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Uniqueness of Topological Solutions and the Structure of Solutions for the Chern-Simons System with Two Higgs Particles

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Abstract: The existence of topological solutions for the Chern-Simons equation with two Higgs particles has been proved by Lin, Ponce and Yang [16]. However, both the uniqueness problem and the existence of non-topological solutions have been left open. In this paper, we consider the case of one vortex at origin. Among others, we prove the uniqueness of topological solutions and give a complete study of the radial solutions, in particular, the existence of some non-topological solutions.

1. Introduction and Main Results

In this paper, we will consider the nonlinear elliptic system

$$\begin{cases} \Delta u + \lambda e^{v}(1 - e^{u}) = 4\pi \sum_{s=1}^{N'} \alpha'_s \delta_{p'_s} \quad \text{in } \mathbf{R}^2, \\ \Delta v + \lambda e^{u}(1 - e^{v}) = 4\pi \sum_{s=1}^{N''} \alpha''_s \delta_{p''_s} \quad \text{in } \mathbf{R}^2, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2}$, λ is a positive constant, N' and N'' are two positive constants which are called the vortex numbers, $\alpha'_s > 0$ and $\alpha''_s > 0$ are constants, and δ_p is the Dirac measure at p. Equation (1.1) arises from a relativistic Abelian Chern-Simons model with two Higgs particles. For any solution (u, v) to Eq. (1.1), we let $z = x^1 + ix^2$

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and define ϕ, χ, $A_r^{(1)}$ and $A_r^{(2)}$, $r = 1, 2$, in the following:

$$\begin{cases} \theta_1(z) = -\sum_{s=1}^{N'} \arg(z - p'_s), \theta_2(z) = -\sum_{s=1}^{N''} \arg(z - p''_s), \\ \phi(z) = e^{\frac{1}{2}u(z)+i\theta_1(z)}, \chi(z) = e^{\frac{1}{2}v(z)+i\theta_2(z)}, \\ A_1^{(1)}(z) = -\mathrm{Re}\{2i\bar{\partial}\ln\phi(z)\}, A_2^{(1)}(z) = -\mathrm{Im}\{2i\bar{\partial}\ln\phi(z)\}, \\ A_1^{(2)}(z) = -\mathrm{Re}\{2i\bar{\partial}\ln\chi(z)\}, A_2^{(2)}(z) = -\mathrm{Im}\{2i\bar{\partial}\ln\chi(z)\}; \end{cases} \tag{1.2}$$

here ϕ and χ are interpreted as two complex scalar fields in $\mathbf{R}^2$ representing two Higgs particles, and $A_r^{(1)}$ and $A_r^{(2)}$, $r = 1, 2$, are two gauge fields. Then $(\phi, \chi, A_r^{(I)})$, $I = 1, 2$, $r = 1, 2$, satisfy the self-dual equation for the Chern-Simons-Higgs model with two Higgs particles. For details of computations, we refer the readers to [8,15,16] and the references therein.

For the past twenty years, the equation of Chern-Simons with one Higgs particle has been intensively studied, e.g., see [2–4,6–8,10–15,17–20,22,23] and references therein. However, the study for the system (1.1) only recently began with the paper [16].

For Eq. (1.1), there are two natural boundary conditions for solutions at ∞, namely,

$$\begin{aligned} &\text{(i)} \lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0, \text{ or} \\ &\text{(ii)} \lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = -\infty. \end{aligned} \tag{1.3}$$

We note that if (u, v) is a solution with the boundary condition either (i) or (ii), then, by the maximum principle, we have $u(x) < 0$ and $v(x) < 0$ for all $x \in \mathbf{R}^2$. In physics literature, a solution (u, v) satisfying boundary condition (i) is called a topological solution. Since the nonlinear term $e^v(1 - e^u) = -e^v u + O(|u|^2)$ for u small and $u(x), v(x) \to 0$ as $|x| \to +\infty$, by the estimates of elliptic PDE, we know that if (u, v) is a topological solution of (1.1), then both $|u|$ and $|v|$ decay exponentially at ∞.

To solve (1.1), one may consider a regularized form:

$$\begin{cases} \Delta u + \lambda e^v(1 - e^u) = \sum_{s=1}^{N'} \frac{4\alpha'_s \varepsilon}{(\varepsilon+|x-p'_s|^2)^2} \\ \Delta v + \lambda e^u(1 - e^v) = \sum_{s=1}^{N''} \frac{4\alpha''_s \varepsilon}{(\varepsilon+|x-p''_s|^2)^2}, \end{cases} \tag{1.4}$$

where ε is a small positive number, and introduce the background functions

$$u_0^\varepsilon(x) = \sum_{s=1}^{N'} 4\alpha'_s \ln\left(\frac{\varepsilon + |x - p'_s|^2}{1 + |x - p'_s|^2}\right), \; v_0^\varepsilon(x) = \sum_{s=1}^{N''} 4\alpha''_s \ln\left(\frac{\varepsilon + |x - p''_s|^2}{1 + |x - p''_s|^2}\right).$$

Then

$$\Delta u_0^\varepsilon(x) = -h_1(x) + \sum_{s=1}^{N'} \frac{4\alpha'_s \varepsilon}{(\varepsilon + |x - p'_s|^2)^2}, \; \Delta v_0^\varepsilon = -h_2(x) + \sum_{s=1}^{N''} \frac{4\alpha''_s \varepsilon}{(\varepsilon + |x - p''_s|^2)^2},$$

where $h_1, h_2 \in W^{1,2}$ do not depend on $\varepsilon > 0$. By letting $u = u_0^\varepsilon + f$, $v = v_0^\varepsilon + g$, the regularized form of (1.1) becomes

$$\begin{cases} \Delta f + \lambda e^{v_0^\varepsilon+g}(1 - e^{u_0^\varepsilon+f}) = h_1 \\ \Delta g + \lambda e^{u_0^\varepsilon+f}(1 - e^{v_0^\varepsilon+g}) = h_2. \end{cases} \tag{1.5}$$

It is clear that (1.5) is the Euler-Lagrange equations of the nonlinear functional:

$$I(f,g)=\int(\nabla f\cdot\nabla g+\lambda e^{u_0^\varepsilon+v_0^\varepsilon+f+g}-\lambda e^{u_0^\varepsilon+f} \\ -\lambda e^{v_0^\varepsilon+g}+h_2 f+h_1 g)\,dx. \tag{1.6}$$

We refer to [16] for the details of arguments. From (1.6), we see that Eq. (1.1) is the so-called skew gradient system in the literature, see [21]. Clearly, the indefinite form of I presents a lot of difficulties for solving Eq. (1.1). Hence, it is remarkable that in Lin-Ponce-Yang [16], they are able to show the existence of topological solutions for Eq. (1.1) for any given set of singularities.

Theorem A. [16] *For any given sets* $\{p_1',\ldots,p_{N'}'\}$ *and* $\{p_1'',\ldots,p_{N''}''\}$ *and* $\alpha_s',\ \alpha_s''>0$*, Eq.* (1.1) *possesses a topological solution* (u,v)*.*

After Theorem A, it is natural to ask the question about the uniqueness of topological solutions for Eq. (1.1). For the single Chern-Simons-Higgs model, the uniqueness result was proved in [6] with only one singularity, and in [3 and 19] for multi-singularity in $\mathbf{R}^2$ and large λ as well as in the periodic case. In this article, we consider the topological solution (u,v) for the case $N'=N''=1$ and p_1' and p_1'' to be the origin O. Then (u,v) satisfies

$$\begin{cases}\Delta u+e^v(1-e^u)=4\pi N_1\delta_0\\ \Delta v+e^u(1-e^v)=4\pi N_2\delta_0\end{cases}\text{in }\mathbf{R}^2, \tag{1.7}$$

with the boundary condition

$$u(x)\to 0,\ v(x)\to 0\text{ as }|x|\to\infty. \tag{1.8}$$

By noting $u(x)<0, v(x)<0$ for $x\in\mathbf{R}^2$, and by applying the standard method of moving planes, we can show that (u,v) is radially symmetric with respect to the origin O. The proof is standard, and will be omitted here. We refer to [1] for the details of the proof.

To a single nonlinear elliptic equation, the uniqueness problem has been extensively studied for the last decades. It is well-known that the uniqueness problem is closely related to non-degeneracy of its linearized equation. See [4,5] and references therein. In this paper, we also want to prove the uniqueness by studying the non-degeneracy of linearized equations. The linearized equation at (u,v) of (1.7) is called *degenerate* if there exists a nonzero bounded solution pair $(A(r),B(r))$ of

$$\begin{cases}\Delta A+e^v(1-e^u)B-e^{u+v}A=0\\ \Delta B+e^u(1-e^v)A-e^{u+v}B=0\end{cases}\text{in }\mathbf{R}^2. \tag{1.9}$$

Comparing to the case of a single equation, there are additional difficulties to be overcome for (1.9). In the proof of the uniqueness for a single equation, some standard techniques such as Sturm-Liouville comparison theorem play important roles. See [4] and [5]. However, these standard tools are no longer available for a system of Eqs. (1.9). Hence, we have to develop new ideas to work out for (1.9), which will be presented in Sect. 2. We believe that the method developed here should be helpful for a general class of nonlinear elliptic systems.

After the non-degeneracy of (1.9) is established, we can prove the following uniqueness theorem.

Theorem 1.1. *Let (u, v) be a topological solution of (1.7). Then the linearized equation (1.9) of (1.7) at (u, v) is non-degenerate. Moreover, Eq. (1.7) possesses one and only one topological solution.*

Now we come back to discuss the case of the boundary condition (ii) of (1.11). In the Abelian Chern-Simons-Higgs model with one particle, a solution $u(x)$ satisfies

$$\Delta u + e^u(1 - e^u) = 4\pi N\delta_0 \quad \text{in } \mathbf{R}^2. \tag{1.10}$$

Suppose $u = u(|x|)$ is a non-topological solution of (1.10), i.e., $u(r) \to -\infty$ as $r \to +\infty$. Then it can be proved that u satisfies

$$\int_{\mathbf{R}^2} e^u(1 - e^u)\, dx < +\infty. \tag{1.11}$$

But, for the system (1.1), (1.11) might not hold even for the radial solution $(u(r), v(r))$. Actually, in Sect. 5, we will show that there exists a solution pair (u, v) of (1.7) satisfying both $u(r)$ and $v(r)$ tend to $-\infty$ as $r \to \infty$ with $\int_{\mathbf{R}^2} e^v(1 - e^u)\, dx < +\infty$ and $\int_{\mathbf{R}^2} e^u(1 - e^v)\, dx = +\infty$. Thus, while compared with (1.10), the structure of solutions for (1.7) could be more complicated. One of our purposes in this paper is to classify solutions according to their behaviors at infinity.

In this paper, we call a solution to be non-topological if (u, v) satisfies the boundary condition (ii) in (1.11), and both $e^u(1 - e^v)$ and $e^v(1 - e^u)$ are in $L^1(\mathbf{R}^2)$. For an entire solution (u, v) of (1.1), we set

$$\beta_1 = \frac{1}{2\pi}\int_{\mathbf{R}^2} e^v(1 - e^u)\, dx, \quad \beta_2 = \frac{1}{2\pi}\int_{\mathbf{R}^2} e^u(1 - e^v)\, dx. \tag{1.12}$$

In order to investigate the structure of all radial solutions of (1.7), we consider the following ODE system:

$$\begin{cases} u''(r) + \frac{1}{r}u'(r) + e^{v(r)}(1 - e^{u(r)}) = 0, \\ v''(r) + \frac{1}{r}v'(r) + e^{u(r)}(1 - e^{v(r)}) = 0, \end{cases} \quad r > 0 \tag{1.13}$$

with the initial value

$$\begin{cases} u(r) = 2N_1 \log r + \alpha_1 + o(1), \\ v(r) = 2N_2 \log r + \alpha_2 + o(1) \end{cases} \quad \text{as } r \to 0^+. \tag{1.14}$$

According to the behaviors at ∞, all entire solutions of (1.13) can be classified into the following five types:

Type (I): $\lim_{r\to\infty}(u(r), v(r)) = (0, 0)$, i.e., (u, v) is the topological solution.
Type (II): $\lim_{r\to\infty}(u(r), v(r)) = (-\infty, -\infty)$ with $\beta_1 < \infty$ and $\beta_2 < \infty$, i.e., (u, v) is a non-topological solution.
Type (III): $\lim_{r\to\infty} u(r) = -\infty$, $\lim_{r\to\infty} v(r) = -\infty$, and

$$\text{either } 2N_1 < \beta_1 \le 2N_1 + 2,\ \beta_2 = \infty \text{ or } \beta_1 = \infty,\ 2N_2 < \beta_2 \le 2N_2 + 2.$$

Type (IV): $\lim_{r\to\infty}(u(r), v(r)) = (-c_u, -\infty)$ or $\lim_{r\to\infty}(u(r), v(r)) = (-\infty, -c_v)$ for some constants $c_u > 0$ and $c_v > 0$.

Type (V): $\lim_{r\to\infty}(u(r), v(r)) = (+\infty, -\infty)$ or $\lim_{r\to\infty}(u(r), v(r)) = (-\infty, +\infty)$.

Our second result is the asymptotic behaviors of all entire solutions.

Theorem 1.2. *Let (u, v) be a solution of (1.13)–(1.14). Then (u, v) must be one of the above five types. Conversely, solutions of all types do exist.*

Let $\alpha = (\alpha_1, \alpha_2)$, and $(u(r,\alpha), v(r,\alpha))$ denote the solution of (1.13)-(1.14). According to the behavior of (u, v), the set of initial data could be classified into the following regions:

$$\begin{aligned}
&\Omega = \{\alpha|(u(r,\alpha), v(r,\alpha)) \text{ is a solution with } \lim_{r\to\infty}(u(r,\alpha), v(r,\alpha)) = (-\infty, -\infty)\},\\
&T = \{\alpha|(u(r,\alpha), v(r,\alpha)) \text{ is the unique topological solution}\},\\
&\Omega_{NT} = \{\alpha|(u(r,\alpha), v(r,\alpha)) \text{ is a non-topological solution}\},\\
&S_u = \{\alpha|(u(r,\alpha), v(r,\alpha)) \text{ is a Type (IV) solution with } \lim_{r\to\infty} u(r) = -c_u\},\\
&S_v = \{\alpha|(u(r,\alpha), v(r,\alpha)) \text{ is a Type (IV) solution with } \lim_{r\to\infty} v(r) = -c_v\},\\
&W_u = \{\alpha|(u(r,\alpha), v(r,\alpha)) \text{ is a Type (V) solution with } \lim_{r\to\infty} u(r) = \infty\},\\
&W_v = \{\alpha|(u(r,\alpha), v(r,\alpha)) \text{ is a Type (V) solution with } \lim_{r\to\infty} v(r) = \infty\}.
\end{aligned}$$

Then the structure of solutions sets is described as follows:

Theorem 1.3. *Both Ω and Ω_{NT} are non-empty and open simply connected. Furthermore, all sets $\Omega\backslash\Omega_{NT}$, S_u, S_v, W_u and W_v are non-empty, and the following statements are valid.*

(i) $\Omega = \Omega_v \bigcup \Omega_{NT} \bigcup \Omega_u$ *is a non-empty and simple connected set, where*

$$\begin{aligned}
\Omega_u &= \{\alpha \in \Omega|(u(r,\alpha), v(r,\alpha)) \textit{ is a Type (III) solution with } \beta_1 < \infty\},\\
\Omega_v &= \{\alpha \in \Omega|(u(r,\alpha), v(r,\alpha)) \textit{ is a Type (III) solution with } \beta_2 < \infty\}.
\end{aligned}$$

(ii) $\partial\Omega = S_u \bigcup T \bigcup S_v$ *and* $\overline{S}_u \bigcap \overline{S}_v = \partial\Omega \bigcap \partial\Omega_{NT} = T$.

(iii) *For any $\alpha \in \Omega_{NT}$ the corresponding (β_1, β_2) satisfies*

$$(\beta_1 - 2(N_1+1))(\beta_2 - 2(N_2+1)) > 4(N_1+1)(N_2+1). \tag{1.15}$$

(iv) *W_u is open. Furthermore, for each $(\theta, \eta) \in S_u$ there exists $\epsilon > 0$ such that $(\alpha_1, \eta) \in W_u$ $\forall \theta < \alpha_1 < \theta + \epsilon$.*

(v) *W_v is open. Furthermore, for each $(\mu, \nu) \in S_v$ there exists $\delta > 0$ such that $(\mu, \alpha_2) \in W_v$ $\forall \nu < \alpha_2 < \nu + \delta$.*

We remark that the uniqueness of topological solutions implies the simple-connectedness of both Ω and Ω_{NT}. The simple-connectedness is important itself, because it allows us to study the linearized equation of (1.7) at any non-topological solution through the argument of continuation. An important question about non-topological solutions arises: given any pair of (β_1, β_2) satisfying (1.15) of Theorem 1.3, is there an unique non-topological solution (u, v) which satisfies (1.12)? We will come back to this issue in a coming paper.

From Theorem 1.3, we note that there are drastic differences between the solutions of (1.10) and (1.7). For Eq. (1.10), if a solution is positive somewhere, then it will blow up in finite $|x|$. But the situations do change for the system of equations. For example, the solution of Type (V) depicts that u might be positive somewhere, but both u and v do not blow up in finite $|x|$. Another consequence of Theorem 1.2 is that if both u and v are positive at some $|x_0|$, then u and v must blow up in finite $|x|$.

The paper is organized as follows. First we investigate the monotone and non-degenerate properties of the linearized equations on the negative solutions of (1.13) in Sect. 2. Based on the results of Sect. 2 and applying the Implicit Function Theorem, we prove the uniqueness of topological solution for (1.13) in Sect. 3. In Sect. 4, we will give the asymptotic behaviors of all entire solutions. Finally, we prove the existences and classification of solutions of all types, Theorems 1.2 and 1.3, in Sect. 5.

2. The Non-Degeneracy of Linearized Equations

In this section, we give the proof about the non-degeneracy of the linearized equation on the topological solution of (1.7). Before going to our proof, we need to state some properties concerning solutions. First, we have the Pohozaev identity as follows.

Lemma 2.1. (Pohozaev identity). *Let* $(u(r), v(r))$ *be a solution of* (1.13)–(1.14) *in* $(0, R]$ *for some* $R > 0$*. Then we have the following identity:*

$$[ru'(r)\cdot rv'(r) + r^2(e^{u(r)} + e^{v(r)}) - r^2 e^{u(r)+v(r)}] - 2\int_0^r s(e^{u(s)} + e^{v(s)})\, ds$$
$$+2\int_0^r s e^{u(s)+v(s)}\, ds = 4N_1N_2 \quad \forall r \in (0, R]. \tag{2.1}$$

Proof. By multiplying rv' and ru' on both sides of the first and second equation of (1.13) respectively, we obtain

$$\begin{cases} rv'(ru')' + rv're^{v}(1-e^{u}) = 0 \\ ru'(rv')' + ru're^{u}(1-e^{v}) = 0 \end{cases} \quad \forall r \in (0, R]. \tag{2.2}$$

Then adding these two equations together and taking the integration from 0 to r, we get

$$[ru'(r)\cdot rv'(r) - \lim_{r\to 0^+}(ru'(r)\cdot rv'(r))] + \int_0^r s^2 d(e^{u(s)}) + \int_0^r s^2 d(e^{v(s)})$$
$$- \int_0^r s^2 d(e^{u(s)+v(s)}) = 0 \quad \forall r \in (0, R].$$

By the above equality, and using the initial value (1.14) and the integration by parts, we can easily obtain (2.1). □

Secondly, we have the following property for solutions with zero boundary value.

Lemma 2.2. *Let* $(u(r), v(r))$ *be a solution of* (1.13)–(1.14) *satisfying* $u(R_0) = v(R_0) = 0$ *for some* $R_0 > 0$ *(or* $R_0 = +\infty$*). Then the following are valid*:

(i) $u < 0, v < 0, u' > 0$ *and* $v' > 0$ *on* $(0, R_0)$*. Furthermore, if* $R_0 = \infty$*, i.e.,* (u, v) *is a topological solution of* (1.7)*, then the corresponding* (β_1, β_2) *satisfies* $\beta_1 = 2N_1$ *and* $\beta_2 = 2N_2$*, where* (β_1, β_2) *is defined in* (1.12)*.*
(ii) *If* $N_1 < N_2$*, then* $u > v$ *on* $(0, R_0)$*.*
(iii) *If* $N_1 > N_2$*, then* $u < v$ *on* $(0, R_0)$*.*
(iv) *If* $N_1 = N_2$*, then* $u \equiv v$*.*

Proof. We shall apply the maximum principle to prove (i). Suppose $u(r_0) = \max_{(0,R_0]} u > 0$. Then $\Delta u(r_0) \le 0$ and thus

$$0 = \Delta u(r_0) + e^{v(r_0)}(1 - e^{u(r_0)}) < 0,$$

which yields a contradiction. Hence, $u(r) \le 0$ on $(0, R_0)$. The strong maximum principle implies $u(r) < 0$ in $(0, R_0)$. Similarly, it holds for v.

Since $u(r) < 0$ and $v(r) < 0$ in $(0, R_0)$, the maximum principle also implies that both u and v can not attain their local minima inside $(0, R_0)$. Since $u'(r) > 0$ and $v'(r) > 0$ for r near 0, we obtain $u'(r) > 0$, $v'(r) > 0$ on $(0, R_0)$.

If $R_0 = \infty$ then, by $u(r) < 0$ on $(0, \infty)$ and (1.7), we have $(ru'(r))' = re^{v(r)}(e^{u(r)} - 1) < 0$ $\forall r \in (0, \infty)$. Thus, by $u'(r) > 0$ on $(0, \infty)$, we get $0 \le \lim_{r\to\infty} ru'(r) = 2N_1 - \int_0^\infty re^v(1 - e^u)dr$ exists $(\equiv c_u)$. If $c_u > 0$ then we easily have $u(r) > 0$ for large r. This contradiction proves $c_u = 0$. From this we get $\beta_1 = 2N_1$. The case of β_2 is similar. Hence (i) holds.

By (1.15), we have

$$\Delta(u - v) = 4\pi(N_1 - N_2)\delta_0 + (e^u - e^v).$$

If $(u - v)(r_0) < 0$ at some $r_0 \in (0, R_0)$, then we can let r_0 satisfy $(u - v)(r_0) = \min_{(0,R_0]} (u - v) < 0$, and we have

$$0 \le \Delta(u - v)(r_0) = e^{u(r_0)} - e^{v(r_0)} < 0,$$

a contradiction. Hence $u(r) \ge v(r)$. By the strong maximum principle, the strict inequality $u(r) > v(r)$ holds for $r \in (0, R_0)$. This proves (ii). Obviously, (iii) and (iv) follow easily. □

In the following, we investigate the monotone property of the negative solution of (1.13)–(1.14). Let, for $i = 1, 2$,

$$\begin{cases} \phi_i(r) = \dfrac{\partial U}{\partial \alpha_i}, \\ \psi_i(r) = \dfrac{\partial V}{\partial \alpha_i}, \end{cases} \tag{2.3}$$

where $U(r; \alpha_1, \alpha_2) = u(r; \alpha_1, \alpha_2) - 2N_1 \log r$ and $V(r; \alpha_1, \alpha_2) = v(r; \alpha_1, \alpha_2) - 2N_2 \log r$. Then (ϕ_i, ψ_i), $i = 1, 2$, satisfy the linearized equations

$$\begin{cases} \Delta\phi_i - e^{u+v}\phi_i + e^v(1 - e^u)\psi_i = 0, \quad r \in (0, R_0), \\ \Delta\psi_i - e^{u+v}\psi_i + e^u(1 - e^v)\phi_i = 0, \quad r \in (0, R_0), \\ \phi_1(0) = 1 = \psi_2(0), \phi_2(0) = 0 = \psi_1(0), \phi_i'(0) = 0 = \psi_i'(0). \end{cases} \tag{2.4}$$

The monotone property of ϕ_i and ψ_i is as follows:

Lemma 2.3. *Let* $(u(r), v(r))$ *be a solution of* (1.13)–(1.14). *If* $u(r) < 0$ *and* $v(r) < 0$ *for* $r \in (0, R_0)$ *for some* $R_0 > 0$ (*or* $R_0 = \infty$), *then the corresponding* (ϕ_i, ψ_i) *satisfy*

$$\begin{cases} \phi_1(r) > 0, \phi_1'(r) > 0, \ \phi_2(r) < 0, \phi_2'(r) < 0, \\ \psi_1(r) < 0, \psi_1'(r) < 0, \ \psi_2(r) > 0, \psi_2'(r) > 0 \end{cases} \quad \forall r \in (0, R_0). \tag{2.5}$$

Proof. By (2.4) and (1.14), we obtain there exists $r_0 \in (0, R_0)$ such that

$$\begin{aligned} r\psi_1'(r) &= -\int_0^r s[e^{u(s)}(1-e^{v(s)})\phi_1(s) - e^{u(s)+v(s)}\psi_1(s)]\,ds \quad \forall r > 0 \\ &\le -\int_0^r s[C_1 s^{2N_1}(1-C_2 s^{2N_2})\phi_1(s) - C_3 s^{2N_1+2N_2}\psi_1(s)]\,ds \quad \forall r \in (0, r_0) \\ &\le -Cr^{2N_1+2} < 0 \quad \forall r \in (0, r_0). \end{aligned} \tag{2.6}$$

By $\psi_1(0) = 0$, $\psi_1'(0) = 0$ and (2.6), we have $\psi_1(r) < 0$ and $\psi_1'(r) < 0$ $\forall r \in (0, r_0)$. On the other hand, by (2.4), (1.14), and the above result, we get

$$\begin{aligned} r\phi_1'(r) &= \int_0^r s[e^{u(s)+v(s)}\phi_1(s) + e^{v(s)}(e^{u(s)}-1)\psi_1(s)]\,ds \quad \forall r > 0 \\ &\ge \int_0^r C_4 s \cdot s^{2N_1+2N_2}\phi_1(s)\,ds \quad \forall r \in (0, r_0) \\ &\ge Cr^{2N_1+2N_2+2} > 0 \quad \forall r \in (0, r_0). \end{aligned} \tag{2.7}$$

By $\phi_1(0) = 1$, $\phi_1'(0) = 0$ and (2.7), we have $\phi_1(r) > 0$ and $\phi_1'(r) > 0$ $\forall r \in (0, r_0)$. These prove that the first inequality of (2.5) holds for $r \in (0, r_0)$. However (2.6) and (2.7) hold as long as the first inequality of (2.5) is true. This shows that the first inequality of (2.5) holds. The proof for the second inequality of (2.5) is similar. The proof is complete. □

Finally, we state and prove the non-degenerate property of the linearized equation at a topological solution in the following:

Lemma 2.4. *Let $(u(r), v(r))$ be a solution of* (1.13)–(1.14) *satisfying $u(R_0) = v(R_0) = 0$ for some $R_0 > 0$ (or $R_0 = +\infty$). If $(\phi_i(r), \psi_i(r))$, $i = 1, 2$, is the respective solution pair of* (2.4)*, then the following statements are valid.*

(i) *If $R_0 = \infty$, i.e., (u, v) is a topological solution, then there exist constants $c_1 > 0$, $c_2 < 0$, $d_1 < 0$ and $d_2 > 0$ such that,*

$$\lim_{r\to\infty} \frac{\phi_i(r)}{r^{-\frac{1}{2}}e^r} = c_i \ \text{and} \ \lim_{r\to\infty} \frac{\psi_i(r)}{r^{-\frac{1}{2}}e^r} = d_i, \ i = 1, 2.$$

(ii) *Let $M_A(r) = -\frac{\phi_1(r)}{\phi_2(r)}$ and $M_B(r) = -\frac{\psi_1(r)}{\psi_2(r)}$. Then $M_A(r) > M_B(r) > 0$ $\forall r \in [0, R_0]$ (resp., $[0, \infty)$ if $R_0 = \infty$) and $M_A'(r) < 0$, $M_B'(r) > 0$ $\forall r \in (0, R_0)$.*

(iii) $\det\begin{pmatrix} \phi_1(r) & \phi_2(r) \\ \psi_1(r) & \psi_2(r) \end{pmatrix} \neq 0$ $\forall r \in [0, R_0]$ *(resp., $[0, \infty)$ if $R_0 = \infty$).*

(iv) *The corresponding linearized equation* (1.9) *is non-degenerate.*

Proof. **(i)** We prove the asymptotic behavior of ϕ_1. The cases of ψ_1, ϕ_2 and ψ_2 are similar. Let $w(r) = \phi_1(r) - \psi_1(r) - e^r$. Then by (2.4), w satisfies

$$\begin{cases} \Delta w(r) = (e^u\phi_1 - e^v\psi_1) - (1 + \frac{1}{r})e^r \\ w(0) = 0,\ w'(0) = -1. \end{cases}$$

Since $u(r) < 0$, $v(r) < 0$, $\phi_1(r) > 0$ and $\psi_1(r) < 0$ $\forall r > 0$, it follows that $\Delta w \le (\phi_1(r) - \psi_1(r)) - (1 + \frac{1}{r})e^r = w(r) - \frac{1}{r}e^r$ $\forall r \in (0, \infty)$. Thus we obtain $w(r) < 0$ $\forall r \in (0, \infty)$, i.e.,

$$\phi_1(r) - \psi_1(r) < e^r \quad \text{on} \quad (0, \infty). \tag{2.8}$$

Let $z(r) = \phi_1(r)r^{\frac{1}{2}}$. Then z satisfies

$$z''(r) + [-1 + q(r)]z(r) = 0, \tag{2.9}$$

where

$$q(r) = 1 - e^{u+v} + \frac{e^v(1-e^u)\psi_1}{\phi_1} + \frac{1}{4r^2}.$$

Since $\lim_{r\to\infty} \frac{e^u-1}{u} = 1$ and $\psi_1 < 0$, we have

$$(e^u - 1)\psi_1 \le C \cdot u(r)\psi_1(r) \text{ for large } r \text{ and some } C > 0. \tag{2.10}$$

By $|u(r)|, |v(r)| \le Cr^{-\frac{1}{2}}e^{-r}$ for large r, (2.8) and (2.10), we easily obtain

$$(e^u - 1)\psi_1 \le Cr^{-\frac{1}{2}} \text{ for large } r.$$

From this and $\phi_1(r) > r$ for large r, we deduce $\frac{-e^v(1-e^u)\psi_1}{\phi_1} \in L^1(R,\infty)$ for $R > 0$ large. Moreover, since $\int_R^\infty (1-e^{u+v})dr < \infty$, we get

$$q(r) \in L^1[R,\infty). \tag{2.11}$$

By (2.11) and applying Corollary 9.2 of [9] to (2.9), we finally obtain

$$\lim_{r\to\infty} \frac{z(r)}{e^r} = c_1 > 0,$$

and hence

$$\lim_{r\to\infty} \frac{\phi_1(r)}{r^{-\frac{1}{2}}e^r} = c_1.$$

This proves the case of ϕ_1. Thus **(i)** holds.
(ii) By (2.4), we have $\lim_{r\to 0^+} M_A(r) = \infty$, $\lim_{r\to 0^+} M_B(r) = 0$, and thus $M_A(r) > M_B(r) \quad \forall r \in (0, r_1)$ for some $r_1 \in (0, R_0]$. We divide the proof of **(ii)** into the following two steps.
Step 1. If $M_A(r) > M_B(r)\ \forall r \in (0, r_0)$ for some $r_0 \le R_0$, then $M_A'(r) < 0$ and $M_B'(r) > 0\ \forall r \in (0, r_0)$.

We prove **Step 1** by contradiction. Suppose $M_A'(r) < 0 \quad \forall r \in (0, r_0)$ is not true. Then there exist $0 < r_1 < r_2 \le r_0$ such that

$$\begin{aligned} &M_A'(r_1) < 0,\, M_A'(r_2) > 0,\, M_A(r_1) = M_A(r_2)(\equiv C_0), \text{ and} \\ &\quad 0 < M_B(r) < M_A(r) < C_0 \quad \forall r \in (r_1, r_2). \end{aligned} \tag{2.12}$$

For any $c > 0$ and $r \in (0, R_0]$, we define

$$A_c(r) = \phi_1(r) + c\cdot\phi_2(r) \text{ and } B_c(r) = \psi_1(r) + c\cdot\psi_2(r). \tag{2.13}$$

Then A_c and B_c satisfy

$$\begin{cases} \Delta A_c - e^{u+v}A_c = e^v(e^u-1)B_c & \forall r \in (0, R_0], \\ \Delta B_c - e^{u+v}B_c = e^u(e^v-1)A_c & \forall r \in (0, R_0], \\ A_c(0) = 1,\ B_c(0) = c > 0. \end{cases} \tag{2.14}$$

From (2.12) and (2.13), we easily obtain

$$A_{C_0}(r) < 0 < B_{C_0}(r)\ \forall r \in (r_1, r_2) \quad \text{and} \quad A_{C_0}(r_1) = 0 = A_{C_0}(r_2), \tag{2.15}$$

which imply that A_{C_0} has a local minimum at some $\bar{r} \in (r_1, r_2)$ and $\Delta A_{C_0}(\bar{r}) \geq 0$. But, from (2.14) and (2.15), we get

$$\Delta A_{C_0}(\bar{r}) = e^{u(\bar{r})+v(\bar{r})} A_{C_0}(\bar{r}) + e^{v(\bar{r})}(e^{u(\bar{r})} - 1) B_{C_0}(\bar{r}) < 0. \tag{2.16}$$

This contradiction proves $M_A'(r) < 0\ \forall r \in (0, r_0)$.

Similarly, suppose $M_B'(r) > 0\ \forall r \in (0, r_0)$ is not true. Then there exist $0 < r_1 < r_2 \leq r_0$ such that

$$\begin{aligned} M_B'(r_1) > 0,\ M_B'(r_2) < 0,\ M_B(r_1) = M_B(r_2) (\equiv C_0), \text{ and} \\ C_0 < M_B(r) < M_A(r)\ \forall r \in (r_1, r_2). \end{aligned} \tag{2.17}$$

By (2.17) and (2.13), we easily obtain

$$B_{C_0}(r) < 0 < A_{C_0}(r)\ \forall r \in (r_1, r_2) \quad \text{and} \quad B_{C_0}(r_1) = 0 = B_{C_0}(r_2), \tag{2.18}$$

and hence B_{C_0} has a local minimum at some $\bar{r} \in (r_1, r_2)$ with $\Delta B_{C_0}(\bar{r}) \geq 0$. However, from (2.14) and (2.15) we get

$$\Delta B_{C_0}(\bar{r}) = e^{u(\bar{r})+v(\bar{r})} B_{C_0}(\bar{r}) + e^{u(\bar{r})}(e^{v(\bar{r})} - 1) A_{C_0}(\bar{r}) < 0. \tag{2.19}$$

This contradiction proves **Step 1**.

Step 2. There does not exist $R \in (0, R_0)$ such that $M_A(R) = M_B(R)$.

Suppose **Step 2** is not true. Then there exists a smallest $R \in (0, R_0]$ such that $M_A(R) = M_B(R) (\equiv C)$ and $M_A(r) > M_B(r) > 0\ \forall r \in (0, R)$. Let A_c and B_c be defined in (2.13). Then, in this case, by **Step 1** we obtain

$$\begin{aligned} &A_C(r) > 0,\ B_C(r) > 0 \quad \forall r \in (0, R), \\ &A_C(R) = B_C(R) = 0, \\ &A_C'(R) < 0,\ B_C'(R) < 0 \quad \text{if } R < \infty. \end{aligned} \tag{2.20}$$

Taking the differentiation w.r.t. α_i, $i = 1, 2$, on both sides of the Pohozaev identity, (2.1), then for any $c > 0$ and $r \in (0, R_0]$, we obtain

$$\begin{aligned} &r^2 A_c'(r) v'(r) + r^2 B_c'(r) u'(r) + r^2 [e^{u(r)} A_c(r) + e^{v(r)} B_c(r)] \\ &\quad - r^2 e^{u(r)+v(r)} (A_c(r) + B_c(r)) 2 \int_0^r s[e^u A_c + e^v B_c] ds + 2 \int_0^r s e^{u+v} (A_c + B_c) ds = 0. \end{aligned} \tag{2.21}$$

If $R < \infty$ then, by replacing c and r with C and R in (2.21) respectively, we easily have

$$\begin{aligned} 0 = &\left[R^2 A_C'(R) v'(R) + R^2 B_C'(R) u'(R) \right] \\ &+ \left[R^2 B_C(R) e^{v(R)} (1 - e^{u(R)}) + R^2 A_C(R) e^{u(R)} (1 - e^{v(R)}) \right] \\ &+ 2 \left[\int_0^R r A_C e^u (e^v - 1) dr + \int_0^R r B_C e^v (e^u - 1) dr \right]. \end{aligned} \tag{2.22}$$

Then, combining (i) of Lemma 2.2, (2.20) and (2.22), we deduce

$$\begin{aligned} 0 &> R^2 A_C'(R)v'(R) + R^2 B_C'(R)u'(R) \\ &= 2\left[\int_0^R rA_C e^u(1-e^v)dr + \int_0^R rB_C e^v(1-e^u)dr\right] > 0, \end{aligned}$$

which yields a contradiction.

If $R = \infty$ then we first claim that

$$\text{one of } A_C \text{ and } B_C \text{ is unbounded.} \tag{†}$$

Suppose (†) is not true. Then A_C and B_C are bounded. By (2.21) we have

$$\begin{aligned} 0 = &\lim_{r\to\infty}\left[rA_C'(r)\cdot rv'(r) + rB_C'(r)\cdot ru'(r)\right] \\ &+ \lim_{r\to\infty}\left[B_C(r)\cdot r^2 e^{v(r)}(1-e^{u(r)}) + A_C(r)\cdot r^2 e^{u(r)}(1-e^{v(r)})\right] \\ &+2\left[\int_0^\infty rA_C e^u(e^v-1)dr + \int_0^\infty rB_C e^v(e^u-1)dr\right]. \end{aligned} \tag{2.23}$$

Moreover, (i) of Lemma 2.2 implies that

$$\begin{aligned} &\lim_{r\to\infty} ru'(r) = 0 = \lim_{r\to\infty} rv'(r), \\ &\lim_{r\to\infty}[r^2 e^{u(r)}(1-e^{v(r)})] = 0 = \lim_{r\to\infty}[r^2 e^{v(r)}(1-e^{u(r)})]. \end{aligned} \tag{2.24}$$

Since $|u|$ and $|v|$ decay exponentially at ∞, and (A_C, B_C) is bounded, by (2.14) and (i) of Lemma 2.2, we get the limits

$$\begin{aligned} \lim_{r\to\infty} rA_C'(r) &= \int_0^\infty [re^{u+v}A_C]dr - \int_0^\infty [re^v(1-e^u)B_c]dr, \\ \lim_{r\to\infty} rB_C'(r) &= \int_0^\infty [re^{u+v}B_C]dr - \int_0^\infty [re^u(1-e^v)A_c]dr \end{aligned} \tag{2.25}$$

all exist. Hence, due to (2.20) and (2.23)–(2.25), we finally obtain

$$0 = 2\left[\int_0^\infty rA_C e^u(1-e^v)dr + \int_0^\infty rB_C e^v(1-e^u)dr\right] > 0.$$

This contradiction shows that A_C or B_C is unbounded, and the claim is proved.

Secondly, suppose A_C is unbounded. From (2.14) we have

$$\Delta(A_C - B_C) - e^u(A_C - B_C) = (e^u - e^v)B_C,$$

and hence, by Lemma 2.2 and the strong maximum principle, we obtain that $A_C(r)$ intersects $B_C(r)$ at most one point on $[0,\infty)$. Thus, w.l.o.g., we may assume that there exists $r_1 > 0$ such that

$$\begin{cases} A_C'(r_1) \ge 0, \\ A_C(r) > B_C(r) > 0 \ \text{ on } \ [r_1,\infty). \end{cases} \tag{2.26}$$

Since (u, v) is a topological solution of (1.7), there exists $r_2 > r_1$ such that

$$e^{u(r)+v(r)} \ge \max\{e^{v(r)}(1-e^{u(r)}), e^{u(r)}(1-e^{v(r)})\} \quad \forall r \ge r_2. \tag{2.27}$$

Therefore, by (2.14) and (2.26)–(2.27), we get $A'_C(r) > 0$ on (r_1, ∞) and thus $\lim_{r\to\infty} A_C(r) = \infty$. Now, by applying the same arguments in the proof of **(i)**, we can obtain

$$\lim_{r\to\infty} \frac{A_C(r)}{r^{-\frac{1}{2}}e^r} = C_A = c_1 + C \cdot c_2 > 0,$$

where c_1 and c_2 are constants in **(i)**. Then there exists $\epsilon > 0$ such that $C_\epsilon = c_1 + (C + \epsilon) \cdot c_2 > 0$ and $\lim_{r\to\infty} \frac{A_{C+\epsilon}(r)}{r^{-\frac{1}{2}}e^r} = C_\epsilon$. But, by **Step 1** we have $A_{C+\epsilon}(r) < 0$ for large r. We get a contradiction. The case of unboundedness for B_C is similar. This shows **Step 2**. According to **Steps 1** and **2**, we easily obtain **(ii)**.

(iii) Suppose $\det \begin{pmatrix} \phi_1(R) & \phi_2(R) \\ \psi_1(R) & \psi_2(R) \end{pmatrix} = 0$ for some $R \in [0, R_0]$ (resp., $R \in [0, \infty)$ if $R_0 = \infty$). Then, w.l.o.g., there exists $C_0 > 0$ such that

$$\begin{pmatrix} \phi_1(R) \\ \psi_1(R) \end{pmatrix} + C_0 \begin{pmatrix} \phi_2(R) \\ \psi_2(R) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.28}$$

By (2.28) we obtain $M_A(R) = C_0 = M_B(R)$ which contradicts the result of **(ii)**. Hence we prove **(iii)**.

(iv) Let (u, v) be a topological solution of (1.7). Then any solution pair $(A(r), B(r))$ of the linearized equations (1.9) can be written as

$$A(r) = c_1\phi_1(r) + c_2\psi_1(r) \quad \text{and} \quad B(r) = c_1\phi_2(r) + c_2\psi_2(r) \ \text{ for some } c_1, c_2 \in \mathbf{R}.$$

By the result of (†) in the proof of (ii), we easily obtain the non-degeneracy result if $c = c_2/c_1 > 0$. When $c \le 0$ or $c_1 = 0$, then by (i), we can also get that both $A_c(r)$ and $B_c(r)$ are unbounded. This proves (iv). □

3. Uniqueness of Topological Solution

In this section, we will use a continuation argument and Lemma 2.4 to establish the uniqueness of topological solutions. As we have seen in Sect. 2, if $N_1 = N_2$, then $u \equiv v$, and the uniqueness follows from the case of scalar equation 1.10. Concerning the uniqueness for the scalar equation, we refer readers to [6 or 8].

Proof of Theorem 1.1. Suppose that for some pair (N_1^0, N_2^0), Eq. (1.15) possesses at least two topological solutions. Without loss of generality, we may assume $0 \le N_1^0 < N_2^0$. Let

$$N_1^* = \inf\{0 \le N_1 | (1.15) \text{ possesses a unique topological solution for all } (\hat{N}_1, N_2^0) \text{ where } N_1 \le \hat{N}_1 \le N_2^0\}.$$

Clearly, $N_1^* \ge N_1^0$. To yield a contradiction, we claim the following:

(∗) Suppose (u_0, v_0) is a topological solution of (1.15) with respect to (N'_1, N'_2). Let $U_0(r) = u_0(r) - 2N'_1 \log r$ and $V_0(r) = v_0(r) - 2N'_2 \log r$. Then there is a neighborhood B of (N'_1, N'_2) such that for any pair of (N_1, N_2) in B, there exists the corresponding (U, V) with respect to (N_1, N_2), which is close to (U_0, V_0) in $C^2\left(\overline{B_R(0)}\right) \times C^2\left(\overline{B_R(0)}\right)$ for any $R > 0$, where $(u(r), v(r)) = (U(r) + 2N_1 \log r, V(r) + 2N_2 \log r)$ is a topological solution of (1.15) with respect to (N_1, N_2).

If the domain is bounded, then claim $(*)$ follows directly from the non-degeneracy of linearized equation and the Implicit Function Theorem. Since our domain is $\mathbf{R}^2$, in order to apply the Implicit function theory, we need to show the linearized equation of (1.7) is an invertible operator from $W_r^{2,2}(\mathbf{R}^2) \times W_r^{2,2}(\mathbf{R}^2)$ to $L^2(\mathbf{R}^2)$, where $W_r^{2,2}(\mathbf{R}^2) = \{z(x) = z(r) | z, z', z'' \in L^2(\mathbf{R}^2)\}$. For the sake of completeness, we will present a proof of the claim $(*)$.

First, let us assume claim $(*)$ holds. The proof of claim $(*)$ will be given later. By the claim $(*)$, at (N_1^*, N_2), Eq. (1.7) possesses a unique topological solution. Thus $N_1^* > N_1^0$. By the definition of N_1^*, there are two sequences of solutions (u_k, v_k), (u_k^*, v_k^*) of (1.7) with (N_1^k, N_2^0) such that $N_1^k \downarrow N_1^*$. The following lemma shows the pre-compactness of (U_k, V_k), where $U_k(r) = u_k(r) - 2N_1^k \log r$ and $V_k(r) = v_k(r) - 2N_2^0 \log r$.

Lemma 3.1. *There exists a subsequence of* (U_k, V_k) *such that it converges to* (U, V) *in* $C^2\left(\overline{B_R(0)}\right) \times C^2\left(\overline{B_R(0)}\right)$ *for any* $R > 0$, *where* $(u(r), v(r)) = (U(r) + 2N_1^* \log r, V(r) + 2N_2^0 \log r)$ *is a topological solution of* (1.15).

Proof. Since (u_k, v_k) is a topological solution of (1.15) with (N_1^k, N_2^0), we have

$$\int_0^\infty e^{v_k}(1 - e^{u_k}) r dr = 2N_1^k \quad \text{and} \quad \int_0^\infty e^{u_k}(1 - e^{v_k}) r dr = 2N_2^0. \tag{3.1}$$

Since

$$2\int_0^r s(1-e^{u_k(s)})(1-e^{v_k(s)})ds = r^2 - 2\int_0^r s(e^{u_k(s)} + e^{v_k(s)})ds + 2\int_0^r s e^{u_k(s)+v_k(s)} ds,$$

by the Pohozaev identity (2.1), we have

$$\int_0^\infty r(1 - e^{u_k(r)})(1 - e^{v_k(r)})dr = 2N_1^k N_2^0.$$

Thus by (3.1), we obtain

$$\begin{aligned}\int_0^\infty [(1 - e^{u_k}) + (1 - e^{v_k})] r\, dr &= 2(N_1^k + N_2^0) + 4N_1^k N_2^0 \\ &\leq C_0 < +\infty \ \ \forall k.\end{aligned} \tag{3.2}$$

Then

$$\begin{cases} \Delta U_k + e^{v_k}(1 - e^{u_k}) = 0 \\ \Delta V_k + e^{u_k}(1 - e^{v_k}) = 0. \end{cases}$$

By integrating the equation, one has

$$-U_k'(r) r = \int_0^r e^{v_k}(1 - e^{u_k}) s\, ds \leq \int_0^\infty (1 - e^{u_k}) s\, ds < C_0 \text{ (by (3.2))}.$$

Thus, $|U_k'(r)|$ is uniformly bounded in any bounded subinterval of $[0, \infty)$. We claim that

$$U_k(1) \text{ is bounded.}$$

Otherwise, since $U_k(r) - U_k(1)$ is uniformly bounded on any bounded subsequence of $[0, \infty)$, we have

$$U_k(r) = U_k(1) + O(1) \quad \text{for } 0 \le r \le R_0,$$

where R_0 is chosen so that

$$\frac{1}{2}\int_0^{R_0} s\, ds > C_0.$$

Suppose $U_k(1) \to -\infty$. Then for large k,

$$1 - e^{u_k(r)} \ge \frac{1}{2} \quad \text{for } 0 \le r \le R_0.$$

Therefore, we have

$$C_0 \ge \int_0^\infty (1 - e^{u_k})r\, dr > \int_0^{R_0} (1 - e^{u_k})r\, dr \ge \frac{1}{2}\int_0^{R_0} r\, dr > C_0,$$

a contradiction. Therefore, $U_k(1)$ is bounded.

Recall that u_k and v_k are increasing in r and both are negative. Thus $|u_k(r)|$ and $|v_k(r)|$ are uniformly bounded in $[r_0, \infty)$ for any $r_0 > 0$. Without loss of generality, we may assume $U_k(r), V_k(r)$ converges to $U(r), V(r)$ in $C^2([0, R])$ for all $R > 0$, and $(u(r), v(r))$ which is defined in Lemma 3.1 satisfies (1.15) with (N_1^*, N_2^0) and $u'(r), v'(r) > 0$. By (3.2) and Fatou's Lemma,

$$\int_0^\infty [(1 - e^u) + (1 - e^v)]r\, dr \le \liminf_{k\to\infty} \int_0^\infty [(1 - e^{u_k}) + (1 - e^{v_k})]r\, dr < c_0,$$

which implies $\lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} v(r) = 0$, that is, (u, v) is a topological solution. This completes the proof. □

Now we go back to the proof of uniqueness. By Lemma 3.1, (u_k, v_k) and (u_k', v_k') converges to (u, v), due to the fact that (1.15) has only one topological solution at (N_1^*, N_2^0). W.l.o.g., we can assume $|(u_k - u_k')(x_k)| = ||u_k - u_k'||_{L^\infty} \ge ||v_k - v_k'||_{L^\infty}\ \forall k$. Set $A_k = \frac{(u_k - u_k')}{||u_k - u_k'||_{L^\infty}}$ and $B_k = \frac{(v_k - v_k')}{||u_k - u_k'||_{L^\infty}}$. Then A_k, B_k satisfies

$$\begin{cases} \Delta A_k + e^{\eta_k(x)}(1 - e^{u_k})B_k - e^{\xi_k(x)+v_k}A_k = 0 \\ \Delta B_k + e^{\xi_k(x)}(1 - e^{v_k})A_k - e^{\eta_k(x)+u_k}B_k = 0, \end{cases}$$

where $\xi_k(x) \in (u_k(x), u_k'(x))$ and $\eta_k(x) \in (v_k(x), v_k'(x))$. Since for any fixed k, $u_k(x) \to 0$, $v_k(x) \to 0$ as $x \to \infty$, we can apply the same argument of (3.13) and Lemma 3.2, to obtain that the maximum points x_k are bounded. Thus A_k and B_k converges to A and B in $C^2(\mathbf{R}^2)$ respectively, where (A, B) satisfies

$$\begin{cases} \Delta A + e^v(1 - e^u)B - e^{u+v}A = 0, \\ \Delta B + e^u(1 - e^v)A - e^{u+v}B = 0. \end{cases}$$

Since A and B are bounded and not all zero in $\mathbf{R}^2$, by Lemma 2.4, we have $A \equiv 0$ and $B \equiv 0$, a contradiction. This completes the proof of Theorem 1.1. □

Now we need to show claim $(*)$. To show it, we define a background function pair (u_0, v_0) by

$$u_0(x) = N_1 \ln\left(\frac{|x|^2}{1+|x|^2}\right) \text{ and } v_0(x) = N_2 \ln\left(\frac{|x|^2}{1+|x|^2}\right). \tag{3.3}$$

Let $(\hat{u} + u_0, \hat{v} + v_0)$ be a topological solution of (1.7). Then (u, v) satisfies

$$\begin{cases} \Delta\hat{u} + e^{v_0+\hat{v}}(1 - e^{u_0+\hat{u}}) - \frac{4N_1}{(1+|x|^2)^2} = 0 & \text{in } \mathbf{R}^2, \\ \Delta\hat{v} + e^{u_0+\hat{u}}(1 - e^{v_0+\hat{v}}) - \frac{4N_2}{(1+|x|^2)^2} = 0 & \text{in } \mathbf{R}^2, \\ \hat{u}(x) \to 0 \text{ and } \hat{v}(x) \to 0 \text{ as } |x| \to \infty. \end{cases} \tag{3.4}$$

To prove our claim $(*)$, we have to prove the linearized equation is an invertible operator from $W_r^{2,2}(\mathbf{R}^2) \times W_r^{2,2}(\mathbf{R}^2) \to L^2(\mathbf{R}^2)$, i.e., Eq. (3.5) below is uniquely solvable in $W_r^{2,2}(\mathbf{R}^2) \times W_r^{2,2}(\mathbf{R}^2)$ for any pair $(f, g) \in L^2$,

$$\begin{cases} \Delta A + e^v(1 - e^u)B - e^{u+v}A = f, \\ \Delta B + e^u(1 - e^v)A - e^{u+v}B = g, \end{cases} \text{ in } \mathbf{R}^2. \tag{3.5}$$

For any pair (f, g), by Lemma 2.4 there is at most one solution $(A, B) \in W_r^{2,2}(\mathbf{R}^2) \times W_r^{2,2}(\mathbf{R}^2)$. Hence, it suffices for us to show the existence of solutions. Since $\mathbf{R}^2$ is an unbounded domain, the existence can not follow directly from the uniqueness of the solution of (3.5), i.e., the Fredholm alternative theorem might not hold always. However, for any $R > 0$, the equation

$$\begin{cases} \Delta A_k + e^v(1 - e^u)B_k - e^{u+v}A_k = f, \\ \Delta B_k + e^u(1 - e^v)A_k - e^{u+v}B_k = g, & \text{in } B_R(O), \\ A_k = B_k = 0 & \text{on } \partial B_R(O), \end{cases} \tag{3.6}$$

has a solution, i.e., the Fredholm alternative theorem is true for each $R > 0$. Then by letting $R = R_n \to +\infty$, we want to prove $(A_n, B_n) = (A_{R_n}, B_{R_n})$ has a convergent subsequence in $W_r^{2,2}(\mathbf{R}^2) \times W_r^{2,2}(\mathbf{R}^2)$.

Lemma 3.2. (A_n, B_n) *has a convergent subsequence in* $W_r^{2,2}(\mathbf{R}^2)$.

Proof. By Sobolev's embedding theorem, (A_n, B_n) is locally Hölder function. We want to show that

$$||A_n||_{L^\infty(B_{R_n})} + ||B_n||_{L^\infty(B_{R_n})} \le C\{||f||_{L^2(\mathbf{R}^2)} + ||g||_{L^2(\mathbf{R}^2)}\} \tag{3.7}$$

for some constant C independent of n.

Suppose (3.7) does not hold. Without loss of generality, one may assume

$$\begin{cases} ||A_n||_{L^\infty} = \max(||A_n||_{L^\infty}, ||B_n||_{L^\infty}) \text{ and} \\ ||A_n||_{L^\infty} \to +\infty \text{ as } n \to \infty. \end{cases} \tag{3.8}$$

Let $x_n \in \mathbf{R}^2$, such that

$$|A_n(x_n)| = ||A_n||_{L^\infty}. \tag{3.9}$$

First, we claim x_n is bounded.

To prove our claim, Eq. (3.4) can be rewritten as

$$\Delta A_n - A_n = f + e^v(e^u - 1)B_n + (e^{u+v} - 1)A_n, \tag{3.10}$$

and let $K_R(x, y)$ be the fundamental solution of $\Delta - I$ with zero boundary value on $\partial B_R(O)$. It is easy to see

$$0 < K_R(x, y) \le C\frac{e^{-|x-y|}}{\sqrt{|x-y|}} \quad \text{for } |x-y| \ge 1. \tag{3.11}$$

Then

$$A_n(x_n) = \int_{B_R} K_R(x_n, y)[e^v(e^u - 1)B_n(y) + (e^{v-u} - 1)A_n(y) + f(y)]dy. \tag{3.12}$$

Since $v(y) \to 0$ and $u(y) \to 0$ as $y \to +\infty$, by (3.12) we have

$$|A_n(x_n)| \le o(1)(||A_n||_{L^\infty} + ||B_n||_{L^\infty}) + C \cdot ||f||_{L^2}, \tag{3.13}$$

which yields a contradiction. Thus, x_n is bounded.

By letting $\hat{A}_n = \frac{A_n}{||A_n||_{L^\infty}}$ and $\hat{B}_n = \frac{B_n}{||A_n||_{L^\infty}}$, then a subsequence of $(\hat{A}_n, \hat{B}_n)$ will converge to $(\hat{A}, \hat{B})$ and $(\hat{A}, \hat{B})$ satisfies

$$\begin{cases} \Delta\hat{A} + e^v(1 - e^u)\hat{B} - e^{u+v}\hat{A} = 0 \\ \Delta\hat{B} + e^u(1 - e^v)\hat{A} - e^{u+v}\hat{B} = 0 \end{cases}$$

and $\hat{A} \not\equiv 0$. Since $\hat{A}$ and $\hat{B}$ are bounded, by Lemma 2.4, we have $\hat{A} \equiv 0$ and $\hat{B} \equiv 0$, which yields a contradiction. Thus (3.7) is established.

By (3.7), a subsequence of (A_n, B_n) will converge to (A, B) and (A, B) satisfies (3.5). Let Eq. (3.4) be rewritten as (3.6). Since A and B are bounded, by (3.5) and a standard argument, we can show that $A(r)$ and $B(r)$ in $L^2(\mathbf{R}^2)$. This proves the linearized equation is 1-1 and onto from $W_r^{2,2}(\mathbf{R}^2) \times W_r^{2,2}(\mathbf{R}^2)$ to $L^2(\mathbf{R}^2)$. Then by the open mapping theorem of functional Analysis (linear), we know the inverse operator of the linearized equation is bounded from $L^2(\mathbf{R}^2)$ to $W_r^{2,2}(\mathbf{R}^2) \times W_r^{2,2}(\mathbf{R}^2)$. By apply the Implicit Function Theorem, our claim $(*)$ is proved, and thus the proof of Theorem 1.1 is completely finished. □

4. Asymptotic Behaviors of All Entire Solutions

In this section, we give the asymptotic behaviors of all entire radial solutions for (1.7) as follows. Here solutions are not necessarily negative in $\mathbf{R}^2$.

Proposition 4.1. *Let (u, v) be an entire solution of (1.13)–(1.14). Then (u, v) satisfies one of the following behaviors:*

(A) $\lim_{r\to\infty} u(r) = 0$ *and* $\lim_{r\to\infty} v(r) = 0$;
(B) $\lim_{r\to\infty} u(r) = -\infty$ *and* $\lim_{r\to\infty} v(r) = -\infty$;
(C) $\lim_{r\to\infty}(u(r), \frac{v(r)}{r^2}) = (-c_u, -\frac{e^{-c_u}}{4})$ *for some* $c_u > 0$;
(D) $\lim_{r\to\infty}(\frac{u(r)}{r^2}, v(r)) = (-\frac{e^{-c_v}}{4}, -c_v)$ *for some* $c_v > 0$;

(E) $\lim_{r\to\infty} u(r) = \infty$, $\lim_{r\to\infty} v(r) = -\infty$;
(F) $\lim_{r\to\infty} u(r) = -\infty$, $\lim_{r\to\infty} v(r) = \infty$.

In order to prove Proposition 4.1, we need the following lemmas.

Lemma 4.1. *Let* (u, v) *be an entire solution of* (1.13)–(1.14). *Then the following statements hold.*

(i) *If* $u(r_1) \geq 0$ *for some* $r_1 > 0$*, then* $u'(r) > 0$ $\forall r > 0$, $\lim_{r\to\infty} u(r) = \infty$ *and* $\lim_{r\to\infty} v(r) = -\infty$.
(ii) *If* $u'(r_1) \leq 0$ *for some* $r_1 > 0$*, then* $u(r_1) < 0$ *and* $\lim_{r\to\infty} u(r) = -\infty$.

Proof. **(i)** We shall use the maximum principle to prove the first part of **(i)**. Suppose $u'(r_2) \leq 0$ for some $r_2 > 0$. Then, since $u(r) \to -\infty$ as $r \to 0^+$ and $u(r_1) \geq 0$, we obtain that u either has a local minimum $u(r_3) < 0$ or a local maximum $u(r_4) > 0$ for some r_3 and r_4 depending on r_1 and r_2. From (1.13) we get

$$0 = \Delta u(r_3) + e^{v(r_3)}(1 - e^{u(r_3)}) > 0 \text{ or}$$
$$0 = \Delta u(r_4) + e^{v(r_4)}(1 - e^{u(r_4)}) < 0.$$

This contradiction shows $u'(r) > 0$ $\forall r \in (0, \infty)$. Furthermore, since

$$(ru'(r))' = re^{v(r)}(e^{u(r)} - 1) > 0 \quad \forall r > r_1,$$

we have $ru'(r) > r_1 u'(r_1) > 0$ $\forall r > r_1$ and thus $\lim_{r\to\infty} u(r) = \infty$.
This proves the first part of **(i)**.
If $\lim_{r\to\infty} v(r) = -\infty$ is not true, then there exist $r_2 > 0$ and a constant C_1 such that $v(r) > C_1$ $\forall r > r_2$ and

$$\Delta u(r) \geq e^{C_1}(e^{u(r)} - 1) \geq Ce^{u(r)} \quad \forall r > r_3, \tag{4.1}$$

for some constants $C > 0$ and $r_3 > r_2$. From (4.1) we easily obtain that u must blow up in finite time. This contradiction shows that $\lim_{r\to\infty} v(r) = -\infty$ and **(i)** holds.
(ii) If $u'(r_1) \leq 0$ for some $r_1 > 0$, then by **(i)** we have $u(r_1) < 0$. From this and (1.13), we obtain $ru'(r) < r_1 u'(r_1) < 0$ $\forall r > r_1$. Hence we get $\lim_{r\to\infty} u(r) = -\infty$. This completes the proof. □

Lemma 4.2. *Let* (u, v) *be an entire solution of* (1.13)–(1.14). *If* $\lim_{r\to\infty} u(r) = -c_u$ *exists and* $v'(r_1) \leq 0$ *for some* $r_1 > 0$*, then* $\lim_{r\to\infty} ru'(r) = 0$, $\lim_{r\to\infty} \frac{v(r)}{r^2} = -\frac{e^{-c_u}}{4}$ *and* $c_u > 0$.

Proof. Since $v'(r_1) \leq 0$, by **(ii)** of Lemma 4.1 in the case of v, we have $\lim_{r\to\infty} v(r) = -\infty$ and

$$\begin{aligned} rv'(r) &= r_1 v'(r_1) - \int_{r_1}^{r} se^{u(s)}(1 - e^{v(s)}) \quad \forall r > r_1 \\ &< -C \int_{r_1}^{r} s\, ds \quad \forall r > r_2 > r_1 \text{ and for some constant } C > 0 \\ &\to -\infty \quad \text{as } r \to \infty. \end{aligned}$$

Then, combining the above inequality and (1.13), we easily obtain

$$\lim_{r\to\infty}\frac{v(r)}{r^2}=\lim_{r\to\infty}\frac{rv'(r)}{2r^2}=\lim_{r\to\infty}\frac{re^u(e^v-1)}{4r}=-\frac{e^{-c_u}}{4}.$$

This proves the second result of this lemma.

Since $\lim_{r\to\infty} u(r)=-c_u$ exists, by Lemma 4.1, we easily obtain that $u(r)<0$, $u'(r)>0\ \forall r\in(0,\infty)$ and thus $-c_u\le 0$. Then, by

$$(ru'(r))'=-re^{v(r)}(1-e^{u(r)})<0\ \forall r>0,$$

it follows that $\lim_{r\to\infty} ru'(r)=0$. Suppose $c_u=0$. We claim the following two statements:

(a) $\lim_{r\to\infty}\frac{u(r)}{e^{v(r)}}=0$;

(b) $\frac{u'(r)}{u(r)}\ge v'(r)\ \forall r\ge r_0$ for some sufficiently large r_0.

Since $\lim_{r\to\infty} u(r)=0$ and $\lim_{r\to\infty} e^{v(r)}=0$, by (1.13) we have

$$\lim_{r\to\infty}\frac{u(r)}{e^{v(r)}}=\lim_{r\to\infty}\frac{ru'(r)}{re^{v(r)}v'(r)}=\lim_{r\to\infty}\frac{re^{v(r)}(e^{u(r)}-1)}{re^{v(r)}(v'(r))^2+re^{u(r)+v(r)}(e^{v(r)}-1)}$$
$$=\lim_{r\to\infty}\frac{e^u-1}{(v')^2+e^u(e^v-1)}=0.$$

This proves claim **(a)**.

In order to prove claim **(b)**, we first show

(i) $\lim_{r\to\infty}\frac{e^{u(r)}-1}{u(r)}=1$; (ii) $\lim_{r\to\infty}\frac{u(r)}{ru'(r)}=0$; (iii) $\lim_{r\to\infty}\frac{u'(r)}{u(r)}=0$.

Since $\lim_{r\to\infty} u(r)=0$, we obtain $\lim_{r\to\infty}\frac{e^u-1}{u}=\lim_{r\to\infty}\frac{e^{u(r)}u'(r)}{u'(r)}=\lim_{r\to\infty} e^{u(r)}$ $=1$ which proves (i). In addition, combining the facts of $\lim_{r\to\infty} ru'(r)=0$ and $\lim_{r\to\infty} r^2e^v=0$ with (1.13), we have

$$\lim_{r\to\infty}\frac{u(r)}{ru'(r)}=\lim_{r\to\infty}\frac{ru'(r)}{r^2e^v(e^u-1)}=\lim_{r\to\infty}\frac{re^v(e^u-1)}{2re^v(e^u-1)+r^2e^v(e^u-1)v'+r^2e^{u+v}u'}$$
$$=\lim_{r\to\infty}\frac{1}{2+rv'+\frac{ru'e^u}{e^u-1}}. \tag{4.2}$$

Since $\lim_{r\to\infty} rv'(r)=-\infty$ and $\frac{ru'e^u}{e^u-1}<0\ \forall r>0$, by (4.2) we obtain (ii).
Using the assertions of (i) and (ii), we get

$$\lim_{r\to\infty}\frac{u'(r)}{u(r)}=\lim_{r\to\infty}\frac{u''(r)}{u'(r)}=\lim_{r\to\infty}\frac{-u'(r)-re^{v(r)}(1-e^{u(r)})}{ru'(r)}$$
$$=\lim_{r\to\infty}\left[\frac{ru(r)e^{v(r)}}{ru'(r)}\cdot\frac{e^{u(r)}-1}{u(r)}\right]=\lim_{r\to\infty}\frac{ru(r)e^{v(r)}}{ru'(r)}\ \text{(by (i))}$$
$$=\lim_{r\to\infty}\left[\frac{u(r)}{ru'(r)}\cdot re^{v(r)}\right]=0\ \text{(by (ii) and }\lim_{r\to\infty} re^{v(r)}=0).$$

This proves (iii).

Applying the results (i)–(iii) and (1.13), we obtain

$$\begin{aligned}
\lim_{r\to\infty}\frac{\frac{ru'(r)}{u(r)}}{rv'(r)} &= \lim_{r\to\infty}\left[\frac{u(u'(r)+ru''(r))-r(u'(r))^2}{u^2(r)}\cdot\frac{1}{re^{u(r)}(e^{v(r)}-1)}\right] \text{ (by (1.13))}\\
&= \lim_{r\to\infty}\frac{-rue^{v}(1-e^{u})-r(u')^2}{ru^2e^{u}(e^{v}-1)} \text{ (by (1.13))}\\
&= \lim_{r\to\infty}\frac{e^{v}(1-e^{u})}{u}+\lim_{r\to\infty}(\frac{u'}{u})^2 = 0 \text{ (by (i) and (iii)).} \qquad (4.3)
\end{aligned}$$

Since $\frac{ru'(r)}{u(r)} < 0$ and $rv'(r) < 0\ \forall r > 0$, by (4.3) we finally have $\frac{ru'(r)}{u(r)} \geq rv'(r)$ for sufficiently large r. Hence we prove claim **(b)**.

From **(b)**, we easily obtain that

$$[\ln(-u(r)) - v(r)]' \geq 0 \quad \forall r \geq r_0. \qquad (4.4)$$

Integrating both sides of (4.4) from r_0 to r, we deduce $\frac{u(r)}{e^{v(r)}} \leq -e^{C_0} < 0\ \forall r \geq r_0$, where $C_0 = \ln(-u(r_0)) - v(r_0)$. This contradicts **(a)**. Therefore $c_u > 0$ and we complete the proof. □

Now we are in a position to prove Proposition 4.1.

Proof of Proposition 4.1. We divide the proof into the following cases.

Case 1. $u(r_1) \geq 0$ (resp. $v(r_1) \geq 0$) for some $r_1 > 0$:

Then, by **(i)** of Lemma 4.1, we obtain that $\lim_{r\to\infty} u(r) = \infty$ and $\lim_{r\to\infty} v(r) = -\infty$ (resp. $\lim_{r\to\infty} v(r) = \infty$ and $\lim_{r\to\infty} u(r) = -\infty$). Hence (E) (resp. (F)) happens in this case.

Case 2. $u(r) < 0$ and $v(r) < 0\ \forall r \in (0, \infty)$:

(i) If $u'(r_1) \leq 0$ and $v'(r_2) \leq 0$ for some $r_1 > 0$ and $r_2 > 0$, then by **(ii)** of Lemma 4.1, we have $\lim_{r\to\infty} u(r) = \lim_{r\to\infty} v(r) = -\infty$. This proves that (B) holds in this case.

(ii) If $u'(r) > 0$ and $v'(r) > 0\ \forall r \in (0, \infty)$, then $\lim_{r\to\infty} u(r) = -C_1 \leq 0$ and $\lim_{r\to\infty} v(r) = -C_2 \leq 0$ all exist. If $C_1 < 0$ then, by (1.13)–(1.14), we have

$$\begin{aligned}
ru'(r) &= 2N_1 - \int_0^r se^{v(s)}(1-e^{u(s)})ds \quad \forall r > 0\\
&< 2N_1 - (1-e^{-C_1})\int_0^r s^{1+2N_2}ds \quad \forall r > 0\\
&< -C < 0 \quad \text{for } r \text{ large,}
\end{aligned}$$

which implies $u'(r) < 0$ for large r. This contradiction shows $C_1 = 0$. Similarly, $C_2 = 0$ as well. Thus (A) occurs under this case.

(iii) If $u'(r) > 0$ (resp. $v'(r) > 0$) $\forall r \in (0, \infty)$ and $v'(r_1) \leq 0$ (resp. $u'(r_1) \leq 0$) for some $r_1 > 0$, then $\lim_{r\to\infty} u(r) = -c_u$ (resp. $\lim_{r\to\infty} v(r) = -c_v$) exists. By Lemma 4.2, we obtain $\lim_{r\to\infty} \frac{v(r)}{r^2} = -\frac{e^{-c_u}}{4}$ (resp. $\lim_{r\to\infty} \frac{u(r)}{r^2} = -\frac{e^{-c_v}}{4}$) and $c_u > 0$ (resp. $c_v > 0$). This shows that (C) (resp. (D)) happens in this case.

According to **Case 1** and **Case 2**, we complete the proof. □

5. The Structure of All Entire Solutions

In this section, we will study the structures of all radial entire solutions for (1.7). Applying this classification, we give the proof of Theorems 1.2 and 1.3. Let the respective set of initial data according to the behaviors of solutions be depicted beneath the statement of Theorem 1.2 in Sect. 1. First, we derive the structure property of Ω in the following.

Proposition 5.1. *Ω is an open subset of $\mathbf{R}^2$ and the following statements are valid.*

(A) *If $(\alpha_1, \alpha_{21}), (\alpha_1, \alpha_{22}) \in \Omega$ with $\alpha_{21} < \alpha_{22}$, then $(\alpha_1, \alpha_2) \in \Omega\ \forall \alpha_{21} < \alpha_2 < \alpha_{22}$. Similarly, if $(\alpha_{11}, \alpha_2), (\alpha_{12}, \alpha_2) \in \Omega$ with $\alpha_{11} < \alpha_{12}$, then $(\alpha_1, \alpha_2) \in \Omega\ \forall \alpha_{11} < \alpha_1 < \alpha_{12}$.*
(B) *There exists $(\tilde{\alpha}_1, \tilde{\alpha}_2) \in \mathbf{R}^2$ such that $(\alpha_1, \alpha_2) \notin \Omega\ \forall \alpha_1 \geq \tilde{\alpha}_1$ or $\forall \alpha_2 \geq \tilde{\alpha}_2$.*
(C) *Ω is a simply connected and unbounded region such that*

$$\begin{cases} \Omega = \Omega_{NT} \bigcup \Omega_u \bigcup \Omega_v, \\ \partial\Omega = S_u \bigcup T \bigcup S_v \text{ and } \overline{S}_u \bigcap \overline{S}_v = T. \end{cases}$$

In particular, both S_u and S_v are nonempty.

Proof. We divide the proof into the following steps.
Step 1. Let $\bar{\alpha} \in \Omega$ be any point. Then there exists $r_0 = r_0(\bar{\alpha}) > 0$ such that $u'(r_0, \bar{\alpha}) < 0$ and $v'(r_0, \bar{\alpha}) < 0$. By the continuity of (u', v') w.r.t. α, there exists $\delta > 0$ such that

$$u'(r_0, \alpha) < 0 \quad \text{and} \quad v'(r_0, \alpha) < 0\ \forall \alpha \in B_\delta(\bar{\alpha}). \tag{5.1}$$

By **(ii)** of Lemma 4.1, we obtain $u(r, \alpha) \to -\infty$ and $v(r, \alpha) \to -\infty$ as $r \to \infty\ \forall \alpha \in B_\delta(\bar{\alpha})$. This proves $B_\delta(\bar{\alpha}) \subset \Omega$ and thus Ω is open.
Step 2. By the monotone property of $(\phi_i, \psi_i), i = 1.2$, Lemma 2.3, we easily obtain (A).
Step 3. We prove (B) by scaling arguments and monotone property. Choose $d > 0$ such that

$$\frac{N_2}{N_1 + 1} < d < \frac{N_2 + 1}{N_1}, \alpha_1 = s \quad \text{and} \quad \alpha_2 = ds. \tag{5.2}$$

Let

$$\begin{cases} \hat{u_s}(r) = U(e^{-\frac{(1+d)s}{2(N_1+N_2+1)}} r, s, ds) - s, \\ \hat{v_s}(r) = V(e^{-\frac{(1+d)s}{2(N_1+N_2+1)}} r, s, ds) - ds, \end{cases} \tag{5.3}$$

where

$$\begin{cases} U(t, \alpha_1, \alpha_2) = u(t, \alpha_1, \alpha_2) - 2N_1 \ln t, \\ V(t, \alpha_1, \alpha_2) = v(t, \alpha_1, \alpha_2) - 2N_2 \ln t. \end{cases}$$

Then $(\hat{u_s}, \hat{v_s})$ satisfies

$$\begin{cases} \Delta\hat{u_s}(r) + e^{-d_1 s} r^{2N_2} e^{\hat{v_s}} = r^{2(N_1+N_2)} e^{\hat{u_s}+\hat{v_s}}, \\ \Delta\hat{v_s}(r) + e^{-d_2 s} r^{2N_1} e^{\hat{u_s}} = r^{2(N_1+N_2)} e^{\hat{u_s}+\hat{v_s}}, \\ \hat{u_s}(0) = 0, \hat{u_s}'(0) = 0, \hat{v_s}(0) = 0, \hat{v_s}'(0) = 0, \end{cases} \tag{5.4}$$

where $d_1 = \frac{N_2+1-dN_1}{N_1+N_2+1}$ and $d_2 = \frac{d(N_1+1)-N_2}{N_1+N_2+1}$. By (5.2), we easily have $d_1 > 0$ and $d_2 > 0$. Now suppose there exists a sequence $(s_j, ds_j) \in \Omega$ with $(s_j, ds_j) \to (\infty, \infty)$.

Set $(\hat{u}_j, \hat{v}_j) = (\hat{u}_{s_j}, \hat{v}_{s_j})$. Then by $e^{\hat{u}_j} \le 1, e^{\hat{v}_j} \le 1$ and $e^{\hat{u}_j+\hat{v}_j} \le 1$, we have that for all $R > 0$, $|\hat{u}_j'(r)|, |\hat{v}_j'(r)| \le M$ on $[0, R]$ for some $M = M(R) > 0$. Then $|\hat{u}_j(r)|, |\hat{v}_j(r)| \le \bar{M}$ on $[0, R]$ for some $\bar{M} = \bar{M}(R) > 0$. From elliptic estimates, we have $(\hat{u}_j, \hat{v}_j) \to (\hat{u}, \hat{v})$ (passing subsequence if necessary) in $C^2([0, R])$ for any $R > 0$ and $(\hat{u}, \hat{v})$ which satisfies

$$\begin{cases} \Delta\hat{u}(r) = r^{2(N_1+N_2)}e^{\hat{u}+\hat{v}} \\ \Delta\hat{v}(r) = r^{2(N_1+N_2)}e^{\hat{u}+\hat{v}} \\ \hat{u}(0) = 0, \hat{u}'(0) = 0, \hat{v}(0) = 0, \hat{v}'(0) = 0. \end{cases} \tag{5.5}$$

Since $U(t, s, ds)$ and $V(t, s, ds)$ are both decreasing in t, we have $\hat{u}$ and $\hat{v}$ are nonincreasing in r. But, any solution pair of (5.5) must be increasing in r. Actually, both $\hat{u}$ or $\hat{v}$ must blow up at finite r. This contradiction shows that there exists $s_0 > 0$ such that $u(r, s, ds)$ or $v(r, s, ds)$ blows up $\forall s > s_0$, and hence, by Lemma 2.3, so does $u(r, \alpha_1, \alpha_2)$ or $v(r, \alpha_1, \alpha_2)$ for any $\alpha_1 \ge s_0 \equiv \tilde{\alpha}_1$ or $\alpha_2 \ge ds_0 \equiv \tilde{\alpha}_2$. This proves **(B)**.
Step 4. In this step, we prove the result of **(C)**. For this purpose we claim the following statements.

(a) Let $\alpha \in \Omega$ and $(u(r), v(r)) = (u(r, \alpha), v(r, \alpha))$. Then the corresponding (β_1, β_2) satisfies either $\beta_1 < \infty$ or $\beta_2 < \infty$.
(b) $\Omega = \Omega_{NT} \bigcup \Omega_u \bigcup \Omega_v$.
(c) $\partial\Omega \subset S_u \bigcup T \bigcup S_v$.
(d) If $\alpha = (\alpha_1, \alpha_2) \in \partial\Omega$ and $(\alpha_1 - \epsilon, \alpha_2) \in \Omega$ for some $\epsilon > 0$, then $\alpha \in S_u$.
(e) If $\alpha = (\alpha_1, \alpha_2) \in \partial\Omega$ and $(\alpha_1, \alpha_2 - \epsilon) \in \Omega$ for some $\epsilon > 0$, then $\alpha \in S_v$.
(f) Let $\Gamma_1 = \partial\Omega \bigcap S_u$ and $\Gamma_2 = \partial\Omega \bigcap S_v$. Then $\overline{\Gamma}_1 \bigcap \overline{\Gamma}_2 = T$.
(g) Ω is a simply connected and unbounded region.
(h) If $\alpha = (\alpha_1, \alpha_2) \in S_u$, then $(\alpha_1 + \epsilon, \alpha_2) \notin S_u$ $\forall\epsilon > 0$.
(i) If $\alpha = (\alpha_1, \alpha_2) \in S_v$, then $(\alpha_1, \alpha_2 + \epsilon) \notin S_v$ $\forall\epsilon > 0$.

(a) Suppose the result is not true. Then $\beta_1 = \infty, \beta_2 = \infty$ and $\lim_{r\to\infty} ru'(r) = -\infty$. Hence, for each $M > 2$ there exists $r_M > 0$ such that $ru'(r) < -M$ $\forall r > r_M$. Furthermore we get $e^{u(r)} < C \cdot r^{-M}$ for large r and

$$\beta_2 = \int_0^\infty re^u(1 - e^v)dr \le \int_0^\infty re^u dr < \infty.$$

This contradiction proves **(a)**.
(b) By **(a)** and the definitions of Ω_u and Ω_v, we easily have **(b)**.
(c) Let $E = S_u \bigcup T \bigcup S_v$, $\alpha \in \partial\Omega$ and (u, v) be the respective solution. Then, by the Hopf lemma we have $u(r) < 0, v(r) < 0$ $\forall r > 0$. By Proposition 4.1, we have $\partial\Omega \subset E$. This proves **(c)**.
(d) If $\alpha = (\alpha_1, \alpha_2) \in \partial\Omega$ and $\alpha_\epsilon = (\alpha_1 - \epsilon, \alpha_2) \in \Omega$ for some $\epsilon > 0$, then, by Lemma 2.3, $v(r, \alpha_\epsilon) \to -\infty$ and $v(r, \alpha) < v(r, \alpha_\epsilon)$ $\forall r > 0$. This proves $\alpha \in S_u$.
(e) The proof is similar to **(d)**.
(f) From **(c)**-**(e)**, $\Gamma_1 \ne \emptyset$ and $\Gamma_2 \ne \emptyset$, then by the continuity w.r.t. initial data, we obtain that

$$\overline{\Gamma}_1 \cap S_v = \emptyset \quad \text{and} \quad \overline{\Gamma}_2 \cap S_u = \emptyset.$$

Therefore, **(f)** is proved.

(g) Suppose Ω is not connected. Then there exist two disjoint open sets O_1 and O_2 satisfying $O_1 \bigcup O_2 = \Omega$. From **(d)–(f)**, O_1 and O_2 possess one initial data of topological solutions at least, respectively, and by **(A)–(B)**, there exist two distinct initial data of topological solutions T_1 and T_2 such that $T_1 \in \partial O_1$ and $T_2 \in \partial O_2$. This contradicts the uniqueness of the topological solution. Hence Ω is connected. By using the similar arguments, we get Ω is unbounded. From **(A)**, we obtain that the set Ω does not have a hole, that is, Ω is a simply connected set. This shows **(g)**.

(h) First we show that if $\alpha = (\alpha_1, \alpha_2) \in S_u$, then $\forall \epsilon > 0$ we have $\alpha_\epsilon = (\alpha_1 + \epsilon, \alpha_2)$ and $u(r_0, \alpha_\epsilon) \geq 0$ for some $r_0 = r_0(\epsilon) > 0$, that is, $\alpha_\epsilon \notin S_u$ $\forall \epsilon > 0$ by Lemma 4.1. Suppose there exists $\epsilon_0 > 0$ such that $u(r, \alpha_{\epsilon_0}) < 0$ $\forall r > 0$. By Lemma 2.3, we have

$$\begin{matrix} u(r,\alpha) < u(r,\alpha_\epsilon) < u(r,\alpha_{\epsilon_0}) < 0 \\ v(r,\alpha_{\epsilon_0}) < v(r,\alpha_\epsilon) < v(r,\alpha) < 0 \end{matrix} \quad \forall r \in (0,\infty). \tag{5.6}$$

From (5.6) and $\alpha \in S_u$, we obtain $u(r, \alpha_{\epsilon_0})$ is bounded below for large r. Then, by **(ii)** of Lemma 4.1, we have $u'(r, \alpha_{\epsilon_0}) > 0$ $\forall r > 0$, and hence $\lim_{r\to\infty} u(r, \alpha_{\epsilon_0}) = -c_u \leq 0$. Furthermore, by Lemma 4.2, we get $\lim_{r\to\infty} ru'(r, \alpha) = \lim_{r\to\infty} ru'(r, \alpha_{\epsilon_0}) = 0$. Combining these facts, we attain $\beta_1(\alpha) = 2N_1 = \beta_1(\alpha_{\epsilon_0})$. But, from (5.6) we have $\beta_1(\alpha) > \beta_1(\alpha_{\epsilon_0})$. This contradiction proves our assertion and thus **(h)** holds.

(i) The proof is similar to **(h)**.

By above claims **(b)–(i)** and the existence of the topological solution, we finally obtain **(C)** and the proof is complete. □

In the following, we let Ω_u and Ω_v be defined in Sect. 1. Then the corresponding (β_1, β_2) of each solution can be classified as follows.

Proposition 5.2. *Let* (u, v) *be an entire solution of* (1.13)–(1.14) *and the corresponding* (β_1, β_2) *be defined in* (1.20). *Then the following statements are valid*:

(a) β_1 *is continuous w.r.t* α_1 *and* α_2 *for all* $(\alpha_1, \alpha_2) \in \Omega_{NT} \cup \Omega_u$. *Similarly,* β_2 *is continuous w.r.t* α_1 *and* α_2 *for all* $(\alpha_1, \alpha_2) \in \Omega_{NT} \cup \Omega_v$.

(b) *If* (u, v) *is a topological solution, then* $(\beta_1, \beta_2) = (2N_1, 2N_2)$.

(c) *If* (u, v) *is a non-topological solution, then the respective* (β_1, β_2) *satisfies*

$$(\beta_1 - 2(N_1+1))(\beta_2 - 2(N_2+1)) > 4(N_1+1)(N_2+1).$$

(d) *For any* $\alpha \in S_u$ *(resp.* $\alpha \in S_v$*),* (u, v) *is a* **Type (IV)** *solution with* $(\beta_1, \beta_2) = (2N_1, \infty)$ *(resp.* $(\beta_1, \beta_2) = (\infty, 2N_2)$*).*

(e) *For any* $\alpha \in \Omega_u$ *(resp.* $\alpha \in \Omega_v$*),* (u, v) *is a* **Type (III)** *solution with* $2N_1 < \beta_1 \leq 2N_1 + 2$ *and* $\beta_2 = \infty$ *(resp.* $\beta_1 = \infty$ *and* $2N_2 < \beta_2 \leq 2N_2 + 2$*).*

Proof. **(a)** We prove the case of β_1. The case of β_2 is similar. Let $D = \Omega_{NT} \bigcup \Omega_u$, $\alpha_0 = (\alpha_{10}, \alpha_{20}) \in D$. First, we want to show that $D = \Omega_{NT} \bigcup \Omega_u$ is open. Because of $\beta_1(\alpha_0) < \infty$, we obtain that

$$\lim_{r\to\infty} rv'(r;\alpha_0) < -2 - \varepsilon$$

for some $\varepsilon > 0$, by continuity and (1.13), there exist $r_0, \delta > 0$ such that for all $|\alpha - \alpha_0| < \delta$,

$$rv'(r;\alpha) < -2 - \frac{\varepsilon}{2} \ \text{ on } \ [r_0, \infty),$$

which imply $\beta_1(\alpha) < \infty$, that is, $\alpha \in D$. Thus the set D is open. Now if $\delta_n > 0$ and $\delta_n \to 0$ as $n \to \infty$ such that $\alpha_{1n} = \alpha_{10}+\delta_n$, $(\alpha_{1n}, \alpha_{20}) \in D$ and $\beta_{1,n} = \beta_1(\alpha_{1n}, \alpha_{20})\ \forall n$. We want to prove $\beta_{1n} \to \beta_1(\alpha_0)$ as $n \to \infty$. By monotone property, Lemma 2.3, and the continuity of (u, v) w.r.t. the initial data, we obtain that

$$\begin{cases} u(r, \alpha_{1n}, \alpha_{20}) \nearrow u(r, \alpha_0) \\ v(r, \alpha_{1n}, \alpha_{20}) \searrow v(r, \alpha_0) \end{cases} \text{pointwise in } r \text{ as } n \to \infty. \tag{5.7}$$

By (5.7), the definition of β_1 and the monotone convergence theorem, we obtain $\beta_{1n} \to \beta_1(\alpha_0)$ as $n \to \infty$. The case of $\delta_n < 0$ is similar. So β_1 is continuous w.r.t. α_1. By using the same arguments, we get β_1 is continuous w.r.t. α_2. This proves **(a)**.
(b) By **(i)** of Lemma 2.2 we obtain the result.
(c) Since (u, v) is a non-topological solution of (1.13), the respective $\beta_1 < \infty$ and $\beta_2 < \infty$. Hence there exists a sequence $\{r_j\}$ such that $r_j \to \infty$ and

$$r_j^2 e^{u(r_j)}(1 - e^{v(r_j)}) \to 0, \quad r_j^2 e^{v(r_j)}(1 - e^{u(r_j)}) \to 0 \text{ as } j \to \infty.$$

By the Pohozaev identity, Lemma 2.1, we easily obtain

$$\begin{aligned} &ru'(r) \cdot rv'(r) - 2\int_0^r e^{v(s)}(1 - e^{u(s)})s\, ds - 2\int_0^r e^{u(s)}(1 - e^{v(s)})s\, ds \\ &\quad + r^2 e^{u(r)}(1 - e^{v(r)}) + r^2 e^{v(r)}(1 - e^{u(r)}) \\ &\quad = 4N_1N_2 + 6\int_0^r s e^{u(s)+v(s)}\, ds \quad \forall r \in (0, \infty). \end{aligned} \tag{5.8}$$

Taking $r = r_j$ on both sides of (5.8) and then letting $j \to \infty$, we have

$$(2N_1 - \beta_1)(2N_2 - \beta_2) - 2\beta_1 - 2\beta_2 = 4N_1N_2 + 6\int_0^\infty r e^{u(r)+v(r)}\, dr$$

which implies

$$(\beta_1 - 2(N_1 + 1))(\beta_2 - 2(N_2 + 1)) = 4(N_1 + 1)(N_2 + 1) + 6\int_0^\infty r e^{u(r)+v(r)}\, dr.$$

This proves **(c)**.
(d) We prove the case of S_u. The proof for S_v is similar. Let $\alpha \in S_u$ and $(u(r), v(r)) = (u(r, \alpha), v(r, \alpha))$. By Lemma 4.2 we have $\lim_{r\to\infty} ru'(r) = 0 = 2N_1 - \beta_1(\alpha)$. This proves **(d)**.
(e) We prove the case of Ω_u. The proof for Ω_v is similar. Let $\alpha \in \Omega_u$ and $(u(r), v(r)) = (u(r, \alpha), v(r, \alpha))$. By (1.21), (1.22) and the definition of Ω_u, we have $\lim_{r\to\infty} ru'(r; \alpha) < 0$ and $\lim_{r\to\infty} ru'(r) = 2N_1 - \beta_1(\alpha)$. Hence $\beta_1(\alpha) > 2N_1$. Now we claim $\beta_1(\alpha) \le 2N_1 + 2\ \forall \alpha \in \Omega_u$. Suppose not, then there exists $\alpha \in \Omega_u$ such that $\beta_1(\alpha) > 2N_1 + 2$ and $\beta_2(\alpha) = \infty$. Then we obtain $\lim_{r\to\infty} ru'(r) = 2N_1 - \beta_1(\alpha) < -2$, and thus $\int_0^\infty re^u dr < \infty$. From this we deduce

$$\infty > \int_0^\infty re^u dr > \int_0^\infty re^u(1 - e^v)dr = \beta_2(\alpha) = \infty.$$

This contradiction proves **(e)** and the proof is complete. □

The following results describe the existence and properties of **Type (V)** solution.

Proposition 5.3. *W_u and W_v are open subsets of $\mathbf{R}^2$. Furthermore, the following statements are valid*:

(i) *For each $(\theta, \eta) \in S_u$, there exists $\epsilon > 0$ such that $(\alpha_1, \eta) \in W_u$ $\forall \theta < \alpha_1 < \theta + \epsilon$ and*

$$\begin{cases} \lim_{r\to\infty}(u(r) - \lambda \log r) = c_u \\ \lim_{r\to\infty} \frac{v(r)}{r^{2+\lambda}} = -\frac{e^{c_u}}{(2+\lambda)^2} \\ \beta_1 = 2N_1 - \lambda,\ \beta_2 = \infty, \end{cases} \tag{5.9}$$

where c_u and $\lambda = \lambda(\alpha_1, \eta) > 0$ are constants.

(ii) *For each $(\mu, \nu) \in S_v$, there exists $\delta > 0$ such that $(\mu, \alpha_2) \in W_v$ $\forall \nu < \alpha_2 < \nu + \delta$ and*

$$\begin{cases} \lim_{r\to\infty} \frac{u(r)}{r^{2+\gamma}} = -\frac{e^{c_v}}{(2+\gamma)^2} \\ \lim_{r\to\infty}(v(r) - \gamma \log r) = c_v, \\ \beta_1 = \infty,\ \beta_2 = 2N_2 - \gamma, \end{cases} \tag{5.10}$$

where c_v and $\gamma = \gamma(\mu, \alpha_2) > 0$ are constants.

(iii) *W_u and W_v are all nonempty.*

In order to prove Proposition 5.3, we need the following lemmas.

Lemma 5.1. *Suppose $(u(r), v(r))$ is a radial solution satisfying $u(r_0) > 0, v(r_0) < 0$ and $v'(r_0) < 0$ (resp. $v(r_0) > 0, u(r_0) < 0$ and $u'(r_0) < 0$). Then $(u(r), v(r))$ is an entire solution and $\lim_{r\to\infty} u(r) = \infty$ and $\lim_{r\to\infty} v(r) = -\infty$ (resp. $\lim_{r\to\infty} v(r) = \infty$ and $\lim_{r\to\infty} u(r) = -\infty$).*

Proof. Suppose (u, v) is not an entire solution. Then there exists $R_0 > 0$ such that $u(r) \to \infty$ as $r \to R_0^-$. Then we claim that

(a) $\lim_{r\to R_0^-} rv'(r) = \lim_{r\to R_0^-} v(r) = -\infty$.

(b) $(u + v)$ is bounded above on $[R_1, R_0]$ for some $0 < R_1 < R_0$.

Since $v(r_0) < 0$ and $v'(r_0) < 0$, we easily have $v'(r) < 0$ $\forall r \in [r_0, R_0)$. Then we obtain

$$rv'(r) = r_0 v'(r_0) + \int_{r_0}^{r} se^u(e^v - 1)ds \le r_0 v'(r_0) + (e^{v(r_0)} - 1)\int_{r_0}^{r} se^u ds \to -\infty,$$

$$\begin{aligned} v(r) &= v(r_0) + v'(r_0)\ln\frac{r}{r_0} + \int_{r_0}^{r} s \ln\frac{r}{s} e^u(e^v - 1)ds \\ &\le v(r_0) + v'(r_0)\ln\frac{r}{r_0} + (e^{v(r_0)} - 1)\int_{r_0}^{r} s\ln\frac{r}{s} e^v(e^u - 1)ds \\ &= v(r_0) + v'(r_0)\ln\frac{r}{r_0} + (e^{v(r_0)} - 1)(u(r) - u(r_0) - u'(r_0)\ln\frac{r}{r_0}) \to -\infty \end{aligned}$$

as $r \to R_0^-$ since $\lim_{r\to R_0^-} u(r) = \lim_{r\to R_0^-} ru'(r) = \infty$. This prove **(a)**.

By **(a)**, there exists $0 < R_1 < R_0$ such that $2e^{v(r)} - 1 \leq -\frac{1}{2}$ $\forall r \in (R_1, R_0)$. By (1.13) we easily have

$$r(u'(r) + v'(r)) = R_1(u'(R_1) + v'(R_1)) + \int_{R_1}^{r} se^u(2e^v - 1 - e^{v-u})ds$$
$$\leq C - \frac{1}{2}\int_{R_1}^{r} se^u ds < 0 \quad \text{for r close to } R_0^-.$$

This proves $(u+v)'(r) < 0$ for r near R_0^- and thus **(b)** follows.

Now by **(a)-(b)** we deduce

$$\infty = \lim_{r \to R_0^-} ru'(r) = R_1 u'(R_1) + \int_{R_1}^{R_0} se^v(e^u - 1)ds < \infty.$$

This contradiction proves the first result. From Lemma 4.1, we obtain the second result and the proof is complete. □

The following lemma depicts the asymptotic behaviors of **Type (V)** solution at infinity.

Lemma 5.2. *Let* (u, v) *be an entire solution of* (1.13)–(1.14) *on* $(0, \infty)$*. If* $u(r_0) \geq 0$ *for some* $r_0 > 0$*, then*

(a) $\lim_{r\to\infty} ru'(r) = \lambda$ *and* $\lim_{r\to\infty}(u(r) - \lambda \log r) = c_u$ *for some constants* $\lambda > 0$ *and* c_u*.*
(b) $r^p e^{u(r)+v(r)} \in L^1(0, \infty)$ *for any* $p \geq 0$*.*
(c) $\lim_{r\to\infty} \frac{rv'(r)}{r^{2+\lambda}} = -\frac{e^{c_u}}{2+\lambda}$ *and* $\lim_{r\to\infty} \frac{v(r)}{r^{2+\lambda}} = -\frac{e^{c_u}}{(2+\lambda)^2}$*, where* λ *and* c_u *are the constants in* (a).

Proof. (a) By Lemma 4.1, we see that

$$u(r) \geq C \quad \text{and} \quad 1 - e^{v(r)} \leq \frac{1}{2} \quad \text{on } [r_0, \infty)$$

for some $C, r_0 > 0$, then

$$\lim_{r\to\infty} rv'(r) = -\infty \quad \text{and} \quad v(r) \leq -Cr^2 \quad \forall r \geq r_0.$$

From (4.1) we obtain $(ru'(r))' > 0$ on $[r_0, \infty)$ and

$$\lim_{r\to\infty} ru'(r) = \lambda, \quad 0 < \lambda \leq \infty. \tag{5.11}$$

To complete this proof, we need the following fact.

Claim. $\lambda < \infty$.

Proof of Claim. Suppose $\lambda = \infty$, then, using (1.13)–(1.14), we obtain

$$\int_0^{\infty} re^{u(r)+v(r)}dr \geq \lim_{r\to\infty} ru'(r) - 2N_1 = \infty, \tag{5.12}$$

and by $\lim_{r\to\infty} rv'(r) = -\infty$,

$$\lim_{r\to\infty} \frac{ru'(r)}{rv'(r)} = \lim_{r\to\infty} \frac{re^{v(r)}(e^{u(r)}-1)}{re^{u(r)}(e^{v(r)}-1)} = \lim_{r\to\infty} \frac{1-e^{-u(r)}}{1-e^{-v(r)}} = 0. \tag{5.13}$$

By (5.13) we easily have, for any $p > 0$,

$$(r^{2+p}e^{u+v})' = r^{1+p}e^{u+v}(2 + p + ru'(r) + rv'(r)) < 0 \quad \text{for large } r. \tag{5.14}$$

From (5.14) we see that $r^{2+p}e^{u+v}$ is bounded from above by a positive constant and hence $re^{u+v} \le Cr^{-(1+p)}$ for all large r and $\int_0^\infty re^{u(r)+v(r)}dr < \infty$. This contradicts (5.12) and thus $\lambda < \infty$.

Next, we show the asymptotic behavior of u at $r = \infty$. Let $y(r) = u(r) - \lambda \log r$. Then, by (5.11), we have $\lim_{r\to\infty} ry'(r) = 0$ and

$$\lim_{r\to\infty} \frac{ry'(r)}{r^{-p}} = \lim_{r\to\infty} \frac{re^{v}(e^{u}-1)}{-pr^{-p-1}} = 0 \quad \text{for any } p > 0. \tag{5.15}$$

Since $\lim_{r\to\infty} ry'(r) = 0$ and $(ry')' = re^{v}(e^{u}-1) > 0$ on $[r_0, \infty)$, we obtain that $y'(r) < 0$ on $[r_0, \infty)$. From (5.15) and $y'(r) < 0$ on $[r_0, \infty)$, we easily obtain $y'(r) > -C_1 r^{-2}$, $y(r) > C_2$ for large r for some $C_2 \in \mathbf{R}$ and thus $\lim_{r\to\infty} y(r) = c_u$ for some $c_u \in \mathbf{R}$. This shows the results of (a).

(b) Now, by Lemma 4.1 we easily obtain $v(r) \le -Cr^2$ $\forall r \ge R$ for some constants $C > 0$ and $R > 0$. From this inequality and the claim in the proof of (a), we have that $r^p e^{u(r)+v(r)} < Cr^{-2}$ for all large $r > 0$ for any $p > 0$. Thus the result (b) is valid.

(c) By Lemma 4.1 and Eq. (1.13) we have $\lim_{r\to\infty} v(r) = -\infty$ and

$$\begin{aligned}\lim_{r\to\infty} \frac{v(r)}{r^{2+\lambda}} &= \lim_{r\to\infty} \frac{rv'(r)}{(2+\lambda)r^{2+\lambda}} = \lim_{r\to\infty} \frac{re^{u}(e^{v}-1)}{(2+\lambda)^2 r^{1+\lambda}} \\ &= \frac{1}{(2+\lambda)^2} \cdot \lim_{r\to\infty} \frac{e^{u}}{r^{\lambda}} \cdot \lim_{r\to\infty} (e^{v}-1) = \frac{1}{(2+\lambda)^2} \cdot e^{c_u} \cdot (-1).\end{aligned}$$

Thus (c) is true and the proof is complete. □

Remark 5.2. If we replace u by v in the condition of Lemma 5.2, we can obtain the following respective results. The proof is similar.

Lemma 5.3. *Let* (u, v) *be an entire solution of* (1.13)– (1.14) *on* $(0, \infty)$. *If* $v(r_0) \ge 0$ *for some* $r_0 > 0$, *then*

(a) $\lim_{r\to\infty} rv'(r) = \eta$ *and* $\lim_{r\to\infty} (v(r) - \eta \log r) = c_v$ *for some constants* $\eta > 0$ *and* c_v.
(b) $r^p e^{u(r)+v(r)} \in L^1(0, \infty)$ *for any* $p \ge 0$.
(c) $\lim_{r\to\infty} \frac{ru'(r)}{r^{2+\eta}} = \frac{e^{c_v}}{2+\eta}$ *and* $\lim_{r\to\infty} \frac{u(r)}{r^{2+\eta}} = -\frac{e^{c_v}}{(2+\eta)^2}$, *where* η *and* c_v *are the constants in* (a) *above.*

Now we are in the position to prove Proposition 5.3.

Proof of Proposition 5.3. We prove the case of (W_u, S_u). The case of (W_v, S_v) is similar. We divide the proof into the following steps.
Step 1. W_u is open.

Let $\bar{\alpha} \in W_u$. Then there exists $r_0 > 0$ such that $u(r_0, \bar{\alpha}) > 0$, $v(r_0, \bar{\alpha}) < 0$ and $v'(r_0, \bar{\alpha}) < 0$. By the continuity of (u, v), (u', v') w.r.t α, there exists $\delta > 0$ such that

$$u(r_0, \alpha) > 0, v(r_0, \alpha) < 0 \quad \text{and} \quad v'(r, \alpha) < 0 \quad \forall \alpha \in B_\delta(\bar{\alpha}). \tag{5.16}$$

By (5.16) and Lemma 5.1, we obtain $(u(r, \alpha), v(r, \alpha)) \to (\infty, -\infty)$ as $r \to \infty$ $\forall \alpha \in B_\delta(\bar{\alpha})$. This proves $B_\delta(\bar{\alpha}) \subset W_u$ and hence W_u is open.
Step 2. Let $\tilde{\alpha} = (\theta, \eta) \in S_u$. Then there exists $r_0 > 0$ such that $v(r_0, \tilde{\alpha}) < 0$ and $v'(r_0, \tilde{\alpha}) < 0$. By continuity, there exists $\epsilon > 0$ such that $v(r_0, \alpha) < 0$ and $v'(r_0, \alpha) < 0$ $\forall \alpha = (\alpha_1, \eta)$ with $\eta < \alpha_1 < \eta + \epsilon$. By **(h)** of **Step 4** in the proof of Proposition 5.1 and (1.13), we have $u(r_1, \alpha) > 0$ and $v'(r_1, \alpha) < 0$ for some $r_1 > r_0$. From Lemma 5.1, we obtain that $(\alpha_1, \eta) \in W_u$ $\forall \eta < \alpha_1 < \eta + \epsilon$. By Lemma 5.2, we also obtain the asymptotic behavior of (u, v) at ∞ and the corresponding (β_1, β_2) which satisfies

$$\beta_1 = 2N_1 - \lim_{r \to \infty} r u'(r) = 2N_1 - \lambda \quad \text{and} \quad \beta_2 = 2N_2 - \lim_{r \to \infty} r v'(r) = \infty.$$

These prove **(i)** and **(ii)**.
Step 3. W_u and W_v are all nonempty.

Since, by **(C)** of Proposition 5.1, S_u and S_v are all nonempty, we get W_u and W_v are also all nonempty from **(i)-(ii)**. This completes the proof. □

Now, we give the proof of Theorems 1.3 and 1.2 in the following.

Proof of Theorem 1.3. We divide the proof into the following steps.
Step 1. By **(C)** of Proposition 5.1, **(c)** of Proposition 5.2 and **(i)-(ii)** of Proposition 5.3, we obtain **(i)**, **(iii)** and **(iv)-(v)** respectively.
Step 2. First we claim the following statements.

(a) For each $\alpha \in (S_v \bigcup \Omega_v)$ (resp. $\alpha \in (S_u \bigcup \Omega_u)$), there does not exist $\{\alpha_k\} \subset \Omega_u$ (resp. $\{\alpha_k\} \subset \Omega_v$) such that $\alpha_k \to \alpha$ as $k \to \infty$.
(b) For each $\alpha = (\alpha_1, \alpha_2) \in S_v$ (resp. $\alpha \in S_u$), there does not exist $\epsilon_k \searrow 0$ with $\{\alpha_k = (\alpha_1 + \epsilon_k, \alpha_2)\} \subset \Omega_{NT}$ (resp. $\{\alpha_k = (\alpha_1 - \epsilon_k, \alpha_2)\} \subset \Omega_{NT}$) such that $\alpha_k \to \alpha$ as $k \to \infty$.
(c) For each $\alpha = (\alpha_1, \alpha_2) \in S_v$ (resp. $\alpha \in S_u$), there does not exist $\epsilon_k \searrow 0$ with $\{\alpha_k = (\alpha_1, \alpha_2 - \epsilon_k)\} \subset \Omega_{NT}$ (resp. $\{\alpha_k = (\alpha_1, \alpha_2 + \epsilon_k)\} \subset \Omega_{NT}$) such that $\alpha_k \to \alpha$ as $i \to \infty$.

(a) We prove the case of $S_v \bigcup \Omega_v$. The case of $S_u \bigcup \Omega_u$ is based on the same arguments. Suppose there exists $\{\alpha_k\} \subset \Omega_u$ such that $\alpha_k \to \alpha$ as $k \to \infty$ for some $\alpha \in (S_v \bigcup \Omega_v)$. Then, by (d)-(e) of Proposition 5.2, we have

$$\beta_1(\alpha) = \infty \quad \text{and} \quad \beta_1(\alpha_k) \le 2N_1 + 2 \ \forall k.$$

Denote $(u(r), v(r)) = (u(r, \alpha), v(r, \alpha))$ and $(u_k(r), v_k(r)) = (u(r, \alpha_k), v(r, \alpha_k))$ $\forall k$. Then, by the definition of β_1, we obtain that there exists $R_0 > 0$ such that $\int_0^{R_0} r e^v (1 - e^u) dr > 2N_1 + 2$, and

$$\begin{aligned} 2N_1 + 2 &\ge \lim_{k \to \infty} \int_0^{R_0} r e^{v_k} (1 - e^{u_k}) dr \\ &= \int_0^{R_0} r e^v (1 - e^u) dr \text{ (by Bounded Convergence Theorem)} \\ &> 2N_1 + 2. \end{aligned}$$

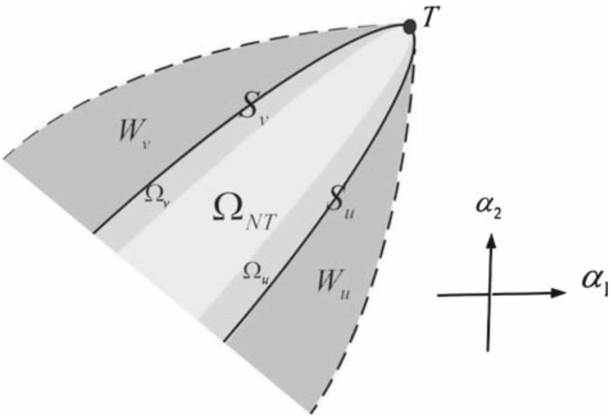

Fig. 1. Structure of all entire solutions

This contradiction proves **(a)**.
(b) We prove the case of S_v. The case of S_u is similar. Let $\alpha = (\alpha_1, \alpha_2) \in S_v$. Suppose there exists $\epsilon_k \searrow 0$ with $\{\alpha_k = (\alpha_1 + \epsilon_k, \alpha_2)\} \subset \Omega_{NT}$ such that $\alpha_k \to \alpha$ as $k \to \infty$. Then, by **(c)–(d)** of 5.2, we have

$$\beta_2(\alpha) = 2N_2 \text{ and } \beta_2(\alpha_k) > 2N_2 + 2 \ \forall k.$$

Then by the continuity of β_2 w.r.t. α_1, i.e., **(a)** of Proposition 5.2, we obtain $\beta_2(\alpha) = \lim_{r\to\infty} \beta_2(\alpha_k) \geq 2N_2 + 2$. This contradiction shows **(b)**.
(c) The proof is similar to **(b)**. We omit the details.

Since S_u and S_v are all nonempty, by **(a)-(c)** above we obtain that $\forall \alpha = (\alpha_1, \alpha_2) \in S_v$ and $\forall \bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2) \in S_u$, there respectively exists $\delta_1 = \delta_1(\alpha) > 0$ and $\delta_2 = \delta_2(\bar{\alpha}) > 0$ such that

$$\begin{aligned} &(\alpha_1 + \delta, \alpha_2), (\alpha_1, \alpha_2 - \delta) \in \Omega_v \quad \forall 0 < \delta < \delta_1 \\ &(\bar{\alpha}_1 - \delta, \bar{\alpha}_2), (\bar{\alpha}_1, \bar{\alpha}_2 + \delta) \in \Omega_u \quad \forall 0 < \delta < \delta_2. \end{aligned} \tag{5.17}$$

By (5.17) we deduce both Ω_u and Ω_v are nonempty and connected. Now, by **(C)** of Proposition 5.1 and **(a)** above, we obtain $\Omega_{NT} \neq \emptyset$. By Proposition 5.2, Ω is simple and open connected. From Lemma 2.3, we also have Ω_u, Ω_{NT} and Ω_v are all simple.

Suppose Ω_{NT} is not open. Then, by (5.17) and Lemma 2.3, w.l.o.g., there exists $\{\alpha_i = (\alpha_{i1}, \alpha_{i2})\} \subset \Omega_v$ such that $\alpha_i \to \bar{\alpha}$ as $i \to \infty$ for some $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2) \in \Omega_{NT}$. Let $\widetilde{\alpha}_i = (\bar{\alpha}_1, \alpha_{i2})$. Then $\widetilde{\alpha}_i \to \bar{\alpha}$ and $\{\widetilde{\alpha}_i\} \subset \Omega_v$. By **(a),(c)** and **(e)** of Proposition 5.2, we finally obtain

$$2N_2 + 2 \geq \lim_{i\to\infty} \beta_2(\widetilde{\alpha}_i) = \beta_2(\bar{\alpha}) > 2N_2 + 2.$$

This contradiction proves that Ω_{NT} is open.

Now, suppose $Z = \partial\Omega_{NT} \bigcap \partial\Omega = \emptyset$, then there exist $\alpha \in \Omega_u$ and a sequence $\{\alpha_i\} \subset \Omega_v$ such that $\alpha_i \to \alpha$ as $i \to \infty$. This contradicts **(a)**. Hence $Z \neq \emptyset$ and $Z = T$. Furthermore, by **(a)** we also get Ω_{NT} is connected. The proof is complete. □

Proof of Theorem 1.2. Let (u, v) be a radial solution of (1.7). Then, by Propositions 4.1 and 5.2, we obtain that (u, v) must be one of the **Types (I)–(V)**. Conversely, by Theorem 1.1, the **Type (I)** solution, i.e., topological solution, exists and is unique. Then, by **(C)** of Proposition 5.1, we have that both $\partial\Omega$ and Ω are nonempty. Thus **Types (II)–(IV)** solutions all exist due to Proposition 5.1 and Theorem 1.3. In particular, the non-topological solution exists. Furthermore, by Proposition 5.3, the **Type (V)** solution exists. We complete the proof. □

Remark 5.3. Combining the results of Theorems 1.1, 1.2 and 1.3, we can sketch the structure of entire solutions as in Fig. 1.

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