as concentrated in degree 0.

A degree $r$ morphism from a graded vector space $V$ to a graded vector space $W$ is a linear map from $V$ to $W$ that sends $V^n$ to $W^{n+r}$, where $r$ could be any integer. The set $\mathrm{Hom}(V, W)$ consisting of such homogeneous morphisms is also a graded vector space. Thus the category of graded vector spaces over $\mathbb{K}$, denoted by $\mathrm{GVS}_{\mathbb{K}}$, is a $\mathbb{K}$-linear category.

The dual of $V$, denoted by $V^\vee$, is the graded vector space whose degree $n$ part is the ordinary dual $(V^{-n})^*$ of $V^{-n}$. If $V$ is of finite dimension, then the dual of $V^\vee$ is isomorphic to $V$. In this paper, we will always assume that $V$ is finite dimensional if $V^\vee$ is involved.

For $k \in \mathbb{Z}$, we denote by $V[k]$ the graded vector space with $k$-shifted gradings $(V[k])^n = V^{n+k}$. Hence $(V[k])^\vee = V^\vee[-k]$.

The category of graded vector spaces is monoidal. The tensor product of two objects $V$ and $W$ is the graded vector space whose degree $n$ part is

$$(V \otimes W)^n = \oplus_{i+j=n} V^i \otimes W^j.$$

We have isomorphisms of graded vector spaces:

$$V^\vee \otimes W \cong W \otimes V^\vee \cong \mathrm{Hom}(V, W),$$
$$\xi \otimes w \mapsto (-1)^{|\xi||w|} w \otimes \xi \mapsto \phi(-),$$

where $\xi \in V^\vee, w \in W$ and $\phi$ is the map $v \mapsto (\xi \otimes w)(v) = (-1)^{|v||w|}\xi(v)w$.

For any homogeneous element $\phi \in \mathrm{Hom}(V, W)$, its dual $\phi^\vee \in \mathrm{Hom}(W^\vee, V^\vee)$, which is also homogeneous of degree $|\phi|$, is defined in the standard manner:

$$\langle \phi^\vee(\alpha), v\rangle = (-1)^{|\phi||\alpha|}\langle \alpha, \phi(v)\rangle, \qquad \alpha \in W^\vee, v \in V.$$

The symmetric algebra of $V$ and its formal completion are, respectively,

$$S^\bullet(V) = \oplus_{n\geq 0} S^n(V), \qquad\qquad \widehat{S}^\bullet(V) = \prod_{n\geq 0} S^n(V).$$

Note that they might be infinite dimensional. The product in $S^\bullet(V)$, as well as that in $\widehat{S}^\bullet(V)$, is denoted by $\odot$. The Koszul sign $\epsilon(\sigma)$ of a permutation $\sigma$ of homogeneous vectors $v_1, \cdots, v_n$ in $V$ is determined by the equality

$$v_1 \odot \cdots \odot v_n = \epsilon(\sigma) v_{\sigma(1)} \odot \cdots \odot v_{\sigma(n)}.$$

Given $v \in V$, there induces two natural contractions, one from left and one from right, denoted respectively $\iota_v$ and $\llcorner v$, on $V^\vee$:

$$\iota_v \xi = (-1)^{|\xi||v|} \xi \llcorner v = (-1)^{|\xi||v|}\xi(v), \ \ \forall \xi \in V^\vee.$$

The left contraction $\iota_v$ is extended to $\iota_v : \ S^\bullet(V^\vee) \to S^{\bullet-1}(V^\vee)$ by the Leibniz rule

$$\iota_v(\xi \odot \eta) = \iota_v(\xi) \odot \eta + (-1)^{|v||\xi|}\xi \odot \iota_v(\eta), \ \ \forall \xi, \eta \in S^\bullet(V^\vee).$$

The extension of the right contraction is similar:

$$(\xi \odot \eta)\llcorner v = (-1)^{|v||\eta|}(\xi \llcorner v) \odot \eta + \xi \odot (\eta \llcorner v), \ \ \forall \xi, \eta \in S^\bullet(V^\vee).$$

We define a duality pairing

$$S^\bullet(V) \times S^\bullet(V^\vee) \to \mathbb{K}$$

by

$$\langle v_1 \odot \cdots \odot v_p, \xi^1 \odot \cdots \odot \xi^q\rangle = \begin{cases} \iota_{v_1} \cdots \iota_{v_p}(\xi^1 \odot \cdots \odot \xi^p), & p = q, \\ 0, & \text{otherwise}. \end{cases}$$

The pairing between $S^\bullet(V^\vee)$ and $S^\bullet(V)$ is similarly defined, and we have

$$\langle v_1 \odot \cdots \odot v_n, \xi^1 \odot \cdots \odot \xi^n\rangle = (-1)^{(\sum_{i=1}^n |v_i|)(\sum_{j=1}^n |\xi^j|)}\langle \xi^1 \odot \cdots \odot \xi^n, v_1 \odot \cdots \odot v_n\rangle.$$

Let

$$\mathcal{O}(V) = \widehat{S}^\bullet(V^\vee)$$

be the space of formal power series on the graded vector space $V$, which is a local algebra with the unique maximal ideal

$$\mathcal{O}^+(V) = \mathcal{O}(V) \odot V^\vee = \widehat{S}^{n\geq 1}(V^\vee) = \prod_{n\geq 1} S^n(V^\vee).$$

Denote by $r^+ : \mathcal{O}(V) \to \mathcal{O}^+(V)$ the obvious projection.

For all $k \geq 0$, consider the product

$$\mu_{k+1}^V : S^k(V) \otimes V \to S^{k+1}(V), \quad x \otimes v \mapsto x \odot v, \quad \forall x \in S^k(V), v \in V. \tag{1.1}$$

The dual map will be denoted by

$$I_{k+1}^V : S^{k+1}(V^\vee) \to S^k(V^\vee) \otimes V^\vee. \tag{1.2}$$

The summation of these $I_{k+1}^V$ defines an operator

$$I^V = \sum_{k \geq 0} I_{k+1}^V \circ r^+ : \mathcal{O}(V) \to \mathcal{O}(V) \otimes V^\vee, \tag{1.3}$$

which is in fact the algebraic de Rham operator of the $\mathbb{K}$-algebra $\mathcal{O}(V)$. It is clear that $I^V$ is an $\mathcal{O}(V)$-derivation valued in the $\mathcal{O}(V)$-bimodule $\mathcal{O}(V) \otimes V^\vee$, i.e., for all $\omega, \omega' \in \mathcal{O}(V)$,

$$I^V(\omega \odot \omega') = \omega \odot I^V(\omega') + I^V(\omega) \odot \omega' = \omega \odot I^V(\omega') + (-1)^{|\omega||\omega'|}\omega' \odot I^V(\omega). \tag{1.4}$$

A degree $n$ derivation $D$ of $\mathcal{O}(V)$ is a degree $n$ $\mathbb{K}$-linear map $D : \mathcal{O}(V) \to \mathcal{O}(V)$ such that the following Leibniz rule holds:

$$D(\xi \odot \eta) = D(\xi) \odot \eta + (-1)^{n|\xi|}\xi \odot D(\eta), \quad \forall \xi, \eta \in \mathcal{O}(V).$$

The space $\mathrm{Der}(\mathcal{O}(V))$ of derivations of $\mathcal{O}(V)$, together with the graded commutator

$$[D_1, D_2] := D_1 \circ D_2 - (-1)^{|D_1||D_2|} D_2 \circ D_1, \quad \forall D_1, D_2 \in \mathrm{Der}(\mathcal{O}(V)),$$

is a graded Lie algebra.

## 1.2. **Strongly homotopy Lie algebras.**

**Definition 1.5.** An SH Lie algebra (or $L_\infty[1]$-algebra) is a pair $(L, \{\lambda_k\}_{k=0}^\infty)$, simply denoted by $(L, \lambda_\bullet)$, where $L$ is a graded vector space and $\lambda_k : S^k(L) \to L, k \geq 0$, called the $k$th-bracket, are degree 1 linear maps satisfying the following generalized Jacobi identities:

$$\sum_{k+l=n} \sum_{\sigma \in \mathrm{sh}(l,k)} \epsilon(\sigma) \lambda_{k+1}(\lambda_l(u_{\sigma(1)}, \cdots, u_{\sigma(l)}), \cdots, u_{\sigma(n)}) = 0, \tag{1.6}$$

for all $k, l, n \geq 0$ and homogeneous elements $u_i \in L, 1 \leq i \leq n$. Here $\mathrm{sh}(s,k)$ is the set of $(s,k)$-unshuffles.

**Example 1.7.** Let $E$ be a graded vector space. Then $\mathrm{End}(E)$ together with the graded commutator $[-,-]$ is a Lie algebra. Thus $\mathrm{End}(E)[1]$ with the shifted commutator $\{-,-\}$:

$$\{\bar{\phi}, \bar{\psi}\} := (-1)^{|\phi|}[\phi, \psi] = (-1)^{|\phi|}(\phi \circ \psi - (-1)^{|\phi||\psi|}\psi \circ \phi), \quad \forall \bar{\phi}, \bar{\psi} \in \mathrm{End}(E)[1], \tag{1.8}$$

is an $L_\infty[1]$-algebra with only one nontrivial bracket $\lambda_2 = \{-,-\}$. Here $\phi$ and $\bar{\phi}$ are the same element with different degrees: $|\bar{\phi}| = |\phi| - 1$. The relation between $\bar{\psi}$ and $\psi$ is similar.

*Remark* 1.9. Our Definition 1.5 of $L_\infty[1]$-algebras is not the more commonly known notion of $L_\infty$-algebras (e.g., see standard texts [19, 20]). In particular, one should notice the different convention of degrees and signs of $L_\infty$ and $L_\infty[1]$-algebras. What we adopt is similar to that in [34], where $\mathbb{Z}_2$ grading is used. For a passage connecting our definition to that in [19, 20], we refer to [34, Remark 2.1].

SH Lie algebras could also be characterized as $Q$-manifolds [2]:

**Definition 1.10.** A homological vector field on $L$ is a degree 1 derivation $Q$ on $\mathcal{O}(L)$ such that $Q^2 = \frac{1}{2}[Q, Q] = 0$.

**Proposition 1.11.** *Let $L$ be a graded vector space. Then there is a one-to-one correspondence between $L_\infty[1]$-algebra structures on $L$ and homological vector fields on $L$.*

In fact, on the one hand, if $(L, \{\lambda_k\}_{k\geq 0})$ is an $L_\infty[1]$-algebra, then we can construct a homological vector field $Q_L$ as follows:

For each $k \geq 0$, the dual of $\lambda_k :\ S^k(L) \to L$ is a map $\lambda_k^\vee :\ L^\vee \to S^k(L^\vee)$, which can be uniquely extended to a degree 1 derivation $\mathcal{O}(L) \to \mathcal{O}(L)$. Then define $Q_L$ by

$$Q_L = \sum_{k\geq 0} (-1)^k \lambda_k^\vee :\ \mathcal{O}(L) \to \mathcal{O}(L),$$

i.e., for all $\xi \in L^\vee, u_i \in L, 1 \leq i \leq k$, we have

$$\langle \xi, \lambda_k(u_1, \cdots, u_k)\rangle = (-1)^{|\xi|+k} \langle Q_L(\xi), u_1 \odot \cdots \odot u_k\rangle. \tag{1.12}$$

On the other hand, given a homological vector field $Q_L$ on $L$, we can define a collection of degree 1 linear maps $\lambda_k :\ S^k(L) \to L, k \geq 0$ by

$$\lambda_k(u_1, \cdots, u_k) = \iota^{-1}\left([[\cdots[[Q, \iota_{u_1}], \iota_{u_2}]\cdots], \iota_{u_k}]\right) \in L, \quad \forall u_i \in L. \tag{1.13}$$

Here the map

$$\iota^{-1} :\ \mathrm{Der}(\mathcal{O}(L)) \to L$$

is defined by

$$\langle \iota^{-1}(D), \xi\rangle = pr_0 \circ D(\xi) \in \mathbb{K}, \quad \forall D \in \mathrm{Der}(\mathcal{O}(L)), \xi \in L^\vee,$$

where

$$pr_0 :\ \mathcal{O}(L) = \widehat{S}^\bullet(L^\vee) \to S^0(L^\vee) = \mathbb{K}$$

is the obvious projection. It is clear that the map $\iota^{-1}$ is the left inverse of the contraction operator $\iota_- : L \to \mathrm{Der}(\mathcal{O}(L))$ in the sense that $\iota^{-1}(\iota_u) = u$, for all $u \in L$.

The fact that $Q_L$ in Equation (1.12) is of square zero and the fact that $\{\lambda_k\}_{k\geq 0}$ in Equation (1.13) defines an $L_\infty[1]$-algebra structure on $L$ are equivalent. Details can be found in [34, 35].

For this reason, an $L_\infty[1]$-algebra can be denoted by any of the notations $(L, \lambda_\bullet)$, $(L, Q_L)$ or $(L, \lambda_\bullet \sim Q_L)$.

Now we recall morphisms of SH Lie algebras:

**Definition 1.14.** Let $(L, \lambda_\bullet \sim Q_L)$ and $(L', \lambda'_\bullet \sim Q_{L'})$ be two $L_\infty[1]$-algebras. An $L_\infty[1]$-morphism from $L$ to $L'$ is a morphism $\phi : \mathcal{O}(L') \to \mathcal{O}(L)$ of $\mathbb{K}$-algebras such that

$$\phi \circ Q_{L'} = Q_L \circ \phi : \mathcal{O}(L') \to \mathcal{O}(L). \tag{1.15}$$

Equivalently, an $L_\infty[1]$-morphism from $L$ to $L'$ is a family of degree zero linear maps

$$f_k :\ S^k(L) \to L', \quad k \geq 0$$

satisfying the following two conditions:

(1) The element $f_0 \in (L')^0$ satisfies

$$\sum_{k\geq 1} \frac{1}{k!} \lambda'_k(f_0, \cdots, f_0) + \lambda'_0 = f_1(\lambda_0). \tag{1.16}$$

(2) For each $n \geq 1$, the relation

$$\sum_{k+l=n} \sum_{\sigma\in \mathrm{sh}(l,k)} \epsilon(\sigma) f_{k+1}(\lambda_l(u_{\sigma(1)}, \cdots, u_{\sigma(l)}) \cdots, u_{\sigma(n)}) \tag{1.17}$$
$$= \sum_{\substack{i_1,\cdots,i_r\geq 1\\ i_1+\cdots+i_r=n}} \sum_{\tau\in \mathrm{sh}(i_1,\cdots,i_r)} \sum_{j\geq 0} \frac{1}{(r+j)!} \epsilon(\tau) \lambda'_{r+j}(f_0^{\odot j}, f_{i_1}(u_{\tau(1)}, \cdots, u_{\tau(i_1)}), \cdots f_{i_r}(\cdots, u_{\tau(n)}))$$

holds, where $k, l \geq 0$ and $u_i \in L$ are homogeneous.

For completeness, we will give a proof on the equivalence of the two definitions of morphisms in Appendix 5.

1.3. **Connections, curvatures and modules of SH Lie algebras.** This part can be thought of as formal differential geometry of $L_\infty[1]$-algebras. What we shall deal with, namely connections and curvatures, are defined in the same manner of super-connections in [29]. Some closely related contents can be found in [1, 19, 24, 33].

Let us fix an $L_\infty[1]$-algebra $(L, \lambda_\bullet \sim Q_L)$ and a graded vector space $E$. A degree $n$ operator $\partial : \mathcal{O}(L) \otimes E \to \mathcal{O}(L) \otimes E$ is called an $E$-derivation if it is $\mathbb{K}$-linear and there exists a degree $n$ derivation $\underline{\partial} \in \mathrm{Der}(\mathcal{O}(L))$, called the symbol of $\partial$, such that

$$\partial(\omega \otimes e) = \underline{\partial}(\omega) \otimes e + (-1)^{n|\omega|}\omega \odot \partial(e), \qquad \omega \in \mathcal{O}(L), e \in E.$$

Let us denote the space of all $E$-derivations by $\mathrm{Der}(\mathcal{O}(L) \otimes E)$. There is a Lie bracket on $\mathrm{Der}(\mathcal{O}(L) \otimes E)$ defined by its graded commutator

$$[\partial, \partial'] = \partial \circ \partial' - (-1)^{|\partial||\partial'|}\partial' \circ \partial, \ \forall \partial, \partial' \in \mathrm{Der}(\mathcal{O}(L) \otimes E).$$

**Lemma 1.18.** *The Lie algebra* $\mathrm{Der}(\mathcal{O}(L) \otimes E)$ *is isomorphic to the semidirect product of* $\mathrm{Der}(\mathcal{O}(L))$ *and* $\mathcal{O}(L) \otimes \mathrm{End}(E)$.

*Proof.* Note that $\mathcal{O}(L) \otimes \mathrm{End}(E)$ consists of $E$-derivations with zero symbol, thus a Lie subalgebra of $\mathrm{Der}(\mathcal{O}(L) \otimes E)$. And $\mathrm{Der}(\mathcal{O}(L))$ is also a Lie subalgebra by the inclusion

$$j : \mathrm{Der}(\mathcal{O}(L)) \to \mathrm{Der}(\mathcal{O}(L) \otimes E), \quad j(X)(\omega \otimes e) = X(\omega) \otimes e, \ \forall X \in \mathrm{Der}(\mathcal{O}(L)), \omega \in \mathcal{O}(L), e \in E,$$

which gives rise to a natural splitting of the short exact sequence of Lie algebras

$$0 \to \mathcal{O}(L) \otimes \mathrm{End}(E) \hookrightarrow \mathrm{Der}(\mathcal{O}(L) \otimes E) \to \mathrm{Der}(\mathcal{O}(L)) \to 0,$$

where the Lie algebra morphism $\mathrm{Der}(\mathcal{O}(L) \otimes E) \to \mathrm{Der}(\mathcal{O}(L))$ is taking symbols. It gives rise to a semidirect product $\mathrm{Der}(\mathcal{O}(L)) \ltimes (\mathcal{O}(L) \otimes \mathrm{End}(E))$ that is isomorphic to $\mathrm{Der}(\mathcal{O}(L) \otimes E)$. $\square$

1.3.1. *Connections and curvatures.*

**Definition 1.19.** An $L$-connection on $E$ is a degree 1 $E$-derivation $\nabla \in \mathrm{Der}(\mathcal{O}(L) \otimes E)$ whose symbol is $Q_L$, i.e., the following Leibniz rule holds:

$$\nabla(\omega \otimes e) = Q_L(\omega) \otimes e + (-1)^{|\omega|}\omega \odot \nabla(e), \ \ \forall \omega \in \mathcal{O}(L), e \in E.$$

The degree 2 $E$-derivation

$$R^\nabla := \nabla^2 = \frac{1}{2}[\nabla, \nabla] : \quad \mathcal{O}(L) \otimes E \to \mathcal{O}(L) \otimes E$$

is of zero symbol, i.e., $\mathcal{O}(L)$-linear, and will be called the curvature of $\nabla$. An $L$-connection $\nabla$ is said to be flat if its curvature $R^\nabla$ vanishes.

The difference of two connections is a degree 1 endomorphism of the $\mathcal{O}(L)$-module $\mathcal{O}(L) \otimes E$. Thus the set of all $L$-connections on $E$ is an affine space over $(\mathcal{O}(L) \otimes \mathrm{End}(E))^1$.

According to Lemma 1.18, any $L$-connection $\nabla$ is determined by an element $D^E \in (\mathcal{O}(L) \otimes \mathrm{End}(E))^1$ so that $\nabla = Q_L + D^E$. An easy computation shows that the curvature has the form

$$R^\nabla = \nabla^2 = (Q_L + D^E) \circ (Q_L + D^E) = Q_L(D^E) + (D^E)^2.$$

**Lemma 1.20.** *We have the Bianchi identity:*

$$Q_L(R^\nabla) + [D^E, R^\nabla] = 0. \tag{1.21}$$

*Proof.* This follows from straightforward computations:

$$[Q_L + D^E, R^\nabla] = [\nabla, \nabla^2] = \nabla \circ \nabla^2 - (-1)^{1\times 2}\nabla^2 \circ \nabla = 0.$$

□

Given an $L$-connection $\nabla$ on $E$, there corresponds a dual $L$-connection $\nabla^\vee : \mathcal{O}(L) \otimes E^\vee \to \mathcal{O}(L) \otimes E^\vee$ on $E^\vee$. Explicitly, we have

$$\langle \nabla^\vee(f), g\rangle = Q_L\langle f, g\rangle - (-1)^{|f|}\langle f, \nabla(g)\rangle \in \mathcal{O}(L), \qquad \forall f \in \mathcal{O}(L) \otimes E, g \in \mathcal{O}(L) \otimes E^\vee. \tag{1.22}$$

For two graded vector spaces with $L$-connections $(E, \nabla^E)$ and $(F, \nabla^F)$, the induced $L$-connection on $E \otimes F$ is given by

$$\nabla^{E\otimes F}(\omega \otimes e \otimes f) = Q_L(\omega) \otimes e \otimes f + (-1)^{|\omega|}\omega \odot (\nabla^E(e) \otimes f) + (-1)^{|\omega|+|e|}(\omega \otimes e) \otimes_{\mathcal{O}(L)} \nabla^F(f). \tag{1.23}$$

Here we used the canonical isomorphism

$$(\mathcal{O}(L) \otimes E) \otimes_{\mathcal{O}(L)} (\mathcal{O}(L) \otimes F) \xrightarrow{\cong} \mathcal{O}(L) \otimes (E \otimes F)$$

to view the last term $(\omega \otimes e) \otimes_{\mathcal{O}(L)} \nabla^F(f)$ as an element in $\mathcal{O}(L) \otimes (E \otimes F)$.

In particular, we have the induced $L$-connection $\nabla^{\mathrm{Hom}(E,F)}$ on $\mathrm{Hom}(E, F) \cong E^\vee \otimes F$.

**Lemma 1.24.** *For each $\Psi \in \mathcal{O}(L) \otimes \mathrm{Hom}(E, F) \cong \mathrm{Hom}_{\mathcal{O}(L)}(\mathcal{O}(L) \otimes E, \mathcal{O}(L) \otimes F)$,*

$$\nabla^{\mathrm{Hom}(E,F)}(\Psi) = \nabla^F \circ \Psi - (-1)^{|\Psi|}\Psi \circ \nabla^E. \tag{1.25}$$

1.3.2. *Modules over SH Lie algebras.*

**Definition 1.26.** An $L$-module $(E, \partial_L^E)$ is a graded vector space $E$ together with a flat $L$-connection $\partial_L^E : \mathcal{O}(L) \otimes E \to \mathcal{O}(L) \otimes E$, which will be called the Chevalley-Eilenberg differential of $E$. The associated cohomology $\mathrm{H}^\bullet(\mathcal{O}(L) \otimes E, \partial_L^E)$ will be denoted by $\mathrm{H}^\bullet_{\mathrm{CE}}(L, E)$.

Recall that as an $L$-connection, $\partial_L^E$ is determined by an element $D^E \in (\mathcal{O}(L) \otimes \mathrm{End}(E))^1$ such that $\partial_L^E = Q_L + D^E$. The flat condition of $\partial_L^E$ becomes a Maurer-Cartan equation:

$$Q_L(D^E) + (D^E)^2 = 0. \tag{1.27}$$

We have alternative descriptions of $L$-modules.

**Proposition 1.28.** *Let $L = (L, \lambda_\bullet \sim Q_L)$ be an $L_\infty[1]$-algebra, and $E$ be a graded vector space. The following data are mutually determined:*

(1) *An $L$-module structure on $E$;*
(2) *An $L$-module structure on $E^\vee$;*
(3) *A degree $1$ derivation $Q \in \mathrm{Der}(\mathcal{O}(L \oplus E))$ such that $Q|_{\mathcal{O}(L)} = Q_L$, $Q(E^\vee) \subset \mathcal{O}(L) \otimes E^\vee$ and $Q^2 = 0$;*
(4) *An abelian extension $L \oplus E$ of $L$ along $E$ (also called a* semi-direct product *of $L$ with $E$): i.e., $L \oplus E$ carries an $L_\infty[1]$-algebra structure $\{\tilde\lambda_k\}_{k\geq 0}$ such that (i) $L$ is an $L_\infty[1]$ subalgebra (in particular, $\tilde\lambda_0 = \lambda_0 \in L^1$); (ii) $E$ is an ideal: $\tilde\lambda_k(E, \cdots) \subset E, k \geq 1$; (iii) $E$ is abelian: $\tilde\lambda_k(E, E, \cdots) = 0, k \geq 2$;*
(5) *A family of degree $1$ linear maps*

$$m_k^E : S^{k-1}(L) \otimes E \to E, \tag{1.29}$$

*$k = 1, 2, \cdots$, such that*

$$\sum_{k+l=n} \sum_{\sigma \in \mathrm{sh}(l,k)} \epsilon(\sigma) m_{k+2}^E(\lambda_l(u_{\sigma(1)}, \cdots, u_{\sigma(l)}), \cdots, u_{\sigma(n)}, e)$$

$$= -\sum_{k+l=n}\sum_{\tau\in \mathrm{sh}(k,l)} \epsilon(\tau)(-1)^{\dagger_k^\tau} m^E_{k+1}(u_{\tau(1)},\cdots,u_{\tau(k)}, m^E_{l+1}(u_{\tau(k+1)},\cdots,u_{\tau(n)},e)) \tag{1.30}$$

*hold for all $k,l,n\geq 0$ and homogeneous elements $u_i\in L, e\in E$, where $\dagger_k^\tau=\sum_{i=1}^k |u_{\tau(i)}|$;*

(6) *An $L_\infty[1]$-morphism $f^E=\{f^E_k\}$ from $L$ to $\mathrm{End}(E)[1]$, where*

$$f^E_k : S^k(L)\to \mathrm{End}(E)[1], \qquad k\geq 0. \tag{1.31}$$

*Proof.* For the equivalence $(1)\Leftrightarrow(2)$, we need to show that an $L$-connection $\partial^E_L$ on $E$ is flat if and only if its dual $L$-connection $\partial^{E^\vee}_L$ on $E^\vee$ is flat. In fact, by Equation (1.22), we have

$$\begin{aligned}\langle(\partial^{E^\vee}_L)^2(e^\vee),e\rangle &= Q_L\langle \partial^{E^\vee}_L(e^\vee),e\rangle - (-1)^{|e^\vee|+1}\langle \partial^{E^\vee}_L(e^\vee),\partial^E_L(e)\rangle\\ &= Q_L\langle \partial^{E^\vee}_L(e^\vee),e\rangle - (-1)^{|e^\vee|+1}Q_L\langle e^\vee,\partial^E_L(e)\rangle - \langle e^\vee,(\partial^E_L)^2(e)\rangle\\ &= -\langle e^\vee,(\partial^E_L)^2(e)\rangle.\end{aligned}$$

Thus $(\partial^E_L)^2=0$ is equivalent to $(\partial^{E^\vee}_L)^2=0$.

The equivalence $(2)\Leftrightarrow(3)$ is obvious.

To see the equivalence $(3)\Leftrightarrow(4)$, we note that: $Q^2=0\Leftrightarrow L\oplus E$ is an $L_\infty[1]$-algebra (by Proposition 1.11); $Q|_{\mathcal{O}(L)}=Q_L\Leftrightarrow$ (i) in (4); $Q(E^\vee)\subset \mathcal{O}(L)\otimes E^\vee\Leftrightarrow$ (ii) and (iii) in (4).

To see $(4)\Leftrightarrow(5)$, we note that by setting

$$m^E_k(u_1,\cdots,u_{k-1},e)=\tilde\lambda_k(u_1,\cdots,u_{k-1},e),$$

Equation (1.30) is equivalent to the generalized Jacobi identity (1.6) of $(L\oplus E,\tilde\lambda_\bullet)$.

Finally, we show that $(5)\Leftrightarrow(6)$. In fact, for all $k\geq 0$, $f^E_k$ and $m^E_{k+1}$ are mutually determined by

$$f^E_k(u_1,\cdots,u_k)(e)=m^E_{k+1}(u_1,\cdots,u_k,e).$$

Now we reformulate Equation (1.30) in terms of $f^E_k$: For $n=0$, it becomes

$$m^E_2(\lambda_0,e)=-m^E_1(m^E_1(e)),\ \ \forall e\in E,$$

which is equivalent to

$$f^E_1(\lambda_0)(e)=-\frac12[f^E_0,f^E_0](e)=\frac12\{f^E_0,f^E_0\}(e),\ \ \forall e\in E.$$

Here $[-,-]$ is the graded commutator in $\mathrm{End}(E)$ and $\{-,-\}$ is the shifted graded commutator in $\mathrm{End}(E)[1]$ (see Example 1.7), which is the only nontrivial bracket in $\mathrm{End}(E)[1]$.

For $n\geq 1$, Equation (1.30) can be reorganized as

$$\begin{aligned}&\sum_{k+l=n}\sum_{\sigma\in\mathrm{sh}(l,k)}\epsilon(\sigma)f^E_{k+1}(\lambda_l(u_{\sigma(1)},\cdots,u_{\sigma(l)}),\cdots,u_{\sigma(n)})\\ &=\frac12\{f^E_0,f^E_n(u_1,\cdots,u_n)\}+\sum_{k+l=n}\sum_{\tau\in\mathrm{sh}(l,k)}\epsilon(\tau)\frac12\{f^E_k(u_{\tau(1)},\cdots,u_{\tau(k)}),f^E_l(u_{\tau(k+1)},\cdots,u_{\tau(n)})\}.\end{aligned} \tag{1.32}$$

These are exactly Equation (1.17). □

*Remark* 1.33. According to this theorem, we can write the explicit relation between $D^E$ (or $\partial^E_L$) and $m^E_\bullet$:

$$m^E_{k+1}(u_1,\cdots,u_k,e)=(-1)^{|e|*_k+k+1}D^E(e)\llcorner u_1\llcorner\cdots\llcorner u_k,\ \forall u_1,\cdots,u_k\in L, e\in E, \tag{1.34}$$

where $*_k=\sum_{i=1}^k|u_i|$.

From now on, we will denote an $L$-module by any of the notations $(E, \partial_L^E)$, $(E, D^E)$, $(E, m_\bullet^E)$, $(E, f_\bullet^E)$ or $(E, \partial_L^E \sim m_\bullet^E)$, etc. The data $\partial_L^E : \mathcal{O}(L) \otimes E \to \mathcal{O}(L) \otimes E$, $D^E \in (\mathcal{O}(L) \otimes \mathrm{End}(E))^1$, $\{m_k^E\}_{k\geq 1}$ in Equation (1.29), and $\{f_k^E\}$ in Equation (1.31) are mutually determined by the above theorem.

**Example 1.35** (Adjoint module)**.** Let $(L, \lambda_\bullet)$ be an $L_\infty[1]$-algebra. Then the maps

$$m_{k+1}^L = \lambda_{k+1} \circ \mu_{k+1}^L : S^k(L) \otimes L \to L, \quad k \geq 0 \tag{1.36}$$

make $L$ an $L$-module, where $\mu_{k+1}^L$ are defined by Equation (1.1). In analogy to Lie algebras, we call it the adjoint $L$-module or the adjoint representation of $L$.

Now we introduce morphisms of $L$-modules.

**Definition 1.37** (Morphisms of $L$-modules)**.** Let $(E, \partial_L^E \sim m_\bullet^E)$ and $(F, \partial_L^F \sim m_\bullet^F)$ be two $L$-modules. A morphism of $L$-modules from $E$ to $F$ is an $\mathcal{O}(L)$-linear map $\phi : \mathcal{O}(L) \otimes E \to \mathcal{O}(L) \otimes F$ of degree 0 which is also a chain map, i.e.,

$$\phi \circ \partial_L^E = \partial_L^F \circ \phi. \tag{1.38}$$

Equivalently, an $L$-module morphism from $E$ to $F$ is an element $\phi = \sum_{k\geq 0} \phi_k \in \mathcal{O}(L) \otimes \mathrm{Hom}(E, F)$, where $\phi_k \in S^k(L^\vee) \otimes \mathrm{Hom}(E, F)$, such that, for all $n \geq 0$,

$$\begin{aligned}
&\sum_{k+l=n} \sum_{\sigma \in \mathrm{sh}(k,l)} \epsilon(\sigma)(-1)^{*_n(\dagger_k^\sigma+1)+k} \phi_l(u_{\sigma(k+1)}, \cdots, u_{\sigma(n)})(m_{k+1}^E(u_{\sigma(1)}, \cdots, u_{\sigma(k)}, e)) \\
&= \sum_{k+l=n} \sum_{\tau \in \mathrm{sh}(k,l)} \epsilon(\tau)(-1)^k \phi_{n-k+1}(\lambda_k(u_{\tau(1)}, \cdots, u_{\tau(k)}), \cdots, u_{\tau(n)})(e) \\
&\quad + \sum_{k+l=n} \sum_{\tau \in \mathrm{sh}(k,l)} \epsilon(\tau)(-1)^{(*_n+1)\dagger_k^\tau+n-k} m_{n-k+1}^F(u_{\tau(k+1)}, \cdots, u_{\tau(n)}, \phi_k(u_{\tau(1)}, \cdots, u_{\tau(k)})(e))
\end{aligned} \tag{1.39}$$

holds for all $e \in E, u_i \in L, k, l \geq 0$, where $*_n = \sum_{i=1}^n |u_i|$, $\dagger_k^\sigma = \sum_{i=1}^k |u_{\sigma(i)}|$ and $\dagger_k^\tau = \sum_{i=1}^k |u_{\tau(i)}|$.

The set of such morphisms will be denoted by $\mathrm{Hom}_L(E, F)$.

Let $(E, f_\bullet^E)$ be an $L$-module. Consider the family of maps

$$\phi_k^{L,E} = f_{k+1}^E \circ \mu_{k+1}^L : S^k(L) \otimes L \to \mathrm{End}(E)[1], \quad k \geq 0, \tag{1.40}$$

where $\mu_{k+1}^L : S^k(L) \otimes L \to S^{k+1}(L)$ is defined by Equation (1.1). It turns out to be a canonical morphism of $L$-modules:

**Lemma 1.41.** *The family of maps $\phi^{L,E} = \{\phi_\bullet^{L,E}\}$ defines an $L$-module morphism from the adjoint $L$-module $L$ to* $\mathrm{End}(E)[1]$.

*Proof.* To see that $\phi^{L,E}$ is an $L$-module morphism, it suffices to show that $\phi^{L,E}$ satisfies Equation (1.39). In fact, note that

$$m^{\mathrm{End}(E)[1]}(\varphi) = \{m^E, \varphi\}, \quad \forall \varphi \in \mathrm{End}(E)[1],$$

where $\{-, -\}$ is the shifted graded commutator (1.8) in $\mathrm{End}(E)[1]$. Then Equation (1.39) follows by reformulating Equation (1.32) via Equations (1.36) and (1.40). $\square$

## 2. ATIYAH CLASSES OF SH LIE PAIRS

### 2.1. SH Lie pairs.

**Definition 2.1.** By an SH Lie pair $(L, A)$, we mean an $L_\infty[1]$-algebra $(L, \lambda_\bullet \sim Q_L)$ with a subalgebra $A \subset L$. The structure maps in $A$ are again denoted by $\{\lambda_k\}_{k\geq 0}$. In particular, $\lambda_0 \in A^1 \subset L^1$.

The homological vector field $Q_A : \mathcal{O}(A) \to \mathcal{O}(A)$ on the subalgebra $A$ is determined by $Q_L$. In fact, the inclusion map $j : A \to L$ gives rise to $j^\vee : L^\vee \to A^\vee$ and a surjective morphism of commutative algebras $j^\vee : \mathcal{O}(L) \to \mathcal{O}(A)$. The condition that $A$ is a subalgebra in $L$ is equivalent to

$$j^\vee \circ Q_L = Q_A \circ j^\vee : \ \mathcal{O}(L) \to \mathcal{O}(A).$$

We fix such an SH Lie pair $(L, A)$. For simplicity, here and in the sequel, we write $B = L/A$.

**Lemma 2.2.** *The quotient space $B$ is a canonical $A$-module.*

*Proof.* There is an exact sequence of graded vector spaces

$$0 \longrightarrow A \xrightarrow{\ j\ } L \xrightarrow{\ p\ } B \longrightarrow 0. \tag{2.3}$$

The canonical $A$-module structure on $B$

$$m_k : \ S^{k-1}(A) \otimes B \to B, k \geq 1$$

is defined by

$$m_k(a_1, \cdots, a_{k-1}, b) = p \circ \lambda_k(a_1, \cdots, a_{k-1}, l), \ \ \forall a_i \in A, l \in L \text{ such that } p(l) = b.$$

These $m_k, k \geq 1$, are well-defined because $A$ is a subalgebra. That $\{m_k\}_{k\geq 1}$ satisfies Equation (1.30) follows from the generalized Jacobi identity (1.6). ☐

It follows that the dual vector space

$$B^\vee = (L/A)^\vee \cong A^\perp = \ker(j^\vee : \ L^\vee \to A^\vee)$$

is also an $A$-module, which will be denoted by $(A^\perp, \partial_A^\perp)$.

Define an operator $J : \mathcal{O}(L) \to \mathcal{O}(A) \otimes L^\vee$ by the commutative diagram:

$$\begin{tikzcd} \mathcal{O}(L) \arrow[r, "I^L"] \arrow[dr, "J"'] & \mathcal{O}(L) \otimes L^\vee \arrow[d, "j^\vee \otimes 1"] \\ & \mathcal{O}(A) \otimes L^\vee. \end{tikzcd}$$

Here $I^L : \mathcal{O}(L) \to \mathcal{O}(L) \otimes L^\vee$ is defined by Equation (1.3). It follows immediately that

$$(1 \otimes j^\vee) \circ J = j^\vee \circ I^L = I^A \circ j^\vee : \ \mathcal{O}(L) \to \mathcal{O}(A) \otimes A^\vee. \tag{2.4}$$

It is also easy to see that $J$ is a derivation valued in the $\mathcal{O}(L)$-bimodule $\mathcal{O}(A) \otimes L^\vee$, i.e., for all $\omega, \omega' \in \mathcal{O}(L)$,

$$J(\omega \odot \omega') = j^\vee(\omega) \odot J(\omega') + J(\omega) \odot j^\vee(\omega') = j^\vee(\omega) \odot J(\omega') + (-1)^{|\omega||\omega'|} j^\vee(\omega') \odot J(\omega). \tag{2.5}$$

For any graded vector space $E$, $\mathcal{O}(L) \otimes \operatorname{End}(E)$ has an obvious associative product $\circ$. By Equation (2.5), the map

$$J \otimes 1 : \mathcal{O}(L) \otimes \operatorname{End}(E) \to \mathcal{O}(A) \otimes L^\vee \otimes \operatorname{End}(E)$$

satisfies, for all $\phi, \psi \in \mathcal{O}(L) \otimes \operatorname{End}(E)$,

$$(J \otimes 1)(\phi \circ \psi) = (j^\vee \otimes 1)(\phi) \circ (J \otimes 1)(\psi) + (J \otimes 1)(\phi) \circ (j^\vee \otimes 1)(\psi). \tag{2.6}$$

**Proposition 2.7.** *Let $\partial_A^\perp = Q_A + D^\perp$ be the dual $A$-module structure on $B^\vee \cong A^\perp$. Then for any $\omega \in \ker(j^\vee) \subset \mathcal{O}(L)$, we have $J(\omega) \in \mathcal{O}(A) \otimes A^\perp$ and*

$$\partial_A^\perp (J(\omega)) = J(Q_L(\omega)). \tag{2.8}$$

*Proof.* Let $(B, D^B \sim m_\bullet)$ be the $A$-module structure on $B$ in Lemma 2.2. We first show that

$$D^\perp(\xi) = J(Q_L(\xi)), \quad \forall \xi \in A^\perp. \tag{2.9}$$

In fact, using Equation (1.34), we have, for all $p(l) \in B, a_1 \odot \cdots \odot a_k \in S^k(A)$,

$$\begin{aligned}
&\langle D^\perp(\xi), p(l)\rangle(a_1 \odot \cdots \odot a_k) = (-1)^{|\xi|+1}\langle \xi, D^B(p(l))(a_1 \odot \cdots \odot a_k)\rangle \\
&= (-1)^{|\xi|+|l|*_k+k}\langle \xi, m_{k+1}(a_1, \cdots, a_k, p(l)) = (-1)^{|\xi|+|l|*_k+k}\langle \xi, p \circ \lambda_{k+1}(j(a_1), \cdots, j(a_k), l)\rangle \\
&= (-1)^{k+1}\langle (j^\vee \circ I^L \circ \lambda_{k+1}^\vee)(\xi), p(l)\rangle(a_1 \odot \cdots \odot a_k) \\
&= \langle (J \circ Q_L)(\xi), p(l)\rangle(a_1 \odot \cdots \odot a_k),
\end{aligned}$$

where $*_k = \sum_{i=1}^k |a_i|$, since $Q_L = \sum_{k\geq 0}(-1)^k \lambda_k^\vee$. This proves Equation (2.9).

To prove Equation (2.8), it suffices to consider elements of the form $\omega \odot \xi \in \mathcal{O}(L) \odot A^\perp \cong \ker(j^\vee)$, where $\omega \in \mathcal{O}(L), \xi \in A^\perp$. Then using Equation (2.9), we have

$$\begin{aligned}
&J(Q_L(\omega \odot \xi)) = J(Q_L(\omega) \odot \xi + (-1)^{|\omega|}\omega \odot Q_L(\xi)) \\
&= j^\vee(Q_L(\omega)) \odot J(\xi) + J(Q_L(\omega)) \odot j^\vee(\xi) + (-1)^{|\omega|}\left(J(\omega) \odot j^\vee(Q_L(\xi)) + j^\vee(\omega) \odot J(Q_L(\xi))\right) \\
&= Q_A\left(j^\vee(\omega)\right) \odot \xi + (-1)^{|\omega|} j^\vee(\omega) \odot D^\perp(\xi) = \partial_A^\perp(J(\omega \odot \xi)).
\end{aligned}$$

This completes the proof. $\square$

2.2. **Construction of Atiyah classes.** In this part, besides the SH Lie pair $(L, A)$, we fix an $A$-module $(E, m_\bullet \sim D^{A,E})$. As usual, assume that the differential is of the form

$$\partial_A^E = Q_A + D^{A,E},$$

where $D^{A,E} \in (\mathcal{O}(A) \otimes \mathrm{End}(E))^1$ is the $A$-module structure on $E$, and it will be treated as an $\mathcal{O}(A)$-linear map $\mathcal{O}(A) \otimes E \to \mathcal{O}(A) \otimes E$.

Meanwhile, $A^\perp \otimes \mathrm{End}(E)$ carries an $A$-module structure, its differential

$$\partial_A^{A^\perp \otimes \mathrm{End}(E)} : \mathcal{O}(A) \otimes (A^\perp \otimes \mathrm{End}(E)) \to \mathcal{O}(A) \otimes (A^\perp \otimes \mathrm{End}(E))$$

is expressed by

$$\partial_A^{A^\perp \otimes \mathrm{End}(E)} = Q_A + D^\perp + [D^{A,E}, -].$$

The corresponding cohomology space is denoted by $\mathrm{H}^\bullet_{\mathrm{CE}}(A, A^\perp \otimes \mathrm{End}(E))$.

Since $j^\vee : \mathcal{O}(L) \to \mathcal{O}(A)$ is surjective, one is able to find some $D^{L,E} \in (\mathcal{O}(L) \otimes \mathrm{End}(E))^1$ such that $(j^\vee \otimes 1)(D^{L,E}) = D^{A,E}$. Thus we get an $L$-connection $\nabla = Q_L + D^{L,E}$ on $E$ subject to the commutative diagram

$$\begin{array}{ccc}
\mathcal{O}(L) \otimes E & \xrightarrow{\nabla} & \mathcal{O}(L) \otimes E \\
\downarrow{\scriptstyle j^\vee \otimes 1} & & \downarrow{\scriptstyle j^\vee \otimes 1} \\
\mathcal{O}(A) \otimes E & \xrightarrow{\partial_A^E} & \mathcal{O}(A) \otimes E.
\end{array}$$

We call $\nabla$ an $L$-connection extending $(A, E)$. However, it is not necessarily flat. The curvature of $\nabla$ is easily available:

$$R^\nabla = Q_L(D^{L,E}) + (D^{L,E})^2 \in \mathcal{O}(L) \otimes \mathrm{End}(E).$$

We observe the following commutative diagram:

$$\begin{tikzcd} E \arrow[r, "R^\nabla"] \arrow[d, equal] & \mathcal{O}(L)\otimes E \arrow[r, "J\otimes 1"] \arrow[d, "j^\vee\otimes 1"] & \mathcal{O}(A)\otimes L^\vee\otimes E \arrow[d, "1\otimes j^\vee\otimes 1"] \\ E \arrow[r, "(\partial_A^E)^2=0"] & \mathcal{O}(A)\otimes E \arrow[r, "I^A\otimes 1"] & \mathcal{O}(A)\otimes A^\vee\otimes E, \end{tikzcd}$$

which implies that

$$(1\otimes j^\vee\otimes 1)(J\otimes 1)(R^\nabla)=0.$$

Hence, we get an element

$$\alpha^E_\nabla := (J\otimes 1)(R^\nabla)\in \mathcal{O}(A)\otimes A^\perp\otimes \mathrm{End}(E) \tag{2.10}$$

of degree 2.

**Theorem-Definition 2.11.** (1) *The element $\alpha^E_\nabla$ defined by Equation* (2.10) *is a cocycle of the Chevalley-Eilenberg complex of $A$ with coefficient in $A^\perp\otimes\mathrm{End}(E)$, which will be called the Atiyah cocycle of the SH Lie pair $(L,A)$ with respect to the $A$-module $E$ and the $L$-connection $\nabla$ extending $(A,E)$.*

(2) *The cohomology class $[\alpha^E]=[\alpha^E_\nabla]\in \mathrm{H}^2_{\mathrm{CE}}(A,A^\perp\otimes\mathrm{End}(E))$ does not depend on the choice of the $L$-connection $\nabla$ extending $(A,E)$. We call it the Atiyah class of the SH Lie pair $(L,A)$ with respect to the $A$-module $E$.*

(3) *For the canonical $A$-module $L/A$, there associates a canonical Atiyah class*

$$[\alpha^{L/A}]\in \mathrm{H}^2_{\mathrm{CE}}(A,A^\perp\otimes\mathrm{End}(L/A))=\mathrm{H}^2_{\mathrm{CE}}(A,\mathrm{Hom}(L/A\otimes L/A,L/A)).$$

Before giving a proof of the above theorem, we prove the following

**Lemma 2.12.** *If $X\in\mathcal{O}(L)\otimes\mathrm{End}(E)$ satisfying $(j^\vee\otimes 1)(X)=0$, then*

$$[D^{A,E},(J\otimes 1)(X)]=(J\otimes 1)[D^{L,E},X].$$

*Proof.* Without lose of generality, we may assume that $X$ is homogeneous. Note that $(j^\vee\otimes 1)(D^{L,E})=D^{A,E}$. We have

$$\begin{aligned}
&(J\otimes 1)[D^{L,E},X]=(J\otimes 1)(D^{L,E}\circ X-(-1)^{|X|}X\circ D^{L,E})\\
&=(j^\vee\otimes 1)(D^{L,E})\circ(J\otimes 1)(X)+(J\otimes 1)(D^{L,E})\circ(j^\vee\otimes 1)(X)\\
&\quad -(-1)^{|X|}((j^\vee\otimes 1)(X)\circ(J\otimes 1)(D^{L,E})+(J\otimes 1)(X)\circ(j^\vee\otimes 1)(D^{L,E})) \qquad \text{(by Equation (2.6))}\\
&=D^{A,E}\circ(J\otimes 1)(X)-(-1)^{|X|}(J\otimes 1)(X)\circ D^{A,E}=[D^{A,E},(J\otimes 1)(X)].
\end{aligned}$$

□

*Proof of Theorem-Definition 2.11.* (1). Note that $(j^\vee\otimes 1)(R^\nabla)=(\partial^E_A)^2=0$. It follows that $(J\otimes 1)(R^\nabla)\in\mathcal{O}(A)\otimes A^\perp\otimes\mathrm{End}(E)$. Thus

$$\begin{aligned}
\partial_A^{A^\perp\otimes\mathrm{End}(E)}(\alpha^E_\nabla)&=(Q_A+D^\perp+[D^{A,E},-])((J\otimes 1)(R^\nabla))\\
&=(Q_A+D^\perp)((J\otimes 1)(R^\nabla))+[D^{A,E},(J\otimes 1)(R^\nabla)]\\
&=(J\otimes 1)(Q_L(R^\nabla)+[D^{L,E},R^\nabla])=0,
\end{aligned}$$

where we have used Equation (2.8), Lemma 2.12, and the Bianchi identity (1.21) in the last two steps.

(2). Let $\tilde\nabla=Q_L+\tilde D^{L,E}$ be another $L$-connection extending $(A,E)$. Then

$$\phi=\nabla-\tilde\nabla=D^{L,E}-\tilde D^{L,E}\in(\mathcal{O}(L)\otimes\mathrm{End}(E))^1$$

satisfies

$$(j^\vee \otimes 1)(\phi) = 0, \qquad\qquad (J \otimes 1)(\phi) \in \mathcal{O}(A) \otimes A^\perp \otimes \mathrm{End}(E).$$

It follows from Equation (2.6) that $(J \otimes 1)(\phi^2) = 0$. Therefore, we have

$$\begin{aligned}
\alpha^E_\nabla - \alpha^E_{\tilde\nabla} &= (J \otimes 1)(R^\nabla - R^{\tilde\nabla}) = (J \otimes 1)\left(Q_L(D^{L,E}) + (D^{L,E})^2 - Q_L(\tilde D^{L,E}) - (\tilde D^{L,E})^2\right) \\
&= (J \otimes 1)(Q_L(\phi) + [\tilde D^{L,E}, \phi] + \phi^2) = (J \otimes 1)(Q_L(\phi) + [\tilde D^{L,E}, \phi]) \\
&= (Q_A + D^\perp)((J \otimes 1)(\phi)) + [D^{A,E}, (J \otimes 1)(\phi)] \qquad \text{(by Equation (2.8) and Lemma 2.12)} \\
&= (Q_A + D^\perp + [D^{A,E}, -])((J \otimes 1)(\phi)) = \partial_A^{A^\perp \otimes \mathrm{End}(E)}((J \otimes 1)(\phi)),
\end{aligned}$$

which implies that $[\alpha^E_\nabla] = [\alpha^E_{\tilde\nabla}]$.

Finally, statement (3) follows from the standard identification $A^\perp \cong (L/A)^\vee$. □

We now characterize the Atiyah cocycle $\alpha^E_\nabla \in \mathcal{O}(A) \otimes A^\perp \otimes \mathrm{End}(E)$ in terms of the brackets $\lambda_\bullet$ coming from $Q_L$. Recall that we started from $D^{L,E} \in (\mathcal{O}(L) \otimes \mathrm{End}(E))^1$ which extends $D^{A,E} \in (\mathcal{O}(A) \otimes \mathrm{End}(E))^1$. This can also be interpreted by a family of degree 1 linear maps $\{\tilde m_k : S^{k-1}(L) \otimes E \to E\}_{k\geq 1}$ extending $\{m_k : S^{k-1}(A) \otimes E \to E\}_{k \geq 1}$ (see Equation (1.34) for the relation between $D^{L,E}$ and $\tilde m_\bullet$).

We further assume that

$$\alpha^E_\nabla = \sum_{k\geq 0} \alpha_k,$$

where $\alpha_k \in S^k(A^\vee) \otimes (L/A)^\vee \otimes \mathrm{End}(E)$. Below is the explicit formula of $\alpha_k$.

**Proposition 2.13.** *For all $a_1, \cdots, a_k \in A, e \in E, b \in B = L/A$, we have*

$$\begin{aligned}
&(-1)^{k+1}\alpha_k(a_1, \cdots, a_k, b, e) \\
=&\sum_{p=0}^{k} \sum_{\sigma \in \mathrm{sh}(p,k-p)} \epsilon(\sigma) \tilde m_{k-p+3}(\lambda_p(a_{\sigma(1)}, \cdots, a_{\sigma(p)}), \cdots, a_{\sigma(k)}, l, e) \\
&+ \sum_{p=0}^{k} \sum_{\sigma \in \mathrm{sh}(p,k-p)} (-1)^{|b|(*_k - \dagger^\sigma_p)} \epsilon(\sigma) \tilde m_{k-p+2}(\lambda_{p+1}(a_{\sigma(1)}, \cdots, a_{\sigma(p)}, l), a_{\sigma(p+1)}, \cdots, a_{\sigma(k)}, e) \\
&+ \sum_{p=0}^{k} \sum_{\sigma \in \mathrm{sh}(p,k-p)} (-1)^{\dagger^\sigma_p} \epsilon(\sigma) m_{p+1}(a_{\sigma(1)}, \cdots, a_{\sigma(p)}, \tilde m_{k-p+2}(a_{\sigma(p+1)}, \cdots, a_{\sigma(k)}, l, e)) \\
&+ \sum_{p=0}^{k} \sum_{\sigma \in \mathrm{sh}(p,k-p)} (-1)^{\dagger^\sigma_p + |b|(*_k - \dagger^\sigma_p + 1)} \epsilon(\sigma) \tilde m_{p+2}(a_{\sigma(1)}, \cdots, a_{\sigma(p)}, l, m_{k-p+1}(a_{\sigma(p+1)}, \cdots, a_{\sigma(k)}, e)),
\end{aligned}$$

*where $l \in L$ satisfies $p(l) = b$, $*_k = \sum_{i=1}^k |a_i|$ and $\dagger^\sigma_p = \sum_{i=1}^p |a_{\sigma(i)}|$.*

The proof follows from some straightforward computations and thus is omitted.

*Remark* 2.14. To construct the Atiyah cocycle $\alpha^E_\nabla$, we need $D^{L,E}$, or $\tilde m_k : S^{k-1}(L) \otimes E \to E, k \geq 1$. Nevertheless, Proposition 2.13 implies that the only information we need is the behavior of $\tilde m_k$ restricted to $S^{k-2}(A) \otimes L \otimes E$. In other words, to compute $\alpha^E_\nabla$, it is enough to do first order extensions $\tilde m_k^{(1)}$ : $S^{k-2}(A) \otimes L \to \mathrm{End}(E)$ of $m^E_k$, for all $k \geq 2$. For this reason, we believe that there should exist other, perhaps "higher" Atiyah classes.

A more convenient way to get $\alpha^E_\nabla$ is to find a complementary subspace to $A$ in $L$. In doing so, one may simply assume that $L = A \oplus B$, where $B$ is only a sub-vector space, not necessarily a subalgebra of $L$.

Then $\mathcal{O}(L) \cong \mathcal{O}(A) \otimes \mathcal{O}(B)$. Let $(E, D^{A,E} \sim m^E_\bullet)$ be an $A$-module, where $D^{A,E} \in \mathcal{O}(A) \otimes \mathrm{End}(E) \subset \mathcal{O}(L) \otimes \mathrm{End}(E)$. Thus $\nabla = Q_L + D^{A,E}$ is an $L$-connection on $E$ extending $(A, E)$. Equivalently, $\nabla$ is determined by $\{\tilde{m}^E_k\}_{k\geq 1} : S^{k-1}(L) \otimes E \to E$:

$$\tilde{m}^E_k = \sum_{p\geq 0} \tilde{m}^E_k \mid_{S^{k-1-p}(A)\otimes S^p(B)\otimes E} = \begin{cases} m^E_k, & p = 0, \\ 0, & p > 0. \end{cases}$$

Then the Atiyah cocycle becomes much simpler:

$$\begin{aligned}&(-1)^{k+1}\alpha_k(a_1, \cdots , a_k, b, e)\\ =&\sum_{p=0}^{k} \sum_{\sigma\in \mathrm{sh}(p,k-p)} (-1)^{|b|(*_k-\dagger^\sigma_p)}\epsilon(\sigma)m^E_{k-p+2}(\mathrm{Pr}_A(\lambda_{p+1}(a_{\sigma(1)}, \cdots , a_{\sigma(p)}, b)), a_{\sigma(p+1)}, \cdots , a_{\sigma(k)}, e),\end{aligned}$$

where $\mathrm{Pr}_A : L \to A$ is the projection.

From now on, when we talk about the Atiyah cocycle of an SH Lie pair $(L, A)$ with respect to an $A$-module $E$, we always assume a splitting of sequence (2.3) and that the Atiyah cocycle is obtained via the trivial $L$-connection on $E$ extending $(A, E)$ as in Remark 2.14.

**Example 2.15.** Let $(\mathfrak{g}, \mathfrak{h})$ be an ordinary Lie algebra pair and $E$ an $\mathfrak{h}$-module, where $\mathfrak{g}, \mathfrak{h}$ and $E$ are all usual ungraded vector spaces. The Atiyah class in [12] can be recovered as follows: In fact, setting $L = \mathfrak{g}[1], A = \mathfrak{h}[1]$, we get an SH Lie pair $(L, A)$ with the obvious $A$-module structure on $E$. Applying Proposition 2.13, we get the Atiyah cocycle

$$\alpha^E_\nabla(a, b, e) = \nabla_{[a,l]}(e) - \nabla_a\nabla_l(e) + \nabla_l\nabla_a(e) = -R^\nabla(a, l)(e),$$

where $a \in A, b \in L/A, e \in E, l \in L$ such that $p(l) = b$ and $\nabla : L \otimes E \to E$ is an $L$-connection extending $(A, E)$. Comparing with the Atiyah cocycle defined in [12], the only difference is a minus sign.

A nontrivial example of Atiyah classes of this type can be found in [9] (see also [12, Example 22]).

**Example 2.16.** Let $(L = L^{-1}, A = A^{-1})$ be a one-term SH Lie pair and $E = \oplus_{n\in\mathbb{Z}}E^n$ be an $A$-module, or a Lie algebra representation up to homotopy [1] of $A[-1]$ on $E$. Assume that $L = A \oplus B$, where $B$ is also concentrated in degree $(-1)$. If the $A$-module structure of $E$ is given by $m_k : S^{k-1}A \otimes E \to E, k \geq 1$, then the Atiyah cocycle $\alpha^E = \sum_{k\geq 0} \alpha_k \in \mathcal{O}(A) \otimes B^\vee \otimes \mathrm{End}(E)$ is given by

$$(-1)^{k+1}\alpha_k(a_1, \cdots , a_k, b, e) = \sum_{i=1}^{k}(-1)^{k+i}m_{k+1}(\mathrm{Pr}_A\lambda_2(a_i, b), \cdots , \hat{a_i}, \cdots , e),$$

for $a_i \in A, e \in E, b \in B = L/A$.

In particular, if the $A$-module structure on $E$ has only two nontrivial actions $m_1 : E \to E$ and $m_2 : A \otimes E \to E$, then the Atiyah cocycle $\alpha^E = \alpha_1 \in A^\vee \otimes B^\vee \otimes \mathrm{End}(E)$ reads

$$\alpha_1(a, b, e) = m_2(\mathrm{Pr}_A\lambda_2(a, b), e).$$

**Example 2.17.** Let $(L = L^{-2} \oplus L^{-1}, A = A^{-2} \oplus A^{-1})$ be a Lie 2-algebra [4] pair with brackets $\lambda_1, \lambda_2, \lambda_3$ and $E = E^{-2} \oplus E^{-1}$ be an $A$-module. Let us fix a splitting $L = A \oplus B$. The Atiyah cocycle $\alpha^E = \alpha_0 + \alpha_1 + \alpha_2$ ($\alpha_i \in S^i(A^\vee) \otimes B^\vee \otimes \mathrm{End}(E)$) is given by:

$$\begin{aligned}\alpha_0(b, e) &= -m_2(\mathrm{Pr}_A\lambda_1(b), e)\\ \alpha_1(a, b, e) &= (-1)^{|b||a|}m_3(\mathrm{Pr}_A\lambda_1(b), a, e) + m_2(\mathrm{Pr}_A\lambda_2(a, b), e)\\ \alpha_2(a_1, a_2, b, e) &= -m_2(\mathrm{Pr}_A\lambda_3(a_1, a_2, b), e) - (-1)^{|b||a_2|}m_3(\mathrm{Pr}_A\lambda_2(a_1, b), a_2, e)\\ &\quad - (-1)^{(|b|+|a_2|)|a_1|}m_3(\mathrm{Pr}_A\lambda_2(a_2, b), a_1, e),\end{aligned}$$

for $a_i \in A, e \in E, b \in B = L/A$.

**Example 2.18.** Let $(L, A, \lambda_1, \lambda_2)$ be a DG Lie algebra pair. Suppose that $L = A \oplus B$. Then the associated $A$-module structure on $B$ consists of two actions: $m_1^B$ and $m_2^B$. Assume that $E$ is an $A$-module with only two nontrivial actions from $A$: $m_1$ and $m_2$. Then the Atiyah cocycle has two terms

$$\alpha^E = \alpha_0 + \alpha_1 \in (B^\vee \otimes \mathrm{End}(E)) \ \oplus \ (A^\vee \otimes B^\vee \otimes \mathrm{End}(E)),$$

where

$$-\alpha_0(b, e) = m_2(\mathrm{Pr}_A \lambda_1(b), e), \qquad \alpha_1(a, b, e) = m_2(\mathrm{Pr}_A \lambda_2(a, b), e).$$

For the $A$-module $F = B^\vee \otimes \mathrm{End}(E)$, we are able to split the differential operator

$$\partial_A = \partial_0 + \partial_1 : \ \mathcal{O}(A) \otimes F \to \mathcal{O}(A) \otimes F,$$

where

$$\partial_0 : S^\bullet(A^\vee) \otimes F \to S^\bullet(A^\vee) \otimes F, \qquad \partial_1 : S^\bullet(A^\vee) \otimes F \to S^{\bullet+1}(A^\vee) \otimes F.$$

Now the Chevalley-Eilenberg cochain complex $(\mathcal{O}(A) \otimes F, \partial_A)$ associated to the $A$-module $F$ becomes a double complex:

$$D^{p,q} = (S^p(A^\vee) \otimes F)^{p+q}, \qquad p \geq 0, q \in \mathbb{Z}$$

with differentials

$$\partial_0 : \ D^{p,q} \to D^{p,q+1}, \qquad \partial_1 : \ D^{p,q} \to D^{p+1,q}.$$

As for the Atiyah cocycle $\alpha^E = \alpha_0 + \alpha_1$, it sits in $D^{0,2} \oplus D^{1,1}$. So, the Atiyah class yields two canonical elements: $[\alpha_0]$ in $\mathrm{H}^2(D^{0,\bullet}, \partial_0)$ and $[\alpha_1]$ in $\mathrm{H}^1(D^{\bullet,1}, \partial_1)$.

We present a particular example with nontrivial Atiyah classes.

**Example 2.19.** Let $A = \mathrm{span}\{a_1, a_2\}$ be a 2-dimensional vector space concentrating in degree $(-1)$ such that $A^\vee = \mathrm{span}\{a_1^\vee, a_2^\vee\}$, and $B = \mathrm{span}\{b\}$ an ordinary 1-dimensional vector space concentrating in degree 0 with dual space $B^\vee = \mathrm{span}\{b^\vee\}$. Then $L = A \oplus B$ together with the homological vector field $Q_L = \delta : \ A^\vee \to S^2(A^\vee) \otimes B^\vee$ defined by

$$\delta(a_1^\vee) = k_1 a_1^\vee \odot a_2^\vee \otimes b^\vee, \qquad \delta(a_2^\vee) = k_2 a_1^\vee \odot a_2^\vee \otimes b^\vee, \quad k_1, k_2 \in \mathbb{K}$$

determine an SH Lie pair $(L, A)$ such that $A \subset L$ is abelian. Note that the only nontrivial structure map is $\lambda_3 : \ S^2(A) \otimes B \to A$ by

$$\lambda_3(a_1, a_2, b) = -k_1 a_1 - k_2 a_2.$$

Let $E$ be another 1-dimensional vector space. Then $D^E : \ E \to A^\vee \otimes E$ defined by, for all $e \in E$,

$$D^E(e) = (k_3 a_1^\vee + k_4 a_2^\vee) \otimes e, \quad \text{where } k_3, k_4 \in \mathbb{K} \text{ such that } k_1 k_3 + k_2 k_4 \neq 0,$$

determines an $A$-module structure on $E$. Equivalently, we have

$$m_2(a_1, e) = -k_3 e, \qquad m_2(a_2, e) = -k_4 e.$$

The only nontrivial part of the Atiyah cocycle is

$$\alpha_2(a_1, a_2, b, e) = -m_2(\mathrm{Pr}_A \lambda_3(a_1, a_2, b), e) = -m_2(-k_1 a_1 - k_2 a_2, e) = -(k_1 k_3 + k_2 k_4) e.$$

Note that the $A$-module structure $\partial_A^{B^\vee \otimes \mathrm{End}(E)}$ on $B^\vee \otimes \mathrm{End}(E)$ is trivial in this case. Thus the Atiyah class $[\alpha_2] \in \mathrm{H}^2_{\mathrm{CE}}(A, B^\vee \otimes \mathrm{End}(E))$ is nontrivial.

2.3. **An equivalent description of Atiyah classes.** Let $(L,A)$ be an SH Lie pair. Then there is a coadjoint $A$-module structure $\partial_A^{A^\vee} = Q_A + D^{A^\vee}$ on $A^\vee$, which is the dual of the adjoint $A$-module structure on $A$ (Example 1.35). Moreover, we have the following commutative diagram:

$$\begin{tikzcd} A^\vee \arrow[r, "Q_A"] \arrow[d, "D^{A^\vee}"'] & \mathcal{O}(A) \arrow[dl, "I^A"] \\ \mathcal{O}(A)\otimes A^\vee, & \end{tikzcd}$$

where $I^A$ is defined in Equation (1.3). In fact, we recall that $Q_A = \sum_{k\geq 0}(-1)^k\lambda_k^\vee$, and for all $\xi \in A^\vee, a_1\odot\cdots\odot a_k\otimes a_{k+1} \in S^k(A)\otimes A$,

$$\begin{aligned}\langle (I^A\circ Q_A)(\xi), a_1\odot\cdots\odot a_k\otimes a_{k+1}\rangle &= (-1)^{k+1}\langle \lambda_{k+1}^\vee(\xi), \mu_{k+1}^A(a_1\odot\cdots\odot a_k\otimes a_{k+1})\rangle \\ &= (-1)^{|\xi|+k}\langle \xi, m_{k+1}^A(a_1,\cdots,a_k,a_{k+1})\rangle,\end{aligned}$$

where $m_{k+1}^A = \lambda_{k+1}\circ\mu_{k+1}^A : S^k(A)\otimes A \to A$ is the adjoint $A$-module structure on $A$. Using Equation (1.34), we have

$$\langle (I^A\circ Q_A)(\xi), a_{k+1}\rangle = (-1)^{|\xi|+1}\langle \xi, D^A(a_{k+1})\rangle = \langle D^{A^\vee}(\xi), a_{k+1}\rangle \in S^k(A^\vee).$$

Hence, we have $D^{A^\vee} = I^A\circ Q_A$, as desired.

Similarly, $L$ carries a natural $A$-module structure

$$m_k^L = \lambda_k\circ\mu_k^L\circ(j^{\odot(k-1)}\otimes 1) : S^{k-1}A\otimes L \to L, k\geq 1,$$

where $\mu_k^L$ is the operator defined by Equation (1.1). And the dual $A$-module structure $\partial_A^{L^\vee} = Q_A + D^{L^\vee}$ on $L^\vee$ fits into the commutative diagram:

$$\begin{tikzcd} L^\vee \arrow[r, "Q_L"] \arrow[d, "D^{L^\vee}"'] & \mathcal{O}(L) \arrow[dl, "J"] \\ \mathcal{O}(A)\otimes L^\vee, & \end{tikzcd}$$

where $J = (j^\vee\otimes 1)\circ I^L$. Moreover, it follows from a simple induction argument that

$$\partial_A^{L^\vee}\circ J = J\circ Q_L : \mathcal{O}(L)\to\mathcal{O}(A)\otimes L^\vee, \tag{2.20}$$

$$\partial_A^{A^\vee}\circ I_A = I_A\circ Q_A : \mathcal{O}(A)\to\mathcal{O}(A)\otimes A^\vee. \tag{2.21}$$

Using Equations (2.8), (2.20) and (2.21), it can be verified that the linear dual of Sequence (2.3) of graded vector spaces

$$0 \longrightarrow A^\perp \longrightarrow L^\vee \xrightarrow{j^\vee} A^\vee \longrightarrow 0$$

is also a short exact sequence of $A$-modules.

Let $(E, m_\bullet \sim D^{A,E})$ be an $A$-module. We have a companion exact sequence of $A$-modules:

$$0\longrightarrow A^\perp\otimes\mathrm{End}(E)\longrightarrow L^\vee\otimes\mathrm{End}(E)\xrightarrow{j^\vee\otimes 1} A^\vee\otimes\mathrm{End}(E)\longrightarrow 0,$$

as well as a short exact sequence of cochain complexes:

$$0\longrightarrow\mathcal{O}(A)\otimes A^\perp\otimes\mathrm{End}(E)\longrightarrow\mathcal{O}(A)\otimes L^\vee\otimes\mathrm{End}(E)\xrightarrow{1\otimes j^\vee\otimes 1}\mathcal{O}(A)\otimes A^\vee\otimes\mathrm{End}(E)\longrightarrow 0. \tag{2.22}$$

A long exact sequence on the cohomology level follows:

$$\cdots \to \mathrm{H}^1_{\mathrm{CE}}(A, A^\perp\otimes\mathrm{End}(E)) \longrightarrow \mathrm{H}^1_{\mathrm{CE}}(A, L^\vee\otimes\mathrm{End}(E)) \longrightarrow \mathrm{H}^1_{\mathrm{CE}}(A, A^\vee\otimes\mathrm{End}(E))$$

$$\xrightarrow{\delta} \mathrm{H}^2_{\mathrm{CE}}(A, A^\perp \otimes \mathrm{End}(E)) \longrightarrow \mathrm{H}^2_{\mathrm{CE}}(A, L^\vee \otimes \mathrm{End}(E)) \longrightarrow \cdots . \tag{2.23}$$

**Lemma 2.24.** *The element* $(I^A \otimes 1)(D^{A,E}) \in \mathcal{O}(A) \otimes A^\vee \otimes \mathrm{End}(E)$ *is a degree* $1$ *cocycle.*

*Proof.* Since $I^A$ is a derivation valued in the $\mathcal{O}(A)$-bimodule $\mathcal{O}(A) \otimes A^\vee$, $(I^A \otimes 1)$ is also a derivation on the associative algebra $(\mathcal{O}(A) \otimes \mathrm{End}(E), \circ)$. Thus

$$(I^A \otimes 1)((D^{A,E})^2) = (I^A \otimes 1)(D^{A,E}) \circ D^{A,E} + D^{A,E} \circ (I^A \otimes 1)(D^{A,E}) = [D^{A,E}, (I^A \otimes 1)D^{A,E}].$$

Using Equation (2.21), we have

$$(\partial_A^{A^\vee} \otimes 1)((I^A \otimes 1)(D^{A,E})) = (I^A \otimes 1)(Q_A(D^{A,E})).$$

Hence,

$$\begin{aligned} &\partial_A^{A^\vee \otimes \mathrm{End}(E)}((I^A \otimes 1)(D^{A,E})) = (\partial_A^{A^\vee} + [D^{A,E}, -])((I^A \otimes 1)(D^{A,E})) \\ &= (I^A \otimes 1)(Q_A(D^{A,E})) + (I^A \otimes 1)((D^{A,E})^2) = (I^A \otimes 1)(Q_A(D^{A,E}) + (D^{A,E})^2) = 0, \end{aligned}$$

where the last equality follows from the Maurer-Cartan equation (1.27). □

It turns out that the element $(I^A \otimes 1)(D^{A,E})$ gives the Atiyah class:

**Theorem 2.25.** *The cohomology class*

$$\delta[(I^A \otimes 1)(D^{A,E})] \in \mathrm{H}^2_{\mathrm{CE}}(A, A^\perp \otimes \mathrm{End}(E))$$

*coincides with the Atiyah class* $[\alpha^E]$.

*Proof.* We chase the connecting map $\delta$ in Equation (2.22): Starting with $(I^A \otimes 1)(D^{A,E})$, one first chooses a degree 1 element $\beta \in \mathcal{O}(A) \otimes L^\vee \otimes \mathrm{End}(E)$ such that $(1 \otimes j^\vee \otimes 1)(\beta) = (I^A \otimes 1)(D^{A,E})$.

Then one is able to find a unique degree 2 element $\alpha \in \mathcal{O}(A) \otimes A^\perp \otimes \mathrm{End}(E)$ such that

$$\partial_A^{L^\vee \otimes \mathrm{End}(E)}(\beta) = (Q_A + D^{L^\vee} + [D^{A,E}, -])(\beta) = \alpha.$$

The cohomology class $[\alpha]$ is the upshot of $\delta[(I^A \otimes 1)(D^{A,E})]$. We now show that there exists an $L$-connection $\nabla = Q_L + D^{L,E}$ extending $(A, E)$, i.e., $(j^\vee \otimes 1)(D^{L,E}) = D^{A,E}$, and the resulting Atiyah cocycle $\alpha^E_\nabla$ equals $\alpha$.

First of all, there exists an element $D^{L,E} \in \mathcal{O}(L) \otimes \mathrm{End}(E)$ such that

$$(J \otimes 1)(D^{L,E}) = \beta, \qquad (j^\vee \otimes 1)(D^{L,E}) = D^{A,E}.$$

In fact, as $J : \mathcal{O}(L) \to \mathcal{O}(A) \otimes L^\vee$ is surjective, we can find some $K^{L,E} \in \mathcal{O}(L) \otimes \mathrm{End}(E)$ such that $(J \otimes 1)(K^{L,E}) = \beta$. Then by Equation (2.4), we have

$$(I^A \otimes 1)(j^\vee \otimes 1)(K^{L,E}) = (1 \otimes j^\vee \otimes 1)(J \otimes 1)(K^{L,E}) = (1 \otimes j^\vee \otimes 1)(\beta) = (I^A \otimes 1)(D^{A,E}).$$

Note that $\ker(I^A) \cong \ker(r^+) = \mathbb{K}$. Thus $(j^\vee \otimes 1)(K^{L,E}) - D^{A,E} = \varphi$ for some $\varphi \in \mathrm{End}(E)$. It follows that $D^{L,E} = K^{L,E} - \varphi$ satisfies the above requirements.

Then, using Equation (2.20),

$$\begin{aligned} \alpha^E_\nabla &= (J \otimes 1)(Q_L(D^{L,E}) + (D^{L,E})^2) = (\partial_A^{L^\vee} \otimes 1) \circ (J \otimes 1)(D^{L,E}) + [(j^\vee \otimes 1)D^{A,E}, (J \otimes 1)(D^{L,E})] \\ &= (Q_A + D^{L^\vee})(J \otimes 1)(D^{L,E}) + [D^{A,E}, (J \otimes 1)(D^{L,E})] \\ &= (Q_A + D^{L^\vee} + [D^{A,E}, -])(\beta) = \alpha, \end{aligned}$$

as required. □

2.4. **Vanishing of Atiyah classes.** Let $A$ be an $L_\infty[1]$-algebra and $B$ an $A$-module. Then the associated abelian extension $L = A \oplus B$ of $A$ along $B$ (see Proposition 1.28) gives rise to an SH Lie pair $(L, A)$, while the Atiyah class $\alpha_E$ with respect to any $A$-module $E$ is always trivial. So apparently the Atiyah class measures the nontriviality of the extension of $A$ to $L$.

It is natural to ask what we can say in general when the Atiyah class vanishes. The following facts are some first stage results. Further investigations of this question will be shown somewhere else.

**Theorem 2.26.** *Let $(L, A)$ be an SH Lie pair and $(E, (\partial_A^E = Q_A + D^{A,E}) \sim f_\bullet^E)$ an $A$-module. Then the following four statements are equivalent:*

*(1) The Atiyah class $[\alpha^E] \in \mathrm{H}^2_{\mathrm{CE}}(A, A^\perp \otimes \mathrm{End}(E))$ vanishes;*
*(2) There exists a degree $1$ cocycle $\phi \in \mathcal{O}(A) \otimes L^\vee \otimes \mathrm{End}(E)$ such that*
$$(1 \otimes j^\vee \otimes 1)(\phi) = (I^A \otimes 1)(D^{A,E}); \tag{2.27}$$
*(3) There exists an $A$-module morphism $\{\phi_k : S^k(A) \otimes L \to \mathrm{End}(E)[1]\}_{k\geq 0}$ from $L$ to $\mathrm{End}(E)[1]$ extending the canonical $A$-module morphism $\phi^{A,E}$ defined in Equation (1.40) from $A$ to $\mathrm{End}(E)[1]$, i.e.,*
$$\phi_k \circ (1 \otimes j) = \phi_k^{A,E} = f_{k+1}^E \circ \mu_{k+1}^A : S^k(A) \otimes A \to \mathrm{End}(E)[1]; \tag{2.28}$$
*(4) There exists an $L$-connection $\nabla$ on $E$ extending $(A, E)$ such that the Atiyah cocycle $\alpha_\nabla^E$ of $E$ relative to $\nabla$ vanishes.*

*Proof.* $(1) \Rightarrow (2)$. Assume that $[\alpha^E] = 0$. It follows from Theorem 2.25 that $\delta[(I^A \otimes 1)(D^{A,E})] = 0$. By chasing the long exact sequence (2.23), there exists some $\partial_A^{L^\vee \otimes \mathrm{End}(E)}$-cocycle $\tilde{\phi} \in \mathcal{O}(A) \otimes L^\vee \otimes \mathrm{End}(E)$ of degree 1 such that
$$[(1 \otimes j^\vee \otimes 1)(\tilde{\phi})] = [(I^A \otimes 1)(D^{A,E})] \in \mathrm{H}^1_{\mathrm{CE}}(A, A^\vee \otimes \mathrm{End}(E)).$$
It follows that there is a degree 0 element $\beta \in \mathcal{O}(A) \otimes A^\vee \otimes \mathrm{End}(E)$ such that
$$(1 \otimes j^\vee \otimes 1)(\tilde{\phi}) - (I^A \otimes 1)(D^{A,E}) = \partial_A^{A^\vee \otimes \mathrm{End}(E)}(\beta).$$
By exactness of Sequence (2.22), one can choose an element $\gamma \in (\mathcal{O}(A) \otimes L^\vee \otimes \mathrm{End}(E))^0$ such that $(1 \otimes j^\vee \otimes 1)(\gamma) = \beta$.

Let $\phi = \tilde{\phi} - \partial_A^{L^\vee \otimes \mathrm{End}(E)}(\gamma) \in (\mathcal{O}(A) \otimes L^\vee \otimes \mathrm{End}(E))^1$. Then
$$\begin{aligned}(1 \otimes j^\vee \otimes 1)(\phi) &= (1 \otimes j^\vee \otimes 1)(\tilde{\phi}) - ((1 \otimes j^\vee \otimes 1) \circ \partial_A^{L^\vee \otimes \mathrm{End}(E)})(\gamma) \\ &= (I^A \otimes 1)(D^{A,E}) + \partial_A^{A^\vee \otimes \mathrm{End}(E)}(\beta) - (\partial_A^{A^\vee \otimes \mathrm{End}(E)} \circ (1 \otimes j^\vee \otimes 1))(\gamma) \\ &= (I^A \otimes 1)(D^{A,E}).\end{aligned}$$

$(2) \Leftrightarrow (3)$. Note that a degree 1 element $\phi \in \mathcal{O}(A) \otimes L^\vee \otimes \mathrm{End}(E)$ consists of a family of degree 0 maps $\phi_k : S^k(A) \otimes L \to \mathrm{End}(E)[1]$. It can be easily seen that Equation (2.27) is equivalent to Equation (2.28).

$(2) \Rightarrow (4)$. Given a degree 1 cocycle $\phi \in \mathcal{O}(A) \otimes L^\vee \otimes \mathrm{End}(E)[1]$ satisfying Equation (2.27), by the argument in the proof of Theorem 2.25, we can find $D^{L,E} \in \mathcal{O}(L) \otimes \mathrm{End}(E)$ such that
$$(J \otimes 1)(D^{L,E}) = \phi, \qquad (j^\vee \otimes 1)(D^{L,E}) = D^{A,E}.$$
Then $\nabla = Q_L + D^{L,E}$ is an $L$-connection on $E$ extending $(A, E)$. The associated Atiyah cocycle
$$\begin{aligned}\alpha_\nabla^E &= (J \otimes 1)(R^\nabla) = (J \otimes 1)(Q_L(D^{L,E}) + D^{L,E} \circ D^{L,E}) \\ &= \partial_A^{L^\vee}((J \otimes 1)(D^{L,E})) + [(j^\vee \otimes 1)(D^{L,E}), (J \otimes 1)(D^{L,E})] \qquad \text{(by Equations (2.20) and (2.6))} \\ &= \partial_A^{L^\vee}(\phi) + [D^{A,E}, \phi] = \partial_A^{L^\vee \otimes \mathrm{End}(E)[1]}(\phi) = 0.\end{aligned}$$
Finally, $(4) \Rightarrow (1)$ is obvious. This completes the proof. $\square$

## 3. ATIYAH CLASSES AS FUNCTORS

3.1. **Atiyah operators.** Let $(L, A)$ be an SH Lie pair and denote by $B = L/A$ the standard $A$-module. The identification of $B^\vee$ and $A^\perp$ is assumed. For simplicity, from now on, we will denote the Chevalley-Eilenberg differential $\partial_A^E$ of any $A$-module $E$ by $\partial_A$.

Recall that the Atiyah cocycle

$$\alpha_\nabla^E \in \mathcal{O}(A) \otimes A^\perp \otimes \operatorname{End}(E) = \mathcal{O}(A) \otimes B^\vee \otimes \operatorname{Hom}(E, E) = \mathcal{O}(A) \otimes \operatorname{Hom}(B \otimes E, E),$$

where $\nabla$ is an $L$-connection on $E$ extending $(A, E)$. The associated $\mathcal{O}(A)$-linear maps

$$\begin{aligned} \boldsymbol{\alpha}_\nabla^E : &\qquad \mathcal{O}(A) \otimes E \longrightarrow \mathcal{O}(A) \otimes B^\vee \otimes E; \\ \boldsymbol{\alpha}_\nabla^E(x) : &\qquad \mathcal{O}(A) \otimes E \to \mathcal{O}(A) \otimes E, \qquad \text{where } x \in \mathcal{O}(A) \otimes B, \end{aligned}$$

will be called Atiyah operators. As $\alpha_\nabla^E$ is a cocycle, we have

**Lemma 3.1.** *If $x \in \mathcal{O}(A) \otimes B$ is a cocycle, i.e. $\partial_A(x) = 0$, then $\partial_A^{\operatorname{End}(E)}(\boldsymbol{\alpha}_\nabla^E(x)) = 0$, i.e., $\boldsymbol{\alpha}_\nabla^E(x)$ is an $A$-module morphism from E to itself.*

Let us fix a splitting of the short exact sequence (2.3) so that $L \cong A \oplus B$ and

$$\mathcal{O}(L) \cong \mathcal{O}(A) \odot \mathcal{O}(B) \cong \mathcal{O}(A) \otimes \mathcal{O}(B).$$

And the associated homological vector field $Q_L \in \operatorname{Der}(\mathcal{O}(L))$ decomposes into a sum of derivations

$$Q_L = Q_A + \delta + \mathcal{R} + D^\perp + \sum_{i \geq 2} T_i, \tag{3.2}$$

where

$$\begin{cases} Q_A &= Q_L|_{:A^\vee \longrightarrow \mathcal{O}(A)}; \\ \delta &= Q_L|_{:A^\vee \longrightarrow \mathcal{O}(A) \otimes B^\vee}; \\ \mathcal{R} &= Q_L|_{:A^\vee \longrightarrow \mathcal{O}(A) \otimes \widehat{S}^{\geq 2}(B^\vee)}; \\ D^\perp &= Q_L|_{:B^\vee \longrightarrow \mathcal{O}(A) \otimes B^\vee}; \\ T_i &= Q_L|_{:B^\vee \longrightarrow \mathcal{O}(A) \otimes S^i(B^\vee)}, \quad i \geq 2, \end{cases}$$

and they are extended as a derivation of $\mathcal{O}(L)$ in a natural manner.

In this situation, there is an $L$-connection $\nabla = Q_L + D^{A,E}$ extending $(A, E)$ (see Remark 2.14). The associated Atiyah cocycle is denoted by $\alpha^E$, where the subscript is omitted. Similarly, the associated Atiyah operator will be denoted by $\boldsymbol{\alpha}^E$.

We extend the operator $\delta$ to a $\mathbb{K}$-linear and degree 1 map for any graded vector space $E$

$$\delta = (\delta \otimes 1) : \mathcal{O}(A) \otimes E \to \mathcal{O}(A) \otimes B^\vee \otimes E,$$

such that the Leibniz rule

$$\delta(\xi \odot \eta \otimes e) = (-1)^{(|\xi|+1)|\eta|} \eta \odot \delta(\xi) \otimes e + (-1)^{|\xi|} \xi \odot \delta(\eta) \otimes e$$

holds for all $\xi, \eta \in \mathcal{O}(A), e \in E$.

**Lemma 3.3.** *As a map $\mathcal{O}(A) \otimes E \to \mathcal{O}(A) \otimes B^\vee \otimes E$, the Atiyah operator*

$$\boldsymbol{\alpha}^E = [\delta, \partial_A] = \delta \circ \partial_A + \partial_A \circ \delta.$$

*Proof.* Recall the definition of Atiyah cocycles:

$$
\begin{aligned}
\alpha^E &= (J\otimes 1)(\nabla^2) = (J\otimes 1)\left(Q_L(D^{A,E}) + (D^{A,E})^2\right) \\
&= \partial_A^{L^\vee}((J\otimes 1)(D^{A,E})) + [D^{A,E}, (J\otimes 1)(D^{A,E})] && \text{(by Equations (2.20) and (2.6))} \\
&= \delta(D^{A,E}) + \partial_A^{A^\vee}((I^A\otimes 1)(D^{A,E})) + [D^{A,E}, (I^A\otimes 1)(D^{A,E})] && \text{(by Equation (1.27))} \\
&= \delta(D^{A,E}) + (I^A\otimes 1)(Q_A(D^{A,E}) + \frac{1}{2}[D^{A,E}, D^{A,E}]) && \text{(by Equations (2.21) and (1.4))} \\
&= \delta(D^{A,E}) \in \mathcal{O}(A)\otimes B^\vee\otimes \mathrm{End}(E).
\end{aligned}
$$

By observing $Q_L^2 = 0$ on the $\mathcal{O}(A)$ to $\mathcal{O}(A)\otimes B^\vee$-part, we have

$$
\partial_A^\perp\circ\delta + \delta\circ Q_A = 0 : \mathcal{O}(A)\to\mathcal{O}(A)\otimes B^\vee.
$$

Thus, for all $\omega\otimes e\in\mathcal{O}(A)\otimes E$,

$$
\begin{aligned}
&\partial_A(\delta(\omega\otimes e)) + \delta(\partial_A(\omega\otimes e)) \\
&= \partial_A^\perp(\delta(\omega))\otimes e + (-1)^{|\omega|+1}\delta(\omega)\odot(D^{A,E}(e)) + \delta(Q_A(\omega))\otimes e + (-1)^{|\omega|}\delta(\omega\odot D^{A,E}(e)) \\
&= \omega\odot\delta(D^{A,E})(e) = \delta(D^{A,E})(\omega\otimes e) = \boldsymbol{\alpha}^E(\omega\otimes e).
\end{aligned}
$$

□

Applying Lemma 3.3, one can easily get the following properties of Atiyah operators:

**Lemma 3.4.** *For all $x\in\mathcal{O}(A)\otimes B$, the Atiyah operator $\boldsymbol{\alpha}^\bullet(x)$ has the following properties:*

*(1) For any $A$-modules $E$ and $F$,*

$$
\boldsymbol{\alpha}^{E\otimes F}(x)(r\otimes_{\mathcal{O}(A)} s) = (\boldsymbol{\alpha}^E(x)r)\otimes_{\mathcal{O}(A)} s + (-1)^{|x||r|} r\otimes_{\mathcal{O}(A)}(\boldsymbol{\alpha}^F(x)s),
$$

*for all $r\in\mathcal{O}(A)\otimes E$, $s\in\mathcal{O}(A)\otimes F$;*

*(2) For any $A$-module $E$ with dual module $E^\vee$, and for all $\varphi\in\mathcal{O}(A)\otimes E^\vee$, $r\in\mathcal{O}(A)\otimes E$,*

$$
\langle\boldsymbol{\alpha}^{E^\vee}(x)\varphi, r\rangle = -(-1)^{|x||\varphi|}\langle\varphi, \boldsymbol{\alpha}^E(x)r\rangle;
$$

*(3) For $A$-modules $E$ and $F$,*

$$
(\boldsymbol{\alpha}^{\mathrm{Hom}(E,F)}(x)\kappa)(r) = \boldsymbol{\alpha}^F(x)(\kappa(r)) - (-1)^{|x||\kappa|}\kappa(\boldsymbol{\alpha}^E(x)r),
$$

*for all $\kappa\in\mathcal{O}(A)\otimes\mathrm{Hom}(E,F)$, $r\in\mathcal{O}(A)\otimes E$.*

3.2. **Atiyah classes as Lie structures.** Let $(L,A)$ be an SH Lie pair, and suppose that $L = A\oplus B$ as graded vector spaces, and $E$ an $A$-module. Note that the Atiyah cocycle $\alpha^E$ is a degree 2 element in $\mathcal{O}(A)\otimes B^\vee\otimes\mathrm{End}(E)$. If we set $\mathbb{B} = B[-2]$ and $\mathbb{E} = E[-2]$, then the associated Atiyah operators are of degree 0:

$$
\begin{aligned}
\boldsymbol{\alpha}^B(-)- : &\quad (\mathcal{O}(A)\otimes\mathbb{B})\otimes_{\mathcal{O}(A)}(\mathcal{O}(A)\otimes\mathbb{B})\to\mathcal{O}(A)\otimes\mathbb{B}; \\
\boldsymbol{\alpha}^E(-)- : &\quad (\mathcal{O}(A)\otimes\mathbb{B})\otimes_{\mathcal{O}(A)}(\mathcal{O}(A)\otimes\mathbb{E})\to\mathcal{O}(A)\otimes\mathbb{E}.
\end{aligned}
$$

Here are the main results in this section:

**Theorem 3.5.** *Let $(L,A)$ be an SH Lie pair with the quotient space $L/A = B$. Then the graded vector space $\mathrm{H}^\bullet_{\mathrm{CE}}(A,\mathbb{B})$ with the binary operation induced by the Atiyah operator $\boldsymbol{\alpha}^B$ is a Lie algebra. Furthermore, if $E$ is an $A$-module, then $\mathrm{H}^\bullet_{\mathrm{CE}}(A,\mathbb{E})$ is a Lie algebra module over $\mathrm{H}^\bullet_{\mathrm{CE}}(A,\mathbb{B})$, with the action induced by the Atiyah operator $\boldsymbol{\alpha}^E$.*

*In particular, $\mathrm{H}^0_{\mathrm{CE}}(A,\mathbb{B})$ is an ordinary Lie algebra and $\mathrm{H}^0_{\mathrm{CE}}(A,\mathbb{E})$ is an ordinary Lie algebra module over $\mathrm{H}^0_{\mathrm{CE}}(A,\mathbb{B})$.*

*Remark* 3.6. By Theorem-Definition 2.11, the Lie algebra and Lie algebra module structures on the cohomology level are all canonical, i.e., they do not depend on the choice of the splitting $L = A \oplus B$.

To proceed the proof, we need some preparations. Let

$$\tau : B^\vee \otimes B^\vee \to B^\vee \otimes B^\vee, \qquad \xi \otimes \eta \mapsto (-1)^{|\xi||\eta|}\eta \otimes \xi.$$

**Lemma 3.7.** *The symmetrization of the Atiyah cocycle $\alpha^{B^\vee}$ vanishes up to homotopy, i.e.,*

$$(1 \otimes \tau)\alpha^{B^\vee} + \alpha^{B^\vee} = \partial_A P, \tag{3.8}$$

*for some $P \in \mathcal{O}(A) \otimes B^\vee \otimes B^\vee \otimes B$.*

*Proof.* First, observe the following commutative diagram

$$\begin{tikzcd} B^\vee \otimes B^\vee \arrow[r, "\frac{1}{2}(1+\tau)"] \arrow[dr, "s"'] & B^\vee \otimes B^\vee \arrow[d, shift left, "s"] \\ & S^2(B^\vee), \arrow[u, shift left, "s^{-1}"] \end{tikzcd}$$

where the operations $s$ and $s^{-1}$ are defined by

$$s:\ \xi \otimes \eta \mapsto \xi \odot \eta, \qquad s^{-1}: \xi \odot \eta \mapsto \frac{1}{2}(1+\tau)(\xi \otimes \eta), \quad \forall \xi, \eta \in B^\vee.$$

It is clear that $s^{-1}$ is right inverse of the symmetrization operator $s$, i.e., $s \circ s^{-1} = id:\ S^2(B^\vee) \to S^2(B^\vee)$. We introduce

$$\delta^0 : \mathcal{O}(A) \otimes B^\vee \to \mathcal{O}(A) \otimes S^2(B^\vee), \qquad \delta^0 = (1 \otimes s) \circ \delta.$$

Then we get the following commutative diagram

$$\begin{tikzcd} B^\vee \arrow[r, "\frac{1}{2}(\boldsymbol{\alpha}^{B^\vee} + (1\otimes\tau)\boldsymbol{\alpha}^{B^\vee})"] \arrow[dr, "\delta^0 \circ D^\perp"'] & \mathcal{O}(A) \otimes (B^\vee \otimes B^\vee) \arrow[d, shift left, "1\otimes s"] \\ & \mathcal{O}(A) \otimes S^2(B^\vee). \arrow[u, shift left, "1\otimes s^{-1}"] \end{tikzcd}$$

Namely,

$$(1 \otimes s) \circ \frac{1}{2}(\boldsymbol{\alpha}^{B^\vee} + (1 \otimes \tau)\boldsymbol{\alpha}^{B^\vee}) = \delta^0 \circ D^\perp.$$

In fact, it amounts to check

$$\boldsymbol{\alpha}^{B^\vee} = \delta \circ D^\perp, \text{ as a map } B^\vee \to \mathcal{O}(A) \otimes B^\vee \otimes B^\vee,$$

which follows from Lemma 3.3 and $\partial_A^{B^\vee}|_{B^\vee} = D^\perp$. Hence

$$(1 \otimes \tau)\alpha^{B^\vee} + \alpha^{B^\vee} = 2(1 \otimes s^{-1} \otimes 1)(\delta^0 \circ D^\perp).$$

To prove Equation (3.8), it suffices to show that

$$\delta^0 \circ D^\perp \in \mathrm{Hom}(B^\vee, \mathcal{O}(A) \otimes S^2(B^\vee)) \cong \mathcal{O}(A) \otimes S^2(B^\vee) \otimes B$$

is a coboundary.

Restricting the condition $Q_L^2 = 0$ on the $B^\vee$ to $\mathcal{O}(A) \otimes S^2(B^\vee)$-part, we get

$$\delta^0 \circ D^\perp + Q_A \circ T_2 + D^\perp \circ T_2 + T_2 \circ D^\perp = 0,$$

where $T_2 \in \mathcal{O}(A) \otimes S^2(B^\vee) \otimes B$ is defined in Equation (3.2). Hence, we have

$$\delta^0 \circ D^\perp = -(Q_A + D^\perp) \circ T_2 - T_2 \circ D^\perp = -[Q_A + D^\perp, T_2] = -\partial_A T_2,$$

as desired. $\square$

**Lemma 3.9.** *For all $x, y \in \mathcal{O}(A) \otimes B, r \in \mathcal{O}(A) \otimes E$, we have*

$$\boldsymbol{\alpha}^E(x)(\boldsymbol{\alpha}^E(y)r) - (-1)^{|x||y|}\boldsymbol{\alpha}^E(y)(\boldsymbol{\alpha}^E(x)r) = \boldsymbol{\alpha}^E(\boldsymbol{\alpha}^B(x)y)r + (\partial_A T)\llcorner(x \otimes_{\mathcal{O}(A)} y \otimes_{\mathcal{O}(A)} r),$$

*where $T = \delta(\alpha^E) \in \mathcal{O}(A) \otimes B^\vee \otimes B^\vee \otimes \mathrm{End}(E)$.*

*Proof.* Since $\partial_A(\alpha^E) = 0$, it follows from Lemma 3.3 that

$$\partial_A T = \partial_A(\delta(\alpha^E)) = (\partial_A \circ \delta + \delta \circ \partial_A)(\alpha^E) = \boldsymbol{\alpha}^{B^\vee \otimes \mathrm{End}(E)}(\alpha^E) = \boldsymbol{\alpha}^{\mathrm{Hom}(B \otimes E, E)}(\alpha^E).$$

Applying Lemma 3.4, we have

$$\begin{aligned}
&\boldsymbol{\alpha}^{\mathrm{Hom}(B\otimes E,E)}(\alpha^E)\llcorner(x \otimes_{\mathcal{O}(A)} y \otimes_{\mathcal{O}(A)} r) = \big(\boldsymbol{\alpha}^{\mathrm{Hom}(B\otimes E,E)}(x)(\alpha^E)\big)\llcorner(y \otimes_{\mathcal{O}(A)} r) \\
&= \boldsymbol{\alpha}^E(x)(\boldsymbol{\alpha}^E(y \otimes_{\mathcal{O}(A)} r)) - (-1)^{2|x|}\boldsymbol{\alpha}^E(\boldsymbol{\alpha}^{B\otimes E}(x)(y \otimes_{\mathcal{O}(A)} r)) \\
&= \boldsymbol{\alpha}^E(x)(\boldsymbol{\alpha}^E(y)r) - \boldsymbol{\alpha}^E((\boldsymbol{\alpha}^B(x)y) \otimes_{\mathcal{O}(A)} r + (-1)^{|x||y|}y \otimes_{\mathcal{O}(A)} (\boldsymbol{\alpha}^E(x)r)) \\
&= \boldsymbol{\alpha}^E(x)(\boldsymbol{\alpha}^E(y)r) - \boldsymbol{\alpha}^E(\boldsymbol{\alpha}^B(x)y)r - (-1)^{|x||y|}\boldsymbol{\alpha}^E(y)(\boldsymbol{\alpha}^E(x)r).
\end{aligned}$$

□

We are now ready to turn to

*Proof of Theorem 3.5.* Lemma 3.7 implies that the bracket $[x, y] = \boldsymbol{\alpha}^B(x)y$ is skew-symmetric on the cohomology level. When $E$ is taken as $B$ in Lemma 3.9, we see that the $[-,-]$-bracket satisfies Jacobi identity on the cohomology level. Again by Lemma 3.9, the operation $x \triangleright r = \boldsymbol{\alpha}^E(x)r$ defines a Lie algebra action of $\mathrm{H}^\bullet_{\mathrm{CE}}(A, \mathbb{B})$ on $\mathrm{H}^\bullet_{\mathrm{CE}}(A, \mathbb{E})$. □

We remark that similar results appeared in [12] for that of Lie pairs, and in [7] of relative Lie algebroids. In the meantime, the work related to some facts in the derived categories claimed in [11] is still going on.

**Example 3.10.** Take $L = A \oplus B$, where $A$ is spanned by three vectors $a_0, a_1$ and $c$, $B$ by one vector $b$. The degrees are assigned:

$$|a_0| = |a_1| = |b| = -1, \qquad |c| = 0.$$

Let the dual vectors be $a_i^\vee$, $c^\vee$ and $b^\vee$, with degrees

$$|a_0^\vee| = |a_1^\vee| = |b^\vee| = 1, \qquad |c^\vee| = 0.$$

The $Q$-structure on $L$ is the sum of three parts

$$Q_L = Q_A + D^\perp + \delta.$$

Here $Q_A$ is determined by

$$Q_A(a_0^\vee) = Q_A(c^\vee) = 0, \qquad Q_A(a_1^\vee) = -a_0^\vee \odot a_1^\vee.$$

The $A$-module structure on $B^\vee$ is given by

$$D^\perp(b^\vee) = a_0^\vee \otimes b^\vee.$$

The $\delta$-operator is given by

$$\delta(\xi) = (-1)^{|\xi|}\Delta(\xi) \otimes b^\vee, \qquad \forall \xi \in \mathcal{O}(A),$$

where $\Delta$ is a degree 0 derivation on $\mathcal{O}(A)$ determined by

$$\Delta(a_0^\vee) = a_1^\vee, \qquad \Delta(a_1^\vee) = a_0^\vee \odot c^\vee, \qquad \Delta(c^\vee) = 0.$$

It follow from some direct computations that $(L, A)$ is an SH Lie pair. The Atiyah operator $\boldsymbol{\alpha}^{B^\vee}$ now reads

$$\boldsymbol{\alpha}^{B^\vee}(b^\vee) = \delta \circ D^\perp(b^\vee) = \delta(a_0^\vee \otimes b^\vee) = -\Delta(a_0^\vee) \otimes b^\vee \otimes b^\vee = -a_1^\vee \otimes b^\vee \otimes b^\vee.$$

Or, the Atiyah cocycle is spelled as

$$\alpha^B = -a_1^\vee \otimes b^\vee \otimes b^\vee \otimes b.$$

The Atiyah class $[\alpha^B] \neq 0$. In fact, any attempt to make $[\alpha^B] = 0$ yields the equation

$$Q_A(\xi) + \xi \odot a_0^\vee = -a_1^\vee,$$

for $\xi \in \mathcal{O}(A)$ with $|\xi| = 0$. It obviously has no solution.

The space $\mathcal{O}(A) \otimes \mathbb{B}$ is generated by one element $b[-2]$, and the Lie bracket on $\mathrm{H}^\bullet(\mathcal{O}(A) \otimes \mathbb{B})$ can be explicitly expressed:

$$\big[b[-2], b[-2]\big] = \boldsymbol{\alpha}^B(b[-2])b[-2] = a_1^\vee \otimes b[-2].$$

3.3. **Atiyah functors.** Let $A$ be an $L_\infty[1]$-algebra. Then taking the Chevalley-Eilenberg cohomology $\mathrm{H}^\bullet_{\mathrm{CE}}(A, -)$ defines a functor

$$\mathrm{H}^\bullet_{\mathrm{CE}}(A; -): \ \mathrm{Mod}_A \to \mathbf{GVS}_{\mathbb{K}}, \ E \mapsto \mathrm{H}^\bullet_{\mathrm{CE}}(A, E) = \mathrm{H}^\bullet(\mathcal{O}(A) \otimes E, \partial_A^E).$$

For a morphism $\phi \in \mathrm{Hom}_A(E, F)$, the functor sends $\phi$ to $[\phi] : \mathrm{H}^\bullet_{\mathrm{CE}}(A, E) \to \mathrm{H}^\bullet_{\mathrm{CE}}(A, F)$.

Recall that given an SH Lie pair $(L, A)$ with quotient space $B$, we get a Lie algebra object

$$\mathfrak{B} = \mathrm{H}^\bullet_{\mathrm{CE}}(A, \mathbb{B})$$

whose Lie bracket is induced by the Atiyah operator $\boldsymbol{\alpha}^B$ (Theorem 3.5).

Let $\mathrm{Mod}_{\mathfrak{B}}$ denote the category of $\mathfrak{B}$-modules. According to Theorem 3.5 again, we are able to introduce

**Definition 3.11.** The Atiyah functor is the composition of $\mathrm{H}^\bullet_{\mathrm{CE}}(A, -)$ with a degree $(-2)$ shifting:

$$\boldsymbol{A} : (E, \partial_A^E) \to \big(\mathrm{H}^\bullet_{\mathrm{CE}}(A, \mathbb{E}), \boldsymbol{\alpha}^E\big)$$

from the category $\mathrm{Mod}_A$ of $A$-modules to the category $\mathrm{Mod}_{\mathfrak{B}}$ of $\mathfrak{B}$-modules, where $\mathbb{E} = E[-2]$. And for all $\phi \in \mathrm{Hom}_A(E, F)$, we have

$$\boldsymbol{A}(\phi) = [\phi] : \mathrm{H}^\bullet_{\mathrm{CE}}(A, \mathbb{E}) \to \mathrm{H}^\bullet_{\mathrm{CE}}(A, \mathbb{F}).$$

That $\boldsymbol{A}$ is well-defined relies on the following fact: given any $\phi \in \mathrm{Hom}_A(E, F)$, the associated $[\phi]$ preserves the $\mathfrak{B}$-actions, i.e., for all $x \in \mathcal{O}(A) \otimes B$, $r \in \mathcal{O}(A) \otimes E$,

$$\boldsymbol{\alpha}^F(x)(\phi(r)) = \phi(\boldsymbol{\alpha}^E(x)r) + (\partial_A W)(x \otimes_{\mathcal{O}(A)} r),$$

for some $W \in \mathcal{O}(A) \otimes \mathrm{Hom}(B \otimes E, F)$. In fact, we have $W = \delta(\phi)$. The proof of this fact is similar to that of Lemma 3.9, and thus omitted.

*Remark* 3.12. Inspired by Lemma 3.4, we may expect the Atiyah functor to enjoy the following natural properties:

$$\boldsymbol{A}(E \otimes F) \cong \boldsymbol{A}(E) \otimes_{\mathcal{H}} \boldsymbol{A}(F),$$
$$\boldsymbol{A}(E^\vee) \cong \mathrm{Hom}_{\mathcal{H}}(\boldsymbol{A}(E), \mathcal{H}),$$

and

$$\boldsymbol{A}(\mathrm{Hom}(E, F)) \cong \mathrm{Hom}_{\mathcal{H}}(\boldsymbol{A}(E), \boldsymbol{A}(F)).$$

However, some condition is needed to fulfill these isomorphisms. Further investigations of this question will be dealt with somewhere else.

## 4. Invariance of Atiyah classes under infinitesimal deformations

In this section, let us fix an SH Lie pair $(L, A; Q_L \sim \lambda_\bullet)$ with the quotient space $B = L/A$, and an $A$-module $E$. We study infinitesimal deformations of the $L_\infty[1]$-structure $Q_L$ on $L$, and how the associated Atiyah classes $[\alpha^E]$ would be affected.

4.1. **Compatible infinitesimal deformations.** In what follows, $\hbar$ denotes a square zero formal parameter. An infinitesimal deformation, or a first order deformation, of the $L_\infty[1]$-algebra structure on $L$, namely that of $Q_L$, is a differential of the form

$$Q(\hbar) = Q_L + \hbar Q_+ : \mathcal{O}(L)[\hbar] \to \mathcal{O}(L)[\hbar].$$

Here $Q_+$ is a degree 1 derivation of $\mathcal{O}(L)$, and both $Q_L$ and $Q_+$ are $\mathbb{K}[\hbar]$-linear. It follows that

$$[Q_L, Q_+] = Q_L \circ Q_+ + Q_+ \circ Q_L = 0.$$

In this circumstance, $L[\hbar]$ has an $L_\infty[1]$-structure $Q(\hbar)$ which is deformed from $Q_L$.

As our motivation is to regard $L$ as a larger object extended from $A$, we only consider deformations of the following type:

**Definition 4.1.** An infinitesimal deformation $Q(\hbar)$ of $Q_L$ is said to be $A$-compatible, if it is subject to the following two conditions:

(1) The $L_\infty[1]$-structure on $A$ is not deformed, i.e., the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}(L)[\hbar] \arrow[r, "j^\vee"] \arrow[d, "Q(\hbar)"] & \mathcal{O}(A)[\hbar] \arrow[d, "Q_A"] \\
\mathcal{O}(L)[\hbar] \arrow[r, "j^\vee"] & \mathcal{O}(A)[\hbar].
\end{tikzcd}$$

(2) The $A$-module structure on $B$ is not deformed. This means the commutativity of

$$\begin{tikzcd}
A^\perp[\hbar] = B^\vee[\hbar] \arrow[rr, "D^\perp"] \arrow[d, hook] & & \mathcal{O}(A) \otimes B^\vee[\hbar] \arrow[d, hook] \\
L^\vee[\hbar] \arrow[r, "Q(\hbar)"] & \mathcal{O}(L)[\hbar] \arrow[r, "J"] & \mathcal{O}(A) \otimes L^\vee[\hbar].
\end{tikzcd}$$

By choosing a splitting of Sequence (2.3), so that $L \cong A \oplus B$ and that $\mathcal{O}(L)$ is identified with $\mathcal{O}(A) \otimes \mathcal{O}(B)$, the two compatible conditions are unraveled: if $Q(\hbar) = Q_L + \hbar Q_+$, then the above

$$\begin{aligned}
&\text{Condition (1)} \ \Leftrightarrow \ Q_+(A^\vee) \subset \mathcal{O}(A) \otimes \mathcal{O}^+(B), \\
&\text{Condition (2)} \ \Leftrightarrow \ Q_+(B^\vee) \subset \mathcal{O}(A) \otimes \widehat{S}^{\geq 2}(B^\vee).
\end{aligned}$$

Similar to the decomposition of $Q_L$ in Equation (3.2), we denote the part of $Q_+$ that sends $A^\vee$ into $\mathcal{O}(A) \otimes B^\vee$ by $\delta_+$, the part that sends $A^\vee$ into $\mathcal{O}(A) \otimes \widehat{S}^{\geq 2}(B^\vee)$ by $\mathcal{R}_+$, and the part that sends $B^\vee$ into $\mathcal{O}(A) \otimes S^i(B^\vee)$ by $T_+^i, i \geq 2$. Then the $A$-compatible infinitesimal deformation $Q(\hbar)$ has the form

$$Q(\hbar) = Q_L + \hbar\delta_+ + \hbar\mathcal{R}_+ + \hbar \sum_{i \geq 2} T_+^i. \tag{4.2}$$

**Definition 4.3.** (1) A gauge equivalence of $\mathcal{O}(L)[\hbar]$ is an automorphism $\sigma = 1 + \hbar\lambda$ of the graded commutative algebra $\mathcal{O}(L)[\hbar]$, where $\lambda : \mathcal{O}(L) \to \mathcal{O}(L)$ is $\mathbb{K}$-linear, such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}(B)[\hbar] \arrow[r] \arrow[d, "1"] & \mathcal{O}(L)[\hbar] \arrow[r, "j^\vee"] \arrow[d, "\sigma = 1 + \hbar\lambda"] & \mathcal{O}(A)[\hbar] \arrow[d, "1"] \\
\mathcal{O}(B)[\hbar] \arrow[r] & \mathcal{O}(L)[\hbar] \arrow[r, "j^\vee"] & \mathcal{O}(A)[\hbar].
\end{tikzcd} \tag{4.4}$$

(2) Two $A$-compatible infinitesimal deformations $Q(\hbar)$ and $\bar{Q}(\hbar)$ of $Q_L$ are said to be gauge equivalent if there exists a gauge equivalence $\sigma = 1 + \hbar\lambda$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}(L)[\hbar] \arrow[r, "\sigma=1+\hbar\lambda"] \arrow[d, "Q(\hbar)"] & \mathcal{O}(L)[\hbar] \arrow[d, "\bar{Q}(\hbar)"] \\ \mathcal{O}(L)[\hbar] \arrow[r, "\sigma=1+\hbar\lambda"] & \mathcal{O}(L)[\hbar], \end{tikzcd}$$

i.e., $\sigma$ is an isomorphism of $L_\infty[1]$-algebras $(L[\hbar], Q[\hbar]) \cong (L[\hbar], \bar{Q}[\hbar])$.

Assume that $Q(\hbar)$ and $\bar{Q}(\hbar)$ are connected by the gauge equivalence $\sigma = 1 + \hbar\lambda$. Since $\sigma$ is an algebra automorphism, it follows that $\lambda$ is a degree 0 derivation of $\mathcal{O}(L)$. Note that $\sigma^{-1} = 1 - \hbar\lambda$. It follows from a simple computation that

$$Q(\hbar) - \bar{Q}(\hbar) = \hbar[Q_L, \lambda].$$

Recall that $\mathcal{O}(L) = \mathcal{O}(A) \otimes \mathcal{O}(B)$. The commutative property of Diagram (4.4) implies that we can write

$$\lambda = \sum_{k\geq 1} \Psi_k, \quad \text{where} \quad \Psi_k : \ A^\vee \to \mathcal{O}(A) \otimes S^k(B^\vee). \tag{4.5}$$

All these $\Psi_k$ are treated as degree 0 derivations of $\mathcal{O}(L)$ which act trivially on $B^\vee$.

4.2. **Gauge invariance of Atiyah classes.** Let $Q(\hbar) = Q_L + \hbar Q_+$ be an $A$-compatible infinitesimal deformation of $Q_L$. Consider the associated Atiyah cocycle $\alpha^{E[\hbar]}$ of the SH Lie pair $(L[\hbar], A[\hbar])$ with respect to the $A[\hbar]$-module $E[\hbar]$. By Lemma 3.3,

$$\alpha^{E[\hbar]} = [\partial_A, \delta + \hbar\delta_+] = \alpha^E + \hbar[\partial_A, \delta_+],$$

where $\delta$ and $\delta_+$ are the components of $Q_L$ and $Q_+$ specified respectively in Equations (3.2) and (4.2).

The main result in this section is the gauge invariance of the Atiyah class $[\alpha^{E[\hbar]}]$.

**Theorem 4.6.** *Let $Q(\hbar) = Q_L + \hbar Q_+$ and $\bar{Q}(\hbar) = Q_L + \hbar\bar{Q}_+$ be two gauge equivalent $A$-compatible infinitesimal deformations of $Q_L$. Then the associated Atiyah classes coincide:*

$$[\alpha^{E[\hbar]}] = [\overline{\alpha^{E[\hbar]}}] \in H^2(A[\hbar], (B[\hbar])^\vee \otimes \mathrm{End}(E[\hbar])) \cong H^2(A, B^\vee \otimes \mathrm{End}(E))[\hbar].$$

*Proof.* Let the gauge equivalence $\sigma$ be as in Definition 4.3. The $\lambda$ operator is defined in Equation (4.5). Further assume that

$$\begin{aligned} Q(\hbar) &= Q_A + \delta + D^\perp + \mathcal{R} + \sum_{j\geq 2} T^j + \hbar\delta_+ + \hbar\mathcal{R}_+ + \hbar\sum_{i\geq 2} T^i_+, \\ \bar{Q}(\hbar) &= Q_A + \delta + D^\perp + \mathcal{R} + \sum_{j\geq 2} T^j + \hbar\bar{\delta}_+ + \hbar\bar{\mathcal{R}}_+ + \hbar\sum_{i\geq 2} \bar{T}^i_+, \end{aligned}$$

are explained as earlier. Applying the equation

$$\sigma \circ Q(\hbar) = \bar{Q}(\hbar) \circ \sigma : \ \mathcal{O}(L)[\hbar] \to \mathcal{O}(L)[\hbar]$$

to an element $\xi \in A^\vee$, we have

$$\begin{aligned} \sigma(Q(\hbar)(\xi)) &= \left(1 + \hbar\sum_k \Psi_k\right)(Q_A(\xi) + \delta(\xi) + \mathcal{R}(\xi) + \hbar\delta_+(\xi) + \hbar\mathcal{R}_+(\xi)) \\ &= Q_A(\xi) + \delta(\xi) + \mathcal{R}(\xi) + \hbar\delta_+(\xi) + \hbar\mathcal{R}_+(\xi) + \hbar\sum_k \Psi_k(Q_A(\xi) + \delta(\xi) + \mathcal{R}(\xi)), \end{aligned}$$

and

$$\bar{Q}(\hbar)(\sigma(\xi)) = \left(Q_A + \delta + D^\perp + \mathcal{R} + \sum_{j\geq 2} T^j + \hbar\bar{\delta}_+ + \hbar\bar{\mathcal{R}}_+ + \hbar\sum_{i\geq 2}\bar{T}^i_+\right)\left(\xi + \hbar\sum_k \Psi_k(\xi)\right)$$

$$= Q_A(\xi) + \delta(\xi) + \mathcal{R}(\xi) + \hbar\bar{\delta}_+(\xi) + \hbar\bar{\mathcal{R}}_+(\xi) + \hbar\sum_k\left(Q_A + \delta + D^\perp + \mathcal{R} + \sum_{j\geq 2} T^j\right)(\Psi_k(\xi)).$$

Comparing the $\hbar\mathcal{O}(A)\otimes B^\vee$-component of both sides, one gets

$$\Psi_1(Q_A(\xi)) + \delta_+(\xi) = \bar{\delta}_+(\xi) + Q_A(\Psi_1(\xi)) + D^\perp(\Psi_1(\xi)),$$

which implies that

$$\delta_+ - \bar{\delta}_+ = Q_A\circ\Psi_1 - \Psi_1\circ Q_A + D^\perp\circ\Psi_1 :\ A^\vee \to \mathcal{O}(A)\otimes B^\vee.$$

Hence, we have, for all $e\in E$,

$$\begin{aligned}
&\alpha^{E[\hbar]}(e) - \overline{\alpha^{E[\hbar]}}(e) = \hbar[\partial_A^E, \delta_+ - \bar{\delta}_+](e) = \hbar(\delta_+ - \bar{\delta}_+)(\partial_A^E(e))\\
&= \hbar(Q_A\circ\Psi_1 - \Psi_1\circ Q_A + D^\perp\circ\Psi_1)\left(D^E(e)\right)\\
&= \hbar(Q_A\circ\Psi_1\circ D^E + \Psi_1\circ D^E\circ D^E + D^\perp\circ\Psi_1\circ D^E)(e). \qquad \text{(by Equation (1.27))}
\end{aligned}$$

Here $D^E : E\to\mathcal{O}(A)\otimes E$ defines the $A$-module structure on $E$.

Now let

$$W = [\Psi_1, D^E] :\ \mathcal{O}(A)\otimes E \to \mathcal{O}(A)\otimes B^\vee\otimes E$$

be the graded commutator of $\Psi_1$ and $D^E$. One easily finds it is an $\mathcal{O}(A)$-linear map.

Then we have

$$\begin{aligned}
\alpha^{E[\hbar]}(e) - \overline{\alpha^{E[\hbar]}}(e) &= \hbar((Q_A + D^\perp + D^E)((\Psi_1\circ D^E)(e)) + (W\circ D^E)(e))\\
&= \hbar(\partial_A\circ W + W\circ\partial_A)(e) = \hbar\partial_A W(e),
\end{aligned}$$

which implies that

$$\alpha^{E[\hbar]} - \overline{\alpha^{E[\hbar]}} = \hbar\partial_A W.$$

This completes the proof. □

## 5. Appendix: Morphisms of SH Lie algebras

We prove the equivalence of the two definitions of morphisms of SH Lie algebras as in Definition 1.14.

Let $\phi : \mathcal{O}(L')\to\mathcal{O}(L)$ be a morphism of $\mathbb{K}$-algebras such that

$$\phi\circ Q_{L'} = Q_L\circ\phi : \mathcal{O}(L')\to\mathcal{O}(L).$$

Assume that $\phi = \sum_{k\geq 0}\phi_k$, where $\phi_k : (L')^\vee \to S^k(L^\vee)$. Define a family of degree zero linear maps

$$f_k = (-1)^{k+1}\phi_k^\vee :\ S^k(L)\to L',\ \ k\geq 0.$$

We show that $\{f_k\}$ satisfies the two requirements as in Definition 1.14.

By assumption, we have

$$\phi\circ Q_{L'}(\xi) = Q_L\circ\phi(\xi)\in\mathcal{O}(L) \tag{5.1}$$

for all homogeneous $\xi\in(L')^\vee$. Note that

$$\text{LHS of Equation (5.1)} = \phi\left(\langle\lambda_0', \xi\rangle + Q_1'(\xi) + \cdots\right) = \langle\lambda_0',\xi\rangle + \sum_{n\geq 1}\langle(\phi_0)^{\odot n}, Q_n'(\xi)\rangle + \cdots;$$

and

$$\text{RHS of Equation (5.1)} = Q_L\left(\langle\phi_0,\xi\rangle + \phi_1(\xi) + \cdots\right) = Q_0(\phi_1(\xi)) + \cdots = \langle\lambda_0, \phi_1(\xi)\rangle + \cdots.$$

Comparing the $\mathbb{K} = S^0(L^\vee)$-component of both sides, one gets

$$\langle\lambda_0', \xi\rangle + \sum_{n\geq 1}\langle\phi_0^{\odot n}, Q_n'(\xi)\rangle = \langle\lambda_0, \phi_1(\xi)\rangle,$$

which is equivalent to

$$\langle\lambda_0' + \sum_{n\geq 1}\frac{1}{n!}(-1)^n\lambda_n'(\phi_0,\cdots,\phi_0), \xi\rangle = \langle\lambda_0, \phi_1(\xi)\rangle,$$

or Equation (1.16).

We further investigate the $S^n(L^\vee)(n \geq 1)$-component of Equation (5.1). For all $u_i \in L, i = 1, \cdots, n$, we have

$$\begin{aligned}
&\text{LHS} = \langle\phi\circ Q_{L'}(\xi), u_1\odot\cdots\odot u_n\rangle \\
&= \langle Q_{L'}(\xi), \sum_{\substack{i_1,\cdots,i_r\geq 1\\ i_1+\cdots+i_r=n}}\ \sum_{\tau\in\mathrm{sh}(i_1,\cdots,i_r)}\ \sum_{j\geq 0}\epsilon(\tau)(-1)^{n+r+j}\frac{1}{r!}f_0^{\odot j}\odot f_{i_1}\odot\cdots\odot f_{i_r}(u_{\tau(1)},\cdots,u_{\tau(n)})\rangle \\
&= \langle\xi, \sum_{\substack{i_1,\cdots,i_r\geq 1\\ i_1+\cdots+i_r=n}}\ \sum_{\tau\in\mathrm{sh}(i_1,\cdots,i_r)}\ \sum_{j\geq 0}\epsilon(\tau)(-1)^{|\xi|+n}\frac{1}{(r+j)!}\lambda_{r+j}'(f_0,\cdots,f_0,f_{i_1}(\cdots),\cdots,f_{i_r}(\cdots))\rangle
\end{aligned}$$

and

$$\begin{aligned}
\text{RHS} &= \langle Q_L\circ\phi(\xi), u_1\odot\cdots\odot u_n\rangle \\
&= \langle\phi(\xi), \sum_{\substack{k,l\geq 0\\ k+l=n}}\ \sum_{\sigma\in\mathrm{sh}(l,k)}\epsilon(\sigma)(-1)^{|\xi|+l}\lambda_l(u_{\sigma(1)},\cdots,u_{\sigma(l)})\odot\cdots\odot u_{\sigma(n)}\rangle \\
&= \langle\xi, \sum_{\substack{k,l\geq 0\\ k+l=n}}\ \sum_{\sigma\in\mathrm{sh}(l,k)}\epsilon(\sigma)(-1)^{|\xi|+n}f_{k+1}(\lambda_l(u_{\sigma(1)},\cdots,u_{\sigma(l)}),\cdots,u_{\sigma(n)})\rangle.
\end{aligned}$$

Thus Equation (1.17) also holds once we assume Equation (1.15).

The inverse implication "Equations (1.16)+(1.17) $\Longrightarrow$ Equation (5.1)" is also clear from the previous argument. This completes the proof.

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