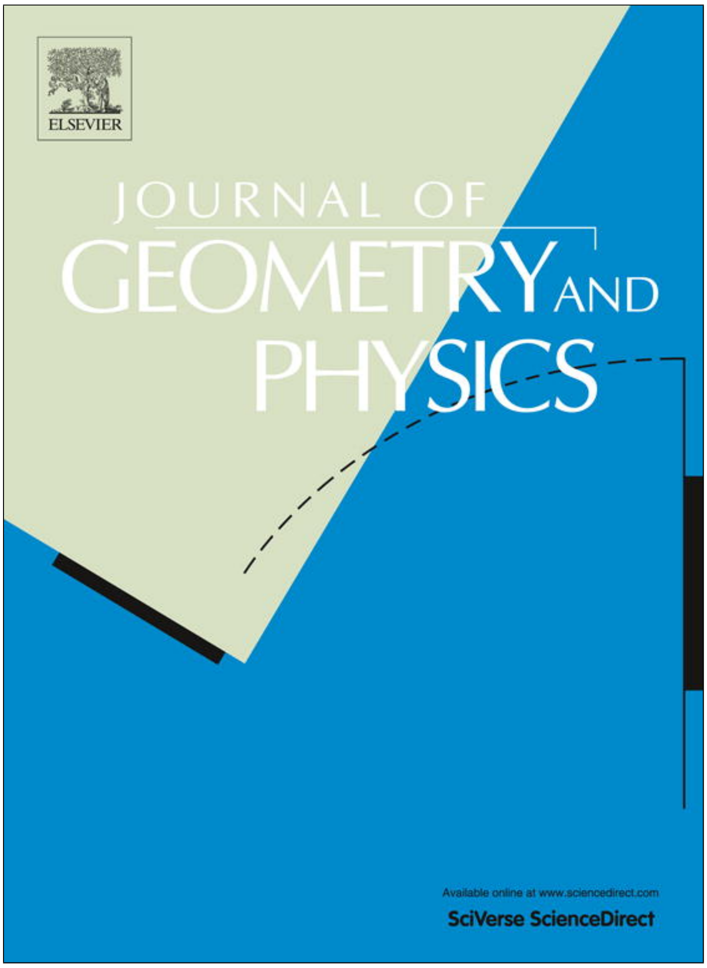
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Weak Lie 2-bialgebras☆

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ABSTRACT

We introduce the notion of weak Lie 2-bialgebra. Roughly, a weak Lie 2-bialgebra is a pair of compatible 2-term L_∞-algebra structures on a vector space and its dual. The compatibility condition is described in terms of the big bracket. We prove that (strict) Lie 2-bialgebras are in one–one correspondence with crossed modules of Lie bialgebras.

1. Introduction

The main purpose of the paper is to develop the notion of weak Lie 2-bialgebras. A Lie bialgebra is a Lie algebra endowed with a compatible Lie coalgebra structure. Lie bialgebras can be regarded as the classical limits of quantum groups. A celebrated theorem from Drinfeld establishes a bijection between Lie bialgebras and connected, simply connected Poisson Lie groups. Poisson 2-groups [1] are a natural first step in the search for an appropriate notion of quantum 2-groups, which can be considered as deformation quantization of ordinary Lie 2-groups. Their infinitesimal counterparts are called Lie 2-bialgebras or crossed modules of Lie bialgebras.

Recall that a Lie algebra crossed module consists of a pair of Lie algebras θ and $\mathfrak{g}$ together with a linear map $\phi : \theta \to \mathfrak{g}$ and an action of $\mathfrak{g}$ on θ by derivations satisfying a certain compatibility condition. A Lie bialgebra crossed module is a pair of Lie algebra crossed modules in duality: $\left(\theta \xrightarrow{\phi} \mathfrak{g}\right)$ and $\left(\mathfrak{g}^* \xrightarrow{-\phi^*} \theta^*\right)$ are both Lie algebra crossed modules, and $(\mathfrak{g} \ltimes \theta, \theta^* \ltimes \mathfrak{g}^*)$ is a Lie bialgebra.

It is well known that Lie algebra crossed modules are a special case of weak Lie 2-algebras (i.e. two-term L_∞ algebras) [2]. It is natural to ask what is a weak Lie 2-bialgebra. Such an object ought to be a weak Lie 2-algebra as well as a weak Lie 2-coalgebra, both structures being compatible with one another in a certain sense. Several notions of L_∞ bialgebras can be found in the existing literature, among which we can mention Kravchenko's homotopy Lie bialgebras [3] and Merkulov's homotopy Lie[1] bialgebras [4]. However, none of them serves our purpose. Although it is a weak Lie 2-algebra, a two-term homotopy Lie bialgebra in the sense of Kravchenko is, for instance, not a weak Lie 2-coalgebra due to the degree convention (see Remark 2.7). To obtain the correct compatibility condition, it turns out that one must shift the degree on the underlying $\mathbb{Z}$-graded vector space V so as to modify the "big bracket", which is a Gerstenhaber bracket on $S^\bullet(V[2] \oplus V^*[1])$. Identifying $S^\bullet(V[2] \oplus V^*[1])$ with the space $\Gamma(\wedge^\bullet T[4]M)$ of polyvector fields on $M = V^*[-2]$, the big bracket can be simply described

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as the Schouten bracket of polyvector fields. *In terms of the big bracket, a weak Lie 2-bialgebra on a graded vector space V is a degree (-4) element ε of $S^\bullet(V[2]\oplus V^*[1])$ such that* $\{\varepsilon,\varepsilon\}=0$. Lie 2-bialgebras arise as a special case of weak Lie 2-bialgebras where certain homotopy terms vanish. Our main theorem establishes a bijection between Lie 2-bialgebras and crossed modules of Lie bialgebras.

This is the first of a series of papers devoted to the study of Poisson 2-groups [1] and their quantization. We are grateful to the organizers of "Journée Quantique" (June 2010), "WAGP 2010" (June 2010), and "Poisson 2010" (July 2010), where we had the pleasure to present our results. Since then drafts of this work and slides of our conference talks have circulated in the community. Some of the results presented here were reproduced in an arXiv preprint posted in 2011 [5]. We would like to thank several institutions for their hospitality while work on this project was being done: Penn State University (Chen), Université du Luxembourg (Chen, Stiénon and Xu), Institut des Hautes Études Scientifiques and Beijing International Center for Mathematical Research (Xu). We would also like to thank Anton Alekseev, Benjamin Enriques, Yvette Kosmann-Schwarzbach, Henrik Strohmayer, Jim Stasheff, and Alan Weinstein for useful discussions and comments. We are grateful to the anonymous referee for carefully reading this paper.

Some notations are in order.

Notations: In this paper, all vector spaces are assumed to be finite dimensional. Given a graded vector space $V=\bigoplus_{k\in\mathbb{Z}}V^{(k)}$, $V[i]$ denotes the graded vector space obtained by shifting the grading on V according to the rule $(V[i])^{(k)}=V^{(i+k)}$, and V^* denotes the dual vector space, which is graded according to the rule $(V^*)^{(-k)}=(V^{(k)})^*$. Note in particular that $(V[i])^*=(V^*)[-i]$. We write $|e|$ for the degree of a homogeneous vector $e\in V$. The symbol $\odot$ is used for the symmetric tensor product: for any homogeneous vectors $e,f\in V$,

$$e\odot f=\frac{1}{2}(e\otimes f+(-1)^{|e||f|}f\otimes e).$$

The symmetric algebra over V will be denoted by $S^\bullet(V)$.

2. Lie 2-bialgebras

2.1. The big bracket

We will introduce a graded version of the big bracket [6,7] involving graded vector spaces.

Let $V=\bigoplus_{k\in\mathbb{Z}}V^{(k)}$ be a $\mathbb{Z}$-graded vector space. Consider the $\mathbb{Z}$-graded manifold $M=V^*[-2]$ and the shifted tangent space

$$T[4]M\cong(M\times V^*[-2])[4]\cong M\times V^*[2].$$

Consider the space of polyvector fields on M with polynomial coefficients:

$$\Gamma(\wedge^\bullet T[4]M)\cong S^\bullet(M^*)\otimes S^\bullet((V^*[2])[-1])\cong S^\bullet(V[2])\otimes S^\bullet(V^*[1])\cong S^\bullet(V[2]\oplus V^*[1]).$$

Here, elements of $V^*[-2]$, when thought of as having degree 1 more than their actual degrees in $V^*[-2]$, are exactly the sections of TM which are constant along M, i.e., the vector fields on M invariant under translation.

In the sequel, let us denote $S^\bullet(V[2]\oplus V^*[1])$ by $\mathcal{S}^\bullet$. The symmetric tensor product on $\mathcal{S}^\bullet$ will be denoted by $\odot$.

There is a standard way to endow $\mathcal{S}^\bullet=\Gamma(\wedge^\bullet T[4]M)$ with a graded Lie bracket, i.e. the Schouten bracket, denoted by $\{\cdot,\cdot\}$. It is a bilinear map $\{\cdot,\cdot\}:\mathcal{S}^\bullet\otimes\mathcal{S}^\bullet\to\mathcal{S}^\bullet$ satisfying the following properties:

(1) $\{v,v'\}=\{\epsilon,\epsilon'\}=0$, $\{v,\epsilon\}=(-1)^{|v|}\langle v\mid\epsilon\rangle$, $\forall v,v'\in V[2],\epsilon,\epsilon'\in V^*[1]$;
(2) $\{e_1,e_2\}=-(-1)^{(|e_1|+3)(|e_2|+3)}\{e_2,e_1\}$, $\forall e_i\in\mathcal{S}^\bullet$;
(3) $\{e_1,e_2\odot e_3\}=\{e_1,e_2\}\odot e_3+(-1)^{(|e_1|+3)|e_2|}e_2\odot\{e_1,e_3\}$, $\forall e_i\in\mathcal{S}^\bullet$.

It is clear that $\{\cdot,\cdot\}$ is of degree 3, i.e.

$$|\{e_1,e_2\}|=|e_1|+|e_2|+3,$$

for all homogeneous $e_i\in\mathcal{S}^\bullet$, and the following graded Jacobi identity holds:

$$\{e_1,\{e_2,e_3\}\}=\{\{e_1,e_2\},e_3\}+(-1)^{(|e_1|+3)(|e_2|+3)}\{e_2,\{e_1,e_3\}\}.$$

Hence $(\mathcal{S}^\bullet,\odot,\{\cdot,\cdot\})$ is a Schouten algebra, also known as an odd Poisson algebra, or a Gerstenhaber algebra [8].

Remark 2.1. Due to our degree convention, when V is a vector space considered as a graded vector space concentrated at degree 0, the big bracket above is different from the usual big bracket in the literature [6].

An element $F \in S^p(V[2]) \odot S^q(V^*[1])$ can be considered as a q-polyvector field on $M = V^*[-2]$, while an element $x \in S^\bullet(V[2])$ can be considered as a function on $V^*[-2]$. Therefore, by applying F to q-tuples of functions on $M = V^*[-2]$, i.e., taking successive Schouten brackets of F with functions on $M = V^*[-2]$, we obtain a multilinear map:

$$D_F : \underbrace{S^\bullet(V[2]) \otimes \cdots \otimes S^\bullet(V[2])}_{q\text{-tuples}} \longrightarrow S^\bullet(V[2]),$$

by

$$D_F(x_1, \ldots, x_q) = \left\{\left\{\cdots\{\{F, x_1\}, x_2\}, \ldots, x_{q-1}\right\}, x_q\right\},$$

for all $x_i \in S^\bullet(V[2])$.

It is easy to see that

$$D_F(x_1, \ldots, x_i, x_{i+1}, \ldots, x_q) = (-1)^{(|x_i|+3)(|x_{i+1}|+3)} D_F(x_1, \ldots, x_{i+1}, x_i, \ldots, x_q). \tag{1}$$

For any $E \in S^k(V[2]) \odot S^l(V^*[1])$, and $F \in S^p(V[2]) \odot S^q(V^*[1])$, we have

$$\begin{aligned} D_{\{E,F\}}(x_1, \ldots, x_n) = & \sum_{\sigma \in \mathfrak{S}(q,l-1)} \epsilon(\breve{\sigma}) D_E(D_F(x_{\sigma(1)}, \ldots, x_{\sigma(q)}), x_{\sigma(q+1)}, \ldots, x_{\sigma(n)}) \\ & - (-1)^{(|E|+3)(|F|+3)} \sum_{\sigma \in \mathfrak{S}(l,q-1)} \epsilon(\breve{\sigma}) D_F(D_E(x_{\sigma(1)}, \ldots, x_{\sigma(l)}), x_{\sigma(l+1)}, \ldots, x_{\sigma(n)}), \end{aligned} \tag{2}$$

for all $x_1, \ldots, x_n \in S^\bullet(V[2])$, where $n = q + l - 1$. Here $\mathfrak{S}(j, n - j)$ denotes the collection of $(j, n - j)$-shuffles and $\epsilon(\breve{\sigma})$ denotes the Koszul sign: switching any two successive elements x_i and x_{i+1} leads to a sign change $(-1)^{(|x_i|+3)(|x_{i+1}|+3)}$.

2.2. Weak Lie 2-algebras, coalgebras and bialgebras

Following Baez–Crans [2], there is an equivalence of 2-categories between the 2-category of weak Lie 2-algebras the 2-category of 2-term L_∞-algebras. Unfolding the L_∞-structure on the 2-term graded vector space $V = \theta \oplus \mathfrak{g}$, where θ is of degree 1 and $\mathfrak{g}$ is of degree 0, we can equivalently define a weak Lie 2-algebra as a pair of vector spaces θ and $\mathfrak{g}$ endowed with the following structures:

(a) a linear map $\phi\colon \theta \to \mathfrak{g}$;
(b) a bilinear skew-symmetric map $[\cdot,\cdot]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$;
(c) a bilinear map $\cdot \succ \cdot\colon \mathfrak{g} \otimes \theta \to \theta$;
(d) a trilinear skew-symmetric map $h\colon \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \to \theta$, called the homotopy map.

These maps are required to satisfy the following compatibility conditions: for all $w, x, y, z \in \mathfrak{g}$ and $u, v \in \theta$,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] + (\phi \circ h)(x, y, z) = 0; \tag{3}$$

$$y \succ (x \succ u) - x \succ (y \succ u) + [x, y] \succ u + h(\phi(u), x, y) = 0; \tag{4}$$

$$\phi(u) \succ v + \phi(v) \succ u = 0; \tag{5}$$

$$\phi(x \succ u) = [x, \phi(u)]; \tag{6}$$

$$\begin{aligned} & -w \succ h(x, y, z) - y \succ h(x, z, w) + z \succ h(x, y, w) + x \succ h(y, z, w) \\ & \quad = h([x, y], z, w) - h([x, z], y, w) + h([x, w], y, z) + h([y, z], x, w) - h([y, w], x, z) + h([z, w], x, y). \end{aligned} \tag{7}$$

If h vanishes, we call it a strict Lie 2-algebra, or simply a Lie 2-algebra.

Now consider the degree-shifted vector spaces $V[2]$ and $V^*[1]$. Under such a degree convention, the degrees of $\mathfrak{g}$, θ, $\mathfrak{g}^*$ and θ^* are specified in the following table:

space	$\mathfrak{g}$	θ	$\mathfrak{g}^*$	θ^*
degree	-2	-1	-1	-2

Remark 2.2. The reason that we are using such a degree convention can be summarized as follows. First, under such assumptions, the elements l in Proposition 2.3, c in Proposition 2.5 and ε in Definition 2.6 all will be of homogeneously degree (-4). Second, we see from the above table that $(\theta, \mathfrak{g})$ and $(\mathfrak{g}^*, \theta^*)$ are symmetric. In fact, when we define the notion of a Lie bialgebra crossed module in the sequel, we are asking $\left(\theta \xrightarrow{\phi} \mathfrak{g}\right)$ and $\left(\mathfrak{g}^* \xrightarrow{-\phi^*} \theta^*\right)$ both to be Lie algebra crossed modules.

We will maintain this convention throughout this paper. Recall that $\mathcal{S}^\bullet = S^\bullet(V^*[1] \oplus V[2])$.

Proposition 2.3. *Under the above degree convention, a weak Lie 2-algebra structure is equivalent to a solution to the equation:*

$$\{l, l\} = 0, \tag{8}$$

where $l = \varepsilon_{01}^{10} + \varepsilon_{00}^{12} + \varepsilon_{11}^{01} + \varepsilon_{10}^{03}$ *in* $\mathcal{S}^{(-4)}$ *such that*

$$\begin{cases} \varepsilon_{01}^{10} \in \theta^* \odot \mathfrak{g}, \\ \varepsilon_{00}^{12} \in (\odot^2 \mathfrak{g}^*) \odot \mathfrak{g}, \\ \varepsilon_{11}^{01} \in \mathfrak{g}^* \odot \theta^* \odot \theta, \\ \varepsilon_{10}^{03} \in (\odot^3 \mathfrak{g}^*) \odot \theta. \end{cases} \tag{9}$$

Here the bracket in Eq. (8) stands for the big bracket as in Section 2.1, and the notation

$$\varepsilon_{kl}^{pq} \in (\odot^q \mathfrak{g}^*) \odot (\odot^l \theta^*) \odot (\odot^k \theta) \odot (\odot^p \mathfrak{g})$$

helps the reader to keep track of its underlying space.

Proof. There is a bijection between the structure maps ϕ, $[\cdot,\cdot]$, $\cdot \succ \cdot$ and h and the data $\varepsilon_{01}^{10}, \varepsilon_{00}^{12}, \varepsilon_{11}^{01}$ and ε_{10}^{03}. They are related by the following equations:

$$\phi(u) = D_{\varepsilon_{01}^{10}}(u),$$
$$[x, y] = D_{\varepsilon_{00}^{12}}(x, y),$$
$$x \succ u = D_{\varepsilon_{11}^{01}}(x, u),$$
$$h(x, y, z) = D_{\varepsilon_{10}^{03}}(x, y, z),$$

$\forall x, y, z \in \mathfrak{g}, u \in \theta$.

Since $l = \varepsilon_{01}^{10} + \varepsilon_{00}^{12} + \varepsilon_{11}^{01} + \varepsilon_{10}^{03} \in \mathcal{S}^{(-4)}$, a simple computation leads to

$$\begin{aligned} \{l, l\} = &\left\{\varepsilon_{00}^{12}, \varepsilon_{00}^{12}\right\} + \left\{\varepsilon_{11}^{01}, \varepsilon_{11}^{01}\right\} + 2\left\{\varepsilon_{01}^{10}, \varepsilon_{00}^{12}\right\} + 2\left\{\varepsilon_{01}^{10}, \varepsilon_{11}^{01}\right\} \\ &+ 2\left\{\varepsilon_{01}^{10}, \varepsilon_{10}^{03}\right\} + 2\left\{\varepsilon_{00}^{12}, \varepsilon_{11}^{01}\right\} + 2\left\{\varepsilon_{00}^{12}, \varepsilon_{10}^{03}\right\} + 2\left\{\varepsilon_{11}^{01}, \varepsilon_{10}^{03}\right\}. \end{aligned}$$

By using Eq. (2), we have, $\forall x, y, z \in \mathfrak{g}$,

$$\begin{aligned} D_{\{l,l\}}(x, y, z) &= D_{\left\{\varepsilon_{00}^{12}, \varepsilon_{00}^{12}\right\}}(x, y, z) + 2D_{\left\{\varepsilon_{01}^{10}, \varepsilon_{10}^{03}\right\}}(x, y, z) \\ &= 2D_{\varepsilon_{00}^{12}}(D_{\varepsilon_{00}^{12}}(x, y), z) + \text{ c.p. } + 2D_{\varepsilon_{01}^{10}}(D_{\varepsilon_{10}^{03}}(x, y, z)) \\ &= 2([[x, y], z] + \text{ c.p. } + \phi \circ h(x, y, z)). \end{aligned} \tag{10}$$

Similarly,

$$D_{\{l,l\}}(x, y, u) = 2\left(y \succ (x \succ u) - x \succ (y \succ u) + [x, y] \succ u\right) + h(\phi(u), x, y), \tag{11}$$

$$D_{\{l,l\}}(u, x) = 2\left(\phi(x \succ u) + [\phi(u), x]\right), \tag{12}$$

$$D_{\{l,l\}}(u, v) = 2\left(\phi(u) \succ v + \phi(v) \succ u\right), \tag{13}$$

$$D_{\{l,l\}}(x, y, z, w) = 2\left(h([x, y], z, w) + w \succ h(x, y, z) + \text{ c.p. }\right). \tag{14}$$

It thus follows that $\{l, l\}$ vanishes if and only if the LHS of Eqs. (10)–(14) vanish. The latter is equivalent to the compatibility conditions defining a weak Lie 2-algebra. This concludes the proof. □

In the sequel, we denote a weak Lie 2-algebra by $(\theta \to \mathfrak{g}, l)$ in order to emphasize the map from θ to $\mathfrak{g}$. Sometimes, we will omit l and denote a weak Lie 2-algebra simply by $(\theta \to \mathfrak{g})$. If $(\mathfrak{g}^* \to \theta^*)$ is a weak Lie 2-algebra, then $(\theta \to \mathfrak{g})$ is called a weak Lie 2-coalgebra.

Remark 2.4. Equivalently, a weak Lie 2-coalgebra underlying $(\theta \to \mathfrak{g})$ is a 2-term L_∞-structure on $\mathfrak{g}^* \oplus \theta^*$, where $\mathfrak{g}^*$ has degree 1 and θ^* has degree 0.

Similarly, we have the following

Proposition 2.5. *A weak Lie 2-coalgebra is equivalent to a solution to the equation:*

$$\{c, c\} = 0,$$

where $c = \varepsilon_{01}^{10} + \varepsilon_{21}^{00} + \varepsilon_{10}^{11} + \varepsilon_{30}^{01} \in \mathcal{S}^{(-4)}$ *such that*

$$\begin{cases} \varepsilon_{01}^{10} \in \theta^* \odot \mathfrak{g}, \\ \varepsilon_{21}^{00} \in \theta^* \odot (\odot^2\theta), \\ \varepsilon_{10}^{11} \in \mathfrak{g}^* \odot \mathfrak{g} \odot \theta, \\ \varepsilon_{30}^{01} \in \mathfrak{g}^* \odot (\odot^3\theta). \end{cases} \tag{15}$$

We denote such a weak Lie 2-coalgebra by $(\theta \to \mathfrak{g}, c)$.

Now we are ready to introduce the main object of this section.

Definition 2.6. A weak Lie 2-bialgebra consists of a pair of vector spaces θ and $\mathfrak{g}$ together with a solution $\varepsilon = \varepsilon_{00}^{12} + \varepsilon_{11}^{01} + \varepsilon_{10}^{03} + \varepsilon_{01}^{10} + \varepsilon_{21}^{00} + \varepsilon_{10}^{11} + \varepsilon_{30}^{01} \in \mathcal{S}^{(-4)}$ to the equation:

$$\{\varepsilon, \varepsilon\} = 0.$$

Here $\varepsilon_{00}^{12}, \varepsilon_{11}^{01}, \varepsilon_{10}^{03}, \varepsilon_{01}^{10}, \varepsilon_{21}^{00}, \varepsilon_{10}^{11}, \varepsilon_{30}^{01}$ are as in Eqs. (9) and (15).

If, moreover, $\varepsilon_{10}^{03} = 0$, it is called a quasi-Lie 2-bialgebra. If both ε_{10}^{03} and ε_{30}^{01} vanish, we say that the Lie 2-bialgebra is strict, or simply a Lie 2-bialgebra.

Remark 2.7. Note that, in the literature, there exist notions of homotopy Lie bialgebras [3] and Lie 2-bialgebras [4]. However, weak Lie 2-bialgebras in our sense are neither of them. The pattern is that any of these notions are given by a homogeneous element h in a (even or odd) Poisson algebra satisfying $\{h, h\} = 0$, whose bracket $\{\cdot, \cdot\}$ are analogues of the usual big bracket of Kosmann-Schwarzbach [6]. The big bracket in [3] is defined on $S^\bullet(V \oplus V^*)$ (without any degree shifting) and generalizes the usual big bracket. Our big bracket in Section 2.1, however, does not reduce to the usual big bracket when V is a vector space considered as a graded vector space concentrated at degree 0. As a result, the usual Lie bialgebras in the sense of Drinfeld [9] are not Lie 2-bialgebras in our sense, but are homotopy Lie bialgebras in the sense of Kravchenko [3].

On the other hand, the big bracket in [4] is an odd Poisson structure, and defined on $S^\bullet(V \oplus V^*[1])$. Although it can be identified with our big bracket in Section 2.1 under some proper degree adjustment, an element h that form a Lie 2-bialgebra in the sense of Merkulov does not define a weak Lie 2-bialgebra in our sense because it has degree 2 as an element in $S^\bullet(V \oplus V^*[1])$, whereas in $S^\bullet(V^*[1] \oplus V[2])$ it is not even homogeneous.

Remark 2.8. The reader may have already noticed that for the degree-(-4) element ε in defining weak Lie 2-bialgebras, we do not consider terms of the form $\odot^2\mathfrak{g}, \odot^2\theta^*, (\odot^2\theta) \odot (\odot^2\mathfrak{g}^*), \odot^4\theta, \odot^4\mathfrak{g}^*, (\odot^2\theta) \odot \mathfrak{g}$ and $(\odot^2\mathfrak{g}^*) \odot \theta^*$. The reason is explained in a companion paper [1], where it is shown that the infinitesimal of a Poisson 2-group corresponds exactly to a Lie 2-bialgebra. Incorporating these missing terms would lead to a notion of quasi-weak Lie 2-bialgebra, by analogy with the interpretation of quasi-Lie bialgebras of the usual big brackets. We expect further studies in this direction.

Proposition 2.9. *Let* $(\theta, \mathfrak{g}, \varepsilon)$ *be a weak Lie* 2*-bialgebra as in Definition* 2.6*. Then* $(\theta \to \mathfrak{g}, l)$*, where* $l = \varepsilon_{01}^{10} + \varepsilon_{00}^{12} + \varepsilon_{11}^{01} + \varepsilon_{10}^{03}$*, is a weak Lie* 2*-algebra, while* $(\theta \to \mathfrak{g}, c)$*, where* $c = \varepsilon_{01}^{10} + \varepsilon_{21}^{00} + \varepsilon_{10}^{11} + \varepsilon_{30}^{01}$*, is a weak Lie* 2*-coalgebra.*

Proof. It is easy to see, by examining each component, that $\{\varepsilon, \varepsilon\} = 0$ implies $\{l, l\} = 0$ and $\{c, c\} = 0$. Hence $(\theta \to \mathfrak{g}, l)$ is a weak Lie 2-algebra and $(\theta \to \mathfrak{g}, c)$ is a weak Lie 2-coalgebra by Propositions 2.3 and 2.5. □

Example 2.10. Assume that $\mathfrak{g}$ is a semisimple Lie algebra. Let $(\cdot, \cdot)^{\mathfrak{g}}$ be its Killing form. Then $h(x, y, z) = \hbar(x, [y, z])^{\mathfrak{g}}$, $\forall\, x, y, z \in \mathfrak{g}$, is a Lie algebra 3-cocycle, where $\hbar$ is a constant. Let $\theta = \mathbb{R}$. Then the trivial map $\mathbb{R} \to \mathfrak{g}$ together with h becomes a weak Lie 2-algebra, called the string Lie 2-algebra [2]. More precisely, the string Lie 2-algebra is as follows:

(a) θ is the abelian Lie algebra $\mathbb{R}$;
(b) $\mathfrak{g}$ is a semisimple Lie algebra;
(c) $\phi : \theta \to \mathfrak{g}$ is the trivial map;
(d) the action map $\triangleright : \mathfrak{g} \otimes \theta \to \theta$ is the trivial map;
(e) $h : \wedge^3 \mathfrak{g} \to \theta$ is given by the map $\hbar(\cdot, [\cdot, \cdot])^{\mathfrak{g}}$, where $\hbar$ is a fixed constant.

Now fix an element $x \in \mathfrak{g}$. We equip a weak Lie 2-coalgebra on $\mathbb{R} \to \mathfrak{g}$ as follows:

(a) $\mathfrak{g}^*$ is an abelian Lie algebra;
(b) $\theta^* \cong \mathbb{R}$ is an abelian Lie algebra;
(c) $\phi^* : \mathfrak{g}^* \to \theta^*$ is the trivial map;
(d) the θ^*-action on $\mathfrak{g}^*$ is given by $\mathbf{1} \triangleright \xi = \mathrm{ad}_x^* \xi$, $\forall\, \xi \in \mathfrak{g}^*$;
(e) $\tilde{\eta} : \wedge^3 \theta^* \to \mathfrak{g}^*$ is the trivial map.

One can verify directly that these relations indeed define a weak Lie 2-bialgebra.

3. Lie bialgebra crossed modules

3.1. Definition

Definition 3.1. A Lie algebra crossed module consists of a pair of Lie algebras θ and $\mathfrak{g}$, and a linear map $\phi : \theta \to \mathfrak{g}$ such that $\mathfrak{g}$ acts on θ by derivations and satisfies, for all $x, y \in \mathfrak{g}, u, v \in \theta$,

(1) $\phi(u) \triangleright v = [u, v]$;
(2) $\phi(x \triangleright u) = [x, \phi(u)]$,

where $\triangleright$ denotes the $\mathfrak{g}$-action on θ.

Remark 3.2. Note that (1) and (2) imply that ϕ must be a Lie algebra homomorphism.

We write $\left(\theta \xrightarrow{\phi} \mathfrak{g}\right)$ to denote a Lie algebra crossed module. The associated semidirect product Lie algebra is denoted by $\mathfrak{g} \ltimes \theta$.

Proposition 3.3. *Given a Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$-module θ and a linear map $\phi : \theta \to \mathfrak{g}$ satisfying the following two conditions:*

$$\phi(x \triangleright u) = [x, \phi(u)], \tag{16}$$

$$\phi(u) \triangleright v = -\phi(v) \triangleright u, \tag{17}$$

for all $u, v \in \theta, x \in \mathfrak{g}$, there exists a unique Lie algebra structure on θ such that $\left(\theta \xrightarrow{\phi} \mathfrak{g}\right)$ is a Lie algebra crossed module.

In other words, any Lie algebra crossed module underlying $\phi : \theta \to \mathfrak{g}$ is determined by a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module θ satisfying Eqs. (16) and (17).

Proof. Define the Lie bracket on θ by $[u, v] = \phi(u) \triangleright v, \quad \forall u, v \in \theta$. The rest of the claim can be easily verified directly. □

Comparing with the definition of a weak Lie 2-algebra at the beginning of Section 2.2, we see that a Lie algebra crossed module is equivalent to a (strict) Lie 2-algebra.

We are now ready to introduce the following

Definition 3.4. A Lie bialgebra crossed module is a pair of Lie algebra crossed modules in duality: $\left(\theta \xrightarrow{\phi} \mathfrak{g}\right)$ and $\left(\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*\right)$, where $\phi^T = -\phi^*$, are both Lie algebra crossed modules such that $(\mathfrak{g} \ltimes \theta, \theta^* \ltimes \mathfrak{g}^*)$ is a Lie bialgebra.

Proposition 3.5. *If $\left(\left(\theta \xrightarrow{\phi} \mathfrak{g}\right), \left(\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*\right)\right)$ is a Lie bialgebra crossed module, so is $\left(\left(\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*\right), \left(\theta \xrightarrow{\phi} \mathfrak{g}\right)\right)$.*

The following proposition justifies our terminology.

Proposition 3.6. *If $\left(\left(\theta \xrightarrow{\phi} \mathfrak{g}\right), \left(\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*\right)\right)$ is a Lie bialgebra crossed module, then both pairs (θ, θ^*) and $(\mathfrak{g}, \mathfrak{g}^*)$ are Lie bialgebras.*

Proof. Since θ and θ^* are Lie subalgebras of $\mathfrak{g} \ltimes \theta$ and $\theta^* \ltimes \mathfrak{g}^*$, respectively, and $(\mathfrak{g} \ltimes \theta, \theta^* \ltimes \mathfrak{g}^*)$ is a Lie bialgebra, it follows that (θ, θ^*) is a Lie bialgebra. Similarly, $(\mathfrak{g}, \mathfrak{g}^*)$ is also a Lie bialgebra. □

Example 3.7. We can construct a Lie bialgebra crossed module from an ordinary Lie bialgebra as follows. Given a Lie bialgebra (θ, θ^*), consider the trivial Lie algebra crossed module $\left(\theta \xrightarrow{\mathbf{1}} \theta\right)$, where the second θ acts on the first θ by the adjoint action. In the mean time, consider the dual Lie algebra crossed module $\left(\theta^* \xrightarrow{-\mathbf{1}} \theta^*\right)$, where the second θ^* is equipped with the opposite Lie bracket: $-[\cdot, \cdot]_*$, and the action of the second θ^* on the first θ^* is by $\kappa_2 \triangleright \kappa_1 = -[\kappa_2, \kappa_1]_*, \ \forall \kappa_1, \kappa_2 \in \theta^*$. It is simple to see that $\left(\left(\theta \xrightarrow{\mathbf{1}} \theta\right), \left(\theta^* \xrightarrow{-\mathbf{1}} \theta^*\right)\right)$ is a Lie bialgebra crossed module.

3.2. Main theorem

We are now ready to state our main theorem.

Theorem 3.8. *There is a bijection between Lie bialgebra crossed modules and (strict) Lie 2-bialgebras.*

Remark 3.9. In fact, the collection of Lie bialgebra crossed modules form a strict 2-category and so does that of Lie 2-bialgebras. The above theorem can be enhanced to an equivalence of these 2-categories. This will be investigated somewhere else.

We need a few lemmas before proving this theorem.

For $k \geq 1$, write

$$W_k = \left\{ w \in \mathfrak{g} \wedge (\wedge^{k-1}\theta) \mid \iota_{\zeta_1}\iota_{\phi^*\zeta_2} w = -\iota_{\zeta_2}\iota_{\phi^*\zeta_1} w,\ \forall \zeta_1, \zeta_2 \in \mathfrak{g}^* \right\}. \tag{18}$$

Let

$$D_\phi : \wedge^\bullet(\mathfrak{g} \ltimes \theta) \to \wedge^\bullet(\mathfrak{g} \ltimes \theta)$$

denote the degree-0 derivation with respect to the wedge product such that $D_\phi(x+u) = \phi(u)$, $\forall x \in \mathfrak{g}, u \in \theta$.

Lemma 3.10. *A Lie algebra crossed module structure on* $\left(\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*\right)$, *where* $\phi^T = -\phi^*$, *is equivalent to a pair of linear maps* $\delta : \mathfrak{g} \to W_2 \subset \mathfrak{g} \wedge \theta$ *and* $\omega : \theta \to \wedge^2\theta$ *satisfying the following conditions*

(1) $D_\phi \circ \omega = \delta \circ \phi$;
(2) $\omega^2 = 0$;
(3) $(\omega + \delta) \circ \delta = 0$.

Here, we consider both ω and δ as degree-1 derivations on the exterior algebra $\wedge^\bullet(\mathfrak{g} \ltimes \theta)$, by letting $\omega|_\mathfrak{g} = 0$ and $\delta|_\theta = 0$.

Moreover in this case, the cobracket $\partial : \mathfrak{g} \ltimes \theta \to \wedge^2(\mathfrak{g} \ltimes \theta)$ corresponding to the Lie algebra structure on $\theta^* \ltimes \mathfrak{g}^*$ is given by:

$$\partial(x+u) = \omega(u) + \delta(x) + \pi(x), \quad \forall x \in \mathfrak{g}, u \in \theta, \tag{19}$$

where π is a linear map $\mathfrak{g} \to \wedge^2\mathfrak{g}$ given by $\pi = -\frac{1}{2}D_\phi \circ \delta$.

Proof. According to Proposition 3.3, a Lie algebra crossed module structure underlying $\left(\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*\right)$ is equivalent to assigning a Lie algebra structure $[\cdot,\cdot]_*$ on θ^* and an action $\triangleright$ of θ^* on $\mathfrak{g}^*$ such that

$$\phi^T(\kappa \triangleright \xi) = \left[\kappa, \phi^T(\xi)\right]_*, \tag{20}$$

$$\phi^T(\xi) \triangleright \zeta = -\phi^T(\zeta) \triangleright \xi, \tag{21}$$

for all $\xi, \zeta \in \mathfrak{g}^*$, $\kappa \in \theta^*$. Introduce linear maps δ and ω by

$$\langle \delta(x) \mid \xi \wedge \kappa \rangle = \langle x \mid \kappa \triangleright \xi \rangle;$$
$$\langle \omega(u) \mid \kappa_1 \wedge \kappa_2 \rangle = -\langle u \mid [\kappa_1, \kappa_2]_* \rangle,$$

$\forall x \in \mathfrak{g}, u \in \theta, \xi \in \mathfrak{g}^*, \kappa, \kappa_1, \kappa_2 \in \theta^*$. It is simple to see that $\omega^2 = 0$ is equivalent to the Jacobi identity for $[\cdot,\cdot]_*$, and $(\omega+\delta)\circ\delta = 0$ is equivalent to the fact that $\triangleright$ is an action of θ^* on $\mathfrak{g}^*$. Moreover, Eq. (20) is equivalent to the condition $D_\phi \circ \omega = \delta \circ \phi$ and Eq. (21) is equivalent to the condition that δ takes values in W_2.

To prove Eq. (19), we have, $\forall x \in \mathfrak{g}, u \in \theta, \xi, \zeta \in \mathfrak{g}^*, \kappa, \kappa_1, \kappa_2 \in \theta^*$,

$$\langle \partial(u) \mid \kappa_1 \wedge \kappa_2 \rangle = -\langle u \mid [\kappa_1, \kappa_2]_* \rangle = \langle \omega(u) \mid \kappa_1 \wedge \kappa_2 \rangle,$$
$$\langle \partial(x) \mid \kappa \wedge \xi \rangle = -\langle x \mid [\kappa, \xi]_* \rangle = -\langle x \mid \kappa \triangleright \xi \rangle = \langle \delta(x) \mid \kappa \wedge \xi \rangle,$$

and

$$\begin{aligned}
\langle \partial(x) \mid \xi \wedge \zeta \rangle &= -\langle x \mid [\xi, \zeta]_* \rangle \\
&= -\left\langle x \mid \phi^T(\xi) \triangleright \zeta \right\rangle \\
&= \left\langle \delta(x) \mid \phi^T(\xi) \wedge \zeta \right\rangle \\
&= \left\langle -\frac{1}{2}(D_\phi \circ \delta)(x) \mid \xi \wedge \zeta \right\rangle.
\end{aligned}$$

The conclusion thus follows. □

Proposition 3.11. *Let* $\left(\theta \xrightarrow{\phi} \mathfrak{g}\right)$ *be a Lie algebra crossed module. It is a Lie bialgebra crossed module if and only if there is a pair of linear maps* (δ, ω) *as in Lemma 3.10 that, in addition, satisfies the following conditions:*

(1) δ *is a Lie algebra* 1*-cocycle;*
(2) $x \triangleright \omega(u) - \omega(x \triangleright u) = \mathrm{Pr}_{\wedge^2\theta}([u, \delta(x)])$, *for all* $x \in \mathfrak{g}$, $u \in \theta$.

Proof. Assume that $\left(\theta \xrightarrow{\phi} \mathfrak{g}\right)$ and $\left(\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*\right)$ are Lie algebra crossed modules.

It suffices to prove that Conditions (1) and (2) are equivalent to $(\mathfrak{g} \ltimes \theta, \theta^* \ltimes \mathfrak{g}^*)$ being a Lie bialgebra. The latter is equivalent to

$$\partial\,[E, F] = [E, \partial(F)] - [F, \partial(E)]\,, \quad \forall\, E, F \in \mathfrak{g} \ltimes \theta, \tag{22}$$

where $\partial : \mathfrak{g} \ltimes \theta \to \wedge^2(\mathfrak{g} \ltimes \theta)$ is the cobracket as given in Eq. (19).

If both E and F are in $\mathfrak{g}$, it is simple to see that Eq. (22) is equivalent to $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \theta$ being a Lie algebra 1-cocycle. On the other hand, we claim that when $E = x \in \mathfrak{g}$ and $F = u \in \theta$, Eq. (22) is equivalent to the second condition in the statement of the proposition. First of all, note that Eq. (22) implies that

$$\begin{aligned}\partial\,[x, u] &= [x, \partial(u)] - [u, \partial(x)] \\ &= x \triangleright (\omega(u)) - \left[u, \delta(x) - \frac{1}{2}(D_\phi \circ \delta)(x)\right].\end{aligned} \tag{23}$$

Since $\partial\,[x, u] = \omega(x \triangleright u)$, it suffices to prove that $\left[u, \delta(x) - \frac{1}{2}(D_\phi \circ \delta)(x)\right] = \mathrm{Pr}_{\wedge^2\theta}([u, \delta(x)])$.

For this purpose, let us assume that $\delta(x) = \sum_i y_i \wedge v_i$, where $y_i \in \mathfrak{g}$ and $v_i \in \theta$. The condition that $\delta(x) \in W_2$ is essentially equivalent to

$$\sum_i (\langle \phi(v_i) \mid \xi \rangle\, y_i + \langle y_i \mid \xi \rangle\, \phi(v_i)) = 0, \quad \forall \xi \in \mathfrak{g}^*. \tag{24}$$

The latter implies that, for any $u \in \theta$,

$$\sum_i ((y_i \triangleright u) \wedge \phi(v_i) - y_i \wedge (\phi(v_i) \triangleright u)) = 0.$$

Indeed, for all $\xi \in \mathfrak{g}^*$,

$$\begin{aligned}\iota_\xi \left(\sum_i ((y_i \triangleright u) \wedge \phi(v_i) - y_i \wedge (\phi(v_i) \triangleright u))\right) &= -\sum_i ((y_i \triangleright u)\, \langle \phi(v_i) \mid \xi \rangle + \langle y_i \mid \xi \rangle\, (\phi(v_i) \triangleright u)) \\ &= -\sum_i (\langle \phi(v_i) \mid \xi \rangle\, y_i + \langle y_i \mid \xi \rangle\, \phi(v_i)) \triangleright u \\ &= 0.\end{aligned}$$

Thus, we have

$$\begin{aligned}\left[u, \delta(x) - \frac{1}{2}(D_\phi \circ \delta)(x)\right] &= \left[u, \sum_i y_i \wedge v_i - \frac{1}{2}\sum_i y_i \wedge \phi(v_i)\right] \\ &= \sum_i \left([u, y_i] \wedge v_i + y_i \wedge [u, v_i] + \frac{1}{2}(y_i \triangleright u) \wedge \phi(v_i) + \frac{1}{2} y_i \wedge (\phi(v_i) \triangleright u)\right) \\ &= \sum_i [u, y_i] \wedge v_i + \frac{1}{2}\sum_i ((y_i \triangleright u) \wedge \phi(v_i) - y_i \wedge (\phi(v_i) \triangleright u)) \\ &= \sum_i [u, y_i] \wedge v_i \\ &= \mathrm{Pr}_{\wedge^2\theta}([u, \delta(x)])\,.\end{aligned}$$

Finally, if both E and F are in θ, Eq. (22) is equivalent to $\omega : \theta \to \wedge^2\theta$ being a Lie algebra 1-cocycle. However, the latter follows from Conditions (1) and (2). To see this, for any $u, v \in \theta$, we have

$$\begin{aligned}\omega\,[u, v] &= \omega(\phi(u) \triangleright v) \\ &= \phi(u) \triangleright \omega(v) - \mathrm{Pr}_{\wedge^2\theta}([v, \delta(\phi(u))]) \\ &= [u, \omega(v)] - \mathrm{Pr}_{\wedge^2\theta}\left(\left[v, D_\phi(\omega(u))\right]\right) \\ &= [u, \omega(v)] - [v, \omega(u)]\,.\end{aligned}$$

Here in the last equality, we used the following identity

$$\mathrm{Pr}_{\wedge^2\theta}\left(\left[v, D_\phi(\zeta)\right]\right) = [v, \zeta], \quad \forall \zeta \in \wedge^2\theta,$$

which can be proved by a direct verification.

This concludes the proof. □

Now we are ready to prove Theorem 3.8.

Proof of Theorem 3.8. Let $\varepsilon = \varepsilon^{12}_{00}+\varepsilon^{01}_{11}+\varepsilon^{10}_{01}+\varepsilon^{00}_{21}+\varepsilon^{11}_{10} \in \mathcal{S}^{(-4)}$, where $\varepsilon^{12}_{00}, \varepsilon^{01}_{11}, \varepsilon^{10}_{01}, \varepsilon^{00}_{21}, \varepsilon^{11}_{10}$ are given as in Eqs. (9) and (15). It is simple to see that the equation $\{\varepsilon, \varepsilon\} = 0$ is equivalent to the following three equations:

$$\left\{\varepsilon^{12}_{00} + \varepsilon^{01}_{11} + \varepsilon^{10}_{01}, \varepsilon^{12}_{00} + \varepsilon^{01}_{11} + \varepsilon^{10}_{01}\right\} = 0; \tag{25}$$

$$\left\{\varepsilon^{10}_{01} + \varepsilon^{00}_{21} + \varepsilon^{11}_{10}, \varepsilon^{10}_{01} + \varepsilon^{00}_{21} + \varepsilon^{11}_{10}\right\} = 0; \tag{26}$$

$$\left\{\varepsilon^{12}_{00}, \varepsilon^{11}_{10}\right\} + \left\{\varepsilon^{01}_{11}, \varepsilon^{00}_{21}\right\} + \left\{\varepsilon^{01}_{11}, \varepsilon^{11}_{10}\right\} = 0. \tag{27}$$

According to Proposition 2.3, Eq. (25) is equivalent to the fact that $\left(\theta \xrightarrow{\phi} \mathfrak{g}\right)$ is a Lie algebra crossed module, where $\phi(u) = D_{\varepsilon^{10}_{01}}(u)$, $\forall\, u \in \theta$, the Lie bracket on $\mathfrak{g}$ is given by $[x, y] = D_{\varepsilon^{12}_{00}}(x, y)$, $\forall\, x, y \in \mathfrak{g}$, and the $\mathfrak{g}$-action on θ is given by $x \triangleright u = D_{\varepsilon^{01}_{11}}(x, u)$, $\forall x \in \mathfrak{g}, u \in \theta$. According to Proposition 2.5, Eq. (26) is equivalent to the fact that $\left(\theta \xrightarrow{\phi} \mathfrak{g}\right)$ is a Lie 2-coalgebra, or $\left(\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*\right)$ is a Lie algebra crossed module. It is simple to show, by a straightforward computation, that the linear maps δ and ω associated to the Lie algebra crossed module $\left(\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*\right)$ are related to ε^{00}_{21} and ε^{11}_{10} by the following relations:

$$\langle \delta(x) \mid \xi \wedge \kappa\rangle = \left\{x, \left\{\left\{\varepsilon^{11}_{10}, \kappa\right\}, \xi\right\}\right\} = -\left\{\left\{D_{\varepsilon^{11}_{10}}(x), \xi\right\}, \kappa\right\};$$

$$\langle \omega(u) \mid \kappa_1 \wedge \kappa_2\rangle = \left\{u, \left\{\left\{\varepsilon^{00}_{21}, \kappa_1\right\}, \kappa_2\right\}\right\} = \left\{\left\{D_{\varepsilon^{00}_{21}}(u), \kappa_1\right\}, \kappa_2\right\},$$

$\forall \xi \in \mathfrak{g}^*, \kappa, \kappa_1, \kappa_2 \in \theta^*$.

Since the left hand side of Eq. (27) belongs to $(\odot^2\mathfrak{g}^*) \odot \mathfrak{g} \odot \theta + (\odot^2\theta) \odot \mathfrak{g}^* \odot \theta^*$, hence we have

$$\begin{aligned}
&\left\{\left\{D_{\left\{\varepsilon^{12}_{00},\varepsilon^{11}_{10}\right\}+\left\{\varepsilon^{01}_{11},\varepsilon^{00}_{21}\right\}+\left\{\varepsilon^{01}_{11},\varepsilon^{11}_{10}\right\}}(x, y), \xi\right\}, \kappa\right\}\\
&\quad = \left\{\left\{D_{\left\{\varepsilon^{12}_{00},\varepsilon^{11}_{10}\right\}}(x, y) + D_{\left\{\varepsilon^{01}_{11},\varepsilon^{11}_{10}\right\}}(x, y), \xi\right\}, \kappa\right\}\\
&\quad = \left\{\left\{D_{\varepsilon^{11}_{10}}([x, y]) + D_{\varepsilon^{12}_{00}+\varepsilon^{01}_{11}}(D_{\varepsilon^{11}_{10}}(x), y) - D_{\varepsilon^{12}_{00}+\varepsilon^{01}_{11}}(D_{\varepsilon^{11}_{10}}(y), x), \xi\right\}, \kappa\right\}\\
&\quad = \langle -\delta\,[x, y] + [x, \delta(y)] - [y, \delta(x)] \mid \xi \wedge \kappa\rangle;
\end{aligned}$$

and

$$\begin{aligned}
&\left\{\left\{D_{\left\{\varepsilon^{12}_{00},\varepsilon^{11}_{10}\right\}+\left\{\varepsilon^{01}_{11},\varepsilon^{00}_{21}\right\}+\left\{\varepsilon^{01}_{11},\varepsilon^{11}_{10}\right\}}(x, u), \kappa_1\right\}, \kappa_2\right\}\\
&\quad = \left\{\left\{D_{\left\{\varepsilon^{01}_{11},\varepsilon^{00}_{21}\right\}}(x, u) + D_{\left\{\varepsilon^{01}_{11},\varepsilon^{11}_{10}\right\}}(x, u), \kappa_1\right\}, \kappa_2\right\}\\
&\quad = \left\{\left\{D_{\varepsilon^{00}_{21}}(x \triangleright u) + D_{\varepsilon^{01}_{11}}(D_{\varepsilon^{00}_{21}}(u), x) - D_{\varepsilon^{01}_{11}}(D_{\varepsilon^{11}_{10}}(x), u), \kappa_1\right\}, \kappa_2\right\}\\
&\quad = \left\langle -x \triangleright \omega(u) + \omega(x \triangleright u) + \mathrm{Pr}_{\wedge^2\theta}([u, \delta(x)]) \mid \kappa_1 \wedge \kappa_2\right\rangle.
\end{aligned}$$

Therefore it follows that Eq. (27) is equivalent to the fact that the pair (δ, ω) satisfies the two compatibility conditions in Proposition 3.11. Hence we conclude that $\{\varepsilon, \varepsilon\} = 0$ is equivalent to the fact that the couple $\left(\theta \xrightarrow{\phi} \mathfrak{g}\right)$ and $\left(\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*\right)$ is a Lie bialgebra crossed module. □

Theorem 3.12. *Assume that* $\left(\theta \xrightarrow{\phi} \mathfrak{g}\right)$ *and* $\left(\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*\right)$ *are Lie algebra crossed modules. Then they form a Lie bialgebra crossed module if and only if* $(\mathfrak{g}, \theta^*)$ *is a matched pair of Lie algebras, where the* $\mathfrak{g}$*-action on* θ^* *is dual to the given* $\mathfrak{g}$*-action on* θ*, while the* θ^**-action on* $\mathfrak{g}$ *is dual to the given* θ^**-action on* $\mathfrak{g}^*$*.*

Proof. By definition, $(\mathfrak{g}, \theta^*)$ is a matched pair of Lie algebras if and only if

$$\kappa \triangleright [x, y] = [x, \kappa \triangleright y] - [y, \kappa \triangleright x] + (y \triangleright \kappa) \triangleright x - (x \triangleright \kappa) \triangleright y, \quad (28)$$

$$x \triangleright [\kappa_1, \kappa_2]_* = [\kappa_1, x \triangleright \kappa_2]_* - [\kappa_2, x \triangleright \kappa_1]_* + (\kappa_2 \triangleright x) \triangleright \kappa_1 - (\kappa_1 \triangleright x) \triangleright \kappa_2, \quad (29)$$

for all $x, y \in \mathfrak{g}, \kappa_1, \kappa_2 \in \theta^*$.

We prove that Eq. (28) is equivalent to δ being a Lie algebra 1-cocycle, while Eq. (29) is equivalent to Condition (2) in Proposition 3.11.

Indeed, a tedious computation leads to

$$\langle \delta([x, y]) \mid \xi, \kappa \rangle - \langle x \triangleright \delta(y) \mid \xi, \kappa \rangle + \langle y \triangleright \delta(x) \mid \xi, \kappa \rangle$$
$$= \langle \xi \mid -\kappa \triangleright [x, y] + [x, \kappa \triangleright y] - [y, \kappa \triangleright x] + (y \triangleright \kappa) \triangleright x - (x \triangleright \kappa) \triangleright y \rangle,$$

$\forall x \in \mathfrak{g}, u \in \theta, \xi \in \mathfrak{g}^*, \kappa \in \theta^*$, and

$$\langle \omega(x \triangleright u) \mid \kappa_1, \kappa_2 \rangle - \langle x \triangleright \omega(u) \mid \kappa_1, \kappa_2 \rangle + \left\langle \mathrm{Pr}_{\wedge^2 \theta}([u, \delta(x)]) \mid \kappa_1, \kappa_2 \right\rangle$$
$$= \langle u \mid x \triangleright [\kappa_1, \kappa_2]_* + [\kappa_2, x \triangleright \kappa_1]_* - [\kappa_1, x \triangleright \kappa_2]_* - (\kappa_2 \triangleright x) \triangleright \kappa_1 + (\kappa_1 \triangleright x) \triangleright \kappa_2 \rangle.$$

This concludes the proof. □

Corollary 3.13. *Let (θ, θ^*) be a Lie bialgebra. Assume that $\mathfrak{I}$ is a subspace of the center $Z(\theta)$ (i.e. $[\theta, \mathfrak{I}] = 0$) and $\omega(\mathfrak{I}) \subset \wedge^2 \mathfrak{I}$, where $\omega : \theta \to \wedge^2 \theta$ is the cobracket on θ. Then there is an induced Lie bialgebra crossed module structure underlying $\left(\theta \xrightarrow{\phi} \mathfrak{g}\right)$, where $\mathfrak{g} = \theta/\mathfrak{I}$ is the quotient Lie algebra and ϕ is the projection.*

Proof. Identifying $\mathfrak{g}^*$ with $\mathfrak{I}^0 \subset \theta^*$, we see that $\mathfrak{g}^*$ is an ideal of θ^*, and the map $\phi^T : \mathfrak{g}^* \to \theta^*$ is the composition of the inclusion with $-I$. Hence $\left(\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*\right)$ is a Lie algebra crossed module. To prove that $\left(\left(\theta \xrightarrow{\phi} \mathfrak{g}\right), \left(\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*\right)\right)$ is a Lie bialgebra crossed module, it suffices to prove that $(\mathfrak{g}, \theta^*)$ is a matched pair of Lie algebras according to Theorem 3.12. Note that (θ, θ^*) is a Lie algebra matched pair since it is a Lie bialgebra. Therefore it remains to show that $\mathfrak{I} \oplus 0$ is an ideal of the double Lie algebra $D = \theta \bowtie \theta^*$. In fact, for any $u \in \mathfrak{I}$ and $\kappa \in \theta^*$, we have $\mathrm{ad}^*_u \kappa = 0$ and $\mathrm{ad}^*_\kappa u \in \mathfrak{I}$. Hence $[u, \kappa]_D = \mathrm{ad}^*_u \kappa - \mathrm{ad}^*_\kappa u \in \mathfrak{I} \oplus 0$.

This concludes the proof. □

Example 3.14. Consider the Lie subalgebra $\mathrm{u}(n) \subset \mathfrak{gl}_n(\mathbb{C})$ of $n \times n$ skew-Hermitian matrices. Let $\theta \subset \mathfrak{gl}_n(\mathbb{C})$ be the Lie subalgebra consisting of upper triangular matrices whose diagonal elements are real numbers. It is standard that $(\theta, \mathrm{u}(n))$ is a Lie bialgebra. Indeed $\theta \oplus \mathrm{u}(n) \cong \mathfrak{gl}_n(\mathbb{C})$, and both θ and $\mathfrak{gl}_n(\mathbb{C})$ are Lagrangian subalgebras of $\mathfrak{gl}_n(\mathbb{C})$ under the nondegenerate pairing $\langle X \mid Y \rangle = \mathrm{Im}(\mathrm{Tr}(XY)), \forall X, Y \in \mathfrak{gl}_n(\mathbb{C})$. Hence $(\mathfrak{gl}_n(\mathbb{C}), \theta, \mathrm{u}(n))$ is a Manin triple and thus $(\theta, \mathrm{u}(n))$ forms a Lie bialgebra.

Let $\mathfrak{I} = \mathbb{R}I$. It is clear that $\mathfrak{I}$ is the center of θ and $\omega(\mathfrak{I}) = 0$. Hence $\mathfrak{g} = \theta/\mathfrak{I}$ can be identified with the Lie algebra of traceless upper triangular matrices whose diagonal elements are real. As a consequence, $\left(\theta \xrightarrow{\phi} \mathfrak{g}\right)$, where ϕ is the map $A \to A - \mathrm{tr}A$, is a Lie bialgebra crossed module.

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