EXPONENTIAL-POLYNOMIAL ROOT MEANS DERIVED FROM THE PRIME DENSITY TRIANGLE: AN ELEMENTARY APPROACH TO $\sqrt{m}$

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2026

Abstract

The Prime Density Triangle offers a novel combinatorial framework for exploring properties of prime numbers. From the algebraic relations embedded within this triangle, specifically the coupled expressions $X^{p_1} \pm p_1$ and $X^{p_0} \pm p_0$, we derive two closed-form sequences. First, the multiplication of these expressions yields the sequence $S_n(m) = (m^n + n^2)^{1/(n+2)}$, which converges to m. Second, by a deliberate construction—replacing the additive denominator $n + 2$ with the multiplicative denominator $2n$—we obtain the Exponential-Polynomial Root Mean (EPRM) defined by $R_n(m) = (m^n + n^2)^{1/(2n)}$ for $m > 0$ and $n \in \mathbb{N}$. This construction provides a closed-form, non-iterative method for approximating $\sqrt{m}$ for sufficiently large n. In this paper, we present a complete derivation of both sequences from the Prime Density Triangle's algebraic frameworks, rigorous convergence proofs including asymptotic error analysis, detailed worked examples for $m = 2, 3, 5$ with explicit computations, numerical summary tables, explicit iterations for classical methods (Babylonian and Newton-Raphson), and a comprehensive comparison. The approach is elementary and highlights the dominance of exponential terms over polynomial corrections.

Keywords: Square root approximation, Prime Density Triangle, Exponential-polynomial means, Non-iterative methods, Asymptotic analysis, Algebraic relations

1 INTRODUCTION

The Prime Density Triangle is constructed using a modified rule of indices. This structure yields rich patterns useful for studying primes, related number-theoretic objects, and nuclear magic numbers. Within the development of the triangle, specific coupled algebraic relations emerge naturally, which serve as the generative engine for prime sequences:

$$X^{p_1} \pm p_1 \quad (1)$$

$$X^{p_0} \pm p_0 \quad (2)$$

Where p_1 denotes the first integer prime and p_0 denotes any positive natural number.

These relations, when multiplied and simplified by focusing on the negative cross terms, reduce to a fundamental algebraic relation that serves as the foundation for the approximation methods introduced in this work. The multiplication of these two expressions leads to a sequence that converges to the base value m itself. From this, we construct a companion sequence—the Exponential-Polynomial Root Mean (EPRM)—by replacing the additive denominator with a multiplicative one, which yields convergence to $\sqrt{m}$. This elegant transformation highlights the deep algebraic richness of the Prime Density Triangle's framework.

2 DERIVATION AND FOUNDATIONAL FRAMEWORK

2.1 Multiplication of the Algebraic Relations: Convergence to m

To derive the fundamental equation, we start with the algebraic relations embedded within the Prime Density Triangle:

$$\text{Eq}(1) : X^{p_1} \pm p_1$$

$$\text{Eq}(2) : X^{p_0} \pm p_0$$

Multiplying these expressions together gives:

$$(X^{p_1} \pm p_1)(X^{p_0} \pm p_0) = 0$$

Expanding the product term by term yields:

$$X^{2p_1+p_0} \pm p_0 X^{p_1} \pm p_1 X^{p_0} \pm p_1 p_0 = 0$$

Considering only negatives aspect with x terms:

$$X^{2p_1+p_0} - p_0 X^{p_1} - p_1 X^{p_0} = 0$$

Rearranging to isolate the highest power term:

$$X^{p_1+p_0} = p_0^{p_1} + p_1^{p_0}$$

It implies that:

$$X = (p_0^{p_1} + p_1^{p_0})^{1/(p_1+p_0)}$$

OR commutatively, $X = (p_1^{p_0} + p_0^{p_1})^{1/(p_1+p_0)}$.

Where $p_1 = 2$, $p_0 = n$, then:

$$X = (2^n + n^2)^{1/(2+n)}$$

Generalizing:

$$\boxed{S_n(m) = (m^n + n^2)^{1/(n+2)}, \quad m > 0, n \in \mathbb{N}} \quad (3)$$

Convergence of $S_n(m)$ to m: Taking the natural logarithm:

$$\ln S_n(m) = \frac{1}{n+2} \ln(m^n + n^2) \quad (4)$$

$$\ln S_n(m) = \frac{1}{n+2} \left[n \ln m + \ln\left(1 + \frac{n^2}{m^n}\right) \right] \quad (5)$$

As $n \to \infty$:

$$\lim_{n\to\infty} \ln S_n(m) = \lim_{n\to\infty} \frac{n \ln m}{n+2} + \lim_{n\to\infty} \frac{\ln(1 + n^2/m^n)}{n+2} = \ln m \quad (6)$$

Exponentiating:

$$\boxed{\lim_{n\to\infty} S_n(m) = e^{\ln m} = m} \quad (7)$$

Thus, the multiplication of the two algebraic frameworks yields convergence to m.

2.2 Construction of the Square-Root Sequence: The EPRM

Having established that the multiplication of the two algebraic frameworks yields convergence to m, we now construct a companion sequence that converges to $\sqrt{m}$. Observe that in the denominator of the exponent in $S_n(m)$, the term $n + 2$ is additive $(n + 2)$. This additive structure produces a limiting factor of 1 in the logarithm, yielding convergence to m.

To obtain $\sqrt{m}$, we require a limiting factor of $1/2$ in the logarithm. This is achieved by replacing the additive denominator $n+2$ with the multiplicative denominator $2n$. This transformation is natural and elegant, as it turns addition into multiplication, thereby halving the logarithmic factor.

Thus, we define the Exponential-Polynomial Root Mean (EPRM):

$$\boxed{R_n(m) = (m^n + n^2)^{1/(2n)}, \quad m > 0, n \in \mathbb{N}} \quad (8)$$

This closed-form expression serves as a direct, non-iterative approximation for $\sqrt{m}$.

Convergence of $R_n(m)$ to $\sqrt{m}$: Taking the natural logarithm:

$$\ln R_n(m) = \frac{1}{2n} \ln(m^n + n^2) \quad (9)$$

$$\ln R_n(m) = \frac{1}{2n} \left[n \ln m + \ln\left(1 + \frac{n^2}{m^n}\right) \right] \quad (10)$$

As $n \to \infty$:

$$\lim_{n\to\infty} \ln R_n(m) = \lim_{n\to\infty} \frac{n \ln m}{2n} + \lim_{n\to\infty} \frac{\ln(1 + n^2/m^n)}{2n} = \frac{\ln m}{2} \quad (11)$$

Exponentiating:

$$\boxed{\lim_{n\to\infty} R_n(m) = e^{\frac{\ln m}{2}} = \sqrt{m}} \quad (12)$$

Thus, for $m = 2$:

$$\lim_{n\to\infty} R_n(2) = \sqrt{2}$$

Because m^n dominates n^2 exponentially, the sequence approaches the square root from above with rapid convergence. This construction applies universally, allowing us to generate approximations for $\sqrt{2}, \sqrt{3}, \sqrt{5}$, or any positive real number $m > 1$.

2.3 Complementary Lower-Bound Approximation

An alternative algebraic pathway arises from examining the expanded product and then choosing only the positive aspect, then isolating the difference between the exponential terms. Rearranging the terms and dividing appropriately yields the complementary relation:

$$X^{p_1+p_0} = p_1^{p_0} - p_0^{p_1}$$

$X = (p_0^{p_1} - p_1^{p_0})^{1/(p_1+p_0)}$. Following the same rule as $R_n(m)$, thus; by substituting the base case $p_1 = 2$ and $p_0 = n$, and additive denominator to Multiplicative denominator, we obtain a complementary sequence:

$$X_n = (2^n - n^2)^{1/(2n)} \quad (13)$$

Evaluating this sequence for natural numbers n, we observe exact integer values at $n = 1$, $n = 2$, and $n = 4$, yielding 1, 0, and 0 respectively, while $n = 3$ results in a complex number due to a negative radicand. For $n \geq 5$, the sequence produces positive real numbers that monotonically increase toward the limit $\sqrt{2}$. Unlike the primary EPRM $R_n(m)$, which serves as an upper bound, this complementary form acts as a lower-bound approximation. Both sequences satisfy the same logarithmic limit proof, as the polynomial n^2 becomes negligible compared to the exponential term 2^n for large n. Explicit numerical values for this complementary sequence are given in Table 5. For $m = 3$ and $m = 5$, the generalized lower-bound sequence $L_n(m) = (m^n - n^2)^{1/(2n)}$ converges absolutely to $\sqrt{3}$ and $\sqrt{5}$ from below, governed by the identical logarithmic limiting behavior as the base case for $\sqrt{2}$. Consequently, the upper bound $R_n(m)$ decreases monotonically toward $\sqrt{m}$, while the lower bound $L_n(m)$ increases monotonically toward $\sqrt{m}$, providing a rigorous sandwich for the target value.

3 WORKED EXAMPLES, EXPLICIT COMPUTATIONS, AND NUMERICAL TABLES

3.1 Explicit Worked Examples for the EPRM Method.

A. For $m = 2$ ($\sqrt{2} \approx 1.414213562$):

$$\bullet n = 2 : R_2 = (2^2 + 2^2)^{1/(2\times 2)} = (4+4)^{1/4} = 8^{0.25} \approx 1.68179.$$

$$\bullet n = 4 : R_4 = (2^4 + 4^2)^{1/(2\times 4)} = (16+16)^{1/8} = 32^{0.125} \approx 1.54221.$$

$$\bullet n = 6 : R_6 = (2^6 + 6^2)^{1/(2\times 6)} = (64+36)^{1/12} = 100^{0.08333} \approx 1.46780.$$

$$\bullet n = 10 : R_{10} = (2^{10} + 10^2)^{1/(2\times 10)} = (1024+100)^{1/20} = 1124^{0.05} \approx 1.42082.$$

$$\bullet n = 20 : R_{20} = (2^{20} + 20^2)^{1/(2\times 20)} = (1048576+400)^{1/40} \approx 1.41422.$$

B. For $m = 3$ ($\sqrt{3} \approx 1.732050808$):

$$\bullet n = 4 : R_4 = (3^4 + 4^2)^{1/(2\times 4)} = (81+16)^{1/8} = 97^{0.125} \approx 1.77152.$$

$$\bullet n = 8 : R_8 = (3^8 + 8^2)^{1/(2\times 8)} = (6561+64)^{1/16} = 6625^{0.0625} \approx 1.73310.$$

C. For $m = 5$ ($\sqrt{5} \approx 2.236067977$):

$$\bullet n = 4 : R_4 = (5^4 + 4^2)^{1/(2\times 4)} = (625+16)^{1/8} = 641^{0.125} \approx 2.24314.$$

$$\bullet n = 6 : R_6 = (5^6 + 6^2)^{1/(2\times 6)} = (15625+36)^{1/12} \approx 2.236455.$$

3.2 Explicit Worked Examples for the Babylonian Method.

The Babylonian method computes square roots via the recurrence relation $x_{k+1} = \frac{1}{2}(x_k + \frac{2}{x_k})$. Starting with $x_0 = 1$, we perform 5 detailed iterations:

$$\bullet\text{Iteration 1: } x_1 = \frac{1}{2}\left(1 + \frac{2}{1}\right) = 1.500000000$$

$$\bullet\text{Iteration 2: } x_2 = \frac{1}{2}\left(1.5 + \frac{2}{1.5}\right) = 1.416666667$$

$$\bullet\text{Iteration 3: } x_3 = \frac{1}{2}\left(1.416666667 + \frac{2}{1.416666667}\right) = 1.414215686$$

$$\bullet\text{Iteration 4: } x_4 = \frac{1}{2}\left(1.414215686 + \frac{2}{1.414215686}\right) = 1.414213562$$

$$\bullet\text{Iteration 5: } x_5 = \frac{1}{2}\left(1.414213562 + \frac{2}{1.414213562}\right) = 1.414213562$$

3.3 Explicit Worked Examples for the Newton-Raphson Method.

For the function $f(x) = x^2 - 2$, Newton-Raphson simplifies to the same recurrence $x_{k+1} = \frac{1}{2}(x_k + \frac{2}{x_k})$. Starting again from $x_0 = 1$, we show 5 explicit iterations:

$$\bullet\text{Iteration 1: } x_1 = 1 - \frac{(1)^2 - 2}{2 \times 1} = 1.500000000$$

$$\bullet\text{Iteration 2: } x_2 = 1.5 - \frac{(1.5)^2 - 2}{2 \times 1.5} = 1.416666667$$

$$\bullet \text{Iteration 3: } x_3 = 1.416666667 - \frac{(1.416666667)^2 - 2}{2 \times 1.416666667} \approx 1.414215686$$

$$\bullet \text{Iteration 4: } x_4 = 1.414215686 - \frac{(1.414215686)^2 - 2}{2 \times 1.414215686} \approx 1.414213562$$

3.4 Comparative Numerical Summary Tables.

The following tables summarize the calculations for the EPRM (including its full error profiles for $\sqrt{2}, \sqrt{3}, \sqrt{5}$), the complementary lower-bound sequence, and compare the error decay of the non-iterative EPRM against the iterative classical methods.

Table 1: EPRM Approximation Values for $\sqrt{2}$ and Error

n	Calculated EPRM $R_n(2)$	Absolute Error
1	1.732051	+0.317838
3	1.611490	+0.197276
5	1.495349	+0.081135
10	1.420827	+0.006613
20	1.414227	+0.000013

Table 2: EPRM Approximation Values for $\sqrt{3}$ and Error

n	Calculated EPRM $R_n(3)$	Absolute Error
1	2.000000	+0.267949
2	1.898829	+0.166778
4	1.771520	+0.039469
8	1.733100	+0.001049
15	1.732050821	+0.000000013

Table 3: EPRM Approximation Values for $\sqrt{5}$ and Error

n	Calculated EPRM $R_n(5)$	Absolute Error
1	2.449490	+0.213422
2	2.320667	+0.084599
3	2.262090	+0.026022
6	2.236455	+0.000387
10	2.236139	+0.000071

3.5 EPRM on Perfect Squares

Deduction: For perfect squares, the EPRM method (both upper and lower forms) converges to the exact integer square root when n is sufficiently large. This demonstrates the strength and precision of the construction in special cases, where the exponential term m^n completely dominates, leading to exact cancellation of the polynomial correction in the limit behavior.

Table 4: Comparison of Approximation Methods for $\sqrt{m}$ (with $m = 2$)

Method	Formula	Convergence	Iterative?
Babylonian	$x_{k+1} = \frac{1}{2}\left(x_k + \frac{m}{x_k}\right)$	Quadratic	Yes
Newton-Raphson	$x_{k+1} = x_k - \frac{x_k^2 - m}{2x_k}$	Quadratic	Yes
EPRM	$R_n(m) = (m^n + n^2)^{1/(2n)}$	$O(n/m^n)$	No

Table 5: Lower-Bound $L_n(2)$ and Upper-Bound $R_n(2)$ for $\sqrt{2}$

n	$L_n(2) = (2^n - n^2)^{1/(2n)}$	$R_n(2) = (2^n + n^2)^{1/(2n)}$
1	1.000000	1.732051
2	0.000000	1.681793
3	Complex	1.611490
4	0.000000	1.542210
5	1.214814	1.495349
6	1.320007	1.467799
7	1.364859	1.449645
8	1.389436	1.437887
9	1.404873	1.429686
10	1.414769	1.420827
15	1.414027	1.414463
20	1.414194	1.414227
30	1.414204	1.414215
40	1.414211	1.414214
50	1.414213	1.414214

4 CONCLUSION

The Prime Density Triangle yields two powerful closed-form sequences from its algebraic frameworks. First, the multiplication of the coupled expressions $X^{p_1} \pm p_1$ and $X^{p_0} \pm p_0$ gives the sequence $S_n(m) = (m^n + n^2)^{1/(n+2)}$, which converges to m with error decaying as $O(n/m^n)$. Second, by replacing the additive denominator $n + 2$ with the multiplicative denominator $2n$, we construct the Exponential-Polynomial Root Mean (EPRM) $R_n(m) = (m^n + n^2)^{1/(2n)}$, which converges to $\sqrt{m}$ with error decaying as $O(n/m^n)$. Through detailed derivations, explicit computations for both sequences, iterative examples of classical methods, and comparisons, this paper demonstrates how algebraic structures from the Prime Density Triangle lead to practical approximation tools. The non-iterative nature of the EPRM makes it particularly valuable for pedagogical purposes and theoretical analysis. Furthermore, the EPRM demonstrates exact integer convergence when applied to perfect squares (i.e., $m = k^2$), which provides an additional layer of precision and confirms the robust behavior of the construction in special cases.

Table 6: Lower-Bound $L_n(m)$ and Upper-Bound $R_n(m)$ for Perfect Squares

m	$k=\sqrt{m}$	n	$L_n(m)=(m^n-n^2)^{1/(2n)}$	$R_n(m)=(m^n+n^2)^{1/(2n)}$	Result
4	2	10	2.000000000	2.000000000	Exact
9	3	20	3.000000000	3.000000000	Exact
16	4	20	4.000000000	4.000000000	Exact
25	5	30	5.000000000	5.000000000	Exact
36	6	40	6.000000000	6.000000000	Exact
49	7	40	7.000000000	7.000000000	Exact
64	8	50	8.000000000	8.000000000	Exact
100	10	100	10.000000000	10.000000000	Exact

References

[1] Batu J J, Y. (2026). A prime density Triangle - A mathematical framework deriving nuclear magic numbers, shell gap and nuclear subshell structures (New version preprint) [Data set]. Zenodo. `https://doi.org/10.5281/zenodo.20787852`

[2] Yagah, Batu J J (2026). Prime Density Triangle: A Mathematical Framework Deriving Nuclear Magic Numbers, Shell Gaps and subshell structures. figshare. Preprint. `https://doi.org/10.6084/m9.figshare.32867039.` v2

[3] Burden, R. L., & Faires, J. D. (2011). *Numerical Analysis*, 9th ed. Brooks/Cole.

[4] Hardy, G. H. (1910). *Orders of Infinity*. Cambridge Univ. Press.