A Multiplicative Combinatorial Array Of A Triangle

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Abstract

We introduce a triangular array of integers defined by $a(n, k) = k(k + 1)^{n-k}$ for $n \geq 1$ and $1 \leq k \leq n$, which we call the researcher's triangle. The construction is multiplicative and differs from classical additive triangles such as Pascal's. We give combinatorial interpretations using a leaders-followers model and a rooted hierarchical structure model. We identify several diagonal sequences as classical figurate numbers, including powers of two (A000079), natural numbers (A000027), and oblong numbers (A002378), and establish connections to triangular and tetrahedral numbers. We derive a bivariate generating function and a recurrence relation, and we discuss the row sums.

1 Introduction

Triangular arrays such as Pascal's triangle, Stirling numbers of the second kind, and Eulerian numbers are central objects in combinatorics. Most classical triangles are defined by additive recurrences. In this note we introduce a multiplicative triangular array motivated by the multiplication rule of indices.

The first three rows are constructed as follows:

$$\text{Row } 1: 1 \times 2^0 = 1,$$
$$\text{Row } 2: (1 \times 2^1) + (2 \times 3^0) = 2 + 2,$$
$$\text{Row } 3: (1 \times 2^2) + (2 \times 3^1) + (3 \times 4^0) = 4 + 6 + 3.$$

Extrapolating this pattern gives the closed-form expression

$$a(n, k) = k \cdot (k + 1)^{n-k}, \qquad n \geq 1,\ 1 \leq k \leq n. \tag{1}$$

We call this array the researcher's triangle. Rows 1 through 8 are displayed in right-angle and equilateral formats in Tables 1 and 2.

2 Combinatorial interpretations

We give two concrete combinatorial models that enumerate the entry $a(n, k) = k(k + 1)^{n-k}$.

*1 Researcher’s triangle in right-angle format (A000079, A025192. Figure 1 . . .)

1							
2	2						
4	6	3					
8	18	12	4				
16	54	48	20	5			
32	162	192	100	30	6		
64	486	768	500	180	42	7	
128	1458	3072	2500	1080	294	56	8

*2 Researcher’s triangle in equilateral format. Figure 2

				1			
			2		2		
		4		6		3	
	8		18		12		4
16		54		48		20	5

(Further rows follow the same pattern.)

2.1 Leaders–followers model

For fixed n and k, consider the following configuration:

- There are k labeled leaders.
- Choose one chief among them: k ways.
- The remaining $n-k$ followers each choose one of $k+1$ “levels”: the chief’s personal level, or the level of any of the k leaders (including the chief as a leader).

Each choice of levels for the followers gives a distinct configuration, and the total number is

$$k \cdot (k+1)^{n-k} = a(n,k).$$

Example ($n = 4, k = 3$):

- There are $k = 3$ labeled leaders.
- Choose one chief among them: 3 ways.
- The remaining $n-k = 1$ follower chooses one of $k+1 = 4$ levels: the chief’s personal level, or the level of any of the 3 leaders (including the chief as a leader), or remain independent.
- Total configurations: $3 \times 4 = 12$.

2.2 Word / alphabet model

Equivalently, form a word of length $n-k$ over an alphabet of size $k+1$, say $\{0, 1, 2, \ldots, k\}$, where 0 represents the chief's personal level and $1, \ldots, k$ represent the leaders. Independently choose a distinguished symbol from $\{1, \ldots, k\}$ (the leaders, excluding 0). The distinguished symbol need not appear in the word; it is merely a marker.

The number of marked words is

$$(k+1)^{n-k} \cdot k = k(k+1)^{n-k} = a(n, k).$$

Example ($n = 4, k = 3$):

- Alphabet of size $k + 1 = 4$: $\{0, 1, 2, 3\}$.
- Form a word of length $n - k = 1$: 4 possible letters $(0, 1, 2, 3)$.
- Independently choose a distinguished symbol from $\{1, 2, 3\}$ (the leaders, excluding 0).
- Total marked words: $4 \times 3 = 12$.

Why exclude 0? In the word model, the symbol 0 represents the chief's personal level, which is not a leader. The distinguished symbol must correspond to a leader (one of the k leaders), so it is chosen from $\{1, \ldots, k\}$. This matches the factor k in the formula.

3 Diagonal sequences and figure numbers

Several diagonals of the researcher's triangle correspond to well-known integer sequences.

Leftmost diagonal ($k = 1$): $a(n, 1) = 2^{n-1}$, the powers of two (A000079).

Rightmost diagonal ($k = n$): $a(n, n) = n$, the natural numbers (A000027).

First sub-diagonal ($k = n - 1$): $a(n, n - 1) = n(n - 1)$, the oblong (pronic) numbers (A002378).

Second sub-diagonal ($k = n - 2$): $a(n, n - 2) = (n - 2)(n - 1)^2$, sequence $k(k + 1)^2$.

3.1 Connection to triangular and tetrahedral numbers

Let $T_m = m(m + 1)/2$ denote the m-th triangular number. Each oblong number satisfies

$$j(j - 1) = 2T_{j-1}.$$

Summing the oblong numbers from $j = 2$ to n gives

$$\sum_{j=2}^{n} j(j - 1) = 2 \sum_{j=2}^{n} T_{j-1}.$$

The sum of the first $n - 1$ triangular numbers is the $(n - 1)$-th tetrahedral number $Te_{n-1} = (n - 1)n(n + 1)/6$ (A000292). Hence

$$\sum_{j=2}^{n} j(j - 1) = 2Te_{n-1} = \frac{(n - 1)n(n + 1)}{3}.$$

Examples:

$$n = 5: \quad 2 + 6 + 12 + 20 = 40 = 2Te_4,$$

$$n = 6: \quad 2 + 6 + 12 + 20 + 30 = 70 = 2Te_5,$$

$$n = 7: \quad 2 + 6 + 12 + 20 + 30 + 42 = 112 = 2Te_6.$$

Here $2 = 1 \cdot 2$, $6 = 2 \cdot 3$, $12 = 3 \cdot 4$, $20 = 4 \cdot 5$, $30 = 5 \cdot 6$, and $42 = 6 \cdot 7$ are the oblong numbers.

4 Row sums

Define the n-th row sum by

$$R(n) = \sum_{k=1}^{n} a(n,k) = \sum_{k=1}^{n} k(k+1)^{n-k}.$$

The first few values are

$$1, 4, 13, 42, 143, 522, 2047, 8596, 38485, 182904,$$
$$919145, 4866870, 27068419, 157693006, 959873707, 6091057008, ..$$

The row sum sequence and the triangle are currently under consideration by the OEIS. Empirically, the growth satisfies $\lim_{n\to\infty} R(n+1)/R(n) \approx 4$, consistent with the dominant term coming from $k \approx n$.

5 Generating function

The bivariate generating function is

$$F(x,y) = \sum_{n\geq 1} \sum_{k=1}^{n} k(k+1)^{n-k} x^n y^k.$$

Setting $m = n - k$,

$$F(x,y) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} k(k+1)^m x^{k+m} y^k = \sum_{k=1}^{\infty} \frac{k(xy)^k}{1-(k+1)x},$$

valid as a formal power series.

Example. The coefficient of x^3y^2 comes from $k = 2$, $m = 1$:

$$\frac{2(xy)^2}{1-3x} = 2x^2y^2(1 + 3x + 9x^2 + \dots) = 2x^2y^2 + 6x^3y^2 + \dots,$$

giving $a(3,2) = 6$. The coefficient of x^4y^3 comes from $k = 3$, $m = 1$: $a(4,3) = 12$.

6 Recurrence relations

Within a fixed column k, entries satisfy a simple multiplicative recurrence. For $n > k$,

$$a(n,k) = (k+1)a(n-1,k), \tag{2}$$

with boundary condition $a(k,k) = k$. For $k = 1$, this reduces to $a(n,1) = 2a(n-1,1)$ with $a(1,1) = 1$, yielding the powers of two.

Example. $a(4,3) = 12$, so $a(5,3) = 4 \cdot 12 = 48 = 3 \cdot 4^2$.

7 Comparison with Pascal's triangle

Pascal's triangle satisfies $P(n,k) = P(n-1,k-1)+P(n-1,k)$ and counts subsets. The researcher's triangle satisfies the multiplicative recurrence (2) and counts functions with a distinguished preimage. Table 1 summarizes the comparison.

*3 Comparison of Pascal and researcher's triangles Table 1

Property	Pascal	Researcher's
Formula	$\binom{n}{k}$	$k(k+1)^{n-k}$
Recurrence	Additive	Multiplicative
$a(n,n)$	1	n
$a(n,1)$	n	2^{n-1}

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