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Geometry of Maurer-Cartan Elements on Complex Manifolds*,**

Zhuo Chen[1], **Mathieu Stiénon**[2], **Ping Xu**[3]

[1] Department of Mathematics, Tsinghua University, Beijing 100084, People's Republic of China.
E-mail: zchen@math.tsinghua.edu.cn
[2] Institut de Mathématiques de Jussieu, Université Paris Diderot, Batiment Chevaleret, case 7012, 75205 Paris cedex 13, France. E-mail: stienon@math.jussieu.fr
[3] Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA.
E-mail: ping@math.psu.edu

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Abstract: The semi-classical data attached to stacks of algebroids in the sense of Kashiwara and Kontsevich are Maurer-Cartan elements on complex manifolds, which we call extended Poisson structures as they generalize holomorphic Poisson structures. A canonical Lie algebroid is associated to each Maurer-Cartan element. We study the geometry underlying these Maurer-Cartan elements in the light of Lie algebroid theory. In particular, we extend Lichnerowicz-Poisson cohomology and Koszul-Brylinski homology to the realm of extended Poisson manifolds; we establish a sufficient criterion for these to be finite dimensional; we describe how homology and cohomology are related through the Evens-Lu-Weinstein duality module; and we describe a duality on Koszul-Brylinski homology, which generalizes the Serre duality of Dolbeault cohomology.

Contents

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1. Introduction

Due to their close connection to mirror symmetry, noncommutative deformations of complex manifolds have recently generated increasing interest [5,20]. The Kashiwara-Kontsevich's stacks of algebroids are one way of substantiating the abstract concept of quantum complex manifolds (or noncommutative deformations of complex manifolds) [6,7,16–18,20,35]. The quantization of the sheaf of holomorphic functions $\mathscr{O}_X$ of a complex manifold X may no longer produce a sheaf of algebras but, instead, lead to a nonabelian gerbe over the complex manifold X [6,34] or, in Kontsevich's terminology, a stack of algebroids. Roughly speaking, an algebroid *à la* Kontsevich consists of an open cover $\{U_i\}_{i\in I}$ of the complex manifold X, a sheaf of associative unital algebras $\mathscr{A}_i$ on each U_i, an isomorphism of algebras $g_{ij}:\mathscr{A}_j|_{U_{ij}}\to\mathscr{A}_i|_{U_{ij}}$ for each nonempty intersection U_{ij}, and an invertible element $a_{ijk}\in\Gamma(U_{ijk},\mathscr{A}_i^{\times})$ for each triple intersection U_{ijk}. The isomorphisms g_{ij} do not satisfy the usual cocycle condition. Instead, the equations $g_{ij}\circ g_{jk}\circ g_{ki}=\operatorname{Ad}_{a_{ijk}^{-1}}$ are satisfied as well as other compatibility conditions (among which a "tetrahedron equation"). In the terminology of [25], an algebroid *à la* Kontsevich would be described as an extension of a Čech groupoid by algebras. A stack of algebroids can be thought of as a Morita equivalence class (see [25]) of algebroids. A canonical abelian category of coherent sheaves can be defined on a quantum complex manifold using its stack of algebroids description [16–18,20].

It is well known that the semi-classical data attached to quantum real manifolds (i.e. star-algebras) are Poisson structures [1,2]. The cotangent bundle of a real Poisson manifold (M,π) is endowed with a canonical Lie algebroid structure denoted by $(T^*M)_\pi$. This Lie algebroid structure plays a central role in Poisson geometry. For instance, the Lichnerowicz-Poisson cohomology is simply the Lie algebroid cohomology of $(T^*M)_\pi$ with trivial coefficients. Evens-Lu-Weinstein discovered a procedure for constructing a canonical module over a given Lie algebroid. With the canonical module of $(T^*M)_\pi$ at hand, they interpreted Koszul-Brylinski homology as a Lie algebroid cohomology. According to Kontsevich's formality theorem and Tsygan's chain formality theorem, the Hochschild cohomology and Hochschild homology of a star algebra are isomorphic to the Lichnerowicz-Poisson cohomology and Koszul-Brylinski homology of the underlying Poisson manifold.

In the context of complex geometry, the semiclassical data associated to quantum complex manifolds are solutions of the Maurer-Cartan equation in the derived global sections $R\Gamma(X,\wedge^\bullet TX[1])$ of the sheaf of graded Lie algebras $\wedge^\bullet TX[1]$ of polyvector fields on X, which, according to Kontsevich's formality theorem, classify the deformations of stacks of algebroids up to gauge transformations [6,20,35]. More precisely, a Maurer-Cartan element is an

$$H=\pi+\theta+\omega\in\Omega^{0,0}(\wedge^2T^{1,0}X)\oplus\Omega^{0,1}(\wedge^1T^{1,0}X)\oplus\Omega^{0,2}(\wedge^0T^{1,0}X)$$

(where $\Omega^{0,p}(\wedge^q T^{1,0}X)$ denotes the space of $\wedge^q T^{1,0}X$-valued $(0, p)$-forms on X) satisfying the following equations:

$$\bar\partial\omega + [\omega, \theta] = 0, \quad \bar\partial\pi + [\theta, \pi] = 0,$$
$$\bar\partial\theta + [\omega, \pi] + \frac{1}{2}[\theta, \theta] = 0, \quad [\pi, \pi] = 0.$$

Holomorphic Poisson bivector fields are special cases of such Maurer-Cartan elements, as are holomorphic (0, 2)-forms. For this reason, complex manifolds endowed with such a Maurer-Cartan element H will be called extended Poisson manifolds. In a recent paper [30], one of the authors studied the Koszul-Brylinski homology of holomorphic Poisson manifolds, and established a duality on it using the general theory developed by Evens-Lu-Weinstein [12].

In this paper, in order to study the geometry of extended Poisson manifolds, we apply the Evens-Lu-Weinstein theory to complex Lie algebroids. Indeed, considering Maurer-Cartan elements as Hamiltonian operators (in the sense of [26]) deforming a Lie bialgebroid [27], we define a complex Lie algebroid, which mimics the role played by the cotangent Lie algebroid in real Poisson geometry. It is not surprising that, for a holomorphic Poisson structure, this complex Lie algebroid is the derived Lie algebroid of the holomorphic cotangent Lie algebroid $(T^*X)_\pi$, i.e. the matched pair $T^{0,1}X \bowtie (T^*X)_\pi^{(1,0)}$ studied in [24,30]. Using this complex Lie algebroid, we introduce a Lichnerowicz-Poisson cohomology and a Koszul-Brylinski homology for extended Poisson manifolds, and study the relation between them. We extend the notion of coisotropic submanifolds of holomorphic Poisson manifolds to the "extended" setting. We give a criterion on the ellipticity of the complex Lie algebroid (in the sense of Block [4]) induced by a Maurer-Cartan element. And in the elliptic case, we obtain a duality, which we call Evens-Lu-Weinstein duality, on the Koszul-Brylinski homology groups. As was pointed out in [30] for the holomorphic Poisson case, this duality generalizes the Serre duality on Dolbeault cohomology.

Note that, modulo gauge equivalences, our extended Poisson structures and Yekutieli's Poisson deformations (see [35]) are equivalent. It would be interesting to explore the connection between our results on Poisson homology and Berest-Etingof-Ginzburg's [3]. It would also be interesting to investigate if one can extend the method in this paper to study the Bruhat-Poisson structures of Evens-Lu on flag varieties [11] and the toric Poisson structures of Caine [8].

2. Preliminaries

2.1. Lie bialgebroids. A complex Lie algebroid [32] consists of a complex vector bundle $A \to M$, a bundle map $a : A \to T_{\mathbb{C}}M$ called anchor, and a Lie algebra bracket $[\cdot, \cdot]$ on the space of sections $\Gamma(A)$ such that a induces a Lie algebra homomorphism from $\Gamma(A)$ to $\mathfrak{X}_{\mathbb{C}}(M)$ and the Leibniz rule

$$[u, fv] = (a(u)f)\, v + f[u, v]$$

is satisfied for all $f \in C^\infty(M, \mathbb{C})$ and $u, v \in \Gamma(A)$.

It is well-known that a Lie algebroid $(A, [\cdot, \cdot], a)$ is equivalent to a Gerstenhaber algebra $(\Gamma(\wedge^\bullet A), \wedge, [\cdot, \cdot])$ [33]. On the other hand, for a Lie algebroid structure on a vector bundle A, there is also a degree 1 derivation d of the graded commutative algebra $(\Gamma(\wedge^\bullet A^*), \wedge)$ such that $d^2 = 0$. The differential d is given by

$$(d\alpha)(u_0, u_1, \dots, u_n) = \sum_{i=0}^{n} (-1)^i a(u_i)\alpha(u_0, \dots, \widehat{u_i}, \dots, u_n) + \sum_{i<j} (-1)^{i+j} \alpha([u_i, u_j], u_0, \dots, \widehat{u_i}, \dots, \widehat{u_j}, \dots, u_n).$$

Indeed, a Lie algebroid structure on A is also equivalent to a differential graded algebra $(\Gamma(\wedge^\bullet A^*), \wedge, d)$.

Let $A \to M$ be a complex vector bundle. Assume that A and its dual A^* both carry Lie algebroid structures with anchor maps $a : A \to T_{\mathbb{C}}M$ and $a_* : A^* \to T_{\mathbb{C}}M$, brackets on sections $\Gamma(A) \otimes_{\mathbb{C}} \Gamma(A) \to \Gamma(A) : u \otimes v \mapsto [u, v]$ and $\Gamma(A^*) \otimes_{\mathbb{C}} \Gamma(A^*) \to \Gamma(A^*) : \alpha \otimes \beta \mapsto [\alpha, \beta]_*$, and differentials $d : \Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^{\bullet+1} A^*)$ and $d_* : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+1} A)$.

This pair of Lie algebroids (A, A^*) is a Lie bialgebroid [22,28,27] if d_* is a derivation of the Gerstenhaber algebra $(\Gamma(\wedge^\bullet A), \wedge, [\cdot,\cdot])$ or, equivalently, if d is a derivation of the Gerstenhaber algebra $(\Gamma(\wedge^\bullet A^*), \wedge, [\cdot,\cdot]_*)$. Since the bracket $[\cdot,\cdot]_*$ (resp. $[\cdot,\cdot]$) can be recovered from the derivation d_* (resp. d), one is led to the following alternative definition.

Proposition 2.1 ([33])**.** *A Lie bialgebroid* (A, A^*) *is equivalent to a differential Gerstenhaber algebra structure on* $(\Gamma(\wedge^\bullet A), \wedge, [\cdot,\cdot], d_*)$ *(or, equivalently, on* $(\Gamma(\wedge^\bullet A^*), \wedge, [\cdot,\cdot]_*, d)$*).*

2.2. Hamiltonian operators. Let (A, A^*) be a complex Lie bialgebroid, and $H \in \Gamma(\wedge^2 A)$. We now replace the differential $d_* : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+1} A)$ by a twist by H:

$$d_*^H : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+1} A), \qquad d_*^H u = d_* u + [H, u]. \tag{1}$$

It follows from a simple verification that if H satisfies the Maurer-Cartan equation:

$$d_* H + \frac{1}{2}[H, H] = 0, \tag{2}$$

then $(d_*^H)^2 = 0$ and $(\Gamma(\wedge^\bullet A), \wedge, [\cdot,\cdot], d_*^H)$ is again a differential Gerstenhaber algebra. Thus one obtains a Lie bialgebroid (A, A_H^*). A solution $H \in \Gamma(\wedge^2 A)$ to Eq. (2) is called a **Hamiltonian operator** [26]. The Lie algebroid structure on A_H^* can be described explicitly: the anchor and the Lie bracket are given, respectively, by

$$a_*^H = a_* + a \circ H^\sharp$$

and

$$[\alpha, \beta]_*^H = [\alpha, \beta]_* + [\alpha, \beta]_H.$$

Here

$$[\alpha, \beta]_H = L_{H^\sharp(\alpha)}\beta - L_{H^\sharp(\beta)}\alpha - d_*\langle H^\sharp(\alpha)|\beta\rangle,$$

for all $\alpha, \beta \in \Gamma(A^*)$. We shall use A_H^* to denote such a Lie algebroid and call it the H-twisted Lie algebroid of A^*. Thus we obtain the following theorem, which was first proved in [26] by a different method.

Theorem 2.2. *If* (A, A^*) *constitutes a Lie bialgebroid, and* $H \in \Gamma(\wedge^2 A)$ *is a Hamiltonian operator, then* (A, A^*_H) *is a Lie bialgebroid.*

3. Maurer-Cartan Elements

3.1. The Lie bialgebroid stemming from a complex manifold. We fix a complex manifold $\boldsymbol{X}$ of complex dimension n with almost complex structure J. We regard the tangent bundle $T\boldsymbol{X}$ as a real vector bundle over $\boldsymbol{X}$. The complexification of $T\boldsymbol{X}$ is denoted $T_{\mathbb{C}}\boldsymbol{X}$, namely: $T_{\mathbb{C}}\boldsymbol{X} = T\boldsymbol{X} \otimes \mathbb{C}$. Similarly, $T^*_{\mathbb{C}}\boldsymbol{X} = T^*\boldsymbol{X} \otimes \mathbb{C}$. Let $\mathbb{J} : T_{\mathbb{C}}\boldsymbol{X} \to T_{\mathbb{C}}\boldsymbol{X}$ be the $\mathbb{C}$-linear extension of the almost complex structure J, and $T^{1,0}\boldsymbol{X}$ and $T^{0,1}\boldsymbol{X}$ its $+i$ and $-i$ eigenbundles, respectively. We adopt the following notations:

$$T^{p,q}\boldsymbol{X} = \wedge^p T^{1,0}\boldsymbol{X} \otimes \wedge^q T^{0,1}\boldsymbol{X},$$
$$(T^{p,q}\boldsymbol{X})^* = \wedge^p (T^{1,0}\boldsymbol{X})^* \otimes \wedge^q (T^{0,1}\boldsymbol{X})^*.$$

Consider the following two vector bundles which are obviously mutually dual:

$$A = T^{1,0}\boldsymbol{X} \oplus (T^{0,1}\boldsymbol{X})^*, \quad A^* = T^{0,1}\boldsymbol{X} \oplus (T^{1,0}\boldsymbol{X})^*. \tag{3}$$

We can endow A with a complex Lie algebroid structure. The anchor is the projection onto the first component:

$$a\left(\frac{\partial}{\partial z^i}\right) = \frac{\partial}{\partial z^i} \qquad a(d\overline{z_j}) = 0.$$

The bracket of two sections of $T^{1,0}\boldsymbol{X}$ is their bracket as vector fields; the bracket of any pair of sections of $(T^{0,1}\boldsymbol{X})^*$ is zero; and the bracket of a holomorphic vector field (i.e. a holomorphic section of the holomorphic vector bundle $T^{1,0}\boldsymbol{X}$) and an anti-holomorphic 1-form (i.e. an anti-holomorphic section of the holomorphic vector bundle $(T^{0,1}\boldsymbol{X})^*$) is also zero. Thus

$$\left[\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right] = 0, \qquad [d\overline{z_i}, d\overline{z_j}] = 0, \quad \text{and} \quad \left[\frac{\partial}{\partial z^i}, d\overline{z_j}\right] = 0.$$

Together with the Leibniz rule, the above three rules completely determine the bracket of any two arbitrary sections of A. Similarly, one endows A^* with a complex Lie algebroid structure as well. It is simple to see that (A, A^*) constitutes a Lie bialgebroid. Indeed A and A^* are transversal Dirac structures of the Courant algebroid $T_{\mathbb{C}}\boldsymbol{X} \oplus T^*_{\mathbb{C}}\boldsymbol{X}$, for they are the eigenbundles of the generalized complex structure on $\boldsymbol{X}$ induced by its complex manifold structure [15,13]. In the sequel we will use the symbols

$$T^{1,0}\boldsymbol{X} \bowtie (T^{0,1}\boldsymbol{X})^* \quad \text{and} \quad T^{0,1}\boldsymbol{X} \bowtie (T^{1,0}\boldsymbol{X})^* \tag{4}$$

to refer to A and A^* when seen as Lie algebroids [24].

Moreover, one has

$$\wedge^k A \cong \bigoplus_{i+j=k} T^{i,0}\boldsymbol{X} \otimes (T^{0,j}\boldsymbol{X})^*,$$
$$\wedge^k A^* \cong \bigoplus_{i+j=k} T^{0,i}\boldsymbol{X} \otimes (T^{j,0}\boldsymbol{X})^*.$$

The Lie algebroid differentials associated to the Lie algebroid structures on A^* and A are the usual $\bar{\partial}$- and ∂-operators, respectively:

$$d_* = \bar{\partial} : \Omega^{0,j}(T^{i,0}\boldsymbol{X}) \to \Omega^{0,j+1}(T^{i,0}\boldsymbol{X}),$$
$$d = \partial : \Omega^{j,0}(T^{0,i}\boldsymbol{X}) \to \Omega^{j+1,0}(T^{0,i}\boldsymbol{X}).$$

3.2. Extended Poisson structures.

Definition 3.1. *An* ***extended Poisson manifold*** $(\boldsymbol{X}, H)$ *is a complex manifold* $\boldsymbol{X}$ *equipped with an* $H \in \Gamma(\wedge^2 A)$ *which is an Hamiltonian operator with respect to* (A, A^*)*, i.e.*

$$\bar{\partial} H + \frac{1}{2}[H, H] = 0. \tag{5}$$

In this case, H *is called an extended Poisson structure.*

Any $H \in \Gamma(\wedge^2 A)$ decomposes as

$$H = \pi + \theta + \omega,$$

where $\pi \in \Gamma(T^{2,0}\boldsymbol{X})$, $\theta \in \Gamma(T^{1,0}\boldsymbol{X} \otimes (T^{0,1}\boldsymbol{X})^*)$ and $\omega \in \Gamma((T^{0,2}\boldsymbol{X})^*)$. We will use the following notations to denote the bundle maps induced by natural contraction:

$$\theta^\flat : T^{0,1}\boldsymbol{X} \to T^{1,0}\boldsymbol{X},$$
$$\theta^\sharp : (T^{1,0}\boldsymbol{X})^* \to (T^{0,1}\boldsymbol{X})^*,$$
$$\pi^\sharp : (T^{1,0}\boldsymbol{X})^* \to T^{1,0}\boldsymbol{X},$$
$$\omega^\flat : T^{0,1}\boldsymbol{X} \to (T^{0,1}\boldsymbol{X})^*.$$

Note that $\theta^\sharp = -(\theta^\flat)^*$.

The following lemma is immediate.

Lemma 3.2. *An element* $H = \pi + \theta + \omega$ *is an extended Poisson structure if and only if the following equations are satisfied:*

$$\bar{\partial}\omega + [\omega, \theta] = 0, \tag{6}$$
$$\bar{\partial}\theta + [\omega, \pi] + \frac{1}{2}[\theta, \theta] = 0, \tag{7}$$
$$\bar{\partial}\pi + [\theta, \pi] = 0, \tag{8}$$
$$[\pi, \pi] = 0. \tag{9}$$

Remark 3.3. When only one of the three terms of H is not zero, we are left with one of the following three special cases:

(a) $H = \pi$ is an extended Poisson if and only if π is a holomorphic Poisson bivector field.
(b) $H = \theta$ is an extended Poisson if and only if $\bar{\partial}\theta + \frac{1}{2}[\theta, \theta] = 0$. Moreover, if $\overline{\theta}^\flat \circ \theta^\flat - \mathrm{id}$ is invertible, θ is equivalent to a deformed complex structure [19].
(c) $H = \omega$ is an extended Poisson if and only if $\bar{\partial}\omega = 0$.

In fact, if $[\omega, \pi] = 0$, Eq. (7) implies that θ defines a deformed complex structure (under the assumption that $\overline{\theta}^\flat \circ \theta^\flat - \mathrm{id}$ is invertible). Then, according to Lemma 3.15 below, Eq. (6) is equivalent to $\bar{\partial}_\theta \omega = 0$, where $\bar{\partial}_\theta = \bar{\partial} + [\theta, \cdot]$, and Eqs. (8)–(9) mean that π is a holomorphic Poisson tensor with respect to the deformed complex structure.

Corollary 3.4. *If* $H = \pi + \theta + \omega$ *is an extended Poisson structure, then so is*

$$\lambda\pi + \theta + \lambda^{-1}\omega,$$

for any $\lambda \in \mathbb{C}^\times$. *In particular,*

$$H^\vee = -\pi + \theta - \omega$$

is an extended Poisson structure.

Note that Maurer-Cartan elements as deformations of Lie bialgebroids or differential Gerstenhaber algebras were already considered by Cleyton-Poon [10] in their study of nilpotent complex structures on real six-dimensional nilpotent algebras.

A natural question is: when will (A, A^*_H) arise from a generalized complex structure in the sense of Hitchin [15, 13]? Let us recall the following:

Lemma 3.5. (*Lemma 6.1 in [29]*). *The graph* $\{H^\sharp\xi + \xi \in A \oplus A^*\}$ *of* H*, which is clearly isomorphic to* A^*_H *as a vector bundle, is the* $+i$*- (or* $-i$*-) eigenbundle of a generalized complex structure on* $\boldsymbol{X}$ *if and only if* $\overline{H}^\sharp \circ H^\sharp - \mathrm{id}_{A^*}$ *is invertible. Here the map* $\overline{H}^\sharp : A \to A^*$ *is defined by* $\overline{H}^\sharp(u) = \overline{H^\sharp(\overline{u})}$*,* $\forall u \in A$.

Again we let $H = \pi + \theta + \omega$ be an extended Poisson structure on $\boldsymbol{X}$. Relative to the direct sum decompositions of A and A^*, the endomorphisms $H^\sharp$ and $\overline{H}^\sharp$ are represented by the block matrices

$$H^\sharp = \begin{pmatrix} \theta^\flat & \pi^\sharp \\ \omega^\flat & \theta^\sharp \end{pmatrix} \quad \text{and} \quad \overline{H}^\sharp = \begin{pmatrix} \overline{\theta}^\flat & \overline{\pi}^\sharp \\ \overline{\omega}^\flat & \overline{\theta}^\sharp \end{pmatrix}.$$

In turn, we have

$$\overline{H}^\sharp H^\sharp = \begin{pmatrix} \overline{\theta}^\flat \circ \theta^\flat + \overline{\pi}^\sharp \circ \omega^\flat & \overline{\theta}^\flat \circ \pi^\sharp + \overline{\pi}^\sharp \circ \theta^\sharp \\ \overline{\omega}^\flat \circ \theta^\flat + \overline{\theta}^\sharp \circ \omega^\flat & \overline{\omega}^\flat \circ \pi^\sharp + \overline{\theta}^\sharp \circ \theta^\sharp \end{pmatrix}. \tag{10}$$

Proposition 3.6. *Given an extended Poisson manifold* $(\boldsymbol{X}, H)$*, let* $A = T^{1,0}\boldsymbol{X} \bowtie (T^{0,1}\boldsymbol{X})^*$. *Then* A^*_H *is the* $(\pm i)$*-eigenbundle of a generalized complex structure if and only if* $\overline{H}^\sharp H^\sharp - \mathrm{id}_{A^*}$ *is invertible.*

Example 3.7. If $H = \pi$ (i.e. H is a holomorphic Poisson bivector field) or $H = \omega$, it is clear that $\overline{H}^\sharp H^\sharp$ is zero. Hence, in these two situations, the extended Poisson structure on $\boldsymbol{X}$ is actually a generalized complex structure.

Here is a simple example of extended Poisson structure, which does not arise from a generalized complex structure.

Example 3.8. Consider the torus $\mathbf{T} = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ with its standard complex structure. Let z be the standard coordinate on $\mathbf{T}$. Obviously, any

$$\theta = f(z, \bar{z})\frac{d}{dz} \wedge d\bar{z}, \tag{11}$$

where f is a smooth $\mathbb{C}$-valued function, is an extended Poisson structure. In this case, $\overline{H}^\sharp H^\sharp = |f|^2\,\mathrm{id}$. Hence A^*_θ does not stem from a generalized complex structure provided that $|f| = 1$.

3.3. Elliptic Lie algebroids. As in [4], we say that a complex Lie algebroid B is **elliptic** if $\mathrm{Re} \circ a_B : B \to T\boldsymbol{X}$ is surjective. Here $a_B : B \to T_{\mathbb{C}}\boldsymbol{X}$ is the anchor map of B and $\mathrm{Re} : T_{\mathbb{C}}\boldsymbol{X} \to T\boldsymbol{X}$ is the projection onto the real part.

Theorem 3.9 ([4])**.** *If B is an elliptic Lie algebroid over a compact complex manifold $\boldsymbol{X}$, and E a finite rank complex vector bundle with a B-action as in* [12]*, then all cohomology groups $\mathrm{H}^\bullet(B, E)$ are finite dimensional.*

It is therefore natural to ask when A_H^* is elliptic. An easy calculation shows the following:

Proposition 3.10. *Let a_*^H denote the anchor of A_H^* and $\mathrm{C} : T^{0,1}\boldsymbol{X} \to T^{1,0}\boldsymbol{X}$ the complex conjugation. The bundle maps $\mathrm{Re} \circ a_*^H$ and*

$$F = (\mathrm{C} + \theta^\flat) \oplus \pi^\sharp : T^{0,1}\boldsymbol{X} \oplus (T^{1,0}\boldsymbol{X})^* \to T^{1,0}\boldsymbol{X}, \tag{12}$$

and the isomorphism of real vector bundles $\mathrm{Re} : T^{1,0}\boldsymbol{X} \to T\boldsymbol{X}$ fit into the commutative diagram

$$\begin{array}{ccc} & T^{0,1}\boldsymbol{X} \oplus (T^{1,0}\boldsymbol{X})^* & \\ {\scriptstyle F}\swarrow & & \searrow{\scriptstyle \mathrm{Re}\circ a_*^H} \\ T^{1,0}\boldsymbol{X} & \xrightarrow{\quad\mathrm{Re}\quad} & T\boldsymbol{X}. \end{array} \tag{13}$$

As a consequence, A_H^* is an elliptic Lie algebroid if and only if F is surjective.

Example 3.11. When $H = \pi$, or ω, it is clear that A_H^* is elliptic. On the other hand, if we consider the torus $\boldsymbol{T}$ endowed with the bivector field θ of Example 3.8, the Lie algebroid A_H^* is elliptic if and only if f is not identically 1.

3.4. Poisson cohomology.

Definition 3.12. *Given an extended Poisson manifold $(\boldsymbol{X}, H)$, the cohomology of the Lie algebroid A_H^* is called the* ***Poisson cohomology*** *of the extended Poisson structure, and denoted $\mathrm{H}^\bullet(\boldsymbol{X}, H)$. In other words, it is the cohomology of the cochain complex:*

$$\cdots \xrightarrow{\bar\partial^H} \Gamma(\wedge^k A) \xrightarrow{\bar\partial^H} \Gamma(\wedge^{k+1} A) \xrightarrow{\bar\partial^H} \ldots, \tag{14}$$

where $\Gamma(\wedge^k A) = \oplus_{i+j=k}\Omega^{0,j}(T^{i,0}\boldsymbol{X})$ and $\bar\partial^H = \bar\partial + [H, \cdot]$.

Poisson cohomology is also called tangent cohomology by Kontsevich [21].

As an immediate consequence of Theorem 3.9 and Proposition 3.10, we have

Corollary 3.13. *If H is an extended Poisson structure on a compact complex manifold $\boldsymbol{X}$ and the map F (given by Eq. (12)) is surjective, then all Poisson cohomology groups are finite dimensional.*

Remark 3.14. When H is a holomorphic Poisson bivector field π, the cochain complex (14) is the total complex of the double complex as discussed in Corollary 4.26 in [24].

On the other hand, if $H = \theta \in \Omega^{0,1}(T^{1,0}\boldsymbol{X})$ is a Maurer-Cartan element such that $\overline{\theta}^\flat \circ \theta^\flat - \mathrm{id}$ is invertible, then θ defines a new complex structure on $\boldsymbol{X}$ according to Kodaira [19].

The following lemma can be verified directly.

Lemma 3.15. *Let $H = \theta \in \Omega^{0,1}(T^{1,0}\boldsymbol{X})$ be a Maurer-Cartan element such that $\overline{\theta}^\flat \circ \theta^\flat - \mathrm{id}$ is invertible. Then the Lie algebroid A_H^* is isomorphic to $T_\theta^{1,0}\boldsymbol{X} \bowtie (T_\theta^{0,1}\boldsymbol{X})^*$, where $T_\theta^{1,0}\boldsymbol{X}$ and $T_\theta^{0,1}\boldsymbol{X}$ are, respectively, the $+i$ and $-i$ eigenbundles of the deformed almost complex structure $J_\theta : T\boldsymbol{X} \to T\boldsymbol{X}$. As a consequence, the differential operator d_*^H in Eq. (1) is equal to $\bar{\partial}_\theta$, the new $\bar{\partial}$-operator of the deformed complex structure.*

Thus we have

Proposition 3.16. *If $H = \theta \in \Omega^{0,1}(T^{1,0}\boldsymbol{X})$ is a Maurer-Cartan element such that $\overline{\theta}^\flat \circ \theta^\flat - \mathrm{id}$ is invertible, then*

$$\mathrm{H}^k(\boldsymbol{X}, H) \cong \oplus_{i+j=k} \mathrm{H}^i(\boldsymbol{X}, \wedge^j T_\theta \boldsymbol{X}),$$

where $T_\theta \boldsymbol{X}$ denotes the holomorphic tangent bundle of the deformed complex manifold $\boldsymbol{X}$.

3.5. Coisotropic submanifolds. Suppose that $\boldsymbol{Y} \subseteq \boldsymbol{X}$ is a complex submanifold [19]. Set

$$N^{1,0}\boldsymbol{Y} = \left\{\xi \in (T^{1,0}\boldsymbol{X}|_{\boldsymbol{Y}})^* \text{ s.t. } \langle \xi | Y \rangle = 0,\ \forall Y \in T^{1,0}\boldsymbol{Y}\right\},$$

and consider the subbundle $K = T^{0,1}\boldsymbol{Y} \oplus N^{1,0}\boldsymbol{Y}$ of A^*.

Definition 3.17. *A complex submanifold $\boldsymbol{Y}$ of $\boldsymbol{X}$ is called **coisotropic** if $H(u, v) = 0$, for all $u, v \in K$.*

Example 3.18. If $H = \pi$ is a holomorphic Poisson bivector field, then $\boldsymbol{Y}$ is coisotropic if and only if it is coisotropic in the usual sense, i.e. $\pi(\xi_1, \xi_2) = 0, \forall \xi_1, \xi_2 \in N^{1,0}\boldsymbol{Y}$, or $\pi^\sharp(N^{1,0}\boldsymbol{Y}) \subseteq T^{1,0}\boldsymbol{Y}$.

Example 3.19. If $H = \omega$, then $\boldsymbol{Y}$ is coisotropic if and only if $\iota^*\omega = 0$, where $\iota : \boldsymbol{Y} \to \boldsymbol{X}$ is the embedding map.

Example 3.20. If $H = \theta$, then $\boldsymbol{Y}$ is coisotropic if and only if $\theta^\flat(T^{0,1}\boldsymbol{Y}) \subseteq T^{1,0}\boldsymbol{Y}$.

It is well known that given a coisotropic submanifold C of a real Poisson manifold (P, π), the conormal bundle $NC = \left\{\xi \in T_c^*P \text{ s.t. } c \in C;\ \langle \xi | X \rangle = 0,\ \forall X \in T_c C\right\}$ is a Lie subalgebroid of the cotangent Lie algebroid $(T^*P)_\pi$ [31]. The following proposition can be considered as an analogue of this fact in the extended Poisson setting.

Proposition 3.21. *Let $\boldsymbol{Y}$ be a coisotropic submanifold of the extended Poisson manifold $(\boldsymbol{X}, H)$. Then the vector subbundle $K = T^{0,1}\boldsymbol{Y} \oplus N^{1,0}\boldsymbol{Y}$ is a Lie subalgebroid of A_H^*. That is, a_*^H maps K into $T_{\mathbb{C}}\boldsymbol{Y}$ and for any smooth extensions $\widetilde{u}, \widetilde{v} \in \Gamma(A_H^*)$ to $\boldsymbol{X}$ of any two sections $u, v \in \Gamma(K)$, the restriction to $\boldsymbol{Y}$ of $[\widetilde{u}, \widetilde{v}]_*^H$ is a section of K which does not depend on the choice of extensions.*

3.6. Poisson relations. Following Weinstein [31], we introduce the following

Definition 3.22. *Let $(\boldsymbol{X}_1, H_1)$ and $(\boldsymbol{X}_2, H_2)$ be extended Poisson manifolds. A Poisson relation from $(\boldsymbol{X}_2, H_2)$ to $(\boldsymbol{X}_1, H_1)$ is a coisotropic submanifold of the product manifold $\boldsymbol{X}_1 \times \boldsymbol{X}_2^{\vee}$ (i.e. $\boldsymbol{X}_1 \times \boldsymbol{X}_2$ endowed with the extended Poisson structure $(H_1, H_2^{\vee})$, see Corollary 3.4).*

We call a holomorphic map $f : \boldsymbol{X}_2 \to \boldsymbol{X}_1$ between extended Poisson manifolds $(\boldsymbol{X}_1, H_1)$ and $(\boldsymbol{X}_2, H_2)$ an **extended Poisson map** if its graph

$$G_f = \{(f(x), x) \text{ s.t. } x \in \boldsymbol{X}_2\} \subset \boldsymbol{X}_1 \times \boldsymbol{X}_2^{\vee}$$

is a Poisson relation.

Proposition 3.23. *Let $(\boldsymbol{X}_1, H_1)$ and $(\boldsymbol{X}_2, H_2)$ be extended Poisson manifolds, where the extended Poisson structures decompose as $H_i = \pi_i + \theta_i + \omega_i$ $(i = 1, 2)$. Then a holomorphic map $f : \boldsymbol{X}_2 \to \boldsymbol{X}_1$ is an extended Poisson map if and only if $f_*\pi_2 = \pi_1$; $f^*\omega_1 = \omega_2$; and $f_* \circ \theta_2^{\flat} = \theta_1^{\flat} \circ f_*$.*

The proof is a direct verification and is left to the reader. As a consequence, we have

Corollary 3.24. *The composition of two extended Poisson maps is again an extended Poisson map.*

4. Koszul-Brylinski Poisson Homology

In this section we will introduce homology groups for extended Poisson manifolds based on the Evens-Lu-Weinstein module of a Lie algebroid.

4.1. Koszul-Brylinski cochain complex. First we recall the notion of Clifford algebras and spin representation. Let V be a vector space of dimension n endowed with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$. Its Clifford algebra $\mathcal{C}(V)$ is defined as the quotient of the tensor algebra $\oplus_{k=0}^{n} V^{\otimes k}$ by the relations $x \otimes y + y \otimes x = 2(x, y)$, with $x, y \in V$. It is naturally an associative $\mathbb{Z}_2$-graded algebra. Up to isomorphisms, there exists a unique irreducible module S of $\mathcal{C}(V)$ called spin representation [9]. The vectors of S are called spinors.

An operator O on S is called even (or of degree 0) if $O(S^i) \subset S^i$ and odd (or of degree 1) if $O(S^i) \subset S^{i+1}$. Here $i \in \mathbb{Z}_2$. If O_1 and O_2 are operators of degree d_1 and d_2 respectively, then their commutator is the operator

$$\lfloor O_1, O_2 \rfloor = O_1 \circ O_2 - (-1)^{d_1 d_2} O_2 \circ O_1.$$

Example 4.1. Let W be a vector space of dimension r. We can endow $V = W \oplus W^*$ with the non-degenerate pairing

$$(u_1 + \xi_1, u_2 + \xi_2) = \frac{1}{2} \left(\xi_1(u_2) + \xi_2(u_1) \right),$$

where $u_1, u_2 \in W$ and $\xi_1, \xi_2 \in W^*$. The representation of $\mathcal{C}(V)$ on $S = \oplus_{k=0}^{r} \wedge^k W$ defined by $u \cdot w = u \wedge w$ and $\xi \cdot w = \iota_\xi w$, where $u \in W$, $\xi \in W^*$ and $w \in S$, is the spin representation. Note that S is $\mathbb{Z}$- and thus also $\mathbb{Z}_2$-graded.

Recall that $\boldsymbol{E} = T_{\mathbb{C}}\boldsymbol{X} \oplus T_{\mathbb{C}}^*\boldsymbol{X}$ admits the standard pseudo-metric

$$(X_1 + \xi_1, X_2 + \xi_2) = \frac{1}{2} \left(\langle \xi_1 | X_2 \rangle + \langle \xi_2 | X_1 \rangle \right),$$

where $X_i \in T_{\mathbb{C}}\boldsymbol{X}$ and $\xi_i \in T^*_{\mathbb{C}}\boldsymbol{X}$. The corresponding Clifford bundle $\mathcal{C}(\boldsymbol{E})$ can be identified with the vector bundle $(\wedge^\bullet T_{\mathbb{C}}\boldsymbol{X}) \otimes (\wedge^\bullet T^*_{\mathbb{C}}\boldsymbol{X})$, under which the Clifford action of $\mathcal{C}(\boldsymbol{E})$ on the spinor bundle

$$\wedge^\bullet T^*_{\mathbb{C}}\boldsymbol{X} = \bigoplus_{p,q}(T^{p,q}\boldsymbol{X})^*$$

is given by

$$(W \otimes \xi) \cdot \lambda = (-1)^{\frac{w(w-1)}{2}} \iota_W(\xi \wedge \lambda).$$

Here $W \in \wedge^w T_{\mathbb{C}}\boldsymbol{X}, \xi, \lambda \in \wedge^\bullet T^*_{\mathbb{C}}\boldsymbol{X}$, and the symbol ι_W denotes the standard contraction

$$\langle \iota_W \xi | X\rangle = \langle \xi | W \wedge X\rangle,$$

for $\xi \in \wedge^p T^*_{\mathbb{C}}\boldsymbol{X}$ and $X \in \wedge^{p-w} T_{\mathbb{C}}\boldsymbol{X}$ with $p \geq w$.

Let $(\boldsymbol{X}, H)$ be an extended Poisson manifold of complex dimension n. Then A^*_H is a Lie algebroid and the **Evens-Lu-Weinstein module** [12] of A^*_H is the complex line bundle

$$Q_{A^*_H} = \wedge^{2n} A^*_H \otimes \wedge^{2n} T^*_{\mathbb{C}}\boldsymbol{X}.$$

The representation of A^*_H on $Q_{A^*_H}$ is given by

$$\nabla^H_\alpha(\alpha_1 \wedge \cdots \wedge \alpha_{2n} \otimes \mu) = \sum_{i=1}^{2n} \Big(\alpha_1 \wedge \cdots \wedge [\alpha, \alpha_i]^H_* \wedge \cdots \wedge \alpha_{2n} \otimes \mu\Big) \\ + \alpha_1 \wedge \cdots \wedge \alpha_{2n} \otimes L_{a^H_*(\alpha)}\mu,$$

where $\alpha,\ \alpha_1, \ldots, \alpha_{2n} \in \Gamma(A^*_H)$, $\mu \in \Gamma(\wedge^{2n} T^*_{\mathbb{C}}\boldsymbol{X})$.

A simple computation yields that $Q_{A^*_H} \cong \wedge^n (T^{1,0}\boldsymbol{X})^* \otimes \wedge^n (T^{1,0}\boldsymbol{X})^*$. Accordingly,

$$\mathscr{L} = Q^{\frac{1}{2}}_{A^*_H} \cong \wedge^n (T^{1,0}\boldsymbol{X})^* = (T^{n,0}\boldsymbol{X})^*$$

is also an A^*_H-module and we use ∇^H again to denote the representation. Equivalently, we have an operator

$$\mathcal{D}^H :\ \Gamma(\mathscr{L}) \to \Gamma(A \otimes \mathscr{L}), \tag{15}$$

such that

$$\iota_\alpha \mathcal{D}^H s = \nabla^H_\alpha s, \quad \forall \alpha \in \Gamma(A^*), \quad s \in \Gamma(\mathscr{L}),$$

which allows us to define a differential operator

$$\breve{d}^H_* : \Gamma(\wedge^k A \otimes \mathscr{L}) \to \Gamma(\wedge^{k+1} A \otimes \mathscr{L})$$

by

$$\breve{d}^H_*(u \otimes s) = (\bar{\partial}^H u) \otimes s + (-1)^k u \wedge \mathcal{D}^H s, \tag{16}$$

for all $u \in \Gamma(\wedge^k A)$ and $s \in \Gamma(\mathscr{L})$.

The following lemma is needed later.

Lemma 4.2. *The relation*

$$\tau(X \otimes s) = X \cdot s,$$

where in the r.h.s. $X \in \wedge^k A$ *is regarded as an element of the Clifford algebra* $\mathcal{C}(\boldsymbol{E})$ *and* $s \in \mathscr{L}$ *is regarded as an element in* $\wedge^\bullet T^*_{\mathbb{C}}\boldsymbol{X}$*, defines an isomorphism of vector bundles*

$$\tau : \wedge^k A \otimes \mathscr{L} \to \bigoplus_{i-j=n-k} (T^{i,j}\boldsymbol{X})^*.$$

Equivalently,

$$\tau\left((W \wedge \xi) \otimes s\right) = (-1)^{\frac{w(w-1)}{2}} \iota_W(\xi \wedge s) = (-1)^{\frac{w(w-1)}{2}+n(k-w)} (\iota_W s) \wedge \xi,$$

for $W \in T^{w,0}\boldsymbol{X}$, $\xi \in (T^{0,k-w}\boldsymbol{X})^*$ and $s \in \mathscr{L}$.

We define the inner product of $H \in \Gamma(\wedge^2 A)$ with $\lambda \in \Gamma(\wedge^\bullet T^*_{\mathbb{C}}\boldsymbol{X})$ as

$$\iota_H \lambda = -H \cdot \lambda.$$

This coincides with the usual inner product of bivector fields with differential forms. Introduce

$$\lfloor \partial, \iota_H \rfloor = \partial \circ \iota_H - \iota_H \circ \partial : \ \Gamma(\wedge^\bullet T^*_{\mathbb{C}}\boldsymbol{X}) \to \Gamma(\wedge^\bullet T^*_{\mathbb{C}}\boldsymbol{X}).$$

Let us denote $\Omega^{i,j}(\boldsymbol{X}) = \Gamma((T^{i,j}\boldsymbol{X})^*)$. The following theorem is the main result in this section.

Theorem 4.3. *The diagram*

$$\begin{array}{ccc} \Gamma(\wedge^k A \otimes \mathscr{L}) & \xrightarrow{\check{d}^H_*} & \Gamma(\wedge^{k+1} A \otimes \mathscr{L}) \\ \downarrow \tau & & \downarrow \tau \\ \bigoplus_{i-j=n-k} \Omega^{i,j}(\boldsymbol{X}) & \xrightarrow[\bar{\partial}+\lfloor\partial,\iota_H\rfloor]{} & \bigoplus_{i-j=n-k-1} \Omega^{i,j}(\boldsymbol{X}) \end{array} \tag{17}$$

commutes.

Definition 4.4. *The cohomology of the cochain complex* $(\bigoplus_{i-j=n-k} \Omega^{i,j}(\boldsymbol{X}), \bar{\partial} + \lfloor \partial, \iota_H \rfloor)$ *is called the* ***Koszul-Brylinski Poisson homology*** *of the extended Poisson manifold* $(\boldsymbol{X}, H)$*, and denoted* $\mathrm{H}_\bullet(\boldsymbol{X}, H)$.

Remark 4.5. (a) If $H = \pi$ is a holomorphic Poisson bivector field, the cochain complex $(\bigoplus_{i-j=n-k} \Omega^{i,j}(\boldsymbol{X}), \bar{\partial} + \lfloor \partial, \iota_H \rfloor)$ is the total complex of a double complex. Its cohomology is the usual Koszul-Brylinski Poisson homology of a holomorphic Poisson manifold, as studied in detail by one of the authors [30].

(b) If $H = \omega \in \Omega^{0,2}(\boldsymbol{X})$ with $\bar{\partial}\omega = 0$, the complex $(\bigoplus_{i-j=n-k} \Omega^{i,j}(\boldsymbol{X}), \bar{\partial} + \lfloor \partial, \iota_H \rfloor)$ becomes $(\bigoplus_{i-j=n-k} \Omega^{i,j}(\boldsymbol{X}), \bar{\partial} + (\partial\omega)\wedge)$. Its cohomology is the twisted Dolbeault cohomology.

(c) If $H = \theta \in \Omega^{0,1}(T^{1,0}\boldsymbol{X})$ is a Maurer-Cartan element such that $\overline{\theta}^\flat \circ \theta^\flat - \mathrm{id}$ is invertible, then θ defines a new complex structure on $\boldsymbol{X}$. According to Lemma 3.15, the cochain complex $(\bigoplus_{i-j=n-k} \Omega^{i,j}(\boldsymbol{X}), \bar{\partial} + \lfloor \partial, \iota_H \rfloor)$ is isomorphic to $(\bigoplus_{i-j=n-k} \Omega^{i,j}_\theta(\boldsymbol{X}), \ \bar{\partial}_\theta)$, where $\bar{\partial}_\theta$ is the $\bar{\partial}$-Dolbeault operator of the deformed complex structure. As a consequence, we have $\mathrm{H}_k(\boldsymbol{X}, \theta) \cong \oplus_{j-i=n-k} \mathrm{H}^{i,j}_\theta(\boldsymbol{X})$, where $\mathrm{H}^{i,j}_\theta(\boldsymbol{X})$ is the Dolbeault cohomology of the deformed complex structure.

4.2. Evens-Lu-Weinstein duality. Consider a compact complex (and therefore orientable) manifold X with $\dim_{\mathbb{C}} X = n$, a complex Lie algebroid B over X with $\mathrm{rk}_{\mathbb{C}} B = r$. According to [12], the complex line bundle $Q_B = \wedge^r B \otimes \wedge^{2n} T^*_{\mathbb{C}} X$ is a module over the complex Lie algebroid B. If $Q_B^{\frac{1}{2}}$ exists as a complex vector bundle, $Q_B^{\frac{1}{2}}$ becomes a B-module as well. There is a natural map

$$\phi : \Gamma(\wedge^k B^* \otimes Q_B^{\frac{1}{2}}) \otimes \Gamma(\wedge^{r-k} B^* \otimes Q_B^{\frac{1}{2}}) \to \Gamma(\wedge^r B^* \otimes Q_B) \cong \Gamma(\wedge^{2n} T^*_{\mathbb{C}} X).$$

Integrating, we get the pairing

$$\Gamma(\wedge^k B^* \otimes Q_B^{\frac{1}{2}}) \otimes \Gamma(\wedge^{r-k} B^* \otimes Q_B^{\frac{1}{2}}) \to \mathbb{C}, \qquad \xi \otimes \eta \mapsto \int_X \phi(\xi \otimes \eta). \tag{18}$$

The following result is essentially due to Evens-Lu-Weinstein [12] for the pairing, and to Block [4] for the non-degeneracy (see also [30]).

Theorem 4.6. *For a complex Lie algebroid B, with $\mathrm{rk}_{\mathbb{C}} B = r$, over a compact manifold X, the pairing (18) induces a pairing*

$$\mathrm{H}^k(B, Q_B^{\frac{1}{2}}) \otimes \mathrm{H}^{r-k}(B, Q_B^{\frac{1}{2}}) \to \mathbb{C}.$$

Moreover, if B is an elliptic Lie algebroid, this pairing is non-degenerate.

Let (X, H) be a compact extended Poisson manifold of complex dimension n. Consider the Lie algebroid $B = (T^{0,1}X \bowtie (T^{1,0}X)^*)_H$. Applying Theorem 4.6 and Proposition 3.10, we obtain

Theorem 4.7. *Let (X, H) be a compact extended Poisson manifold of complex dimension n, with $H = \pi + \theta + \omega$. Then the map*

$$\Omega^{i,j}(X) \otimes \Omega^{k,l}(X) \to \mathbb{C} : \zeta \otimes \eta \mapsto \int_X (\zeta \wedge \eta)^{top}$$

induces a pairing on the Koszul-Brylinski Poisson homology:

$$\mathrm{H}_k(X, H) \otimes \mathrm{H}_{2n-k}(X, H) \to \mathbb{C}. \tag{19}$$

Moreover, if the bundle map $F = (\mathrm{C} + \theta^\flat) \oplus \pi^\sharp$ maps $T^{0,1}X \oplus (T^{1,0}X)^*$ surjectively onto $T^{1,0}X$, then all homology groups $\mathrm{H}_\bullet(X, H)$ are finite dimensional vector spaces and the pairing (19) is non-degenerate.

4.3. Proof of Theorem 4.3. The following lemmas are needed.

Lemma 4.8. *For any $u \in \Gamma(\wedge^p A)$, $\lambda \in \Omega^{\bullet,\bullet}(X)$, one has*

$$\bar\partial(u \cdot \lambda) = (\bar\partial u) \cdot \lambda + (-1)^p u \cdot \bar\partial \lambda. \tag{20}$$

Lemma 4.9. *For any $u \in \Gamma(\wedge^p A)$, $v \in \Gamma(\wedge^q A)$, the Schouten bracket $[u, v]$ is determined by*

$$[u, v] \cdot \lambda = (-1)^{q+1} \lfloor u, \lfloor v, \partial \rfloor \rfloor \lambda, \quad \forall \lambda \in \Omega^{\bullet,\bullet}(X). \tag{21}$$

Both lemmas can be proved by induction; this is left to the reader.

Lemma 4.10. *For any $u \in \Gamma(\wedge^i A)$ and $\lambda \in \Omega^{\bullet,\bullet}(\boldsymbol{X})$, one has*

$$\lfloor \partial, \iota_H \rfloor (u \cdot \lambda) = [H, u] \cdot \lambda + (-1)^i u \cdot (\lfloor \partial, \iota_H \rfloor \lambda). \tag{22}$$

In particular, for any smooth function $f \in C^\infty(\boldsymbol{X}, \mathbb{C})$, one has

$$\lfloor \partial, \iota_H \rfloor (f \lambda) = [H, f] \cdot \lambda + f \lfloor \partial, \iota_H \rfloor \lambda. \tag{23}$$

Proof. According to Eq. (21), we have

$$\begin{aligned}
[H, u] \cdot \lambda =& (-1)^{i+1} \lfloor H, \lfloor u, \partial \rfloor \rfloor \lambda \\
=& (-1)^i (u \cdot \partial (H \cdot \lambda) - H \cdot u \cdot (\partial \lambda)) + (H \cdot (\partial (u \cdot \lambda)) - \partial (u \cdot H \cdot \lambda)) \\
=& (-1)^i (u \cdot \partial (H \cdot \lambda) - u \cdot H \cdot (\partial \lambda)) + (H \cdot (\partial (u \cdot \lambda)) - \partial (H \cdot u \cdot \lambda)) \\
=& - (-1)^i u \cdot (\lfloor \partial, \iota_H \rfloor \lambda) + \lfloor \partial, \iota_H \rfloor (u \cdot \lambda).
\end{aligned}$$

□

A straightforward (though lengthy) computation shows the following:

Lemma 4.11. *Suppose that $(z^1, \ldots, z^n)$ is a local holomorphic chart and $H = \pi + \theta + \omega$ is given by*

$$H = \pi^{i,j} \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^j} + \theta_q^p \frac{\partial}{\partial z^p} \wedge d\bar{z}^q + \omega_{k,l} d\bar{z}^k \wedge d\bar{z}^l, \tag{24}$$

where $\pi^{i,j}$, θ_q^p, and $\omega_{k,l}$ are complex valued smooth functions on $\boldsymbol{X}$. Then the H-twisted Lie algebroid structure on $A_H^* \cong T^{0,1}\boldsymbol{X} \oplus (T^{1,0}\boldsymbol{X})^*$ can be expressed by:

$$a_*^H \left(\frac{\partial}{\partial \bar{z}^i} \right) = \frac{\partial}{\partial \bar{z}^i} - \theta_i^p \frac{\partial}{\partial z^p}, \qquad a_*^H \left(dz^i \right) = 2\pi^{i,q} \frac{\partial}{\partial z^q}, \tag{25}$$

$$\left[\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j} \right]_*^H = 2\partial \omega_{i,j}, \quad \left[dz^i, dz^j \right]_*^H = 2\partial \pi^{i,j}, \quad \left[dz^j, \frac{\partial}{\partial \bar{z}^i} \right]_*^H = \partial \theta_i^j. \tag{26}$$

Lemma 4.12. *Making the same assumptions as in Lemma 4.11, consider the local section*

$$s = dz^1 \wedge \cdots \wedge dz^n \tag{27}$$

of $\mathscr{L} = Q_{A_H^*}^{\frac{1}{2}}$. The representation of A_H^* on $\mathscr{L}$ is given by

$$\nabla^H_{\frac{\partial}{\partial \bar{z}^i}} s = - \frac{\partial \theta_i^p}{\partial z^p} s, \qquad \nabla^H_{dz^i} s = 2 \frac{\partial \pi^{i,p}}{\partial z^p} s. \tag{28}$$

Proof. Using Eq. (25), we compute

$$\begin{aligned}
L_{a_*^H(\frac{\partial}{\partial \bar{z}^i})} dz^j &= -d\theta_i^j, \qquad L_{a_*^H(\frac{\partial}{\partial \bar{z}^i})} d\bar{z}^j = 0, \\
L_{a_*^H(dz^i)} dz^j &= 2d\pi^{i,j}, \quad L_{a_*^H(dz^i)} d\bar{z}^j = 0.
\end{aligned} \tag{29}$$

Write

$$s^2 = (\frac{\partial}{\partial \bar{z}^1} \wedge \cdots \wedge \frac{\partial}{\partial \bar{z}^n} \wedge dz^1 \wedge \cdots \wedge dz^n) \otimes (dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n).$$

Then, using Eqs. (26) and (29), one obtains

$$\nabla^H_{\frac{\partial}{\partial \bar{z}^i}} s^2 = -2\frac{\partial \theta_i^p}{\partial z^p} s^2, \qquad \nabla^H_{dz^i} s^2 = 4\frac{\partial \pi^{i,p}}{\partial z^p} s^2.$$

The conclusion thus follows immediately. □

Corollary 4.13. *Locally, the operator $\mathcal{D}^H$ in Eq.* (15) *is given by*

$$\mathcal{D}^H s = \left(2\frac{\partial \pi^{i,p}}{\partial z^p}\frac{\partial}{\partial z^i} - \frac{\partial \theta_i^p}{\partial z^p} d\bar{z}^i \right) \otimes s, \tag{30}$$

where s is defined in Eq. (27).

We are now ready to prove Theorem 4.3.

Proof of Theorem 4.3. We adopt an inductive approach. First we prove the commutativity of Diagram (17) for $k = 0$.

Note that for any $f \in C^\infty(\boldsymbol{X}, \mathbb{C})$, $u \in \Gamma(\wedge^k A)$ and $s \in \Gamma(\mathscr{L})$, one has

$$\begin{aligned} \tau \breve{d}_*^H (fu \otimes s) &= \tau \left(f\breve{d}_*^H (u \otimes s) + ((\bar{\partial} f + [H, f]) \wedge u) \otimes s \right) \quad \text{by Eq. (16)} \\ &= f\tau\breve{d}_*^H (u \otimes s) + (\bar{\partial} f + [H, f]) \cdot \tau(u \otimes s). \end{aligned}$$

On the other hand, if we write $\lambda = \tau(u \otimes s)$, one has

$$\begin{aligned} &(\bar{\partial} + \lfloor \partial, \iota_H \rfloor)\tau(fu \otimes s) \\ &= (\bar{\partial} + \lfloor \partial, \iota_H \rfloor)(f\lambda) \\ &= \bar{\partial} f \wedge \lambda + f\bar{\partial}\lambda + [H, f] \cdot \lambda + f\lfloor \partial, \iota_H \rfloor \lambda \quad \text{by Eq. (23)} \\ &= f(\bar{\partial} + \lfloor \partial, \iota_H \rfloor)\tau(u \otimes s) + (\bar{\partial} f + [H, f]) \cdot \tau(u \otimes s). \end{aligned}$$

It thus follows that the map $\tau \circ \breve{d}_*^H - (\bar{\partial} + \lfloor \partial, \iota_H \rfloor) \circ \tau$ is $C^\infty(\boldsymbol{X})$-linear. Take a local holomorphic chart $(z^1, \ldots, z^n)$ and write H locally as in Eq. (24) in Lemma 4.11. Again take s as in Eq. (27). For $k = 0$, we have $\breve{d}_*^H s = \mathcal{D}^H s$, which is given locally by Eq. (30). Then, we compute

$$\begin{aligned} \tau\left(\breve{d}_*^H s\right) &= \left(2\frac{\partial \pi^{i,p}}{\partial z^p}\frac{\partial}{\partial z^i} - \frac{\partial \theta_i^p}{\partial z^p} d\bar{z}^i \right) \cdot (dz^1 \wedge \cdots \wedge dz^n) \\ &= 2\sum_{i=1}^n (-1)^{i+1} \frac{\partial \pi^{i,p}}{\partial z^p} dz^1 \wedge \cdots \wedge \widehat{dz^i} \wedge \cdots \wedge dz^n - \frac{\partial \theta_i^p}{\partial z^p} d\bar{z}^i \wedge dz^1 \wedge \cdots \wedge dz^n. \end{aligned}$$

Thus we have

$$\begin{aligned} (\bar{\partial} + \lfloor \partial, \iota_H \rfloor)s &= \partial\iota_H(dz^1 \wedge \cdots \wedge dz^n) \\ &= \partial \left(2\sum_{i<j} (-1)^{i+j-1} \pi^{i,j} dz^1 \wedge \cdots \wedge \widehat{dz^i} \wedge \cdots \wedge \widehat{dz^j} \wedge \cdots \wedge dz^n \right. \end{aligned}$$

$$+\sum_{p=1}^{n}(-1)^{p+1}\theta_i^p d\bar{z}^i \wedge dz^1 \wedge \cdots \wedge \widehat{dz^p} \wedge \cdots \wedge dz^n$$

$$\left. -\omega_{k,l} d\bar{z}^k \wedge d\bar{z}^l \wedge dz^1 \wedge \cdots\cdots \wedge dz^n \right)$$

$$= \tau(\check{d}_*^H s).$$

It thus follows that Diagram (17) indeed commutes when $k = 0$.

Now assume that we have proved the commutativity of Diagram (17) when $k \le m$ (where $0 \le m \le 2n-1$). To prove the $k = m+1$ case, we consider a section $(u \wedge w) \otimes s \in \Gamma(\wedge^{m+1} A \otimes \mathscr{L})$, where $u \in \Gamma(A)$, $w \in \Gamma(\wedge^m A)$ and $s \in \Gamma(\mathscr{L})$. Then

$$\begin{aligned}
&(\bar\partial + \lfloor \partial, \iota_H \rfloor)\tau((u \wedge w) \otimes s) \\
&= (\bar\partial + \lfloor \partial, \iota_H \rfloor)(u \cdot \lambda) && \text{where } \lambda = w \cdot s \\
&= \bar\partial u \cdot \lambda - u \cdot \bar\partial \lambda + [H, u] \cdot \lambda - u \cdot (\lfloor \partial, \iota_H \rfloor \lambda) && \text{by Eqs. (20) and (22)} \\
&= \bar\partial^H u \cdot \lambda - u \cdot (\bar\partial + \lfloor \partial, \iota_H \rfloor)\lambda \\
&= \tau\Big((\bar\partial^H u \wedge w) \otimes s\Big) - u \cdot \tau \check{d}_*^H (w \otimes s) && \text{by assumption} \\
&= \tau \check{d}_*^H ((u \wedge w) \otimes s).
\end{aligned}$$

This concludes the proof. □

4.4. Modular classes. The modular class of a Lie algebroid was introduced by Evens-Lu-Weinstein [12]. The following version for complex Lie algebroids appeared in the preprint version of [12] but not in the published paper. It is also implied in [14]. The presentation which we give below was communicated to us by Camille Laurent-Gengoux [23].

Let B be a complex Lie algebroid over a real manifold M, with $\mathrm{rk}_{\mathbb{C}}\, B = r$ and $\dim M = m$. Its Evens-Lu-Weinstein module is $Q_B = \wedge^r B \otimes \wedge^m T^*_{\mathbb{C}} M$.

Consider the complex of sheaves

$$\widetilde{\mathcal{S}}^0 \xrightarrow{\tilde{d}_B} \mathcal{S}^1 \xrightarrow{d_B} \mathcal{S}^2 \cdots \xrightarrow{d_B} \mathcal{S}^r, \tag{31}$$

where $\widetilde{\mathcal{S}}^0$ is the sheaf of nowhere vanishing smooth complex valued functions on M; $\mathcal{S}^\bullet$ is the sheaf of sections of $\wedge^\bullet B^*$; d_B is the usual Lie algebroid cohomology differential; and $\widetilde{\mathrm{d}}_B f = \mathrm{d}_B \log f = \frac{\mathrm{d}_B f}{f}$, for all $f \in C^\infty(U, \mathbb{C}^\times)$, where U is an arbitrary open subset of M. We denote its hypercohomology by $\widetilde{\mathrm{H}}^\bullet(B, \mathbb{C})$. Note that in Eq. (31), if we replace $\widetilde{\mathcal{S}}^0$ by $\mathcal{S}^0$, the sheaf of smooth complex valued functions on M, and $\tilde{d}_B$ by the usual Lie algebroid differential d_B, the hypercohomology of the resulting complex of sheaves

$$\mathcal{S}^0 \xrightarrow{d_B} \mathcal{S}^1 \xrightarrow{d_B} \mathcal{S}^2 \cdots \xrightarrow{d_B} \mathcal{S}^r, \tag{32}$$

is isomorphic to the usual Lie algebroid cohomology $\mathrm{H}^\bullet(B, \mathbb{C})$ of the complex Lie algebroid B with trivial coefficients $\mathbb{C}$ since each $\mathcal{S}^\bullet$ is a soft sheaf. The exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{S} \to \widetilde{\mathcal{S}} \to 0,$$

where $\mathcal{S}$ (resp. $\widetilde{\mathcal{S}}$) stands for the complex of sheaves (32) (resp. (31)) and the locally constant sheaf $\mathbb{Z}$ is regarded as a complex of sheaves concentrated in degree 0, induces the long exact sequence

$$\cdots \to \mathrm{H}^i(M, \mathbb{Z}) \to \mathrm{H}^i(B, \mathbb{C}) \to \widetilde{\mathrm{H}}^i(B, \mathbb{C}) \to \mathrm{H}^{i+1}(M, \mathbb{Z}) \to \cdots .$$

Note that $\widetilde{\mathrm{H}}^\bullet(B, \mathbb{C})$ can be computed as the total cohomology of the Čech double complex

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
\uparrow \delta & & \uparrow \delta & & \uparrow \delta & & \\
\check{C}^2(\mathcal{U}; \widetilde{\mathcal{S}}^0) & \xrightarrow{\tilde{\mathrm{d}}_B} & \check{C}^2(\mathcal{U}; \mathcal{S}^1) & \xrightarrow{\mathrm{d}_B} & \check{C}^2(\mathcal{U}; \mathcal{S}^2) & \xrightarrow{\mathrm{d}_B} & \cdots \\
\uparrow \delta & & \uparrow \delta & & \uparrow \delta & & \\
\check{C}^1(\mathcal{U}; \widetilde{\mathcal{S}}^0) & \xrightarrow{\tilde{\mathrm{d}}_B} & \check{C}^1(\mathcal{U}; \mathcal{S}^1) & \xrightarrow{\mathrm{d}_B} & \check{C}^1(\mathcal{U}; \mathcal{S}^2) & \xrightarrow{\mathrm{d}_B} & \cdots \\
\uparrow \delta & & \uparrow \delta & & \uparrow \delta & & \\
\check{C}^0(\mathcal{U}; \widetilde{\mathcal{S}}^0) & \xrightarrow{\tilde{\mathrm{d}}_B} & \check{C}^0(\mathcal{U}; \mathcal{S}^1) & \xrightarrow{\mathrm{d}_B} & \check{C}^0(\mathcal{U}; \mathcal{S}^2) & \xrightarrow{\mathrm{d}_B} & \cdots ,
\end{array} \tag{33}$$

where $\mathcal{U} = \{U_i\}_{i \in I}$ is a good open cover of M and δ is the usual Čech coboundary operator.

Let $(U_i)_{i \in I}$ be a good open cover of M, and ω_i a nowhere vanishing section of Q_B over U_i. For all $i, j \in I$, there exists a unique nowhere vanishing function $f_{ij} \in C^\infty(U_{ij}, \mathbb{C}^\times)$ such that $\omega_i = f_{ij}\omega_j$. It is clear from the construction that

$$f_{ij} f_{jk} f_{ki} = 1.$$

Let $\xi_i \in \Gamma(B^*|_{U_i})$ be the modular 1-form on U_i corresponding to ω_i. That is, we have $\nabla_X \omega_i = \langle \xi_i | X \rangle \omega_i$ for all $X \in \Gamma(B|_{U_i})$, where ∇ denotes the canonical representation of B on Q_B of [12]. It thus follows that

$$\xi_i = \xi_j + \frac{d_B f_{ij}}{f_{ij}} = \xi_j + \tilde{d}_B f_{ij}.$$

As a consequence, (ξ_i, f_{ij}) is a 1-cocycle of the double complex (33), and therefore defines a class in $\widetilde{\mathrm{H}}^1(B, \mathbb{C})$.

Definition 4.14. *The class in* $\widetilde{\mathrm{H}}^1(B, \mathbb{C})$ *defined by* $[(\xi_i, f_{ij})]$ *is called the* **modular class** *of the complex Lie algebroid B, and denoted* $\mathrm{mod}(B)$.

Lemma 4.15. *Consider the long exact sequence*

$$\cdots \to \mathrm{H}^1(B, \mathbb{C}) \to \widetilde{\mathrm{H}}^1(B, \mathbb{C}) \xrightarrow{\tau} \mathrm{H}^2(M, \mathbb{Z}) \to \cdots .$$

The image of the modular class $\mathrm{mod}(B)$ *under* τ *is the first Chern class* $c_1(Q_B)$ *of* Q_B*. When* $c_1(Q_B) = 0$*, the modular class* $\mathrm{mod}(B)$ *is the image of a class in* $\mathrm{H}^1(B, \mathbb{C})$*, which is defined exactly in the same way using a global nowhere vanishing section, as the usual modular class in [12].*

A complex Lie algebroid B is said to be **unimodular** if its modular class vanishes. The following result follows immediately from Lemma 4.15.

Corollary 4.16. *A complex Lie algebroid B is unimodular if and only if $c_1(Q_B) = 0$ and for any fixed nowhere vanishing section $\omega \in \Gamma(Q_B)$, the modular section $\xi \in \Gamma(B^*)$ defined by*

$$\nabla_X \omega = \langle \xi | X \rangle \omega \qquad (\forall X \in \Gamma(B))$$

is a coboundary, i.e. $\xi = \mathrm{d}_B f$ for some $f \in C^\infty(M, \mathbb{C})$.

As a consequence, a complex Lie algebroid B is unimodular if and only if Q_B is isomorphic to the trivial module $\mathbb{C}$.

Proposition 4.17. *When $B = T^{0,1}\boldsymbol{X} \bowtie A^{1,0}$ is the derived complex Lie algebroid [24, 30] of a holomorphic Lie algebroid A over $\boldsymbol{X}$, B is a unimodular complex Lie algebroid if and only if A is a unimodular holomorphic Lie algebroid, i.e. Q_A is trivial as a holomorphic line bundle and there exists a holomorphic global section ω of Q_A such that $\nabla_X \omega = 0$ for all $X \in A$.*

Definition 4.18. *An extended Poisson manifold $(\boldsymbol{X}, H)$ is unimodular if its corresponding complex Lie algebroid A_H^* is unimodular.*

According to Theorem 4.3, we have

Proposition 4.19. *An extended Poisson manifold $(\boldsymbol{X}, H)$ is unimodular if and only if there exists a nowhere vanishing $(n, 0)$-form $\omega \in \Omega^{n,0}(\boldsymbol{X})$ such that*

$$\bar{\partial}\omega + \lfloor \partial, \iota_H \rfloor \omega = \bar{\partial}\omega + \partial \iota_H \omega = 0.$$

Remark 4.20. It is clear that, when $H = 0$, $(\boldsymbol{X}, H)$ is unimodular means that $\boldsymbol{X}$ is Calabi-Yau. Thus one can consider a unimodular extended Poisson manifold $(\boldsymbol{X}, H)$ as a generalized Calabi-Yau manifold.

As an immediate consequence of the discussion above, we have

Corollary 4.21. *For any unimodular extended Poisson manifold $(\boldsymbol{X}, H)$ of complex dimension n, we have*

$$\mathrm{H}_k(\boldsymbol{X}, H) \cong \mathrm{H}^{2n-k}(\boldsymbol{X}, H).$$

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