Teichmüller Mapping (T-Map) and Its Applications to Landmark Matching Registration*

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Abstract. Registration, which aims to find an optimal 1-1 correspondence between shapes, is an important process in different research areas. Landmark-based surface registration has been widely studied to obtain a mapping between shapes that matches important features. Obtaining a unique and bijective surface registration that matches features consistently is generally challenging, especially when a large number of landmark constraints are enforced. This motivates us to search for a unique landmark matching surface diffeomorphism, which minimizes the local geometric distortion. For this purpose, we propose a special class of diffeomorphisms called the *Teichmüller mappings* (T-Maps). Under suitable conditions on the landmark constraints, a unique T-Map between two surfaces can be obtained, which minimizes the maximal conformality distortion. The conformality distortion measures how far the mapping deviates from a conformal mapping, and hence it measures the local geometric distortion. In this paper, we propose an efficient iterative algorithm, called the quasi-conformal (QC) iteration, to compute the T-Map. The basic idea is to represent the set of diffeomorphisms using Beltrami coefficients (BCs) and look for an optimal BC associated to the desired T-Map. The associated diffeomorphism can be efficiently reconstructed from the optimal BC using the linear Beltrami solver (LBS). Using BCs to represent diffeomorphisms guarantees the diffeomorphic property of the registration, even with very large deformation. Using our proposed method, the T-Map can be accurately and efficiently computed. The obtained registration is guaranteed to be bijective. The proposed algorithm can also be extended to compute T-Map with soft landmark constraints. We applied the proposed algorithm to real applications, such as brain landmark matching registration, constrained texture mapping, and human face registration. Experimental results shows that our method is both effective and efficient in computing a nonoverlap landmark matching registration with the least amount of conformality distortion.

Key words. Teichmüller mapping, Beltrami coefficient, conformality distortion, linear Beltrami solver, landmark matching registration

AMS subject classifications. 65D, 68U, 68W

DOI. 10.1137/120900186

1. Introduction. Registration refers to the process of finding an optimal one-to-one (1-1) correspondence between images or surfaces. It has been extensively applied to different areas such as medical imaging, computer graphics, and computer vision. For example, in medical

^{*}Received by the editors November 27, 2012; accepted for publication (in revised form) November 1, 2013; published electronically February 27, 2014.

http://www.siam.org/journals/siims/7-1/90018.html

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imaging, registration is always needed for statistical shape analysis, morphometry, and processing of signals on brain surfaces (e.g., denoising or filtering), while in computer graphics, surface registration is needed for texture mapping, which aligns each vertex to a position of the texture image, to improve the visualization of the surface mesh. Developing an effective algorithm for registration is therefore very important.

Landmark-based registration has been widely studied to obtain a smooth 1-1 correspondence between different domains that matches important features. This kind of registration, with good feature alignment, is particularly crucial in medical imaging, computer vision, and computer graphics. For example, in medical imaging, anatomical features in brain cortical surfaces can be systematically delineated, such as sulci (the fissures in the brain surface). Landmark matching brain registration is often required to obtain a meaningful 1-1 correspondence between brain surfaces, so that further analysis can be carried out (e.g., building surface average of many subjects). However, obtaining a unique and bijective registration that matches features consistently is generally challenging, especially when a large number of landmark constraints are enforced. Motivated by this, we are interested in searching for a *unique* and *bijective landmark matching* diffeomorphism, associated with given landmark constraints, which minimizes the local geometric distortion.

In this paper, we propose a special class of diffeomorphisms called the *Teichmüller mappings* (T-Maps), which have uniform conformality distortion over the whole domain (see Figure 4). Under suitable conditions on the landmark constraints, there exists a unique T-Map between two surfaces, which minimizes the maximal conformality distortion. The conformality distortion measures how far the mapping deviates from a conformal mapping, and hence it measures the local geometric distortion. To compute this T-Map, we propose an efficient and effective iterative algorithm, which is called the *quasi-conformal (QC) iteration*. The basic idea is to represent the set of diffeomorphisms using Beltrami coefficients (BCs) and look for an optimal BC associated with the desired T-Map. The associated T-Map can then be efficiently computed from the optimal BC using the linear Beltrami solver (LBS).

Using the T-Map for landmark matching registration is advantageous for the following reasons:

1. Optimized conformality distortion: Given a set of landmark constraints, our algorithm is able to determine an optimal 1-1 correspondence between shapes automatically, which minimizes the conformality distortion. In the case of open surfaces with boundaries, the proposed algorithm can also automatically determine the optimal boundary correspondence that minimizes the maximal conformality distortion. Hence, a Dirichlet boundary condition is not required (which is usually needed for other algorithms such as harmonic registration).

2. Bijectivity: Another major advantage of using T-Maps for landmark matching registrations is that the bijectity (1-1, onto) of the registrations can be guaranteed. Obtaining a bijective landmark matching registration is generally difficult, especially when a large number of landmark constraints are enforced. Using our proposed method, a bijective T-Map can be computed, even with large deformation or a large number of landmarks (see Figure 15).

3. Uniqueness: In addition, the mapping is uniquely determined. In other words, every prescribed landmark constraints is associated with a unique T-Map.

4. Extension to soft landmark constraints: The proposed algorithm can also be extended to compute T-Maps with soft landmark constraints. This becomes necessary when

landmark features cannot be accurately located, and hence it is better to compute registration with landmarks approximately (but not exactly) matched (see Figure 16).

5. Independence of mesh structure: The proposed QC iterations rely on the LBS. The LBS computes the associated piecewise linear map between meshes with a given BC defined on each face. The solver is independent of the mesh structure. Hence, our algorithm can compute a bijective landmark matching registration between meshes even with irregular mesh structure (see Figure 8).

6. Fast computation: The QC iterations involve solving a sparse symmetric positive definite linear system in each iteration. The linear system can be solved quickly, and the iterations converge quickly. Using our proposed method, the T-Map can be efficiently computed, even with dense meshes.

To test the effectiveness of our method, we applied the proposed algorithm to real applications, such as brain registration, constrained texture mapping, and human face registration. Experimental results show that our method is both effective and efficient in computing a nonoverlap landmark matching registration with the least amount of conformality distortion.

In short, the contributions of this paper are threefold. First, we propose an efficient algorithm for obtaining the unique T-Map between shapes with landmark constraints enforced. The mapping is guaranteed to be bijective and minimizes the maximal conformality distortion. Second, we propose an algorithm to compute the T-Map with soft landmark constraints. Landmarks are not exactly matched, but less conformality distortion will be introduced. Third, we apply the proposed algorithms to real applications, namely, constrained texture mapping, medical image registration, and human face registration.

2. Previous work. Surface registration has been extensive studied and various algorithms have been proposed by different research groups. In this section, we will extract some previous work most closely related to our paper.

• Landmark-free surface registration: Landmark-free registration has been proposed to obtain 1-1 correspondences between shapes without feature landmarks. Different algorithms have been proposed to obtain registrations based on the shape information (such as curvatures) defined on the surfaces. Lyttelton et al. [33] computed surface parameterizations with surface curvature matching. Fischl et al. [4] proposed an algorithm for brain registration that better aligns cortical folding patterns, by minimizing the mean squared difference between the convexity of the surface and the average convexity across a set of subjects. Lord et al. [23] proposed matching surfaces by minimizing the deviation of the registration from isometry. Yeo et al. [44] proposed the spherical demons method, which adopted the diffeomorphic demons algorithm [39], to drive surfaces into correspondence based on the mean curvature and average convexity.

Conformal surface registration, which minimizes angular distortions, has also been widely used to obtain a smooth 1-1 correspondence between surfaces [13, 11, 12, 41, 15, 16, 43, 48]. An advantage of conformal registrations is that they preserve local geometry well. However, they cannot map landmark features, such as sulcal landmarks on brain surfaces, consistently.

Sometimes, deformations between objects might not be conformal. Instead, a certain amount of angular distortion could be introduced. To tackle this situation, quasiconformal mappings have been applied to obtain smooth 1-1 correspondences with bounded conformality distortion [32, 30, 31, 47, 25, 42]. The obtained registration can match the geometric quantities (such as curvature), while minimizing the maximal dilation of the mapping.

Landmark-based surface registration: Most of the above registration algorithms cannot match feature landmarks, such as sulcal landmarks on the human brain, consistently. To alleviate this issue, landmark matching registration algorithms are proposed by various research groups. Bookstein [3] proposed obtaining a registration that matches landmarks as much as possible using a thin plate spline (TPS) regularization (or biharmonic regularization). Wang and coworkers [40, 26, 27, 28] proposed computing the optimized conformal parameterizations of brain surfaces by minimizing a compounded energy [40, 28]. This registration can obtain an optimized harmonic map that better aligns the features; however, landmarks cannot be exactly matched. Besides, bijectivity cannot be ensured when a large number of landmark constraints are enforced. Tosun, Rettman, and Prince [38] proposed combining iterative closest point registration, parametric relaxation, and inverse stereographic projection to align cortical sulci across brain surfaces. These diffeomorphisms can better match landmark features, although, not perfectly. Later, Lin et al. [22] proposed a unified variational approach for registration of gene expression data to neuroanatomical mouse atlas in two dimensions that matches feature landmarks. Again, landmarks cannot be exactly matched. Note that inexact landmark matching registrations are sometimes beneficial. In the case when landmark points/curves are not entirely accurate, this method is more tolerant of errors in labeling landmarks and gives better parameterization.

In the situation when exact landmark matching is required, smooth vector fields have been applied to obtain surface registration. Lui et al. [26, 27] proposed the use of vector fields to represent surface maps and reconstruct them through integral flow equations. They obtained shape-based landmark matching harmonic maps by looking for the best vector fields minimizing a shape energy. The use of vector fields to compute the registration makes the optimization easier, although it cannot describe all surface maps. An advantage of this method is that exact landmark matching can be guaranteed. Time-dependent vector fields can also be used [17, 9, 10, 7, 8]. For example, Glaunés, Vaillant, and Miller [9] proposed generating large deformation diffeomorphisms of a sphere, with given displacements of a finite set of template landmarks. The time-dependent vector fields facilitate the optimization procedure, although the computational cost of the algorithm is comparatively more expensive.

Quasi-conformal mapping that matches landmarks consistently has also been proposed [32, 46]. In [32], the authors proposed computing the brain landmark matching registration, which minimizes the L^2 norm of the BCs. Zeng and Gu [46] also proposed computing quasi-conformal mappings for feature matching face registration. The BC associated with a landmark point matching parameterization is approximated. However, either exact landmark matching or the bijectivity of the mapping cannot be guaranteed, especially when very large deformations occur. **3.** Mathematical background. In this section, we describe some basic mathematical concepts related to our algorithms. For details, we refer the reader to [6, 18, 36].

A surface S with a conformal structure is called a *Riemann surface*. Given two Riemann surfaces M and N, a map $f : M \to N$ is *conformal* if it preserves the surface metric up to a multiplicative factor called the *conformal factor*. An immediate consequence is that every conformal map preserves angles. With the angle-preserving property, a conformal map effectively preserves the local geometry of the surface structure.

A generalization of conformal maps is the quasi-conformal maps, which are orientationpreserving homeomorphisms between Riemann surfaces with bounded conformality distortion, in the sense that their first order approximations take infinitesimal circles to infinitesimal ellipses of bounded eccentricity [6]. Mathematically, $f: \mathbb{C} \to \mathbb{C}$ is quasi-conformal provided that it satisfies the Beltrami equation,

(3.1)
$$\frac{\partial f}{\partial \overline{z}} = \mu(z) \frac{\partial f}{\partial z},$$

for some complex-valued function μ satisfying $||\mu||_{\infty} < 1$. μ is called the *Beltrami coefficient* (BC), which is a measure of nonconformality. It measures how far the map at each point deviates from a conformal map. In particular, the map f is conformal at p when $\mu(p) = 0$. In other words, f is angle-preserving at p when $\mu(p) = 0$. Infinitesimally, around a point p, f may be expressed with respect to its local parameter as follows:

(3.2)
$$f(z) = f(p) + f_z(p)z + f_{\overline{z}}(p)\overline{z}$$
$$= f(p) + f_z(p)(z + \mu(p)\overline{z}).$$

Obviously, f is not conformal if and only if $\mu(p) \neq 0$. Inside the local parameter domain, f may be considered as a map composed of a translation to f(p) together with a stretch map $S(z) = z + \mu(p)\overline{z}$, which is postcomposed by a multiplication of $f_z(p)$, which is conformal. All of the conformal distortion of S(z) is caused by $\mu(p)$. S(z) is the map that causes f to map a small circle to a small ellipse. From $\mu(p)$, we can determine the angles of the directions of maximal magnification and shrinking and the amount of them as well. Specifically, the angle of maximal magnification is $\arg(\mu(p))/2$ with magnifying factor $1 + |\mu(p)|$; the angle of maximal shrinking is the orthogonal angle $(\arg(\mu(p)) - \pi)/2$ with shrinking factor $1 - |\mu(p)|$. Thus, the BC μ gives us all the information about the properties of the map (see Figure 1(D)).

The maximal dilation of f is given by

(3.3)
$$K(f) = \frac{1 + ||\mu||_{\infty}}{1 - ||\mu||_{\infty}}.$$

Let $f = u + \sqrt{-1}v$. From the Beltrami equation (3.1),

(3.4)
$$\mu(f) = \frac{(u_x - v_y) + \sqrt{-1} (v_x + u_y)}{(u_x + v_y) + \sqrt{-1} (v_x - u_y)}$$



Figure 1. (A) shows a human face with circle packing texture. Under the conformal parameterization, infinitesimal circles are mapped to circles as shown in (B). Under quasi-conformal parameterization, infinitesimal circles are mapped to ellipses as shown in (C). (D) illustrates how the BC measures the conformality distortion of a quasi-conformal map.

Let $\mu(f) = \rho + \sqrt{-1} \tau$. We can write v_x and v_y as linear combinations of u_x and u_y ,

$$(3.5) \qquad -v_y = \alpha_1 u_x + \alpha_2 u_y, \\ v_z = \alpha_2 u_z + \alpha_2 u_z,$$

where $\alpha_1 = \frac{(\rho-1)^2 + \tau^2}{1 - \rho^2 - \tau^2}$, $\alpha_2 = -\frac{2\tau}{1 - \rho^2 - \tau^2}$, $\alpha_3 = \frac{1 + 2\rho + \rho^2 + \tau^2}{1 - \rho^2 - \tau^2}$. Similarly,

$$\begin{array}{l} -u_y = \alpha_1 v_x + \alpha_2 v_y, \\ u_x = \alpha_2 v_x + \alpha_3 v_y. \end{array}$$

Since $\nabla \cdot \begin{pmatrix} -v_y \\ v_x \end{pmatrix} = 0$, we obtain

(3.7)
$$\nabla \cdot \left(A \left(\begin{array}{c} u_x \\ u_y \end{array} \right) \right) = 0 \text{ and } \nabla \cdot \left(A \left(\begin{array}{c} v_x \\ v_y \end{array} \right) \right) = 0,$$

where $A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix}$. It can be proven that A is symmetric positive definite.

A certain boundary condition is required for the well-posedness of (3.7). According to the quasi-conformal Teichmüller theory, a quasi-conformal map can be uniquely determined up to a Mobius transformation. In other words, three-point correspondence is required to fix the ambiguity of the Mobius transformation, in order to have a unique solution of (3.7).

In this paper, a discrete version of (3.7) for a piecewise linear mapping between meshes will be developed to compute the quasi-conformal mapping (see section 5.1).

Quasi-conformal mapping between two Riemann surfaces S_1 and S_2 can also be defined. Instead of the BC, the *Beltrami differential* is used. A Beltrami differential $\mu(z)\frac{\overline{dz}}{dz}$ on a Riemann surface S is an assignment to each chart $(U_{\alpha}, \phi_{\alpha})$ of an L_{∞} complex-valued function μ_{α} , defined on local parameter z_{α} such that

(3.8)
$$\mu_{\alpha}(z_{\alpha})\frac{d\overline{z_{\alpha}}}{dz_{\alpha}} = \mu_{\beta}(z_{\beta})\frac{d\overline{z_{\beta}}}{dz_{\beta}}$$

on the domain which is also covered by another chart $(U_{\beta}, \phi_{\beta})$. Here, $\frac{dz_{\beta}}{dz_{\alpha}} = \frac{d}{dz_{\alpha}}\phi_{\alpha\beta}$ and $\phi_{\alpha\beta} = \phi_{\beta} \circ \phi_{\alpha}^{-1}$ (see Figure 2).



Figure 2. Illustration of how Beltrami differential is defined on general Riemann surfaces.



Figure 3. Illustration of quasi-conformal mapping between Riemann surfaces.

An orientation-preserving diffeomorphism $f: S_1 \to S_2$ is called quasi-conformal associated with $\mu(z)\frac{dz}{dz}$ if for any chart $(U_{\alpha}, \phi_{\alpha})$ on S_1 and any chart $(V_{\beta}, \psi_{\beta})$ on S_2 , the mapping $f_{\alpha\beta} := \psi_{\beta} \circ f \circ f_{\alpha}^{-1}$ is quasi-conformal associated with $\mu_{\alpha}(z_{\alpha})\frac{dz_{\alpha}}{dz_{\alpha}}$ (see Figure 3). In the case when S_1 and S_2 are simply connected, conformal mapping between S_1 and S_2

In the case when S_1 and S_2 are simply connected, conformal mapping between S_1 and S_2 always exists. However, conformal mapping may not exist between surfaces with complicated topology. For example, there is generally no conformal mapping between multiply connected open surfaces (e.g., two annuli with different radii of inner circles). One would therefore be interested in finding an optimal quasi-conformal mapping that minimizes the conformality distortion. More specifically, it is desirable to obtain an extremal quasi-conformal mapping, which is extremal in the sense of minimizing the $|| \cdot ||_{\infty}$ over all Beltrami differentials corresponding to quasi-conformal mappings between S_1 and S_2 . Extremal mapping always exists but need not be unique. Mathematically, an extremal quasi-conformal mapping can be defined as follows.



Figure 4. Difference between a general quasi-conformal map and a Teichmüller map. (A) shows the original textured mesh. It is mapped to another disk by a general quasi-conformal map. Note that the distribution of the norm of BC is spread out. (C) shows the Teichmüller map, whose BC norm is concentrated near 0.4.

Definition 3.1. Let $f: S_1 \to S_2$ be a quasi-conformal mapping between S_1 and S_2 . f is said to be an extremal mapping if for any quasi-conformal mapping $h: S_1 \to S_2$ isotopic to f relative to the boundary,

$$(3.9) K(f) \le K(h).$$

It is uniquely extremal if the inequality (3.9) is strict when $h \neq f$.

Another kind of mapping, called the *Teichmüller mapping* (T-Map), is closely related to the extremal mapping. T-Map is defined as follows.

Definition 3.2. Let $f: S_1 \to S_2$ be a quasi-conformal mapping. f is said to be a T-Map associated to the quadratic differential $q = \varphi dz^2$, where $\varphi: S_1 \to \mathbb{C}$ is a holomorphic function, if its associated Beltrami differential is of the form

(3.10)
$$\mu(f) = k \frac{\overline{\varphi}}{|\varphi|}$$

for some constant k < 1 and quadratic differential $q \neq 0$ with $||q||_1 = \int_{S_1} |\varphi| < \infty$.

It means a T-Map is a quasi-conformal mapping whose BC has a constant norm. Thus it has a uniform conformality distortion over the whole domain (see Figure 4).

Extremal mapping might not be unique. However, if there is a T-Map in a homotopic class, then it is unique. Strebel's theorem explains the relationship between the T-Map and extremal mapping.

Definition 3.3 (boundary dilation). Suppose S_1 and S_2 are open Riemann surfaces with the same topology. The boundary dilation $K_1[f]$ of f is defined as

(3.11)
$$K_1[f] = \inf_C \{ K(h|_{S_1 \setminus C}) : h \in \mathfrak{F}, C \subseteq S_1, C \text{ is compact} \},$$

where \mathfrak{F} is the family of quasi-conformal homeomorphisms of S_1 onto S_2 which are homotopic to f modulo the boundary.

Theorem 3.4 (Strebel's theorem; see [37, p. 319]). Let f be an extremal quasi-conformal mapping with K(f) > 1. If $K_1[f] < K(f)$, then f is a T-Map associated with an integrable holomorphic quadratic differential on S_1 . Hence, f is also an unique extremal mapping.

T-MAP AND ITS APPLICATIONS

In other words, an extremal mapping between S_1 and S_2 with a suitable boundary condition is a T-Map. For simply connected open surfaces, the required boundary conditions are some properties on the derivatives of the boundary correspondence. More specifically, the first derivative of the boundary correspondence should be nonzero everywhere and the second derivative should be bounded.

Theorem 3.5 (see [35, p. 110]). Let $h : \partial \mathbb{D} \to \partial \mathbb{D}$ be an orientation-preserving diffeomorphism of $\partial \mathbb{D}$. Suppose further that $h'(e^{i\theta}) \neq 0$ and $h''(e^{i\theta})$ is bounded. Then there is a T-Map f that is the unique extremal extension of h to \mathbb{D} . That is, $f : \mathbb{D} \to \mathbb{D}$ is an extremal mapping with $f|_{\partial \mathbb{D}} = h$.

Thus, if the boundary correspondence satisfies certain conditions on its derivatives, the extremal map of the unit disk must be a T-Map.

Now, in the case when interior landmark constraints are further enforced, the existence of a unique T-Map can be guaranteed if the boundary and landmark correspondence satisfy suitable conditions. The unique T-Map is extremal, which minimizes the maximal conformality distortion. The following theorem can be derived immediately from Strebel's theorem (Theorem 3.4).

Theorem 3.6 (landmark matching T-Map). Let S_1 and S_2 be open Riemann surfaces with the same topology. Let $\{p_i\}_{i=1}^n \in S_1$ and $\{q_i\}_{i=1}^n \in S_2$ be the corresponding interior landmark constraints. Let $f : (S_1, \{p_i\}_{i=1}^n) \to (S_2, \{q_i\}_{i=1}^n)$ be the extremal quasi-conformal mapping, such that p_i corresponds to q_i for all $1 \le i \le n$. If $K_1[f] < K(f)$, then f is a T-Map associated with an integrable holomorphic quadratic differential on $(S_1, \{p_i\}_{i=1}^n)$. Hence, f is a unique extremal mapping.

In particular, a unique T-Map $f : \mathbb{D} \to \mathbb{D}$ between unit disks with interior landmark constraints enforced exists if the boundary map $f|_{\partial \mathbb{D}}$ satisfies suitable conditions. The following theorem can be obtained directly from Theorem 3.5.

Theorem 3.7 (landmark matching T-Map of D). Let $h : \partial \mathbb{D} \to \partial \mathbb{D}$ be an orientationpreserving diffeomorphism of $\partial \mathbb{D}$. Suppose further that $h'(e^{i\theta}) \neq 0$ and $h''(e^{i\theta})$ is bounded. Let $\{p_i\}_{i=1}^n \in \mathbb{D}$ and $\{q_i\}_{i=1}^n \in \mathbb{D}$ be the corresponding interior landmark constraints. Then there is a T-Map $f : (\mathbb{D}, \{p_i\}_{i=1}^n) \to (\mathbb{D}, \{q_i\}_{i=1}^n)$ matching the interior landmarks, which is the unique extremal extension of h to D. That is, $f : (\mathbb{D}, \{p_i\}_{i=1}^n) \to (\mathbb{D}, \{q_i\}_{i=1}^n)$ is an extremal T-Map with $f|_{\partial \mathbb{D}} = h$ matching the interior landmarks.

Theorems 3.6 and 3.7 play the fundamental role of obtaining a unique T-Map between surfaces that matches feature landmarks consistently. We can therefore obtain a unique landmark matching registration by searching for an optimal BC whose maximal dilatation is the minimum, while its norm is constant everywhere.

It turns out that in most situations, an extremal quasi-conformal mapping is a T-Map (even for domains with nontrivial topologies). In some rare situations when an extremal mapping is not exactly a T-Map, one can still get a T-Map whose dilation is arbitrarily close to the extremal dilation.

Theorem 3.8 (see [37, p. 320]). Let \mathfrak{F} be a class of quasi-conformal mappings between the open Riemann surfaces S_1 and S_2 , which are homotopic modulo the boundary. Let K_0 be the smallest maximal dilation of the mappings in \mathfrak{F} . Then there are T-Maps in \mathfrak{F} , associated with a meromorphic quadratic differential with at most one simple pole, whose dilation is arbitrarily close to K_0 .

4. Variational formulation. In this section, we give a variational formulation for obtaining the T-Map with the least amount of conformality distortion. We propose using the BC to represent the mapping, instead of the commonly used representations by deformation fields or coordinate functions. The diffeomorphic property of the mapping can then be effectively controlled. Our goal is to formulate the problem into a variational framework to obtain an optimal BC, $\mu(f)$, associated with the desired T-Map f.

Suppose D_1 and D_2 are two domains in the complex plane with the same topology. D_1 and D_2 can be either simply connected or multiply connected. Suppose the boundary condition of the desired T-Map $f: D_1 \to D_2$ is known. Denote it by $f|_{\partial D_1}: \partial D_1 \to \partial D_2 = g$. Mathematically, the T-Map can be described as

(4.1)
$$\frac{\partial f}{\partial \overline{z}} = k \frac{\overline{\varphi}}{|\varphi|} \frac{\partial f}{\partial z} \text{ and } f|_{\partial D_1} = g \text{ on } \partial D_1$$

for some constant k and integrable holomorphic function $\varphi: D_1 \to \mathbb{C} \ (\varphi \neq 0)$.

Recall that a T-Map is extremal in the sense that it minimizes the $\|\cdot\|_{\infty}$ over all Beltrami differentials corresponding to quasi-conformal mappings in the Teichmüller equivalence class. In other words, for any $h: D_1 \to D_2$ satisfying $h|_{\partial D_1} = g$, we have

(4.2)
$$||\mu(f)||_{\infty} \le ||\mu(h)||_{\infty},$$

where $\mu(f)$ and $\mu(h)$ are the BCs of f and h, respectively. Hence, our original problem (4.1) can be formulated into a variational problem as follows:

(4.3)
$$f = \operatorname{argmin}_{f:D_1 \to D_2} E_1(f)$$
$$:= \operatorname{argmin}_{f:D_1 \to D_2} \{ || \mu(f) ||_{\infty} \}$$

subject to

f|_{∂D1} = g (boundary condition),
μ(f) = k φ/|φ| for some constant 0 ≤ k < 1 and holomorphic function φ : D₁ → C.

Theoretically, a diffeomorphism f is associated with a unique BC $\mu(f)$ with $||\mu(f)||_{\infty} < 1$. The BC $\mu(f)$ measures the conformality distortion of the map f. It can be considered as a unique representation of f. The energy functional E_1 aims to minimize the maximal conformality distortion of the mapping.

However, minimizing $E_1(f)$ with respect to the space of all diffeomorphisms between D_1 and D_2 is difficult. More specifically, the variational problem (4.3) can be expressed as the following complicated form:

(4.4)
$$f = \operatorname{argmin}_{f} \{ ||\mu(f)||_{\infty} \}$$
$$= \operatorname{argmin}_{f} \left\| \frac{\partial f / \partial \overline{z}}{\partial f / \partial \overline{z}} \right\|_{\infty}$$

subject to $f|_{\partial D_1} = g$ and $\mu(f) = k \frac{\overline{\varphi}}{|\varphi|}$ for some constant $0 \le k < 1$ and holomorphic function $\varphi: D_1 \to \mathbb{C}.$

In order to minimize the above constrained minimization problem effectively, we propose to reformulate the energy functional with respect to space of all BCs:

(4.5)
$$(\nu, f) = \operatorname{argmin}_{\nu:D_1 \to \mathbb{C}} E_2(\nu) \\ := \operatorname{argmin}_{\nu:D_1 \to \mathbb{C}} \{ ||\nu||_{\infty} \}$$

subject to

- $\nu = \mu(f)$ and $||\nu||_{\infty} < 1$,
- $\nu = k \frac{\overline{\varphi}}{|\varphi|}$ for some constant $0 \le k < 1$ and holomorphic function $\varphi : D_1 \to \mathbb{C}$,
- $f|_{\partial D_1} = g$ (boundary condition).

In other words, the minimization problem (4.3) is reformulated into an optimization problem over the space of all BCs, which are complex-valued functions defined on D_1 . Minimizing the energy functional with respect to BCs is advantageous since the diffeomorphic property of the mapping can be easily controlled [32]. Every diffeomorphism is associated to a smooth BC $\mu(f)$. $\mu(f)$ measures the bijectivity (1-1 and onto) of f. In fact, $\mu(f)$ is related to the Jacobian J(f) of f by the following formula:

(4.6)
$$|J(f)|^2 = \left|\frac{\partial f}{\partial z}\right|^2 (1 - |\mu(f)|^2).$$

Therefore, the map f is bijective if $|\mu(f)|$ is everywhere less than 1. When solving the minimization problem (4.5), the bijectivity of the mapping in each iteration can be ensured by enforcing $||\nu||_{\infty} < 1$. Our goal is to look for an optimal BC, ν , such that its associated quasi-conformal map is our desired T-Map.

Besides, theoretically, the second condition in the variational formulation (4.5) is unnecessary. More specifically, minimizing the supreme norm of ν subject to the first and third constraints will automatically give the second condition, according to Theorem 3.6 and 3.7. However, in practice, minimizing the optimization problem (which involves the L^{∞} norm) subject to the first and third constraints using standard methods, such as linear programming (LP) approaches, is challenging. In this paper, we propose that the QC iteration search for a path that approaches an admissible BC, ν^* , of Teichmüller type from an initial BC. The optimal BC, ν^* , solves the optimization problem (4.5). The second constraint is added in our variational formulation because it gives one of the useful criteria to drive the initial BC to the admissible BC, ν^* , of Teichmüller type. More details will be discussed in section 5.2.

The boundary condition in the variational problem (4.5) can be relaxed. The Dirichlet condition defined on the whole boundary is not required. Also, interior landmark constraints can be enforced. Our goal is to solve the variational problem with these landmark constraints, which determines the optimal 1-1 correspondence (including the boundary correspondence) automatically. In other words, the boundary condition in the problem (4.5) can be reformulated as

(4.7)
$$f(a_i) = b_i; f(p_j) = q_j \text{ for } i = 1, \dots, n, \quad j = 1, \dots, m,$$

where a_i and b_i are corresponding landmark points or curves defined on ∂D_1 and ∂D_2 , respectively, and p_j and q_j are corresponding interior landmark points or curves in D_1 and D_2 ,

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respectively. By optimizing the energy functional (4.5), the landmark matching T-Map can be obtained, which matches landmark features consistently while minimizing the maximal conformality distortion.

Note also that the above formulation is designed for obtaining the T-Map of two-dimensional (2D) connected domains. However, it can easily be extended to simply connected or multiply connected open surfaces. Let S_1 and S_2 be two connected open surfaces with the same topology. We can conformally parameterize S_1 and S_2 by $\phi_1 : S_1 \to D_1 \subset \mathbb{C}$ and $\phi_2 : S_2 \to D_2 \subset \mathbb{C}$, respectively. Then the T-Map $f : S_1 \to S_2$ between S_1 and S_2 induces the T-Map $\tilde{f} := \phi_2 \circ f \circ \phi_1^{-1} : D_1 \to D_2$. All of the above formulation applies to \tilde{f} . In other words, the computation of the T-Map between the conformal domains in \mathbb{C} .

In the subsequent section, we propose an algorithm, called the *quasi-conformal* (QC) *iteration*, to solve the above minimization problems (4.5).

5. Main algorithm. In this section, we describe an iterative scheme, called the *quasi-conformal* (QC) *iteration*, for solving the variational problem (4.5). The QC iteration is based on the LBS. The LBS will first be explained in detail. QC iteration will then be described.

Practically speaking, 2D domains or surfaces in \mathbb{R}^3 are usually represented discretely by triangular meshes. Suppose K_1 and K_2 are two surface meshes with the same topology representing S_1 and S_2 . We define the set of vertices on K_1 and K_2 by $V^1 = \{v_i^1\}_{i=1}^n$ and $V^2 = \{v_i^2\}_{i=1}^n$, respectively. Similarly, we define the set of triangular faces on K_1 and K_2 by $F^1 = \{T_j^1\}_{j=1}^m$ and $F^2 = \{T_j^2\}_{j=1}^m$. Our goal is to look for a piecewise linear homeomorphism between K_1 and K_2 that approximates the T-Map between S_1 and S_2 .

5.1. Linear Beltrami solver. Our goal is to look for an optimal BC associated to the desired T-Map. Every quasi-conformal mapping is associated to a unique BC. Given a BC, it is important to have an algorithm to reconstruct the associated quasi-conformal diffeomorphism.

Suppose $f: K_1 \to K_2$ is an orientation-preserving piecewise linear homeomorphism between K_1 and K_2 . We can assume K_1 and K_2 are both embedded in \mathbb{R}^2 . In the case when K_1 and K_2 are surface meshes in \mathbb{R}^3 , we first parameterize them conformally by $\phi_1: K_1 \to D_1 \subseteq \mathbb{R}^2$ and $\phi_2: K_2 \to D_2 \subseteq \mathbb{R}^2$. The composition of f with the conformal parameterizations, $\tilde{f} := \phi_2 \circ f \circ \phi_1^{-1}$, is then an orientation-preserving piecewise linear homeomorphism between D_1 and D_2 embedded in \mathbb{R}^2 . In this paper, we assume the topology of the surface mesh is either a connected open surface or a genus-0 closed surface. In other words, the conformal domain D_i (i = 1, 2) can either be a 2D rectangle, unit disk, punctual disk, or unit sphere.

To compute the quasi-conformal mapping, the key idea is to discretize (3.7) with two linear systems.

Given a map $f = (u + \sqrt{-1}v) : K_1 \to K_2$, we can easily compute its associated BC μ_f , which is a complex-valued function defined on each triangular face of K_1 . To compute μ_f , we simply need to approximate the partial derivatives on every face T. We denote them by $D_x f(T) = D_x u + \sqrt{-1}D_x v$ and $D_y f(T) = D_y u + \sqrt{-1}D_y v$, respectively. Note that f is piecewise linear. The restriction of f on each triangular face T can be written as

(5.1)
$$f|_T(x,y) = \begin{pmatrix} a_T x + b_T y + r_T \\ c_T x + d_T y + s_T \end{pmatrix}.$$

Hence, $D_x u(T) = a_T$, $D_y u(T) = b_T$, $D_x v(T) = c_T$, and $D_y v(T) = d_T$. Now, the gradient $\nabla_T f := (D_x f(T), D_y f(T))^t$ on each face T can be computed by solving the linear system

(5.2)
$$\begin{pmatrix} \vec{v}_1 - \vec{v}_0 \\ \vec{v}_2 - \vec{v}_0 \end{pmatrix} \nabla_T \tilde{f}_i = \begin{pmatrix} \frac{\tilde{f}_i(\vec{v}_1) - \tilde{f}_i(\vec{v}_0)}{|\vec{v}_1 - \vec{v}_0|} \\ \frac{\tilde{f}_i(\vec{v}_2) - \tilde{f}_i(\vec{v}_0)}{|\vec{v}_2 - \vec{v}_0|} \end{pmatrix},$$

where $[\vec{v_0}, \vec{v_1}]$ and $[\vec{v_0}, \vec{v_2}]$ are two edges on T. By solving (5.2), a_T , b_T , c_T , and d_T can be obtained. The BC $\mu_f(T)$ of the triangular face T can then be computed from the BC (3.1) by

(5.3)
$$\mu_f(T) = \frac{(a_T - d_T) + \sqrt{-1}(c_T + b_T)}{(a_T + d_T) + \sqrt{-1}(c_T - b_T)}$$

Equations (3.5) and (3.6) are both satisfied on every triangular face. Let $\mu_f(T) = \rho_T + \sqrt{-1} \tau_T$. The following discrete versions of (3.5) and (3.6) can be obtained:

(5.4)
$$\begin{aligned} -d_T &= \alpha_1(T)a_T + \alpha_2(T)b_T, \\ c_T &= \alpha_2(T)a_T + \alpha_3(T)b_T \end{aligned}$$

and

(5.5)
$$\begin{aligned} -b_T &= \alpha_1(T)c_T + \alpha_2(T)d_T, \\ a_T &= \alpha_2(T)c_T + \alpha_3(T)d_T, \end{aligned}$$

where $\alpha_1(T) = \frac{(\rho_T - 1)^2 + \tau_T^2}{1 - \rho_T^2 - \tau_T^2}$, $\alpha_2(T) = -\frac{2\tau_T}{1 - \rho_T^2 - \tau_T^2}$, $\alpha_3(T) = \frac{1 + 2\rho_T + \rho_T^2 + \tau_T^2}{1 - \rho_T^2 - \tau_T^2}$. In order to discretize (3.7), we need to introduce the *discrete divergence*. The discrete

In order to discretize (3.7), we need to introduce the *discrete divergence*. The discrete divergence can be defined as follows. Let $T = [v_i, v_j, v_k]$ and $w_I = f(v_I)$, where I = i, j, or k. Suppose $v_I = g_I + \sqrt{-1} h_I$ and $w_I = s_I + \sqrt{-1} t_I$ (I = i, j, k). Using (5.2), a_T, b_T, c_T , and d_T can be written as

(5.6)
$$a_{T} = A_{i}^{T}s_{i} + A_{j}^{T}s_{j} + A_{k}^{T}s_{k}, \qquad b_{T} = B_{i}^{T}s_{i} + B_{j}^{T}s_{j} + B_{k}^{T}s_{k}, c_{T} = A_{i}^{T}t_{i} + A_{j}^{T}t_{j} + A_{k}^{T}t_{k}, \qquad d_{T} = B_{i}^{T}t_{i} + B_{j}^{T}t_{j} + B_{k}^{T}t_{k},$$

where

(5.7)
$$A_i^T = (h_j - h_k) / Area(T), \ A_j^T = (h_k - h_i) / Area(T), \ A_k^T = (h_i - h_j) / Area(T), B_i^T = (g_k - g_j) / Area(T), \ B_j^T = (g_i - g_k) / Area(T), \ B_k^T = (g_j - g_i) / Area(T).$$

Suppose $\vec{V} = (V_1, V_2)$ is a discrete vector field defined on every triangular faces. For each vertex v_i , let N_i be the collection of neighborhood faces attached to v_i . We define the discrete divergence Div of \vec{V} as follows:

(5.8)
$$\operatorname{Div}(\vec{V})(v_i) = \sum_{T \in N_i} A_i^T V_1(T) + B_i^T V_2(T)$$

By careful checking, one can prove that

(5.9)
$$\sum_{T \in N_i} A_i^T b_T = \sum_{T \in N_i} B_i^T a_T, \qquad \sum_{T \in N_i} A_i^T d_T = \sum_{T \in N_i} B_i^T c_T.$$

This gives

(5.10)
$$\operatorname{Div}\left(\begin{array}{c}-D_y u\\D_x u\end{array}\right) = 0 \text{ and } \operatorname{Div}\left(\begin{array}{c}-D_y v\\D_x v\end{array}\right) = 0.$$

As a result, (3.7) can be discretized as

(5.11)
$$\operatorname{Div}\left(A\left(\begin{array}{c}D_{x}u\\D_{y}u\end{array}\right)\right) = 0 \text{ and } \operatorname{Div}\left(A\left(\begin{array}{c}D_{x}v\\D_{y}v\end{array}\right)\right) = 0,$$

where $A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix}$. This is equivalent to

(5.12)
$$\sum_{T \in N_i} A_i^T [\alpha_1(T)a_T + \alpha_2(T)b_T] + B_i^T [\alpha_2(T)a_T + \alpha_3(T)b_T] = 0,$$

(5.13)
$$\sum_{T \in N_i} A_i^T [\alpha_1(T)c_T + \alpha_2(T)d_T] + B_i^T [\alpha_2(T)c_T + \alpha_3(T)d_T] = 0$$

for all vertices $v_i \in D$. Note that a_T and b_T can be written as a linear combination of the x-coordinates of the desired quasi-conformal map f. Hence, (5.12) gives us the linear systems to solve for the x-coordinate function of f. Similarly, c_T and d_T can also be written as a linear combination of the y-coordinates of the desired quasi-conformal map f. Therefore, (5.13) gives us the linear systems to solve for the y-coordinate function of f.

Besides, f has to satisfy certain constraints on the boundary. One common situation is to give the Dirichlet condition on the whole boundary. That is, for any $v_b \in \partial K_1$,

$$(5.14) f(v_b) = w_b \in \partial K_2.$$

Note that the Dirichlet condition is not required to be enforced on the whole boundary. The proposed algorithm also allows a free boundary condition. In the case that K_1 and K_2 are rectangles, the desired quasi-conformal map should satisfy

(5.15)
$$f(0) = 0; f(1) = 1 \ f(i) = i \ f(1+i) = 1+i,$$

$$\mathbf{Re}(f) = 0 \ \text{on arc} \ [0, i], \qquad \mathbf{Re}(f) = 1 \ \text{on arc} \ [1, 1+i],$$

$$\mathbf{Imag}(f) = 0 \ \text{on arc} \ [0, 1], \qquad \mathbf{Imag}(f) = 1 \ \text{on arc} \ [i, 1+i].$$

When K_i (i = 1, 2) is a unit disk, we can parameterize it onto a domain D_i , which is a triangle with boundary vertices p_0^i , p_1^i , and p_2^i . p_0^i is on the y-axis, whereas p_1^i and p_2^i are on the x-axis. This can be done by removing a triangular face at the point 1 and map K_i to the upper half plane using a Mobius transformation $\psi(z) = \sqrt{-1\frac{1+z}{1-z}}$ (see Figure 5). In this case,



Figure 5. Illustration of how quasi-conformal mapping between surface meshes can be transformed into a quasi-conformal mapping between big triangles in \mathbb{R}^2 . (A) shows a quasi-conformal mapping between unit disks (or simply connected open meshes of disk topology). The two meshes are conformally mapped to a big triangle after cutting a triangular face near 1. Through composition, the quasi-conformal mapping can be transformed into a quasi-conformal mapping between the big triangles in \mathbb{R}^2 . (B) shows the case of a homeomorphism between genus-0 closed surface meshes. The two meshes are conformally parameterized onto a big triangle in \mathbb{R}^2 , after cutting away a triangular face on each mesh. Again, through composition, the quasi-conformal mapping can be transformed into a quasi-conformal mapping between the big triangles in \mathbb{R}^2 .

the desired quasi-conformal map f should satisfy

(5.16)
$$f(p_0^1) = p_0^2; f(p_1^1) = p_1^2 \text{ and } \operatorname{Imag}(f) = 0 \text{ on arc } [p_0^1, p_1^1].$$

When K_i (i = 1, 2) is a genus-0 closed surface mesh, we can again parameterize it onto a domain D_i , which is a triangle with boundary vertices p_0^i , p_1^i , and p_2^i . This can be done by removing a triangular face on the surface and mapping K_i to the 2D plane using stereographic projection (see Figure 5). In this case, the desired quasi-conformal map \tilde{f} should satisfy

(5.17)
$$f(p_0^1) = p_0^2; f(p_1^1) = p_1^2 \text{ and } f(p_2^1) = p_2^2.$$

Suppose interior landmark correspondences $\{p_i\}_{i=1}^n \leftrightarrow \{q_i\}_{i=1}^n$ are also enforced; one should add this constraint to the linear system. Mathematically, it is described as $f(p_i) = q_i$ (i = 1, 2, ..., n).

Equations (5.12) and (5.13), together with the above boundary conditions, give a nonsingular linear system to solve for f. In fact, solving (5.12) and (5.13) is equivalent to finding the minimizers of

(5.18)
$$E_{LBS}^{1}(u) = \sum_{T \in Faces} (D_{x}u(T), D_{y}u(T)) \begin{pmatrix} \alpha_{1}(T) & \alpha_{2}(T) \\ \alpha_{2}(T) & \alpha_{3}(T) \end{pmatrix} \begin{pmatrix} D_{x}u(T) \\ D_{y}u(T) \end{pmatrix},$$
$$E_{LBS}^{2}(v) = \sum_{T \in Faces} (D_{x}v(T), D_{y}v(T)) \begin{pmatrix} \alpha_{1}(T) & \alpha_{2}(T) \\ \alpha_{2}(T) & \alpha_{3}(T) \end{pmatrix} \begin{pmatrix} D_{x}v(T) \\ D_{y}v(T) \end{pmatrix}.$$

The matrix $\begin{pmatrix} \alpha_1(T) & \alpha_2(T) \\ \alpha_2(T) & \alpha_3(T) \end{pmatrix}$ is symmetric positive definite for each face T. $E_{LBS}^1(u)$ and $E_{LBS}^2(v)$ are strictly convex subject to the boundary and landmark constraints, which are linear.

Therefore, the equivalent linear systems (5.12) and (5.13) are symmetric positive definite. They can be solved effectively by the conjugate gradient method. We call this algorithm the linear Beltrami solver (LBS). Given a BC ν , we denote the obtained quasi-conformal map from LBS by $\mathbf{LBS}(\nu)$. If landmark constraints are enforced, we denote it by $\mathbf{LBS}_{LM}(\nu)$.

We note that given an arbitrary BC ν and arbitrary landmark correspondences, a quasiconformal mapping associated to ν might not exist. However, the LBS looks for the best quasi-conformal mapping whose BC closely resembles ν .

5.1.1. Discrete formulation of the variational problem (4.5). With the above discretization of Beltrami's equation, we can now give a discrete formulation of the variational problem (4.5). Suppose $\vec{f} := \vec{u} + i\vec{v} : V^1 \to V^2$ is the discrete quasi-conformal map with the discrete BC $\vec{\mu}: F^1 \to \mathbb{C}$. The LBS solves the following linear systems:

(5.19)
$$\mathcal{A}_1(\vec{\mu})\vec{u} = \mathbf{b}_1(\vec{\mu}), \qquad \mathcal{A}_2(\vec{\mu})\vec{v} = \mathbf{b}_2(\vec{\mu}),$$

where $\mathcal{A}_1(\vec{\mu})$ and $\mathcal{A}_2(\vec{\mu})$ are matrices depending on $\vec{\mu}$; $\mathbf{b}_1(\vec{\mu})$ and $\mathbf{b}_2(\vec{\mu})$ are column vectors, which also depend on $\vec{\mu}$.

The variational problem (4.5) can now be discretized as follows. In the discrete setting, we search for a pair of vectors, $\vec{\nu}^*: F^1 \to \mathbb{C}$ and $\vec{f}^*:= \vec{u}^* + i\vec{v}^*: V^1 \to V^2$, which solves

(5.20)
$$(\vec{\nu}^*, \vec{f}^*) = \operatorname{argmin}_{\vec{\nu}: F^1 \to \mathbb{C}} \{ ||\vec{\nu}||_{\infty} \}$$

subject to the constraints that

• $\mathcal{A}_1(\vec{\nu}^*)\vec{u}^* = \mathbf{b}_1(\vec{\nu}^*), \ \mathcal{A}_2(\vec{\nu}^*)\vec{v}^* = \mathbf{b}_2(\vec{\nu}^*), \ \text{and} \ ||\vec{\nu}^*||_{\infty} < 1;$

• $\vec{\nu}^* = k \frac{\vec{\varphi}}{|\vec{\varphi}|}$ for $0 \le k < 1$ and $\vec{\varphi}$ is holomorphic. Solving this variational problem with conventional approaches, such as LP methods, is challenging. In the next subsection, we describe an iterative method, called the QC iteration, to solve the problem.

5.2. Quasi-conformal (QC) iterations. With the LBS, one can easily obtain the best quasi-conformal mapping associated with a given BC. In order to obtain the T-Map f, our goal is to iteratively search for the unique admissible BC of Teichmüller type associated to f. With the optimal BC, the desired T-Map f can be easily reconstructed using the LBS.

Recall that our problem of computing the T-Map can be formulated into an optimization problem

(5.21)
$$(\nu, f) = \operatorname{argmin}_{\nu: D_1 \to \mathbb{C}} \{ \|\nu\|_{\infty} \}$$

subject to (1) $\nu = \mu(f)$ with $||\nu||_{\infty} < 1$, (2) $\nu = k \frac{\overline{\varphi}}{|\varphi|}$ for some constant k and holomorphic function $\varphi: D_1 \to \mathbb{C}$, and (3) f satisfying certain boundary conditions and/or landmark

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constraints. Note that the boundary condition in (3) can be either a Dirichlet condition defined on the whole boundary or a free boundary condition with only a few points on the boundary fixed.

Solving the above variational problem using conventional optimization techniques, such as LP approaches, is challenging. In this subsection, we introduce the QC iteration to solve the above optimization problem.

The QC iteration starts with an initial map $f_0: D_1 \to D_2$ satisfying the given boundary condition and landmark constraints. The initial map is chosen to be the quasi-conformal mapping obtained from the LBS associated to the initial BC $\mu_0 = 0$. In other words,

(5.22)
$$f_0 = \mathbf{LBS}_{LM}(\mu_0 := 0).$$

Note that with the enforced landmark constraints, the BC associated to f_0 might not be equal to $\mu_0 = 0$. The LBS simply looks for the best quasi-conformal mapping whose BC resembles μ_0 as much as possible. Let ν_0 be the actual BC associated to f_0 . This gives us a pair (f_0, ν_0) , for which $\nu_0 = \mu(f_0)$.

With the initial BC ν_0 , our goal is to find a path in the space of all BCs, which approaches from ν_0 to the unique admissible BC ν^* of Teichmüller type. The optimal BC, ν^* , minimizes the variational problem (4.5). To find such a path, we consider two crucial properties of a T-Map to drive an initial BC to an admissible BC of Teichmüller type, given by the variational formulation (4.5). The two properties are

- $\nu = k \frac{\overline{\varphi}}{|\varphi|}$, where $0 \le k < 1$ and $\varphi : S_1 \to \mathbb{C}$ is a holomorphic function; and
- $\nu = \operatorname{argmin}_{\mu:S_1 \to \mathbb{C}} \{ ||\nu||_{\infty} \}$, where the μ 's are admissible BCs associated with surface diffeomorphisms satisfying the given boundary and landmark constraints.

Given the initial BC, ν_0 , we first apply the Laplace smoothing \mathcal{L} on both the norm $|\nu_0|$ and the argument $\arg(\nu_0)$ of ν_0 to obtain a new BC, $\tilde{\mu}_1$. The Laplace smoothing \mathcal{L} is applied on the norm and the argument of ν_0 independently. In other words,

(5.23)
$$\mathcal{L}(\nu_0)(T) = L_0(T)e^{i\theta_0(T)},$$

where

(5.24)
$$L_0(T) = \sum_{T_i \in \text{Nbhd}(T)} \frac{|\nu_0|}{|\text{Nbhd}(T)|} \text{ and } \theta_0(T) = \sum_{T_i \in \text{Nbhd}(T)} \frac{\arg(\nu_0)}{|\text{Nbhd}(T)|},$$

where T is a triangular face of K_1 , Nbhd(T) is the set of neighborhood faces of T, and |Nbhd(T)| is the number of neighborhood faces in the set Nbhd(T).

When ν_0 is not a constant, the Laplace smoothing on $|\nu_0|$ diffuses the norm of ν_0 and hence decreases the L^{∞} norm of ν_0 . Thus, $||\tilde{\mu}_1||_{\infty} < ||\nu_0||_{\infty}$. When $|\nu_0|$ is a constant, we have $||\tilde{\mu}_1||_{\infty} = ||\nu_0||_{\infty}$. It is necessary since we require the iteration to stop when the BC, ν_0 , is of Teichmüller type (and hence with a constant norm).

Next, another crucial requirement is that upon convergence, the argument of the associated BC, ν^* , of the optimal map, f^* , should be equal to the argument of a holomorphic function φ .

In other words, $\arg(\nu^*) = \frac{\overline{\varphi}}{|\varphi|}$. Having this property implies that the argument of ν^* should be harmonic. The reason is as follows. Suppose $\varphi = |\varphi|e^{i\theta}$ and $\nu^* = |\nu^*|e^{i\theta}$. Then, $\log \varphi = \log |\varphi| + i\theta$. Since $\log \varphi$ is holomorphic in one branch, θ should be harmonic, and hence $\Delta \theta = 0$. Motivated by this observation, in our iteration we also apply the Laplace smoothing on $\arg(\nu_0)$.

At the optimal state, $\Delta \theta^* = 0$, and hence θ^* is harmonic. We can therefore find the harmonic conjugate of $-\theta^*$; denote it by ζ . We get a holomorphic function $\zeta - i\theta^*$. Consider $\varphi = e^{\zeta - i\theta^*}$, which is holomorphic. We get that the optimal BC $\mu^* = ke^{i\theta^*} = k\frac{\overline{\varphi}}{|\varphi|}$. Hence, μ^* is of Teichmüller type.

Besides, a BC of Teichmüller type must have a constant norm. We apply the averaging operator \mathcal{A} on $\tilde{\mu}_1$ to project $\tilde{\mu}_0$ to a BC with constant norm. We first change the modulus (or the norm) of $\tilde{\mu}_0$ to a positive constant k. In other words, we consider $\mu_1 = k \arg(\tilde{\mu}_1)$. Of course, one crucial question to ask is what positive constant k should be chosen. Since the T-Map minimizes the L^{∞} norm of the BC, one natural choice is to let $k = ||\tilde{\mu}_1||_{\infty} - \epsilon$ (where ϵ is a small positive constant). Again, an important requirement is that the iteration should stop when the BC is of Teichmüller type (and hence with a constant norm). Therefore, a better choice for k should be

(5.25)
$$k := \frac{\int_{S_1} |\tilde{\mu}_1| dS_1}{\operatorname{Area}(S_1)}.$$

In other words, k is the mean of the norm of $\tilde{\mu}_1$ over the whole domain. When $\tilde{\mu}_1$ does not have a constant norm, $k < ||\tilde{\mu}_1||_{\infty}$. When $\tilde{\mu}_1$ has a constant norm, the norm of $\tilde{\mu}_1$ will be kept unchanged. In summary, the averaging operator \mathcal{A} is defined as follows:

(5.26)
$$\mu_1(T) = \mathcal{A}(\tilde{\mu_1})(T) := \left(\frac{\sum_{T \in \text{ all faces of } K_1} |\tilde{\mu_1}|(T)}{\text{No. of faces of } K_1}\right) \frac{\tilde{\mu_1}(T)}{|\tilde{\mu_1}(T)|}$$

With the Laplace smoothing and averaging on ν_0 , we obtain a new BC, $\mu_1 := \mathcal{A}(\mathcal{L}(\nu_n))$. An updated quasi-conformal map, f_1 , can then be obtained by the LBS: $f_1 = \mathbf{LBS}_{LM}(\mu_1)$. And an updated BC, $\nu_1 := \mu(f_1)$, can be computed. Thus, we get a new pair (f_1, ν_1) .

Therefore, given an initial BC, ν_0 (which is the BC of an initial map f_0), we adjust ν_0 to $\mu_1 := \mathcal{A}(\mathcal{L}(\nu_0))$. μ_1 is a BC with constant norm and its argument is closer to being harmonic. We then look for a new quasi-conformal map f_1 , whose BC is close to μ_1 , using LBS. We obtain a new BC $\nu_1 := \mu(f_1)$, which is closer to the Teichmüller type. We keep the iteration going until the algorithm converges to the desired T-Map. More specifically, given the pair (f_n, ν_n) obtained at the *n* iteration, we can obtain a new pair (f_{n+1}, ν_{n+1}) as follows:

(5.27)
$$\begin{aligned} \mu_{n+1} &:= \mathcal{A}(\mathfrak{L}(\nu_n)), \\ f_{n+1} &:= \mathbf{LBS}_{LM}(\mu_{n+1}), \\ \nu_{n+1} &:= \mu(f_{n+1}). \end{aligned}$$

Consequently, we get a sequence of pairs (f_n, ν_n) , which converges to the optimal BC associated to the T-Map. In practice, we stop the iteration when $||\nu_{n+1} - \nu_n|| < \epsilon$.

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We summarize the QC iteration as follows.

Algorithm 5.1 (QC iteration for open surfaces).

Input : Triangular meshes: K_1 and K_2 ; the desired landmark constraints and/or boundary condition.

Output : Optimal BC ν and the T-Map f.

- 1. Obtain the initial mapping $f_0 = \mathbf{LBS}_{LM}(\mu_0 := 0)$. Set $\nu_0 = \mu(f_0)$;
- 2. Given ν_n , compute $\mu_{n+1} := \mathcal{A}(\mathfrak{L}(\nu_n))$; Compute $f_{n+1} := \mathbf{LBS}_{LM}(\mu_{n+1})$ and set $\nu_{n+1} := \mu(f_{n+1})$;
- 3. If $||\nu_{n+1} \nu_n|| \ge \epsilon$, continue. Otherwise, stop the iteration.

The QC iteration can also be applied to the case when D_i (i = 1, 2) is a unit sphere. In other words, given a set of landmark constraints between the unit sphere, our goal is to look for the T-Map $f: D_1 \to D_2$. However, special attention has to be paid in this case.

Denote the landmark correspondence by $\{p_i\}_{i=1}^n \leftrightarrow \{q_i\}_{i=1}^n$. We can assume that the north pole is fixed. If not, it can also be achieved by a Mobius transformation. The LBS can be applied to unit spheres, by stereographically projecting D_i onto a big triangles in \mathbb{R}^2 (see Figure 5). However, a numerical error near the north pole is inevitable. We therefore propose an alternating scheme to fix this problem.

For the initial map, we add the vertices near the north pole $\{n_j\}_{j=1}^m$ (z > 0.99) as landmarks and fix all $\{n_j\}_{j=1}^m$. We then compute the T-Map f_0 using Algorithm 5.1. Numerical error will be introduced near the north pole. To fix it, in our next step, we consider the vertices $\{s_j\}_{j=1}^m$ near the south pole (z < -0.99) as landmarks. The correspondence is given by $s_j \leftrightarrow f_1(s_j)$. Rotate the south pole of D_i to the north pole by a Mobius transformation. We can again compute the T-Map f_2 using Algorithm 5.1.

We continue this process until the iteration converges. More specifically, at the *n* iteration where *n* is an even integer, we add vertices $\{s_j\}_{j=1}^m$ around the south pole as landmarks. Set the correspondence as $s_j \leftrightarrow f_n(s_j)$. Rotate the south pole of D_i to the north pole by a Mobius transformation, and obtain the T-Map f_{n+1} using Algorithm 5.1. When *n* is an odd integer, we add vertices $\{n_j\}_{j=1}^m$ around the north pole as landmarks. Set the correspondence as $n_j \leftrightarrow f_n(n_j)$ and obtain the T-Map f_{n+1} using Algorithm 5.1. Set $\nu_{n+1} = \mu(f_n)$.

This alternating process between the north pole and the south pole continues until $||\nu_{n+1} - \nu_n|| < \epsilon$.

The detailed algorithm can be summarized as follows.

Algorithm 5.2 (QC iteration for genus-0 closed surfaces).

Input : Triangular meshes: K_1 and K_2 ; the desired landmark constraints and/or boundary condition.

Output : Optimal BC ν and the T-Map f.

- 1. Add vertices around the north pole as landmarks and fix their positions. Obtain the initial T-Map f_0 using Algorithm 5.1. Set $\nu_0 = \mu(f_0)$;
- 2. Given f_n and ν_n . When n is even, add vertices $\{s_j\}_{j=1}^m$ around the south pole as landmarks. Set the correspondence as $s_j \leftrightarrow f_n(s_j)$. Rotate the south pole of D_i to the north pole. When n is odd, add vertices $\{s_j\}_{j=1}^m$ around the south pole as landmarks.

Set the correspondence as $s_j \leftrightarrow f_n(s_j)$. Obtain the T-Map f_{n+1} using Algorithm 5.1. Set $\nu_{n+1} = \mu(f_n)$;

3. If $||\nu_{n+1} - \nu_n|| \ge \epsilon$, continue. Otherwise, stop the iteration.

When D_i (i = 1, 2) is a unit disk, the LBS would also introduce a numerical error near 1. To fix it, the same alternating algorithm between 1 and -1 can be applied.

The QC iterations can also be extended to compute T-Map with soft landmark constraints. It will become useful when landmark features cannot be accurately located, and hence it is better to compute registration with landmarks approximately (but not exactly) matched. Denote the landmark correspondence by $\{p_i\}_{i=1}^n \leftrightarrow \{q_i\}_{i=1}^n$. Instead of enforcing a hard landmark constraint, soft constraints can be introduced by requiring $||p_i - q_i|| < \delta$ for all $i = 1, 2, \ldots, n$. This can be done easily by first computing a rough guess of a landmark matching T-Map g using Algorithm 5.1 or 5.2. The BC, μ_g , of g can be computed. Starting with μ_g , Algorithm 5.1 or 5.2 can be applied to compute a T-Map without enforcing the interior landmark constraints. A T-Map with soft landmark constraints can then be obtained. The detailed algorithm can be described as follows.

Algorithm 5.3 (QC iteration for soft landmark constraints).

Input : Triangular meshes: K_1 and K_2 ; desired landmark constraints and/or boundary condition; landmark constraint tolerance δ .

Output : Optimal Beltrami coefficient ν and the T-Map f.

- 1. Obtain an initial guess of landmark matching T-Map g_0 using Algorithm 5.1 or 5.2. Set the stopping criteria to be $||\nu_{n+1} - \nu_n|| < 100\epsilon$.
- 2. Given g_n and $\mu_n = BC$ of g_n . Starting from μ_n , compute the T-Map g_{n+1} using Algorithm 5.1 or 5.2, without setting the interior landmark constraints. Set the stopping criteria to be $||\nu_{n+1} \nu_n|| < \epsilon$. Let $\mu_{n+1} = BC$ of g_{n+1} .
- 3. If $||p_i q_i|| \ge \delta$ for some i = 1, 2, ..., n, continue. Otherwise, stop the iteration.

Numerical results have demonstrated the effectiveness of the proposed iterative scheme to compute the T-Map. The convergence of the QC iteration to the T-Map on general Riemann surfaces with arbitrary topologies can also be theoretically proven [24]. In [24], we discuss the theoretical proof of the convergence in detail. We have the following key results. Basically, **LBS**(μ) computes the generalized harmonic map under the auxiliary metric, $|dz + \mu dz|^2$, given by the BC, μ . From the QC iteration, we obtain a sequence of BCs, $\{\mu_n := \mathcal{A}(\mathcal{L}(\nu_n))\}_{n=1}^{\infty}$, where $\nu_n = \mu(f_n)$. In [24], we show that μ_n converges to an admissible BC, μ^* . Also, $\mu^* = \nu^* := \mu(f^*)$, where f^* is the optimal quasi-conformal map obtained and ν^* is its associated BC. An important question is whether the algorithm converges to the unique admissible BC, ν^* , of Teichmüller type, which is associated to our desired T-Map, f^* . This can be guaranteed due to the following observations.

Theorem 5.1 (convergence of the QC iteration). The QC iteration gives a convergent sequence of pairs (f_n, ν_n) , where ν_n is the BC of f_n , whose limit point is (f^*, ν^*) . Here, ν^* is the unique admissible BC of Teichmüller type associated with the extremal Teichmüller map.

Proof. Suppose (f_n, ν_n) is obtained at the *n*th iteration. $f_n = \mathbf{LBS}_{LM}(\mu_n)$, where $\mu_n = \mathcal{A}(\mathcal{L}(\nu_{n-1}))$. Note that the BC of f_n is generally not equal to μ_n . In fact, f_n is the harmonic map between S_1 and S_2 under the auxiliary metric, $g(\mu_n) = |dz + \mu_n dz|^2$. An energy functional



Figure 6. Example of the T-Map of the disk with fixed Dirichlet boundary condition. (A) shows the boundary condition. (B) shows the obtained T-Map, visualized using the texture mapping. (C) shows the histogram of the BC norm. (D) shows the sup-norm of the BC in each QC iteration.

 E_{BC} can then be defined on the space of all BCs, $\mathcal{B}(S_1)$, by letting $E_{BC}(\mu)$ be the harmonic energy of the generalized harmonic map under the auxiliary metric $g(\mu)$. In [24], we prove that $E_{BC}(\mu) \ge A_2$ (= surface area of S_2) and equality holds if and only if μ is admissible. It is also proven that μ_n converges to a minimizer of E_{BC} , μ^* . Since E_{BC} attains its minimum at μ^* when μ^* is admissible, it implies that the QC iteration converges to a pair (f^*, ν^*) , where $\nu^* = \mu(f^*) = \mu^*$.

Next, we show that ν^* is of Teichmüller type. At each step of the QC iteration, a Laplace smoothing \mathcal{L} and averaging \mathcal{A} are applied on ν_n . At the optimal state, $\mathcal{A}(\mathcal{L}(\nu^*)) = \nu^*$. This implies that $\Delta((\mathbf{arg})(\nu^*)) = 0$ and $|\nu^*| = k$, where k is a positive constant. Since $\mathbf{arg}(\nu^*)$ is harmonic, there exists a harmonic conjugate ζ of $\mathbf{arg}(\nu^*)$ such that $\zeta - i\mathbf{arg}(\nu^*)$ is holomorphic. Let $\varphi = e^{\zeta - i\mathbf{arg}(\nu^*)}$. φ is holomorphic and $\nu^* = k \frac{\overline{\varphi}}{|\varphi|}$. Hence, ν^* is an admissible BC of Teichmüller type. By Theorem 3.6, given the prescribed landmark and boundary constraints, there exists only one admissible BC of Teichmüller type. We conclude that ν^* is the unique BC associated to the unique extremal Teichmüller map.

6. Numerical experiments. In this section, we evaluate our proposed algorithm numerically by synthetic examples.

6.1. T-Map of simply connected domains. In our first numerical experiment, we test our method to compute the T-Map of the unit disk with a given Dirichlet boundary condition. A Dirichlet condition on the whole boundary is given as shown in Figure 6(A). The obtained T-Map is shown in (B), which is visualized using the texture mapping. Note that the original texture is deformed under the T-Map. However, the dilations of the ellipses deformed from the infinitesimal circles are the same. It means that the norm of the BC is constant everywhere. The histogram of the norm of the BC is also shown in (C), which again demonstrates that the norm of the BC is equal to a constant k = 0.15016. The standard deviation of the BC norm is 0.0034373. Under the QC iteration, we iteratively obtain a sequence of pair $\{f_n, \nu_n = \mu(f_n)\}_{n=0}^{\infty}$. (D) shows the supreme norm of BC, $||\nu_n||_{\infty}$, in each QC iteration. It decreases as iteration increases, indicating that the algorithm converges to an optimized T-Map minimizing the maximal conformality distortion. Also, the algorithm converges quickly in less than 20 iterations.



Figure 7. Another example of the T-Map of the disk with fixed Dirichlet boundary condition. (A) shows the obtained T-Map, visualized using the texture mapping. (B) shows the histogram of the norm of the BC. (C) shows the sup-norm of the BC in each QC iteration.



Figure 8. Computation of T-Map on an irregular mesh. The QC iteration is independent of the mesh structure. The boundary condition is given and the associated T-Map is constructed, as shown in (A). (B) shows the histogram of the norm of the BC.

Besides, the Dirichlet boundary condition can be of arbitrary shape. Figure 7 shows a T-Map between the unit disk and the amoeba shape with given boundary conditions. The norm of BC is also constant everywhere, as demonstrated in (B), indicating that the obtained mapping is indeed a T-Map. (C) shows the sup-norm of BC in each QC iteration.

Our proposed QC iteration is also independent of the mesh structure. Figure 8(A) shows a mesh with an irregular mesh structure. The boundary condition is given and the associated T-Map is constructed. (B) shows the histogram of the norm of the BC, which is accumulated at 0.28. It demonstrates that the obtained map is indeed a T-Map. This result shows that our proposed method is effective for computing a bijective T-Map between meshes, even with irregular mesh structure.

Our algorithm can also be applied to the situation when only a few landmark constraints are enforced on the boundary (instead of the Dirichlet condition defined on the whole boundary). In Figure 9, we test our algorithm to compute the T-Map of the unit disk with only eight landmark point constraints on the boundary. (A) shows the T-Map. Again, the dilations of the ellipses deformed from the small circles are the same, meaning that the norm of the BC is constant everywhere. (B) shows the histogram of the BC norm. The norm k of the BC is equal to 0.201. (C) shows the sup-norm of the BC in each QC iteration. Again, it converges



Figure 9. *T-Map of the disk with only eight landmark point constraints on the boundary.* (A) shows the *T-Map.* (B) shows the histogram of the BC norm. (C) shows the sup-norm of the BC in each QC iteration. (D) shows the automatically obtained optimal boundary correspondence. (E) shows the histogram of the BC norm under the T-Map with arc-length parameterized boundary condition. (F) shows the histogram of the BC norm under harmonic map with arc-length parameterized boundary condition.

quickly in less than 60 iterations. Our algorithm automatically detects the optimal boundary correspondence. (D) shows the obtained optimal boundary correspondence. (E) shows the histogram of the BC norm under the T-Map with arc-length parameterized boundary condition (of which the prescribed constraints on the boundary are satisfied). Although a T-Map can still be obtained, the norm of the BC is equal to 0.23, which is higher than the case when only eight landmark point constraints are enforced. Hence, the obtained T-Map is not extremal. (F) shows the histogram of the BC norm under the harmonic map with arc-length correspondence on the boundary. Note that the distribution of the conformality distortion is highly nonuniform.

6.2. T-Map of multiply connected domains. Our method can also be applied to multiply connected domains. In Figure 10, we test our method to compute the T-Map of an annulus with Dirichlet boundary condition. (A) shows the obtained T-Map. (B) shows the histogram of the norm of BC, which illustrates that the obtained map is indeed a T-Map. Besides, the BC of a T-Map is of the form $\mu = k \frac{\overline{\varphi}}{|\varphi|}$, where φ is holomorphic. The imaginary part of $\log(\mu)$, which is equal to the argument of μ , must be harmonic. (C) shows the Laplacian of the argument of the BC. It is accumulated at 0, meaning that the argument of the BC is indeed harmonic. (D) shows the supreme norm of the BC in each iteration. Again, it decreases as iteration increases and converges quickly in just six iterations.



Figure 10. T-Map of the annulus with fixed Dirichlet boundary condition. (A) shows the obtained T-Map, visualized using the texture mapping. (B) shows the histogram of the BC norm. (C) shows the histogram of the Laplacian of the argument of the BC. (D) shows the sup-norm of the BC in each QC iteration.



Figure 11. T-Map of the multiply connected domain containing three holes with fixed Dirichlet boundary condition. (A) shows the obtained T-Map. (B) shows the histogram of the norm of the BC. (C) shows the sup-norm of the BC in each QC iteration.

In Figure 11, we compute the T-Map between multiply connected domains having three holes with a given boundary condition. (A) shows the obtained T-Map. The histogram of the BC norm and the supreme norm of BC in each iteration are shown in (B) and (C), respectively. Note that the algorithm converges quickly in 10 iterations. Figure 12 demonstrates the result of the T-Map between more complicated multiply connected domains having six holes. The obtained T-Map is shown in (A). (B) and (C) show the histogram of the BC norm and the supreme norm of the BC in each iteration, respectively. Again, the algorithm converges quickly in 30 iterations.

6.3. T-Map with interior landmark constraints. Our algorithm can compute the T-Map with interior landmark constraints enforced. Figure 13 shows the T-Map between the unit disk with 24 interior landmark constraints and few point constraints on the boundary enforced. (A) shows the 24 landmark constraints and 4 point constraints on the boundary. (B) shows the obtained T-Map, visualized using the texture mapping. (C) shows the sup-norm of the BC in each QC iteration, which converges within 30 iterations. (D) shows the histogram of



Figure 12. T-Map of the multiply connected domain containing six holes with fixed Dirichlet boundary condition. (A) shows the obtained T-Map. (B) shows the histogram of the norm of the BC. (C) shows the sup-norm of the BC in each QC iteration.



Figure 13. T-Map between the disks with 24 interior landmark constraints and 4 point constraints on the boundary enforced. (A) shows the 24 landmark constraints and 4 point constraints on the boundary. (B) shows the T-Map. (C) shows the sup-norm of the BC in each QC iterations. (D) shows the histogram of the norm of the BC. (E) shows the histogram of the BC norm under the T-Map with arc-length parameterized boundary condition.

the norm of the BC. The norm of the BC is uniformly equal to 0.2. (E) shows the histogram of the BC norm of a T-Map with the arc-length boundary correspondence enforced (of which the prescribed four point constraints on the boundary are satisfied). As expected, the norm of the BC is larger (= 0.28), which means the T-Map is not an extremal one.

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Figure 14. T-Map between the disks with three interior landmark curve constraints enforced. (A) shows the T-Map with three landmark curve constraints enforced. (B) shows the histogram of the norm of the BC. (C) shows the sup-norm of the BC in each QC iteration.



Figure 15. T-Map between the unit disks with 20 interior landmark constraints enforced (without boundary constraints). (A) shows the 20 landmark constraints. (B) shows the T-Map. (C) shows the histogram of the norm of the BC.

Besides, interior landmark curve constraints can be enforced. In Figure 14, we test our algorithm to compute the T-Map of the unit disk with three interior landmark curve constraints enforced. (A) shows the obtained T-Map. (B) shows the histogram of the norm of the BC. The norm of the BC is accumulated at 0.53. (C) shows the sup-norm of the BC in each QC iteration. The sup-norm of the BC is decreasing, indicating that the algorithm converges to an optimized T-Map, minimizing the maximal conformality distortion.

We also test our algorithm for the case when only interior landmark constraints are enforced (without boundary condition). Figure 15 shows the computed T-Map with 20 interior landmark point constraints enforced. (A) shows the constraints of the feature points. (B) shows the obtained T-Map. (C) shows the histogram of the norm of the BC. Although no boundary constraint is enforced, our algorithm is able to automatically determine an optimal boundary correspondence of the T-Map minimizing the geometric distortion.

Besides hard landmark constraints, our algorithm can also be applied to compute a T-Map with soft landmark constraints. Figure 16 shows an example of the T-Map of the unit disk with soft landmark constraints. (A) shows the T-Map with hard landmark constraints enforced. The histogram of the norm of the BC is plotted, which is accumulated at 0.41. Landmarks are perfectly matched. (B) shows the obtained T-Map with soft landmark constraints. Landmarks



Figure 16. T-Map with soft landmark constraints. (A) shows the obtained T-Map with hard landmark constraints. The histogram of the norm of BC is plotted, which is accumulated at 0.4193. (B) shows the obtained T-Map with soft landmark constraints. Landmarks cannot be exactly matched, but less conformality is introduced, as shown in the histogram of the norm of the BC.



Figure 17. *T-Map between the spheres.* (A) *shows landmark constraints on the sphere.* (B) *shows the* T-Map. (C) *shows the histogram of the norm of the BC.*

cannot be exactly matched, but less conformality distortion is introduced. As shown in the histogram of the BC, the norm of the BC is accumulated at 0.35. It is less than that of the case when hard landmark constraints are enforced.

Our algorithm can be applied as well to computing the T-Map between the unit sphere with interior landmark constraints enforced. Figure 17(A) shows 20 landmark constraints on the sphere. (B) shows the obtained T-Map. (C) shows the histogram of the norm of the BC. The norm of the BC is concentrated near 0.21, meaning that the mapping is indeed a T-Map.

	Vertex number	Time	$ \mu _{\infty}$
Analytic example	8936	$1.297 \ {\rm s}$	0.4122
4 points on boundary $+$ 3 landmark curves	8257	$4.46 \mathrm{\ s}$	0.4154
Disk (Dirichlet boundary)	8257	$5.420 \mathrm{~s}$	0.2295
8 points on boundary	8257	$6.645~\mathrm{s}$	0.2120
4 points on boundary $+$ 20 landmarks	8257	$8.467 \ s$	0.2843
Arbitrary shape	8257	$10.056~{\rm s}$	0.3475
Disk free boundary $+$ 20 landmarks	8257	$18.579 \ s$	0.1855
3 holes disk	17746	$15.030 \ {\rm s}$	0.4088
6 holes disk	22979	$17.680 \ s$	0.4433
Sphere	10242	$34.679 \ s$	0.3086

Table 1Computational time for QC iterations.



Figure 18. Comparison of the landmark matching T-Map with the harmonic map, thin plate spline (TPS), and LDDMM. (A) shows seven landmark point constraints. (B) shows the obtained landmark matching T-Map. (C) shows the landmark matching harmonic map. (D) shows the mapping obtained from TPS. (E) shows the mapping obtained from LDDMM.

6.4. Computational time. To test the efficiency of the QC iterations, we record the computational time of the proposed QC iterations. All our experiments were done on a laptop with an Intel Core i7 2.70 GHz CPU and 8 GB RAM. In Table 1, we list the computational time of the QC iterations for the experiments we have done. Note that for a reasonably dense mesh (~ 8 K vertices), the computational time is generally less than 10 seconds. For more complicated domains with denser meshes, the computational time is longer but the whole computation can still be done within 35 seconds. We note that the implementation of the QC iterations has not been optimized. It is currently implemented using MATLAB. The implementation of the algorithm can be further improved. Also, the algorithm can be parallelized. Using GPU, it is believed that the computational time can be dramatically sped up.

6.5. Comparison with existing methods. To test the effectiveness of our proposed method, we compare our algorithm with other existing methods. In Figure 18, we compare the land-mark matching T-Map with three other existing methods, namely, (1) the harmonic map

=

[40], (2) thin plate spline (TPS) [3], and (3) LDDMM [17, 9, 10, 7, 8]. Seven landmark correspondences are enforced, as shown in (A). (B) shows the obtained landmark matching T-Map. It is bijective, and landmarks are perfectly matched. (C) shows the landmark matching harmonic map. Landmarks are exactly matched, but overlaps (flips) occur. (D) shows the mapping obtained from TPS. Landmarks are not exactly matched and overlaps occur. (E) shows the mapping obtained from LDDMM. A smooth bijective mapping can be obtained. Also, landmarks can be better matched (although not perfectly).

Tables 2 and 3 give the quantitative comparison between different methods. Table 2 gives the comparisons under different sizes of deformation. The computational time and number of overlapping faces are captured. Harmonic map and TPS are the fastest algorithms. However, both methods cannot obtain a bijective mapping that matches landmark exactly. LDDMM can compute a smooth bijective mapping that closely matches landmarks, under different sizes of deformation. However, the computational cost is comparatively more expensive (382 seconds for tiny deformation and 409 seconds for large deformation). Using our algorithm, we can obtain a bijective T-Map that matches landmarks perfectly. The computational time is also reasonably fast, taking about 2.89 seconds for tiny deformation and 4.61 for large deformation. Zero overlapping faces are observed under all sizes of deformation, meaning that the T-Maps obtained are indeed bijective. Table 3 gives the comparisons under different numbers of landmarks. Again, harmonic map and TPS are the fastest algorithms, but both methods cannot obtain a bijective mapping. For example, harmonic map gives 18 overlapping faces if three interior landmark constraints are enforced and 74 overlapping faces if 20 interior landmarks are used. LDDMM can compute a smooth bijective mapping that closely matches landmarks, under different numbers of landmark constraints. Again, the computational cost is comparatively more expensive. For example, it takes 377 seconds with 20 landmark constraints. Using our algorithm, we can obtain a bijective T-Map that matches landmarks perfectly, under different numbers of landmark constraints. The computational time is reasonably fast, which takes about 3.4 seconds with 7 landmarks and 4.15 seconds with 20 landmarks.

(Time / Overlap)	Tiny	Moderate	Large
T-Map	2.892 s / 0	4.412 s / 0	4.610 s / 0
T-Map	2.892 s / 0	4.412 s / 0	4.610 s / 0
Harmonic map	1.633 s / 13	1.665 s / 42	1.652 s / 110
TPS	0.308 s / 20	0.339 s / 27	0.253 s / 27
LDDMM	382.316 s / 0	396.240 s / 0	$409.902~{\rm s}~/~0$

 Table 2

 Comparison with other methods with different sizes of deformation.

 Table 3

 Comparison with other methods under different numbers of interior landmarks.

(Time / Overlap)	3	7	20
T-Map	5.623 s / 0	3.405 s / 0	4.152 s / 0
Harmonic map	1.626 s / 18	1.637 s / 42	1.610 s / 74
TPS	0.299 s / 17	0.299 s / 27	0.278 s / 21
LDDMM	504.727 s / 0	353.963 s / 0	377.400 s / 0



Figure 19. (A) shows two brain surfaces with three corresponding landmarks. (B) shows the T-Map with three landmark constraints enforced. (C) shows the histogram of the norm of the BC.

7. Applications. In this section, we apply our proposed algorithms for computing landmark matching T-Maps to practical problems. More specifically, we will consider the problems of computing brain landmark matching registrations, constrained texture mappings, and human face registrations.

7.1. Brain landmark matching registration. Landmark-based surface registrations are commonly applied for finding meaningful 1-1 correspondences between human brain cortical surfaces [40, 38, 26, 27, 28]. On cortical surfaces, sulcal landmarks can be labeled either manually by neuroscientists or automatically based on various geometric quantities [29]. The sulcal landmarks are important anatomical features. It is therefore desirable to obtain a registration between the cortical surfaces with least geometric distortion, which matches the sulcal landmarks as much as possible. Our algorithms for computing landmark matching T-Maps can be applied. In Figure 19, we apply our algorithm to compute the T-Map between two different brain surfaces with three corresponding landmarks labeled. (A) shows the corresponding sulcal landmarks, indicated by different colors. (B) shows the obtained T-Map with three landmark constraints enforced, visualized by the circle packing textures. The sulcal landmarks are exactly matched under the mapping. (C) shows the histogram of the norm of the associated BC. The norm is a constant showing that the obtained registration is indeed a T-Map. We also test the method to register cortical surfaces with more sulcal landmarks. In Figure 20, we compute the T-Map between two brain surfaces with six corresponding sulcal landmarks labeled. The obtained registration and the norm of its associated BC are shown in (B) and (C), respectively. The landmarks are exactly matched. Again, the norm of the BC is a constant, showing that the obtained registration is a T-Map which minimizes the conformality distortion.



Figure 20. (A) shows two brain surfaces with six corresponding landmarks. (B) shows the T-Map with six landmark constraints enforced. (C) shows the histogram of the norm of the BC.

7.2. Constrained texture mapping. Texture mapping is one of the major photorealistic techniques in computer graphics for generating realistic and visually rich three-dimensional (3D) surfaces [5, 14]. It is usually done by putting each surface mesh in correspondence with a 2D image [45, 13, 20, 19, 34, 21, 2, 1]. Such a correspondence between the surface mesh and the image is called the texture mapping. The use of constrained texture mappings is popular, in which the texture mapping should have minimum distortion, while matching the landmark points exactly. We apply our algorithms to compute the landmark matching T-Map between the surface mesh and the image onto the human face surface. (A) shows the corresponding landmark points labeled manually on the human face and the texture image. The landmark matching T-Map is computed and is used as texture mapping to project the cat image onto the human face. The textured surface is shown in (B). (C) shows the norm of the associated BC of the texture mapping. The norm is approximately a constant. It means the texture mapping computed is indeed a T-Map, which minimizes the conformality distortion.

In Figure 22, we further test our algorithm on a multiply connected human face. (A) and (B) show the texture (tiger) image and a multiply connected human face. Corresponding landmark points are labeled manually on the texture image and the surface mesh. In (C), the surface mesh is mapped to a multiply connected domain in two dimensions by a T-Map matching the landmark points exactly. (D) shows the textured surface. (E) shows the norm of the associated BC of the texture mapping. The norm is approximately a constant, which means the texture mapping computed is indeed a T-Map.

7.3. Human face registration. In face recognition, finding accurate spatial correspondences between human faces is a crucial process in comparing and recognizing faces effectively



Figure 21. (A) shows a human face and a texture image of a cat. Corresponding landmark points are labeled on the surface and the texture image. We compute the T-Map that matches the landmark points. The T-Map is used as constrained texture mapping to project the texture image onto the surface, as shown in (B). (C) shows the histogram of the norm of the BC.



Figure 22. (A) a texture image of a tiger. (B) shows a multiply connected human face. Corresponding landmark points are labeled on the surface and the texture image. In (C), the surface mesh is mapped to a multiply connected domain in two dimensions by a T-Map matching the landmark points exactly. The T-Map is used as constrained texture mapping to project the texture image onto the surface, as shown in (D). (E) shows the histogram of the norm of the BC.

[43, 46]. Corresponding features can be extracted on human faces based on curvatures, such as high curvature points near nose tips and lips. Accurate face registration can then be obtained by computing a mapping that matches the corresponding features. Landmark matching T-Map, which minimizes the geometric distortion, can then be used. In Figure 23, we apply our algorithm to compute the registration between male and female human faces. The human faces are both simply connected open surfaces. Corresponding feature points are labeled on both faces. The associated T-Map is obtained, which is visualized by texture mapping. The corresponding features are exactly matched. (C) shows the histogram of the norm of the BC, which is almost a constant. This demonstrates that the obtained registration is a T-Map.

Our algorithm can also be applied to obtain registration between multiply connected human faces. Figure 24 shows two multiply connected human faces. Corresponding feature landmarks are labeled. T-Map matching the features exactly is computed, as shown in (B). It is again visualized by texture mapping. (C) shows the histogram of the norm of the BC.



Figure 23. *T*-Map of the simply connected domain with landmark point constraints. (A) shows the two faces with landmark point constraints. (B) shows the *T*-Map of the two faces. The resultant mapping is illustrated by texture mapping. (C) shows the histogram of the norm of the BC.



Figure 24. T-Map of the multiply connected domain with landmark point constraints. (A) shows the two faces with landmark point constraints. (B) shows the T-Map of the two faces. The resultant mapping is illustrated by texture mapping. (C) shows the histogram of the norm of the BC.

Again, it is almost a constant, which demonstrates that the obtained registration is a T-Map minimizing the conformality distortion.

8. Conclusion. We address the problem of computing Teichmüller mappings (T-Maps) between surfaces, which minimizes the maximal conformality distortion. The proposed algorithms can be applied to obtain a landmark matching registration between surface meshes. Given a set of corresponding landmark points or curves defined on both surfaces, a unique landmark matching T-Map can be obtained, which minimizes the conformality distortion. In this paper, we propose an efficient iterative algorithm, called the quasi-conformal (QC) iteration, to compute the T-Map. The key idea is to represent the set of diffeomorphisms by Beltrami coefficients (BCs). We then look for an optimal BC associated to the desired T-Map. The associated T-Map can be efficiently reconstructed from the optimal BC using the linear Beltrami solver (LBS). Using our proposed method, the T-Map between reasonably dense meshes can be accurately and efficiently computed. The obtained registration is guaranteed to be bijective. Besides, T-Map with soft landmark constraints can also be computed using our proposed algorithm. It becomes useful when landmark features cannot be accurately located, and hence it is better to compute registration with landmarks approximately (but not exactly) matched. We applied the proposed algorithm to real applications, such as brain

landmark matching registration, constrained texture mapping, and human face registration. Experimental results shows that our method is effective in computing a nonoverlap landmark matching registration with the least amount of conformality distortion.

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