PETTY'S CONJECTURED PROJECTION INEQUALITY IN DIMENSION THREE

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Abstract. Let $K \subset \mathbb{R}^3$ be a convex body and let ΠK be its projection body. We prove

$$\frac{V_3(\Pi K)}{V_3(K)^2} \geq \frac{3\pi^2}{4},$$

with equality exactly for ellipsoids. The proof combines a section estimate of Saroglou with a sharp one-dimensional functional inequality. The latter is proved by passing to the distribution function and approximating it by step functions. We also prove the complementary sharp upper bound $V_3(\Pi K)/V_3(K)^2 \leq 18$, which is attained by tetrahedra.

1. Introduction

Projection bodies, introduced by Minkowski, play a central role in convex geometry, affine geometry, and geometric tomography; see, for example, [3, 9, 16]. Throughout, V_j denotes j-dimensional volume, h_L the support function of L, and B_2^j the Euclidean unit ball. We write $L|v^\perp$ for the orthogonal projection of L onto $v^\perp$ and L° for the polar body of L.

If $K \subset \mathbb{R}^n$ is a convex body, its projection body ΠK is the origin-symmetric convex body whose support function is the brightness function of K:

$$h_{\Pi K}(v) = V_{n-1}(K|v^\perp), \qquad v \in \mathbb{S}^{n-1}. \tag{1.1}$$

The change under affine transformation

$$\Pi(\varphi K) = |\det \varphi|\, \varphi^{-t}\Pi K, \qquad \varphi \in GL(n),$$

shows that the quotient $V_n(\Pi K)/V_n(K)^{n-1}$ is invariant under nonsingular affine transformations. **Petty's conjectured projection inequality** asks, for $n \geq 3$, whether

$$V_n(\Pi K) \geq \frac{\omega_{n-1}^n}{\omega_n^{n-2}} V_n(K)^{n-1}, \tag{1.2}$$

with equality only for ellipsoids, where $\omega_m = V_m(B_2^m)$; see [8,11,12]. In the plane, ΠK is a rotation of the difference body $K + (-K)$, so the Brunn–Minkowski inequality gives $V_2(\Pi K) \geq 4V_2(K)$, with equality if and only if K is centrally symmetric up to translation. Thus dimension three is the first nontrivial case in which ellipsoids are conjectured to be the unique minimizers.

The well-known Petty projection inequality and Zhang's projection inequality for polar projection bodies state that

$$\frac{1}{n^n}\binom{2n}{n} \leq V_n(K)^{n-1}V_n((\Pi K)^\circ) \leq \left(\frac{\omega_n}{\omega_{n-1}}\right)^n. \tag{1.3}$$

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Equality in the right-hand inequality holds precisely for ellipsoids, while equality in the left-hand inequality holds precisely for simplices. The lower bound was proved by Zhang [19]. Gardner and Zhang later extended it to a monotone family of inequalities involving radial mean bodies [4].

Petty's conjectured inequality (1.2) implies the right-hand inequality in (1.3). In fact, if (1.2) holds, then the Blaschke–Santaló inequality applied to the origin-symmetric body ΠK gives

$$V_n(K)^{n-1}V_n\big((\Pi K)^\circ\big) \le \frac{\omega_n^2 V_n(K)^{n-1}}{V_n(\Pi K)} \le \left(\frac{\omega_n}{\omega_{n-1}}\right)^n.$$

The affine isoperimetric inequality for polar projection bodies has generated several further developments. Lutwak, Yang, and Zhang introduced L_p-projection bodies and proved the L_p Petty projection inequality [9]. Haberl and Schuster established sharp inequalities for general L_p-projection bodies [5]. Ludwig characterized the projection body operator within valuation theory [7].

Several approaches to Petty's conjectured projection inequality use class reduction, the second projection body, and local analysis near the Euclidean ball. Lutwak developed equivalent formulations and consequences of the conjecture in [8]. In dimension three, Saroglou obtained the general lower bound $V_3(\Pi K) \ge 6V_3(K)^2$ and isolated the relevant sectional geometry [13]. Saroglou and Zvavitch proved local minimality under a uniform curvature density hypothesis, while Ivaki obtained local rigidity near the ball for the equation $\Pi^2 K = cK$; see [6, 15].

We establish the global conjecture in dimension three. Since $\omega_2 = \pi$ and $\omega_3 = 4\pi/3$, the conjectured constant is $3\pi^2/4$.

Theorem 1.1. *For every convex body $K \subset \mathbb{R}^3$,*

$$V_3(\Pi K) \ge \frac{3\pi^2}{4}\, V_3(K)^2.$$

Equality holds if and only if K is an ellipsoid.

We briefly describe the proof. From now on, we write

$$P(K) := \frac{V_3(\Pi K)}{V_3(K)^2},$$

and aim to show

$$P(K) \ge \frac{3\pi^2}{4}.$$

Write $\Pi^2 K = \Pi(\Pi K)$. For an origin-symmetric body, set

$$M(K) = \max_{v\in\mathbb{S}^2} \frac{h_{\Pi^2 K}(v)}{h_K(v)V_3(K)}.$$

For $u \in \mathbb{S}^2$, let

$$A_u(t) = V_2\big(K \cap (tu + u^\perp)\big), \qquad -h_K(u) \le t \le h_K(u),$$

and define

$$R_K(u) = \frac{\left(\displaystyle\int_{-h_K(u)}^{h_K(u)} A_u(t)^{1/2}\,dt\right)^2}{h_K(u)V_3(K)}. \tag{1.4}$$

The directional quantity $R_K(u)$ already appears in Saroglou's work [13]. Indeed, Saroglou [13, Conjecture 1.5] conjectured that $\max_{u\in\mathbb{S}^2} R_K(u)$ is minimized, among origin-symmetric convex bodies in $\mathbb{R}^3$, precisely by ellipsoids. Saroglou [13, Proposition 2.1] established the following class-reduction identity

$$\min_{K=-K} P(K) = \min_{K=-K} M(K).$$

By Saroglou's fixed-direction symmetrization estimate, we have

$$\frac{h_{\Pi^2 K}(e)}{h_K(e)V_3(K)} \geq 4R_K(e)$$

for every direction e. Choose e to be a direction of minimum support and put $r = h_K(e)$. The corresponding normalized section function is

$$q(s) = \sqrt{\frac{A_e(rs)}{\pi r^2}}, \qquad 0 \leq s \leq 1.$$

It is non-increasing and satisfies

$$q(s) \geq \sqrt{1-s^2}, \qquad \int_0^1 q(s)^4\, ds \leq 4\int_0^1 s^2 q(s)^2\, ds.$$

The required sharp estimate is

$$\left(\int_0^1 q\right)^2 \geq \frac{3\pi^2}{32}\int_0^1 q^2,$$

and it is proved by establishing a slightly stronger inequality with an additional moment term. It follows that $M(K) \geq 3\pi^2/4$ for every symmetric body. The class reduction and Blaschke symmetrization then give the global bound.

The analytic argument passes to the distribution function of q. On each fixed-mass slice, the resulting functional is concave. Monotone approximation by non-increasing piecewise-constant functions therefore reduces the problem to the extreme points of finite-dimensional ordered polytopes, whose constant intervals have a rigid contact structure. The remaining interval is controlled by an analytic interpolation between its two contact configurations, and a final one-variable estimate handles the terminal interval. Equality is possible only when $q(s) = \sqrt{1-s^2}$ almost everywhere. In the geometric equality case, equality in the mixed-volume comparison used in the class reduction forces $\Pi^2 K = \lambda K$. The resulting rigidity of the sections implies that the normalized body is a ball, and hence the original body is an ellipsoid.

The maximization problem for the same invariant is distinct from both Petty's conjectured lower bound and the Petty–Zhang inequalities for the polar projection body. It goes back to Schneider [18]. Brannen conjectured the sharp bound $\frac{V_n(\Pi K)}{V_n(K)^{n-1}} \leq (n+1)n^n/n!$ with simplices as extremizers [1]. Some special cases in dimension three were previously established by Saroglou [14]. We prove the conjecture for arbitrary three-dimensional convex bodies.

Theorem 1.2. *For every convex body $K \subset \mathbb{R}^3$,*

$$V_3(\Pi K) \leq 18\, V_3(K)^2.$$

The constant is sharp, and every tetrahedron attains equality.

The proof of Theorem 1.2 uses Schneider's mixed functional $V^{(0)}_{2,2,2}$ [17], which is denoted by $\mathcal{B}$ in the present paper (see the definition in Section 5). We establish two stronger inequalities for $\mathcal{B}$, which together imply Brannen's conjectured reverse projection inequality in dimension three. Their proofs reduce to elementary mixed-volume comparisons in dimensions six and nine.

The paper is organized as follows. Section 2 reduces the geometric problem to a sharp one-dimensional functional inequality. Section 3 proves this inequality and characterizes equality. Section 4 combines these to prove Theorem 1.1 and classify all equality cases. Section 5 proves Theorem 1.2. Finally, the appendix contains the elementary scalar estimates used in Section 3.

2. Reduction to a one-dimensional inequality

This section collects the projection-body facts used later. Throughout, a convex body is a compact convex set with nonempty interior. We follow the notation of Gardner and Schneider.

Let $S_2(K,\cdot)$ be the surface area measure of $K \in \mathcal{K}^3$. Our normalization of the projection body is

$$h_{\Pi K}(v) = \frac{1}{2}\int_{\mathbb{S}^2} |v\cdot w|\, dS_2(K,w), \qquad v \in \mathbb{S}^2. \tag{2.1}$$

This is equivalent to (1.1). For $\varphi \in GL(3)$,

$$\Pi(\varphi K) = |\det\varphi|\, \varphi^{-t}\Pi K. \tag{2.2}$$

This follows from Gardner [3, Theorem 4.1.5]. Hence P is affine invariant.

We first reduce the problem to the centrally symmetric case. Let ∇K be the origin-symmetric representative of the Blaschke body determined by

$$S_2(\nabla K,\cdot) = \frac{1}{2}\big(S_2(K,\cdot) + S_2(-K,\cdot)\big).$$

By (2.1) and the maximal volume property of the Blaschke body we have

$$\Pi(\nabla K) = \Pi K, \qquad V_3(\nabla K) \ge V_3(K). \tag{2.3}$$

Equality in the volume inequality holds precisely when K is centrally symmetric up to translation. These are Gardner [3, Theorems 4.1.3 and 3.3.9]. Consequently,

$$P(K) \ge P(\nabla K). \tag{2.4}$$

It is enough to prove the lower bound on the class $\mathcal{K}^3_e$ of origin-symmetric convex bodies.

For $K \in \mathcal{K}^3_e$, set

$$M(K) = \max_{v\in\mathbb{S}^2} \frac{h_{\Pi^2 K}(v)}{h_K(v)V_3(K)}.$$

Applying (2.2) twice, we have

$$\Pi^2(\varphi K) = |\det\varphi|\, \varphi\, \Pi^2 K, \qquad \varphi \in GL(3). \tag{2.5}$$

Consequently, M is invariant under nonsingular linear maps.

The following identity is the specialization to $\mathcal{C} = \mathcal{K}^3_e$ of Saroglou's minimization principle [13, Proposition 2.1], whose proof is based on Schneider's class-reduction argument [16, pp. 570–571].

Lemma 2.1. *The following holds*

$$\min_{K\in\mathcal{K}^3_e} P(K) = \min_{K\in\mathcal{K}^3_e} M(K). \tag{2.6}$$

Proof. For the reader's convenience, we sketch the proof. First,

$$V_3(\Pi K) = V_3(\Pi^2 K, K, K). \tag{2.7}$$

Indeed,

$$\begin{aligned} 3V_3(\Pi^2 K, K, K) &= \int_{\mathbb{S}^2} h_{\Pi^2 K}(v)\, dS_2(K,v) \\ &= \frac{1}{2}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2} |v\cdot w|\, dS_2(\Pi K, w)\, dS_2(K,v) \\ &= \int_{\mathbb{S}^2} h_{\Pi K}(w)\, dS_2(\Pi K, w) = 3V_3(\Pi K). \end{aligned}$$

By the definition of $M(K)$,

$$\Pi^2 K \subset M(K)V_3(K)K.$$

Monotonicity of mixed volumes and (2.7) therefore imply $P(K) \le M(K)$.

Applying Minkowski's inequality to (2.7), we have

$$V_3(\Pi K)^3 \ge V_3(\Pi^2 K) V_3(K)^2,$$

which is equivalent to

$$P(\Pi K) \le P(K). \tag{2.8}$$

We now justify attainment. Let (K_j) be a minimizing sequence for P in $\mathcal{K}^3_e$. By linear invariance and John's theorem, we may assume

$$B^3_2 \subset K_j \subset \sqrt{3} B^3_2.$$

Then, by Blaschke's selection theorem, we can find $K_0 \in \mathcal{K}^3_e$ such that

$$P(K_0) = \min_{K \in \mathcal{K}^3_e} P(K).$$

Since $\Pi K_0 \in \mathcal{K}^3_e$, minimality and (2.8) force equality in Minkowski's inequality. Thus $\Pi^2 K_0$ and K_0 are homothetic. Since both bodies are origin-symmetric, we have

$$\Pi^2 K_0 = \lambda K_0$$

for some $\lambda > 0$. Equation (2.7) then yields

$$V_3(\Pi K_0) = \lambda V_3(K_0), \qquad P(K_0) = \frac{\lambda}{V_3(K_0)} = M(K_0).$$

Since $P \le M$ pointwise,

$$\min_{K \in \mathcal{K}^3_e} P(K) \le \inf_{K \in \mathcal{K}^3_e} M(K) \le M(K_0) = P(K_0) = \min_{K \in \mathcal{K}^3_e} P(K).$$

Thus the infimum of M is attained at K_0, and (2.6) follows. □

The next ingredient concerns parallel sections. Recall the functional $R_K(u)$ defined in (1.4).

Proposition 2.2. *[13, Corollary 5.5] For every $K \in \mathcal{K}^3_e$ and $e \in \mathbb{S}^2$,*

$$\frac{h_{\Pi^2 K}(e)}{h_K(e) V_3(K)} \ge 4 R_K(e) \tag{2.9}$$

and hence $M(K) \ge 4R_K(e)$.

Proof. Put

$$h = h_K(e), \qquad A_e(t) = V_2\big(K \cap (te + e^\perp)\big), \qquad -h \le t \le h.$$

Let $\mathcal{S}_e K$ be the Schwarz symmetral of K about the line $\mathbb{R}e$. It has the same section areas, volume, and support value in direction e. The fixed-direction Steiner-symmetrization estimate of Saroglou [13, Lemmas 5.2–5.3] may be iterated along a sequence of symmetrizations generating $\mathcal{S}_e K$. Passing to the Schwarz symmetral by Hausdorff convergence and continuity of the projection body operator, we have

$$h_{\Pi^2(\mathcal{S}_e K)}(e) \le h_{\Pi^2 K}(e). \tag{2.10}$$

For completeness, we compute the remaining normalization. Put

$$J_e = \int_{-h}^{h} A_e(t)^{1/2}\, dt.$$

For every $v \perp e$, the planar projection of the body of revolution $\mathcal{S}_e K$ has area

$$h_{\Pi(\mathcal{S}_e K)}(v) = \frac{2}{\sqrt{\pi}} J_e.$$

Thus $(\Pi(\mathcal{S}_e K))|e^\perp$ is the disk of radius $2J_e/\sqrt{\pi}$, and

$$h_{\Pi^2(\mathcal{S}_e K)}(e) = V_2((\Pi(\mathcal{S}_e K))|e^\perp) = \pi\left(\frac{2J_e}{\sqrt{\pi}}\right)^2 = 4J_e^2.$$

Combining this identity with (2.10) and dividing by $h_K(e)V_3(K)$, we have (2.9). $\square$

By a nonsingular linear transformation, an origin-symmetric body may be placed in inertia-isotropic position,

$$\int_K x \otimes x\, dx = \tau I_3 \tag{2.11}$$

for some $\tau > 0$. Let

$$r = \min_{v\in\mathbb{S}^2} h_K(v) \tag{2.12}$$

and choose $e \in \mathbb{S}^2$ with $h_K(e) = r$. Since K is origin-symmetric,

$$rB_2^3 \subset K \subset \{x \in \mathbb{R}^3 : |x\cdot e| \le r\}. \tag{2.13}$$

For $-r \le t \le r$, put

$$A(t) = V_2(K \cap (te + e^\perp))$$

and define

$$q(s) = \sqrt{\frac{A(rs)}{\pi r^2}}, \qquad -1 \le s \le 1. \tag{2.14}$$

By Brunn's concavity principle (see Schneider [16, Theorem 7.1.1]), $A^{1/2}$ is concave. Hence q is even, nonnegative and concave. In particular, it is non-increasing on $[0,1]$. The first inclusion in (2.13) yields

$$q(s) \ge \sqrt{1-s^2}, \qquad -1 \le s \le 1. \tag{2.15}$$

The second condition follows from the isotropic normalization. Let

$$C_t = \big(K \cap (te + e^\perp)\big) - te \subset e^\perp.$$

Among planar measurable sets of area $A(t)$, the disk centered at the origin has the least polar moment. Therefore

$$\int_{C_t} |z|^2\, dz \ge \frac{A(t)^2}{2\pi}. \tag{2.16}$$

Indeed, let D_t be the centered disk with $V_2(D_t) = A(t)$. The sets $C_t \setminus D_t$ and $D_t \setminus C_t$ have the same area, while $|z|$ is no smaller on the first set than on the second. This proves (2.16).

On the other hand, (2.11) implies

$$\begin{aligned}\int_{-r}^{r}\int_{C_t} |z|^2\, dz\, dt &= 2\int_K (x\cdot e)^2\, dx \\ &= 2\int_{-r}^{r} t^2 A(t)\, dt. \end{aligned}\tag{2.17}$$

After integration of (2.16) and comparison with (2.17),

$$\int_{-r}^{r} A(t)^2\, dt \le 4\pi \int_{-r}^{r} t^2 A(t)\, dt. \tag{2.18}$$

Substitution of (2.14) into (2.18) yields

$$\int_{-1}^{1} q(s)^4\, ds \le 4\int_{-1}^{1} s^2 q(s)^2\, ds.$$

Since q is even,

$$\int_0^1 \left(q(s)^4 - 4s^2 q(s)^2\right) ds \le 0. \tag{2.19}$$

3. A sharp one-dimensional inequality

Let $q : [0,1] \to [0,\infty)$ be bounded and non-increasing, and suppose that it satisfies (2.15) on $[0,1]$ and (2.19). These are precisely the properties of the section function obtained in Section 2 that will be used below. We shall prove

$$\left(\int_0^1 q(t)\,dt\right)^2 \ge \frac{3\pi^2}{32}\int_0^1 q(t)^2\,dt. \tag{3.1}$$

Equality holds if and only if $q(t) = \sqrt{1-t^2}$ almost everywhere.

It is enough to establish the stronger estimate

$$\left(\int_0^1 q(t)\,dt\right)^2 - \frac{3\pi^2}{32}\int_0^1 q(t)^2\,dt + \frac{\pi}{32}\int_0^1 \left(q(t)^4 - 4t^2 q(t)^2\right) dt \ge 0. \tag{3.2}$$

Let $\alpha(s) = |\{t \in [0,1] : q(t) > s\}|$ for $s \ge 0$, and let $\beta(s) = \sqrt{(1-s^2)_+}$. Since q is nonincreasing, function α is also nonincreasing and we have

$$1 \ge \alpha(s) \ge \beta(s). \tag{3.3}$$

By the layer cake formula and Fubini we have

$$\int_0^1 q(t)\,dt = \int_0^\infty \alpha(s)\,ds, \tag{3.4}$$

$$\int_0^1 t^2 q(t)^2\,dt = \frac{2}{3}\int_0^\infty s\alpha(s)^3\,ds. \tag{3.5}$$

The last identity follows from

$$\int_0^1 t^2 q(t)^2\,dt = \int_0^\infty 2s \int_0^{\alpha(s)} t^2\,dt\,ds.$$

Define $\Psi(s,z) = \left(-\frac{3\pi^2}{16}s + \frac{\pi}{8}s^3\right)z - \frac{\pi}{12}sz^3$ and

$$H(\alpha) = \left(\int_0^\infty \alpha(s)\,ds\right)^2 + \int_0^\infty \Psi(s,\alpha(s))\,ds. \tag{3.6}$$

Equations (3.4)–(3.5) show that $H(\alpha)$ is the left hand side of (3.2). It is straightforward to check that $H(\beta) = 0$. Moreover,

$$\frac{\partial^2 \Psi}{\partial z^2}(s,z) = -\frac{\pi}{2}sz \le 0. \tag{3.7}$$

It follows that H is concave on every convex family on which $\int \alpha$ is fixed.

We first consider non-increasing step functions. Fix a partition $0 = t_0 < t_1 < \cdots < t_N = T$ containing 1. Let α be the step function having value x_i on $[t_{i-1}, t_i)$ and vanish on $[T,\infty)$. The admissible vectors satisfy

$$1 \ge x_1 \ge \cdots \ge x_N \ge 0, \qquad x_i \ge \beta(t_{i-1}). \tag{3.8}$$

We call a step function admissible if its value vector satisfies (3.8). After imposing

$$\int_0^\infty \alpha = \sum_{i=1}^N (t_i - t_{i-1})x_i = m, \tag{3.9}$$

collecting all vectors $(x_1, \cdots, x_N)$ satisfying this constraint and (3.8) we obtain a compact polytope. By (3.7), the minimum of H on this polytope is attained at an extreme point.

A block is a maximal consecutive set of intervals on which α is constant. If a block begins at t_{i-1}, we call it a contact block when its value equals $\beta(t_{i-1})$. The initial block has value 1 and is also regarded as a contact block.

Proposition 3.1. *An extreme point of the polytope determined by* (3.8) *and* (3.9) *has at most one noncontact block. If such a block occurs, the block immediately to its left is a contact block.*

Proof. Let the maximal constant blocks have lengths $\ell_1, \dots, \ell_k > 0$ and distinct values $c_1 > \cdots > c_k$. Suppose that two blocks, indexed by i and j, are noncontact blocks. Their values have positive distance from the neighboring block values and from the obstacle at their left boundary points. For sufficiently small $\varepsilon > 0$, both perturbations

$$c_i \mapsto c_i + \frac{\varepsilon}{\ell_i}, \qquad c_j \mapsto c_j - \frac{\varepsilon}{\ell_j}$$

and

$$c_i \mapsto c_i - \frac{\varepsilon}{\ell_i}, \qquad c_j \mapsto c_j + \frac{\varepsilon}{\ell_j}$$

preserve (3.8) and (3.9). Hence, the original vector is the midpoint of two distinct admissible vectors, contrary to extremality, which implies that at most one block is noncontact. Since $x_1 \geq \beta(0) = 1$ and $x_1 \leq 1$, the initial block is a contact block. Every block apart from the possible noncontact block is therefore a contact block, which proves the second assertion. $\square$

To estimate these blocks, write

$$\delta(s) = \alpha(s) - \beta(s) \geq 0.$$

Since $\mathscr{H}(\beta) = 0$ and

$$\int_0^\infty \beta(s)\, ds = \frac{\pi}{4},$$

subtracting $\mathscr{H}(\beta)$ from $\mathscr{H}(\alpha)$, we obtain

$$H(\alpha) = \left(\int_0^\infty \delta(s)\, ds\right)^2 + \int_0^\infty W(s)\delta(s)\, ds - \frac{\pi}{12}\int_0^\infty s\delta(s)^2\big(\alpha(s) + 2\beta(s)\big)\, ds, \tag{3.10}$$

where $W(s) = \frac{\pi}{2} - \frac{3\pi^2}{16}s + \frac{\pi}{8}s^3 - \frac{\pi}{4}s\beta(s)^2$.

Let $I := [\bar{a}, \bar{b}] \subset [0,1]$ be a constant block on which the height $\alpha(s) = c$. Set $z = \beta(s)$, so that

$$s = \sqrt{1 - z^2}, \qquad ds = -\frac{z}{\sqrt{1-z^2}}\, dz.$$

Suppose that $\beta(\bar{a}) = \rho$ and $\beta(\bar{b}) = \sigma$, where $0 \leq \sigma \leq \rho \leq c$. Since $H(\alpha)$ consists of a global positive square, a positive linear term, and a nonlinear negative term. we define

$$D(c;\rho,\sigma) = \int_\sigma^\rho \frac{z(c-z)}{\sqrt{1-z^2}}\, dz, \tag{3.11}$$

$$L(c;\rho,\sigma) = \int_\sigma^\rho z(c-z)\left(1 + \frac{z^2}{2} + \frac{3z^4}{8}\right) dz,$$

$$J(c;\rho,\sigma) = \int_\sigma^\rho z(c-z)^2(c+2z)\, dz. \tag{3.12}$$

Since

$$\frac{1}{\sqrt{1-z^2}} \geq 1 + \frac{z^2}{2} + \frac{3z^4}{8}, \qquad 0 \leq z < 1,$$

we have

$$D(c;\rho,\sigma) \geq L(c;\rho,\sigma) \geq 0. \tag{3.13}$$

A direct computation shows

$$\int_I \delta(s)\ ds = D(c;\rho,\sigma).$$

Hence D is exactly the excess mass of the block above the obstacle β.

The same change of variables gives

$$\int_I s\delta(s)^2(\alpha(s) + 2\beta(s))\ ds = \int_\sigma^\rho z(c-z)^2(c+2z)\ dz = J(c;\rho,\sigma).$$

Thus J is exactly the contribution of this block to the nonlinear term appearing in H with the negative coefficient $-\pi/12$.

The first two terms in (3.10) are favorable, but the last one is negative. The constant that will separate these two effects is $\frac{\pi^2}{192}$, which is dictated by the identity

$$D^2 + \frac{\pi^2}{192}D - \frac{\pi\sqrt{3}}{12}D^{3/2} = D\left(\sqrt{D} - \frac{\pi\sqrt{3}}{24}\right)^2. \tag{3.14}$$

Note that this is also the reason that the coefficient choosen in (3.2) is $\pi/32$. Hence, the proof on the support of the obstacle has two steps. We first show that W is uniformly larger than $\frac{\pi^2}{192}$. We then prove $J \leq \sqrt{3}\, L^{3/2} \leq \sqrt{3}\, D^{3/2}$.

Lemma 3.2. *For $0 \leq s \leq 1$, we have*

$$W(s) > \frac{\pi^2}{192}. \tag{3.15}$$

Proof. Since $\beta(s)^2 = 1 - s^2$ for $0 \leq s \leq 1$, then we have

$$W(s) = \frac{\pi}{2} - \left(\frac{3\pi^2}{16} + \frac{\pi}{4}\right)s + \frac{3\pi}{8}s^3. \tag{3.16}$$

It is easy to see that the minimum of (3.16) occurs at the point s_0 with $s_0^2 = \frac{\pi}{6} + \frac{2}{9}$. By a straightforward computation, we have the desired estimate. $\square$

Fix $\varepsilon = \min_{0\leq s\leq 1}\left(W(s) - \frac{\pi^2}{192}\right) > 0$, and we have

$$W(s) \geq \frac{\pi^2}{192} + \varepsilon \qquad \text{for } 0 \leq s \leq 1.$$

Then the contribution of the linear term on this block satisfies

$$\int_I W(s)\delta(s)\ ds \geq \left(\frac{\pi^2}{192} + \varepsilon\right) D.$$

The first term of H is global and couples the different blocks. Here and below, a group consists either of a single contact block or, when a noncontact block is present, of that block together with the contact block immediately to its left. Assume for the moment that α is supported in $[0,1]$. If the blocks are divided into groups and D_i denotes the excess mass of the i-th group, then

$$\left(\int_0^\infty \delta(s)\ ds\right)^2 = \left(\sum_i D_i\right)^2 = \sum_i D_i^2 + 2\sum_{i<j} D_i D_j.$$

We assign D_i^2 to the i-th group, while the cross terms are nonnegative and may be discarded. Consequently, if D and J denote the corresponding sums over one group, its assigned contribution to H is bounded from below by

$$D^2 + \left(\frac{\pi^2}{192} + \varepsilon \right) D - \frac{\pi}{12} J.$$

It remains to control the negative quantity J in terms of D. The square root factor in the definition of D is inconvenient for this purpose. We therefore introduce the polynomial lower bound

$$L(c; \rho, \sigma) = \int_\sigma^\rho z(c-z) \left(1 + \frac{z^2}{2} + \frac{3z^4}{8} \right) \, dz,$$

for which

$$L \leq D.$$

The choice of L allows us later to prove

$$J \leq \sqrt{3}\, L^{3/2} \leq \sqrt{3}\, D^{3/2}.$$

We need two elementary scalar inequalities.

Lemma 3.3. *For $0 \leq u \leq 1$, set*

$$m(u) = \frac{3(4u^3 + 3u^2 + 2u + 1)}{10(2u+1)}, \qquad \gamma(u) = \frac{27(8u^2 + 9u + 3)^2}{50(2u+1)^3}. \tag{3.17}$$

Then

$$\gamma(u) \leq 9\sqrt{m(u)}. \tag{3.18}$$

Moreover, for every $r \geq 0$,

$$\left(1 + \frac{r}{2} + \frac{3r^2}{8} \right)^3 \geq 3\sqrt{r}. \tag{3.19}$$

The proof of this lemma is elementary and is deferred to the appendix.

Lemma 3.4. *If $0 \leq a \leq p \leq 1$, then*

$$J(p; p, a) \leq \sqrt{3}\, L(p; p, a)^{3/2}. \tag{3.20}$$

Proof. The assertion is immediate when $a = p$. Assume $a < p$ and set $u = a/p$. Write

$$L_0 = \int_a^p z(p-z)\, dz$$

and introduce the probability measure

$$d\mu_u(t) = \frac{t(1-t)\, dt}{\displaystyle\int_u^1 t(1-t)\, dt}, \qquad u \leq t \leq 1.$$

With $\varphi(r) = 1 + r/2 + 3r^2/8$, the substitution $z = pt$ yields

$$\frac{L(p; p, a)}{L_0} = \int_u^1 \varphi(p^2 t^2)\, d\mu_u(t) \geq \varphi(p^2 m(u)), \tag{3.21}$$

where Jensen's inequality was used and $m(u)$ is from (3.17). By a direct computation, we have

$$\frac{J(p; p, a)^2}{L_0^3} = p\gamma(u). \tag{3.22}$$

Combining (3.21), (3.22) and Lemma 3.3, we obtain

$$\frac{J(p;p,a)^2}{L(p;p,a)^3} \le \frac{p\gamma(u)}{\varphi(p^2m(u))^3} \le \frac{9p\sqrt{m(u)}}{3\sqrt{p^2m(u)}} = 3.$$

This is (3.20). □

We next consider the only configuration not covered by Lemma 3.4. Let a contact block have height p and end where the obstacle equals a. Let the adjacent noncontact block have height c and end where the obstacle equals e. Thus

$$0 \le e \le a \le c \le p \le 1. \tag{3.23}$$

For this pair, put

$$\begin{aligned} L(c) &= L(p;p,a) + L(c;a,e), \\ J(c) &= J(p;p,a) + J(c;a,e). \end{aligned} \tag{3.24}$$

The boundary values $c = a$ and $c = p$ are contact configurations. This observation leads to the following interpolation.

Proposition 3.5. *Under* (3.23),

$$J(c) \le \sqrt{3}\, L(c)^{3/2}. \tag{3.25}$$

Proof. For fixed p, a, e, the function $L(c)$ is affine. Differentiation under the integral sign in (3.24) yields

$$\begin{aligned} J'(c) &= 3\int_e^a z(c^2 - z^2)\, dz, \\ J''(c) &= 6c\int_e^a z\, dz. \end{aligned} \tag{3.26}$$

The pointwise inequality

$$(c^2 - z^2)^2 \le 2c(c-z)^2(c+2z), \qquad 0 \le z \le c, \tag{3.27}$$

follows after division by $(c-z)^2$ from $(c+z)^2 \le 2c(c+2z)$. By Cauchy–Schwarz, (3.27) and (3.26) we have

$$\begin{aligned} J'(c)^2 &\le 9J(c)\int_e^a \frac{z(c^2-z^2)^2}{(c-z)^2(c+2z)}\, dz \\ &\le 18cJ(c)\int_e^a z\, dz = 3J(c)J''(c). \end{aligned}$$

Consequently,

$$\frac{d^2}{dc^2} J(c)^{2/3} = \frac{2}{9} J(c)^{-4/3}\left(3J(c)J''(c) - J'(c)^2\right) \ge 0$$

whenever $J(c) > 0$. The degenerate cases follow by continuity. Thus $J(c)^{2/3}$ is convex on $[a, p]$.

When $c = a$, both blocks are contact blocks. Lemma 3.4 and the inequality $x^{3/2} + y^{3/2} \le (x+y)^{3/2}$ show that

$$J(a) \le \sqrt{3}\, L(a)^{3/2}.$$

When $c = p$, the two blocks merge into one contact block, so the same estimate follows directly from Lemma 3.4. Since $J(c)^{2/3}$ is convex and $L(c)$ is affine, interpolation between a and p yields

$$J(c)^{2/3} \le 3^{1/3} L(c).$$

This is equivalent to (3.25). □

We now return to (3.10). Consider either one contact block or a contact block paired with the noncontact block immediately to its right. Let D, L and J be the corresponding sums of (3.11) and (3.12). The contribution assigned to this group, including the square of its excess mass, satisfies

$$\begin{aligned} & D^2 + \left(\frac{\pi^2}{192} + \varepsilon\right) D - \frac{\pi}{12} J \\ & \geq D^2 + \left(\frac{\pi^2}{192} + \varepsilon\right) D - \frac{\pi\sqrt{3}}{12} D^{3/2} \\ & = D\left(\sqrt{D} - \frac{\pi\sqrt{3}}{24}\right)^2 + \varepsilon D. \end{aligned} \tag{3.28}$$

Here we used (3.13), Lemma 3.4 and Proposition 3.5. Thus every group contained in $[0, 1]$ contributes at least εD.

It remains to remove portions of the last positive block lying to the right of the obstacle support. The following estimate is analytic and uniform in the height of the block.

Lemma 3.6. *Suppose that the last positive block $[\bar{a}, y)$ of a non-increasing admissible step function has height $c \in (0, 1]$ and right boundary point $y \geq 1$. Increasing y increases H at a rate at least*

$$c\left(\frac{\pi}{128} - \frac{1}{45}\right) > 0. \tag{3.29}$$

Proof. Set $x = \sqrt{1 - c^2}$. Monotonicity and the obstacle condition imply

$$\int_0^y \alpha(s)\, ds \geq \int_0^x \sqrt{1 - s^2}\, ds + c(y - x). \tag{3.30}$$

Keeping all preceding blocks fixed, for each $y \geq x$ define

$$\alpha_y(s) := \begin{cases} \alpha(s), & 0 \leq s < \bar{a}, \\ c, & \bar{a} \leq s < y, \\ 0, & s \geq y. \end{cases}$$

Thus the last positive block of α_y is $[x, y)$ and has height c. Differentiating (3.6) and using (3.30), we find

$$\frac{d}{dy} H(\alpha_y) \geq c\Theta(c, y), \tag{3.31}$$

where

$$\begin{aligned} \Theta(c, y) = \frac{\pi}{8} y^3 + \left(2c - \frac{3\pi^2}{16} - \frac{\pi}{12} c^2\right) y \\ + \arccos c - c\sqrt{1 - c^2}. \end{aligned}$$

Write

$$\Theta(c, y) = \Theta_0(y) + \left(2c - \frac{\pi}{12} c^2\right)(y - 1) + E(c), \tag{3.32}$$

where

$$\Theta_0(y) = \frac{\pi}{2} - \frac{3\pi^2}{16}y + \frac{\pi}{8}y^3, \tag{3.33}$$

$$E(c) = 2c - \frac{\pi}{12}c^2 - \arcsin c - c\sqrt{1-c^2}.$$

The middle term in (3.32) is nonnegative.

It is easy to show that

$$\Theta_0(y) > \frac{\pi}{128}, \qquad y \geq 1. \tag{3.34}$$

We next estimate E. Put $t = \pi/12$. The only positive critical point of E is

$$c_* = \frac{2t}{1+t^2}.$$

Indeed, it follows from $E'(c) = 2(1-\sqrt{1-c^2}) - 2tc$. Observe that the derivative changes from negative to positive there. A direct computation shows that

$$\min_{0\leq c\leq 1} E(c) = \frac{2t}{1+t^2} - 2\arctan t > -\frac{1}{45}. \tag{3.35}$$

Combining (3.32), (3.34) and (3.35),

$$\Theta(c,y) > \frac{\pi}{128} - \frac{1}{45} > 0.$$

In view of (3.31), this proves (3.29). □

We can now complete the proof of the one-dimensional estimate. Starting from the rightmost positive block, move its right endpoint to the left until it reaches 1, deleting the whole block if it lies entirely to the right of 1, and then continue with the preceding block. In this way we obtain the truncation $\bar{\alpha} = \alpha\,\mathbf{1}_{[0,1]}$.

Let $G_1, \ldots, G_m$ be the groups contained in $[0,1]$, where each group consists either of a single contact block or of a contact block together with the adjacent noncontact block to its right. Denote by

$$D_i = \int_{G_i} (\alpha(s) - \beta(s))\ ds$$

the excess mass of the i-th group. By (3.28), the contribution of G_i is at least εD_i. Moreover, the cross terms in the global square are nonnegative, since

$$\left(\sum_{i=1}^m D_i\right)^2 = \sum_{i=1}^m D_i^2 + 2\sum_{i<j} D_i D_j.$$

It follows that the truncation $\bar{\alpha} = \alpha\mathbf{1}_{[0,1]}$ satisfies

$$H(\bar{\alpha}) \geq \varepsilon \sum_{i=1}^m D_i = \varepsilon \int_0^1 (\alpha(s) - \beta(s))\ ds.$$

On the other hand, Lemma 3.6 gives

$$H(\alpha) \geq H(\bar{\alpha}) + \left(\frac{\pi}{128} - \frac{1}{45}\right)\int_1^\infty \alpha(s)\ ds.$$

Since $\beta(s) = 0$ for $s \geq 1$, combining the preceding two estimates and setting

$$c_0 = \min\left\{\varepsilon, \frac{\pi}{128} - \frac{1}{45}\right\} > 0$$

yields

$$H(\alpha) \geq c_0 \int_0^\infty (\alpha(s) - \beta(s))\ ds. \tag{3.36}$$

Every point of a fixed mass polytope is a convex combination of its extreme points. Since H is concave on that polytope and the right hand side of (3.36) depends only on the fixed mass, the same estimate holds for every admissible step function.

Finally, let α satisfy (3.3). Choose $T > 0$ such that $\alpha = 0$ on $[T, \infty)$. Let

$$0 = t_0^{(k)} < t_1^{(k)} < \cdots < t_{N_k}^{(k)} = T$$

be a sequence of increasingly fine partitions of $[0, T]$, each containing 1, such that the largest length of their subintervals tends to zero. Define

$$\alpha_k(s) = \alpha(t_{i-1}^{(k)}) \qquad \text{for } s \in [t_{i-1}^{(k)}, t_i^{(k)}),$$

and set $\alpha_k = 0$ on $[T, \infty)$.

Since α is non-increasing,

$$\alpha_k(s) \geq \alpha(s)$$

for every s. Hence $\alpha_k \geq \beta$, and each α_k is admissible. Moreover, a monotone function is continuous except at countably many points, so

$$\alpha_k(s) \longrightarrow \alpha(s) \qquad \text{for almost every } s.$$

Finally, $0 \leq \alpha_k \leq 1$, and all the functions α_k vanish outside the fixed interval $[0, T]$. Therefore, dominated convergence applies to the terms defining H. Hence,

$$H(\alpha) \geq c_0 \int_0^\infty (\alpha(s) - \beta(s))\, ds \geq 0. \tag{3.37}$$

This proves (3.2).

The layer cake formula also shows that

$$\int_0^\infty (\alpha - \beta)\, ds = \int_0^1 \left(q(t) - \sqrt{1 - t^2}\right)\ dt. \tag{3.38}$$

By (2.19),

$$\left(\int_0^1 q\right)^2 - \frac{3\pi^2}{32} \int_0^1 q^2 = H(\alpha) - \frac{\pi}{32} \int_0^1 (q^4 - 4t^2 q^2)\, dt \geq 0.$$

This proves (3.1). If equality holds, both nonnegative terms on the right vanish. Hence $H(\alpha) = 0$, and (3.37) and (3.38) imply $q(t) = \sqrt{1 - t^2}$ almost everywhere. The converse is immediate. This completes the proof of the required one-dimensional estimate and its equality statement.

4. Proof of the main theorem

In this section, we prove Theorem 1.1. We apply the one-dimensional result to the section function constructed in Section 2 and then trace all equality conditions. Since M is invariant under nonsingular linear maps, we may assume (2.11) holds. Choose the minimum support direction e as in (2.12). Denote

$$A_0 = \int_0^1 q(s)\, ds, \qquad B_0 = \int_0^1 q(s)^2\, ds.$$

The one-dimensional estimate (3.1) gives

$$A_0^2 \geq \frac{3\pi^2}{32} B_0. \tag{4.1}$$

Since q is even,

$$\frac{\left(\int_{-1}^{1} q(s)\,ds\right)^2}{\int_{-1}^{1} q(s)^2\,ds} = \frac{2A_0^2}{B_0} \geq \frac{3\pi^2}{16}. \tag{4.2}$$

The substitutions $t = rs$ and $A(t) = \pi r^2 q(s)^2$ show that

$$V_3(K) = \pi r^3 \int_{-1}^{1} q(s)^2\,ds,$$

$$\int_{-r}^{r} A(t)^{1/2}\,dt = \sqrt{\pi}\,r^2 \int_{-1}^{1} q(s)\,ds.$$

Consequently,

$$R_K(e) = \frac{\left(\int_{-1}^{1} q(s)\,ds\right)^2}{\int_{-1}^{1} q(s)^2\,ds}. \tag{4.3}$$

Combining (2.9), (4.2), and (4.3), we have

$$M(K) \geq 4R_K(e) \geq \frac{3\pi^2}{4}.$$

Hence

$$\min_{K\in\mathcal{K}_e^3} P(K) = \min_{K\in\mathcal{K}_e^3} M(K) \geq \frac{3\pi^2}{4}$$

by the reduction (2.6). Consequently,

$$P(K) \geq \frac{3\pi^2}{4}, \qquad K \in \mathcal{K}_e^3. \tag{4.4}$$

For an arbitrary convex body, apply (4.4) to its Blaschke body. Equations (2.3) and (2.4) imply

$$P(K) \geq P(\nabla K) \geq \frac{3\pi^2}{4}.$$

Since $\Pi B_2^3 = \pi B_2^3$,

$$P(B_2^3) = \frac{3\pi^2}{4}.$$

Affine invariance shows that every ellipsoid attains equality.

Finally, we characterize the equality case. By (2.7) and (2.8), we have

$$P(\Pi K) \leq P(K).$$

By the equality condition in Minkowski's inequality [16, Theorem 7.6.8], equality holds precisely when $\Pi^2 K$ and K are homothetic.

Suppose first that $K \in \mathcal{K}_e^3$ and $P(K) = 3\pi^2/4$. Then,

$$\frac{3\pi^2}{4} \leq P(\Pi K) \leq P(K) = \frac{3\pi^2}{4}.$$

Hence

$$\Pi^2 K = \lambda K \tag{4.5}$$

for some $\lambda > 0$. Equation (2.5) shows that relation (4.5) is preserved by nonsingular linear transformations, up to changing λ. We may therefore place K in the inertia position as before.

Equation (2.7) now implies

$$\frac{h_{\Pi^2 K}(v)}{h_K(v)V_3(K)} = \frac{\lambda}{V_3(K)} = P(K), \qquad v \in \mathbb{S}^2.$$

For the minimum support direction e, by (2.9), (4.3) and (4.2) we have

$$\frac{3\pi^2}{4} = \frac{h_{\Pi^2 K}(e)}{h_K(e)V_3(K)} \geq 4R_K(e) = \frac{8A_0^2}{B_0} \geq \frac{3\pi^2}{4}.$$

Thus equality holds in (4.1). By the equality statement proved in Section 3, we have

$$q(s) = \sqrt{1-s^2}$$

almost everywhere on $[-1,1]$. Therefore

$$V_3(K) = \pi r^3 \int_{-1}^{1} q(s)^2\, ds = \frac{4\pi r^3}{3} = V_3(rB_2^3).$$

Combining this identity with the inclusion $rB_2^3 \subset K$ from (2.13) yields $K = rB_2^3$. It follows that every origin-symmetric body attaining the equality is an ellipsoid.

Finally, let K be any convex body attaining the equality. Then

$$\frac{3\pi^2}{4} = P(K) \geq P(\nabla K) \geq \frac{3\pi^2}{4}.$$

Equality in the maximal volume property of the Blaschke body implies that K is centrally symmetric up to translation [3, Theorems 4.1.3 and 3.3.9]. The symmetric classification applies after translation. This completes the proof of Theorem 1.1.

5. The sharp upper projection inequality

We now prove Theorem 1.2. For compact convex sets in $\mathbb{R}^m$, let $V_m(L_1,\dots,L_m)$ denote their mixed volume and write $L[r]$ for r repeated arguments. We allow the sets to lie in proper subspaces. We use the following elementary factorization. If $A_1,\dots,A_p$ lie in the first coordinate subspace of $\mathbb{R}^p \oplus \mathbb{R}^q$ and pr_2 is the projection onto the second one, then

$$\binom{p+q}{p} V_{p+q}(A_1,\dots,A_p,B_1,\dots,B_q) = V_p(A_1,\dots,A_p)V_q(\mathrm{pr}_2 B_1,\dots,\mathrm{pr}_2 B_q). \tag{5.1}$$

Indeed, this follows by comparing the coefficient of $s_1\cdots s_p t_1\cdots t_q$ in the corresponding volume polynomial.

For $u_1,u_2,u_3 \in \mathbb{S}^2$, let

$$\Gamma(u_1,u_2,u_3) = \frac{\mathcal{H}^2(\mathrm{pos}\{u_1,u_2,u_3\}\cap \mathbb{S}^2)}{4\pi}$$

be the *normalized solid angle* of their positive hull. Schneider's mixed translative functional in the present case is defined by

$$\mathcal{B}(K_1,K_2,K_3) := \int_{(\mathbb{S}^2)^3} \Gamma(u_1,u_2,u_3)|\det(u_1,u_2,u_3)| \prod_{i=1}^{3} dS_2(K_i,u_i). \tag{5.2}$$

It is symmetric in its arguments and is unchanged when all three bodies are reflected. In $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$, put

$$\Delta_2 K = \{(x,x) : x \in K\}, \qquad \iota_1 K = \{(x,0) : x \in K\}, \qquad \iota_2 K = \{(0,x) : x \in K\}.$$

Schneider's mixed-volume representation [17, Equations (18) and (30)] gives the following.

Lemma 5.1. *There holds*

$$\mathcal{B}(K_1, K_2, K_3) = 90V_6\big(\Delta_2 K_1[2], \iota_1(-K_2)[2], \iota_2(-K_3)[2]\big). \tag{5.3}$$

Proof. Note that, in Schneider's notation, the kernel γ and the functional β are precisely our Γ and $\mathcal{B}$, respectively. Moreover, the mixed functional $\mathcal{B}$ is exactly $V^{(0)}_{2,2,2}$ defined in [17, Equation (18)].

Apply [17, Equation (18)] with $n = k = 3$, and set

$$I(\lambda_1, \lambda_2, \lambda_3) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \chi\big(\lambda_1 K_1 \cap (\lambda_2 K_2 + x_2) \cap (\lambda_3 K_3 + x_3)\big)\, dx_2\, dx_3.$$

The set of (x_2, x_3) for which the intersection is nonempty is

$$\lambda_1 \Delta_2 K_1 + \lambda_2 \iota_1(-K_2) + \lambda_3 \iota_2(-K_3).$$

Hence

$$I(\lambda_1, \lambda_2, \lambda_3) = V_6\big(\lambda_1 \Delta_2 K_1 + \lambda_2 \iota_1(-K_2) + \lambda_3 \iota_2(-K_3)\big).$$

Schneider's expansion shows that the coefficient of $\lambda_1^2\lambda_2^2\lambda_3^2$ on the left is $V^{(0)}_{2,2,2}(K_1, K_2, K_3)$. On the right, the mixed-volume expansion gives

$$\frac{6!}{2!\,2!\,2!} V_6\big(\Delta_2 K_1[2], \iota_1(-K_2)[2], \iota_2(-K_3)[2]\big).$$

Together with [17, Equation (30)], this yields (5.3). □

Fix $K \subset \mathbb{R}^3$ and set

$$b = \mathcal{B}(K, K, K), \qquad c = \mathcal{B}(K, K, -K), \qquad v = V_3(K).$$

Cauchy's projection formula and the volume formula for zonoids yield (see e.g. [16, Equation (5.82)])

$$V_3(\Pi K) = \frac{1}{6}\int_{(\mathbb{S}^2)^3} |\det(u_1, u_2, u_3)| \prod_{i=1}^{3} dS_2(K, u_i). \tag{5.4}$$

For every linearly independent triple, the eight cones generated by $\varepsilon_1 u_1, \varepsilon_2 u_2, \varepsilon_3 u_3$, with $\varepsilon_i \in \{-1, 1\}$, partition $\mathbb{R}^3$ up to their boundaries. Thus their normalized solid angles, namely the eight values of Γ, sum to one. Since the determinant vanishes on dependent triples and $S_2(-K, \omega) = S_2(K, -\omega)$, summing (5.2) over the eight sign choices gives

$$\int_{(\mathbb{S}^2)^3} |\det(u_1, u_2, u_3)| \prod_{i=1}^{3} dS_2(K, u_i) = \sum_{\varepsilon_i \in \{-1,1\}^3} \mathcal{B}(\varepsilon_1 K, \varepsilon_2 K, \varepsilon_3 K) = 2b + 6c.$$

Consequently,

$$V_3(\Pi K) = \frac{b}{3} + c. \tag{5.5}$$

We first estimate c. In $\mathbb{R}^6$ set

$$A = \Delta_2 K, \qquad B = \iota_1(-K), \qquad C = \iota_2 K.$$

For $x \in K$ we have $(0, x) = (x, x) + (-x, 0)$, and hence $C \subset A + B$. By (5.3), monotonicity and multilinearity,

$$\begin{aligned} c &= 90V_6(A[2], B[2], C[2]) \\ &\le 90V_6(A[2], B[2], (A + B)[2]) = 180V_6(A[3], B[3]). \end{aligned}$$

Here the terms $V_6(A[4], B[2])$ and $V_6(A[2], B[4])$ vanish, because both A and B are three-dimensional.

Moreover, $B \subset \mathbb{R}^3 \times \{0\}$ and $\operatorname{pr}_2 A = K$. Hence (5.1) gives

$$\binom{6}{3} V_6(A[3], B[3]) = V_3(B)V_3(\operatorname{pr}_2 A) = v^2.$$

Consequently,

$$c \le 9v^2. \tag{5.6}$$

It remains to estimate b. In $(\mathbb{R}^3)^3$, let

$$D = \Delta_3 K = \{(x, x, x) : x \in K\},$$

and

$$J_0 = K \times \{0\} \times \{0\}, \qquad J_1 = \{0\} \times K \times \{0\}, \qquad J_2 = \{0\} \times \{0\} \times K.$$

Put $Q = J_0 + J_1 + J_2 = K^3$. The map $\varphi \in \mathrm{SL}(9)$, defined by

$$\varphi(x_0, x_1, x_2) = (x_0, x_1 - x_0, x_2 - x_0),$$

sends D onto the first coordinate subspace

$$E_0 = \mathbb{R}^3 \times \{0\} \times \{0\}.$$

Let

$$\operatorname{pr}_2(y_0, y_1, y_2) = (y_1, y_2)$$

be the projection onto the last two coordinate factors. Then

$$\varphi(D) = K \times \{0\} \times \{0\},$$

and

$$\operatorname{pr}_2 \varphi(J_0) = -\Delta_2 K, \qquad \operatorname{pr}_2 \varphi(J_1) = \iota_1 K, \qquad \operatorname{pr}_2 \varphi(J_2) = \iota_2 K.$$

Since $\det \varphi = 1$, (5.1), (5.3), and simultaneous reflection give

$$\binom{9}{3} V_9(D[3], J_0[2], J_1[2], J_2[2]) = vV_6((-\Delta_2 K)[2], \iota_1 K[2], \iota_2 K[2]) = \frac{vb}{90}.$$

Consequently,

$$V_9(D[3], J_0[2], J_1[2], J_2[2]) = \frac{vb}{90 \cdot \binom{9}{3}}. \tag{5.7}$$

Since $D \subset Q$, monotonicity and multilinearity imply

$$V_9(D[3], J_0[2], J_1[2], J_2[2]) \le V_9(Q[3], J_0[2], J_1[2], J_2[2]). \tag{5.8}$$

Expanding the three copies of $Q = J_0 + J_1 + J_2$, every term containing at least four copies of some J_i vanishes, since J_i is three-dimensional. Hence the only nonzero terms contain exactly three copies of each J_i. There are 6 such terms, and therefore

$$V_9(Q[3], J_0[2], J_1[2], J_2[2]) = 6V_9(J_0[3], J_1[3], J_2[3]).$$

Applying (5.1) twice gives

$$\binom{9}{3}\binom{6}{3} V_9(J_0[3], J_1[3], J_2[3]) = \binom{6}{3} vV_6(J_1[3], J_2[3]) = v^3.$$

Consequently,

$$V_9(Q[3], J_0[2], J_1[2], J_2[2]) = 6V_9(J_0[3], J_1[3], J_2[3]) = \frac{6v^3}{\binom{9}{3}\binom{6}{3}}.$$

Equations (5.7) and (5.8) now give

$$b \le 27v^2. \tag{5.9}$$

Combining (5.5), (5.6), and (5.9), we obtain

$$V_3(\Pi K) = \frac{b}{3} + c \le 9v^2 + 9v^2 = 18V_3(K)^2.$$

Finally, the constant is attained. By affine invariance, it is enough to take the standard tetrahedron

$$T = \operatorname{conv}\{0, e_1, e_2, e_3\}.$$

Then $V_3(T) = 1/6$. By (5.4) and a direct computation, we obtain

$$V_3(\Pi T) = \frac{1}{2}.$$

Thus $P(T) = (1/2)/(1/6)^2 = 18$.

Appendix A. Auxiliary scalar estimates

The following is an elementary proof of Lemma 3.3.

Proof. Since $m(u), \gamma(u) \ge 0$, we may square (3.18). After clearing the positive denominator, the resulting inequality is equivalent to the nonnegativity on $[0,1]$ of the polynomial

$$\begin{aligned} &19712u^8 + 48704u^7 + 44256u^6 + 15808u^5 - 35u^4 \\ &\quad - 1072u^3 + 36u^2 + 84u + 7. \end{aligned}$$

If $0 \le u \le 1/4$, then

$$84u - 1072u^3 \ge 17u, \qquad 7 - 35u^4 > 0,$$

and all remaining terms are nonnegative. If $1/4 \le u \le 1$, then

$$15808u^5 + 44256u^6 \ge \left(\frac{15808}{16} + \frac{44256}{64}\right)u^3 = \frac{3359}{2}u^3,$$

whereas

$$1072u^3 + 35u^4 \le 1107u^3.$$

Again, the displayed polynomial is positive. This proves (3.18).

For the second inequality, put

$$\varphi(r) = 1 + \frac{r}{2} + \frac{3r^2}{8}.$$

A binomial expansion has the form

$$\varphi(r)^6 - 9r = 1 - 6r + 6r^2 + \frac{65}{8}r^3 + \frac{555}{64}r^4 + r^5R(r),$$

where R has nonnegative coefficients. Hence

$$\begin{aligned} \varphi(r)^6 - 9r &\ge 1 - 6r + 6r^2 + 8r^3 + 8r^4 \\ &= 8\left(r^2 + \frac{r}{2} - \frac{1}{4}\right)^2 + 8\left(r - \frac{1}{4}\right)^2 \ge 0. \end{aligned}$$

Since both sides of (3.19) are nonnegative, taking square roots proves that inequality. $\square$

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