

L_p -Integral Affine Surface Area-Variational Capacity Inequalities

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ABSTRACT. For $n \geq 2$ and $1 < p < n$, Ludwig-Xiao-Zhang [16] asked whether the L_p -integral affine surface area of every convex body containing the origin in its interior is bounded above by the ball-normalized multiple of L_p -variational capacity, and whether the analogous analytic inequality holds for L_p -convexified level sets. We answer both questions negatively.

1. INTRODUCTION

The setting of this paper is the n -dimensional Euclidean space $\mathbb{R}^n$. A convex body in $\mathbb{R}^n$ is a compact convex set that has a non-empty interior. The unit sphere in $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1}$. Denote by $\mathcal{K}_o^n$ the set of all convex bodies that contain the origin in their interiors. Let $B^n \subset \mathbb{R}^n$ denote the Euclidean unit ball, and write $\omega_n = V(B^n)$ for its volume. We use $\mathcal{H}^k$ to denote the k -dimensional Hausdorff measure and ∂K the boundary of a convex body K .

A convex body K is uniquely determined by its support function $h_K: \mathbb{R}^n \rightarrow [0, \infty)$, given by

$$h_K(x) = \sup\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $\cdot$ denotes the standard Euclidean inner product on $\mathbb{R}^n$.

For $p \geq 1$, the Minkowski-Firey L_p -sum of two convex bodies K and L , denoted by $K +_p L$, is defined through its support function by

$$h(K +_p L, \cdot)^p = h(K, \cdot)^p + h(L, \cdot)^p.$$

For $K, L \in \mathcal{K}_o^n$, their L_p -mixed volume is defined, in [18], by the following variational formula:

$$V_p(K, L) = \frac{p}{n} \lim_{\varepsilon \rightarrow 0^+} \frac{V(K +_p \varepsilon^{1/p} L) - V(K)}{\varepsilon}.$$

A compact set $M \subset \mathbb{R}^n$ is called a star body with respect to the origin if $[0, x] \subset M$ for every $x \in M$. If, in addition, ∂M is Lipschitz, then M is called a Lipschitz star body. In this case, the unit outer normal $\nu_M(x)$ exists for $\mathcal{H}^{n-1}$ -almost every $x \in \partial M$. The radial function of M is the map $\rho_M: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$, which is given by

$$\rho_M(x) = \max\{\lambda \geq 0 : \lambda x \in M\}, \quad x \neq 0.$$

For a Lipschitz star body M , define its core by

$$D_M = \{tx : t > 0, x \in \partial M, |x \cdot \nu_M(x)| > 0\}.$$

The restriction of the radial function ρ_M to D_M is locally Lipschitz continuous.

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Let $M \subset \mathbb{R}^n$ be a Lipschitz star body and $L \subset \mathbb{R}^n$ a convex body. Following [16, Section 5.2], define

$$V_p(M, L) = \frac{1}{n} \int_{\partial M} h_L(\nu_M(x))^p |x \cdot \nu_M(x)|^{1-p} dx.$$

Although $V_p(M, L)$ is well defined, it may be infinite.

Let $M \subset \mathbb{R}^n$ be a Lipschitz star body. For $p \geq 1$, the L_p -integral affine surface area $\Phi_p(M)$ is defined by (see [16, Section 5.2])

$$\Phi_p(M) = \frac{(n\omega_n)^{\frac{n+p}{n}}}{\beta_{n,p}} \left(\int_{\mathbb{S}^{n-1}} V_p(M, \bar{u})^{-\frac{n}{p}} du \right)^{-\frac{p}{n}},$$

where the normalizing constant $\beta_{n,p}$ is given by

$$\beta_{n,p} = V_p(B^n, \bar{u}) = \frac{1}{2^p n} \int_{\mathbb{S}^{n-1}} |u \cdot v|^p dv$$

with $\bar{u} = [-u/2, u/2]$.

The L_p -integral affine surface area is a projection affine functional. Its case $p = 1$ reduces to the classical integral affine surface area associated with the polar projection body, whose origins lie in Minkowski's theory of projection bodies; see [26, Section 9.1]. The L_p framework of this theory originated with Firey's introduction of L_p -addition [7]. Further, Lutwak [18] developed this framework by establishing the theory of L_p -mixed volumes and posing the L_p -Minkowski problem. Based on these developments, Lutwak-Yang-Zhang [19] introduced the notion of L_p projection bodies and established the corresponding affine isoperimetric inequalities, providing a natural geometric interpretation of the L_p -integral affine surface area.

The functional counterpart of the theory of projection bodies began with Zhang's sharp affine Sobolev inequality [33]. Subsequently, Lutwak-Yang-Zhang [20] extended Zhang's result to the sharp L_p affine Sobolev inequality. Further developments include affine Moser-Trudinger and Morrey-Sobolev inequalities [3], asymmetric affine L_p Sobolev inequalities [10], and the asymmetric affine Pólya-Szegő principle [11]. Ludwig-Xiao-Zhang [16] employed the L_p convexification of level sets to establish sharp convex Lorentz-Sobolev inequalities involving the L_p -integral affine surface area.

Let $E \subset \mathbb{R}^n$ be compact. For $p \geq 1$, the L_p -variational capacity of E , denoted by $C_p(E)$, is defined by

$$C_p(E) := \inf \left\{ \|\nabla f\|_p^p : f \in C_0^\infty(\mathbb{R}^n), f \geq \mathbf{1}_E \right\},$$

where $\mathbf{1}_E$ denotes the indicator function of E , $C_0^\infty(\mathbb{R}^n)$ denotes the set of functions on $\mathbb{R}^n$ that are smooth and have compact support, and $\|\cdot\|_p$ denotes the usual L_p norm for functions on $\mathbb{R}^n$.

The L_p -variational capacity originates in nonlinear potential theory. Although Φ_p and C_p arise from different theories, their normalizations on the unit ball satisfy

$$\Phi_p(B^n) = \gamma_{n,p} C_p(B^n) = n\omega_n$$

for $1 < p < n$, where

$$\gamma_{n,p} = \left(\frac{p-1}{n-p} \right)^{p-1}.$$

For a Lipschitz star body $M \subset \mathbb{R}^n$ and $p \geq 1$, the L_p -surface area $S_p(M)$ of M is defined, in [16], by

$$S_p(M) = nV_p(M, B^n) = \int_{\partial M} |x \cdot \nu_M(x)|^{1-p} dx.$$

Note that for $p > 1$, the L_p -surface area $S_p(M)$ is not finite for all Lipschitz star bodies. The normalization is chosen so that $S_p(B^n) = n\omega_n$.

The following result, established in [16], gives sharp comparisons between the L_p -surface area, the L_p -variational capacity, the L_p -integral affine surface area, and volume.

Theorem. [16, Lemma 1] *If M is a Lipschitz star body in $\mathbb{R}^n$ and $1 \leq p < n$, then*

$$S_p(M) \geq \left(\frac{p-1}{n-p}\right)^{p-1} C_p(M) \geq n\omega_n^{\frac{p}{n}} V(M)^{\frac{n-p}{n}}, \quad (1.1)$$

and

$$S_p(M) \geq \Phi_p(M) \geq n\omega_n^{\frac{p}{n}} V(M)^{\frac{n-p}{n}}. \quad (1.2)$$

Equality in (1.1) holds if M is a ball centered at the origin. Equality in the right-hand inequality of (1.2) holds for $p > 1$ if and only if M is an ellipsoid centered at the origin, and for $p = 1$ if and only if M is an ellipsoid containing the origin.

Taking $p = 1$ in inequality (1.1) yields

$$S_1(M) \geq C_1(M) \geq n\omega_n^{1/n} V(M)^{(n-1)/n}. \quad (1.3)$$

For every compact set $E \subset \mathbb{R}^n$ with rectifiable boundary, inequality (1.3) holds, with equality when E is a closed ball in $\mathbb{R}^n$ (see [22, p. 105]). Moreover, as shown in [16, Section 6.1], every convex body E satisfies

$$S_1(E) = C_1(E).$$

Taking $p = 1$ in inequality (1.2) yields

$$S_1(E) \geq \Phi_1(E) \geq n\omega_n^{1/n} V(E)^{(n-1)/n}. \quad (1.4)$$

If $E \subset \mathbb{R}^n$ has Lipschitz boundary, then equality in the second inequality of (1.4) holds if and only if E is an ellipsoid. For a convex body E , the second inequality in (1.4) coincides with the classical Petty projection inequality; see, for example, [8, 17, 24, 26]. Subsequently, Zhang [33] extended this inequality to compact domains with piecewise C^1 boundary.

The second inequality in (1.1) is due to Maz'ya [21], and its special case $p = 2$ and $n = 3$ was established by Pólya and Szegő [25]. In [25], they also proved the first inequality in (1.1) for $p = 2$ and $n = 3$. For compact convex sets, the second inequality in (1.2) was obtained in [19], and an alternative proof can be found in [2].

For a Lipschitz star body, the first inequality in (1.1) is obtained by choosing suitable test functions; see [16, Section 6]. In contrast, the first inequality in (1.2) follows immediately from Jensen's inequality. The second inequality in (1.2) is proved using the L_p convexification of star bodies, which is defined via the L_p convexification of level sets; see [16, Section 4.2].

The geometric inequalities in the preceding theorem provide a key ingredient in the proof of the sharp convex Lorentz-Sobolev inequalities established by Ludwig-Xiao-Zhang [16]. Subsequent work established several extensions of the convex Lorentz-Sobolev inequalities and introduced relative, general, and higher-order affine capacities; see [23, 6, 15, 32, 27, 12, 28, 31, 30, 13, 34].

Since

$$S_1(K) = C_1(K) \quad \text{for all } K \in \mathcal{K}_o^n,$$

it follows from (1.4) that

$$\Phi_1(K) \leq C_1(K) \quad \text{for all } K \in \mathcal{K}_o^n,$$

and

$$\int_0^\infty \Phi(\langle f \rangle_t) dt \leq \int_0^\infty C(\langle f \rangle_t) dt \quad \text{for all } f \in C_0^\infty(\mathbb{R}^n).$$

Here, $f: U \rightarrow \mathbb{R}$ is locally Lipschitz on an open set $U \subset \mathbb{R}^n$, $t > 0$, and $\langle f \rangle_t$ denotes the L_p convexification of the level set $[f]_t = \{x \in U : |f(x)| \geq t\}$; see Section 2 for its precise definition.

Motivated by these observations, Ludwig-Xiao-Zhang [16] posed the following question.

Question. *Are the geometric inequality*

$$\Phi_p(K) \leq \left(\frac{p-1}{n-p} \right)^{p-1} C_p(K) \quad \text{for all } K \in \mathcal{K}_o^n \quad (1.5)$$

and the analytic inequality

$$\int_0^\infty \Phi_p(\langle f \rangle_t) dt \leq \left(\frac{p-1}{n-p} \right)^{p-1} \int_0^\infty C_p(\langle f \rangle_t) dt \quad \text{for all } f \in C_0^\infty(\mathbb{R}^n) \quad (1.6)$$

true for $1 < p < n$?

Our main result gives a negative answer to the question posed above, showing that both proposed inequalities fail for all admissible values of n and p .

Theorem 1.1. *For every integer $n \geq 2$ and every $1 < p < n$, the geometric inequality (1.5) and the analytic inequality (1.6) both fail. More precisely:*

- (i) *there exists an origin-symmetric convex body $K \in \mathcal{K}_o^n$ with C^∞ boundary and everywhere positive Gauss curvature such that*

$$\Phi_p(K) > \left(\frac{p-1}{n-p} \right)^{p-1} C_p(K);$$

- (ii) *there exists a nonnegative function $f \in C_c^\infty(\mathbb{R}^n)$ such that*

$$\int_0^\infty \Phi_p(\langle f \rangle_t) dt > \left(\frac{p-1}{n-p} \right)^{p-1} \int_0^\infty C_p(\langle f \rangle_t) dt.$$

Part (ii) relies on the geometric construction established in part (i). In fact, a smooth cutoff applied to its Minkowski functional allows the strict geometric inequality to be carried over exactly to the associated L_p -convexified level sets.

Organization of the paper. Section 2 collects the notation and preliminary results used throughout the paper. In Section 3, high-degree even spherical-harmonic perturbations of the Euclidean unit ball are used to construct smooth, origin-symmetric convex bodies with everywhere positive Gauss curvature that do not satisfy the proposed geometric inequality, thereby proving part (i) of Theorem 1.1. Section 4 derives the functional construction from their geometric counterparts by applying a smooth cutoff to the Minkowski functional and using the exact dilation identity for L_p -convexified level sets, thereby proving part (ii). Finally, the appendix presents a concrete counterexample for $(n, p) = (3, 2)$.

2. PRELIMINARIES

In this section, we collect some basic facts and notation. For further background, we refer the reader to the monographs by Schneider [26] and Gardner [8] and the paper [16].

Let $d\sigma$ denote the standard surface measure on $\mathbb{S}^{n-1}$, and write

$$M := \sigma(\mathbb{S}^{n-1}) = |\mathbb{S}^{n-1}| = n\omega_n.$$

For $q \in L^1(\mathbb{S}^{n-1})$, define its normalized spherical average by

$$\langle q \rangle_\sigma := \frac{1}{M} \int_{\mathbb{S}^{n-1}} q \, d\sigma.$$

Let $K \in \mathcal{K}_o^n$. Its Minkowski functional is defined by

$$r_K(x) := \inf\{s > 0 : x \in sK\}, \quad x \in \mathbb{R}^n.$$

For every $u \in \mathbb{S}^{n-1}$, we have

$$r_K(\rho_K(u)u) = 1.$$

Moreover,

$$r_K(x) = \frac{1}{\rho_K(x)}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

For $K, L \in \mathcal{K}_o^n$, Lutwak [18] established the representation

$$V_p(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L(u)^p \, dS_p(K, u),$$

where $S_p(K, \cdot)$ denotes the L_p -surface area measure of K , defined by

$$dS_p(K, u) = h_K(u)^{1-p} \, dS_K(u).$$

Here S_K is the classical surface area measure of K ; see [26]. Equivalently, S_K is the Borel measure on $\mathbb{S}^{n-1}$ characterized by

$$\int_{\mathbb{S}^{n-1}} g(u) \, dS_K(u) = \int_{\partial K} g(\nu_K(x)) \, d\mathcal{H}^{n-1}(x)$$

for every $g \in C(\mathbb{S}^{n-1})$.

Let $K \in \mathcal{K}_o^n$. For each $u \in \mathbb{S}^{n-1}$, denote by $\bar{u} = [-u/2, u/2]$ the centered line segment in the direction u . Since

$$h_{\bar{u}}(v) = \frac{|u \cdot v|}{2}, \quad v \in \mathbb{S}^{n-1},$$

the L_p -mixed volume of K and $\bar{u}$ is given by

$$V_p(K, \bar{u}) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \left(\frac{|u \cdot v|}{2} \right)^p \, dS_p(K, v). \quad (2.1)$$

Let

$$\beta_{n,p} = V_p(B^n, \bar{u}).$$

By rotational invariance, $\beta_{n,p}$ is independent of the choice of u . With the normalization adopted in [16], we have

$$S_p(K) = \frac{1}{\beta_{n,p}} \int_{\mathbb{S}^{n-1}} V_p(K, \bar{u}) \, d\sigma(u), \quad (2.2)$$

$$\Phi_p(K) = \frac{M^{(n+p)/n}}{\beta_{n,p}} \left(\int_{\mathbb{S}^{n-1}} V_p(K, \bar{u})^{-n/p} \, d\sigma(u) \right)^{-p/n}. \quad (2.3)$$

In particular,

$$\Phi_p(B^n) = S_p(B^n) = M.$$

Define the normalized p -cosine transform of an integrable function q on $\mathbb{S}^{n-1}$ by

$$(\mathcal{T}_p q)(u) := \frac{\int_{\mathbb{S}^{n-1}} |u \cdot v|^p q(v) d\sigma(v)}{\int_{\mathbb{S}^{n-1}} |e_1 \cdot v|^p d\sigma(v)}.$$

This normalization ensures that

$$\mathcal{T}_p 1 = 1, \quad \langle \mathcal{T}_p q \rangle_\sigma = \langle q \rangle_\sigma.$$

For a smooth strictly convex body K , let

$$dS_p(K, u) = q_K(u) d\sigma(u)$$

and

$$v_K(u) = \frac{V_p(K, \bar{u})}{\beta_{n,p}}.$$

Then

$$v_K = \mathcal{T}_p q_K, \quad \frac{S_p(K)}{M} = \langle v_K \rangle_\sigma, \quad \frac{\Phi_p(K)}{M} = \left\langle v_K^{-n/p} \right\rangle_\sigma^{-p/n}. \quad (2.4)$$

For $1 \leq p < n$ and a compact set $E \subset \mathbb{R}^n$, define its L_p -variational capacity by

$$C_p(E) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla \varphi|^p dx : \varphi \in C_c^\infty(\mathbb{R}^n), \varphi \geq 1 \text{ on } E \right\}. \quad (2.5)$$

For the unit ball B^n ,

$$C_p(B^n) = \begin{cases} M, & p = 1, \\ M \left(\frac{n-p}{p-1} \right)^{p-1}, & 1 < p < n. \end{cases}$$

The L_p -variational capacity is monotone with respect to set inclusion and invariant under translations. Both Φ_p and C_p are homogeneous of degree $n-p$, and hence

$$\gamma_{n,p} C_p(B^n) = \Phi_p(B^n) = M.$$

Let $f \in C_0^\infty(\mathbb{R}^n)$, and define its superlevel sets by

$$[f]_t = \{x \in \mathbb{R}^n : |f(x)| \geq t\}.$$

For almost every $t > 0$, both t and $-t$ are regular values of f . For such t with $[f]_t \neq \emptyset$, the outer unit normal to $[f]_t$ is

$$\nu_f(x) = -\frac{\nabla |f|(x)}{|\nabla f(x)|} = -\operatorname{sgn}(f(x)) \frac{\nabla f(x)}{|\nabla f(x)|}, \quad x \in \partial[f]_t.$$

The L_p convexification $\langle f \rangle_t$ is defined as the unique origin-symmetric convex body whose L_p -surface area measure satisfies

$$\int_{\mathbb{S}^{n-1}} \phi(u) dS_p(\langle f \rangle_t, u) = \int_{\partial[f]_t} \phi(\nu_f(x)) |\nabla f(x)|^{p-1} d\mathcal{H}^{n-1}(x) \quad (2.6)$$

for every even function $\phi \in C(\mathbb{S}^{n-1})$; see [16, Section 4.1].

3. PROOF OF (i) IN THEOREM 1.1

Fix an integer $n \geq 2$ and a real number $p \in (1, n)$, and take

$$k = n - p, \quad \mu = \frac{k}{p-1}.$$

Choose a nonzero real spherical harmonic Y_ℓ of even degree $\ell \geq 2$. It satisfies

$$-\Delta_{\mathbb{S}^{n-1}} Y_\ell = \lambda_\ell Y_\ell \quad \text{where} \quad \lambda_\ell := \ell(\ell + n - 2).$$

For an even integer $\ell \geq 2$ and sufficiently small $|\varepsilon|$, let

$$h_\varepsilon = 1 + \varepsilon Y_\ell.$$

Then $h_\varepsilon > 0$ and

$$\nabla_{\mathbb{S}^{n-1}}^2 h_\varepsilon + h_\varepsilon I > 0$$

on $\mathbb{S}^{n-1}$. Thus, h_ε is the support function of a C^∞ convex body K_ε with everywhere positive Gauss curvature. Moreover, since Y_ℓ is even, we have $h_\varepsilon(-u) = h_\varepsilon(u)$. Hence, K_ε is origin-symmetric.

The following lemma gives the second-order expansion of $S_p(K_\varepsilon) - \Phi_p(K_\varepsilon)$.

Lemma 3.1. *Let $\ell = 2j \geq 2$ be a fixed even integer. Define*

$$a_{p,2j} = \prod_{r=0}^{j-1} \frac{p-2r}{n+p+2r}, \quad A_\ell = (k - \lambda_\ell) a_{p,\ell}. \quad (3.1)$$

Then as $\varepsilon \rightarrow 0$,

$$S_p(K_\varepsilon) - \Phi_p(K_\varepsilon) = \frac{n+p}{2p} A_\ell^2 \varepsilon^2 \int_{\mathbb{S}^{n-1}} Y_\ell^2 d\sigma + o(\varepsilon^2). \quad (3.2)$$

Proof. Take

$$H = \nabla_{\mathbb{S}^{n-1}}^2 Y_\ell + Y_\ell I, \quad \tau = \text{tr } H = (n-1 - \lambda_\ell) Y_\ell.$$

Since Y_ℓ and H are bounded on $\mathbb{S}^{n-1}$, there exists $\varepsilon_0 > 0$ such that whenever $|\varepsilon| < \varepsilon_0$,

$$h_\varepsilon \geq \frac{1}{2}, \quad I + \varepsilon H \geq \frac{1}{2} I \quad \text{on } \mathbb{S}^{n-1}.$$

It follows that all expansions below are uniform for $u \in \mathbb{S}^{n-1}$. Using the Gauss parametrization, the L_p surface area measure of K_ε can be written as

$$dS_p(K_\varepsilon, u) = q_\varepsilon(u) d\sigma(u), \quad q_\varepsilon(u) := h_\varepsilon(u)^{1-p} \det(I + \varepsilon H(u)).$$

Using the expansions

$$h_\varepsilon^{1-p} = 1 + (1-p)\varepsilon Y_\ell + \frac{p(p-1)}{2} \varepsilon^2 Y_\ell^2 + O_{C^0}(\varepsilon^3),$$

and

$$\det(I + \varepsilon H) = 1 + \varepsilon \tau + \frac{\varepsilon^2}{2} (\tau^2 - \text{tr}(H^2)) + O_{C^0}(\varepsilon^3),$$

we obtain

$$q_\varepsilon = 1 + \varepsilon(k - \lambda_\ell) Y_\ell + \varepsilon^2 R_0 + O_{C^0}(\varepsilon^3), \quad (3.3)$$

where

$$R_0 = \frac{1}{2} (\tau^2 - \text{tr}(H^2)) + (1-p) Y_\ell \tau + \frac{p(p-1)}{2} Y_\ell^2.$$

The Funk-Hecke formula [9, Chapter 3] (or a direct Fourier computation when $n = 2$) shows that Y_{2j} is an eigenfunction of $\mathcal{T}_p$ with eigenvalue $a_{p,2j}$. That is,

$$\mathcal{T}_p Y_{2j} = a_{p,2j} Y_{2j}.$$

For $j \geq 0$, a direct computation of the corresponding beta integrals gives

$$a_{p,0} = 1, \quad a_{p,2j+2} = \frac{p-2j}{n+p+2j} a_{p,2j}.$$

Repeated application of this recurrence yields the product formula in (3.1). In particular,

$$a_{p,2} = \frac{p}{n+p}.$$

Applying $\mathcal{T}_p$ to the expansion in (3.3), and using the boundedness of $\mathcal{T}_p : C^0(\mathbb{S}^{n-1}) \rightarrow C^0(\mathbb{S}^{n-1})$, we obtain

$$v_{K_\varepsilon} = 1 + \varepsilon A_\ell Y_\ell + \varepsilon^2 B_0 + O_{C^0}(\varepsilon^3), \quad B_0 := \mathcal{T}_p R_0.$$

Since $\mathcal{T}_p$ preserves normalized spherical averages, we have

$$\langle B_0 \rangle_\sigma = \langle R_0 \rangle_\sigma.$$

Moreover, $\langle Y_\ell \rangle_\sigma = 0$. Therefore, the formula for S_p in (2.4) gives

$$\frac{S_p(K_\varepsilon)}{M} = 1 + \varepsilon^2 \langle B_0 \rangle_\sigma + O(\varepsilon^3).$$

Let $\alpha = n/p$. Taylor expanding $v_{K_\varepsilon}^{-\alpha}$, and applying the power $-1/\alpha$ to its spherical average as in (2.4), we obtain

$$\langle v_{K_\varepsilon}^{-\alpha} \rangle_\sigma = 1 - \alpha \varepsilon^2 \langle B_0 \rangle_\sigma + \frac{\alpha(\alpha+1)}{2} A_\ell^2 \varepsilon^2 \langle Y_\ell^2 \rangle_\sigma + O(\varepsilon^3),$$

and

$$\frac{\Phi_p(K_\varepsilon)}{M} = 1 + \varepsilon^2 \left(\langle B_0 \rangle_\sigma - \frac{\alpha+1}{2} A_\ell^2 \langle Y_\ell^2 \rangle_\sigma \right) + O(\varepsilon^3).$$

Combining the second expansion with the preceding expansion of $S_p(K_\varepsilon)/M$, and using

$$\alpha + 1 = \frac{n+p}{p} \quad \text{and} \quad M \langle Y_\ell^2 \rangle_\sigma = \int_{\mathbb{S}^{n-1}} Y_\ell^2 d\sigma,$$

we give (3.2), in fact with an $O(\varepsilon^3)$ remainder for fixed ℓ . □

Let ρ_ε be the radial function of K_ε . The support parametrization

$$x(u) = h_\varepsilon(u)u + \nabla_{\mathbb{S}} h_\varepsilon(u)$$

gives

$$\rho_\varepsilon = 1 + \varepsilon Y_\ell + O_{C^2}(\varepsilon^2) \quad \text{and} \quad b_\varepsilon = \nabla_{\mathbb{S}} \log \rho_\varepsilon = \varepsilon \nabla_{\mathbb{S}} Y_\ell + O_{C^1}(\varepsilon^2).$$

We next establish the corresponding second-order lower bound for the capacity deficit $S_p(K_\varepsilon) - \gamma_{n,p} C_p(K_\varepsilon)$.

Lemma 3.2. *Let $s_\ell > \mu$ be defined by*

$$(p-1)s_\ell(s_\ell - \mu) = \lambda_\ell, \quad s_\ell = \frac{\mu + \sqrt{\mu^2 + 4\lambda_\ell/(p-1)}}{2}. \quad (3.4)$$

Then

$$S_p(K_\varepsilon) - \gamma_{n,p} C_p(K_\varepsilon) \geq \frac{p(p-1)}{2} (s_\ell - \mu)^2 \varepsilon^2 \int_{\mathbb{S}^{n-1}} Y_\ell^2 d\sigma + o(\varepsilon^2). \quad (3.5)$$

Proof. For $\theta \in \mathbb{S}^{n-1}$ and $t \geq 1$, we parametrize the exterior of K_ε by

$$x = \rho_\varepsilon(\theta)t\theta.$$

Under this parametrization, write

$$v(\rho_\varepsilon(\theta)t\theta) = F(t, \theta).$$

The corresponding Jacobian is

$$dx = \rho_\varepsilon(\theta)^n t^{n-1} dt d\sigma(\theta),$$

while the chain rule gives

$$\nabla v = \rho_\varepsilon(\theta)^{-1} \left[F_t e_r + \frac{1}{t} (\nabla_{\mathbb{S}} F - t F_t b_\varepsilon) \right], \quad e_r = \theta.$$

Thus a direct polar-coordinate calculation gives

$$\begin{aligned} E_\varepsilon(F) &:= \int_{\mathbb{R}^n \setminus K_\varepsilon} |\nabla v|^p dx \\ &= \int_{\mathbb{S}^{n-1}} \int_1^\infty \rho_\varepsilon^k t^{n-1} [F_t^2 + t^{-2} |\nabla_{\mathbb{S}} F - t F_t b_\varepsilon|^2]^{p/2} dt d\sigma. \end{aligned} \quad (3.6)$$

Consider the function

$$F_T(t, \theta) := t^{-\mu}, \quad t \geq 1,$$

extended by the constant 1 on K_ε . It satisfies

$$E_\varepsilon(F_T) = \mu^{p-1} S_p(K_\varepsilon). \quad (3.7)$$

To verify this, observe that on ∂K_ε ,

$$|\nabla r_{K_\varepsilon}(x)| = h_{K_\varepsilon}(\nu_{K_\varepsilon}(x))^{-1}.$$

Therefore, the radial representation of ∂K_ε yields

$$S_p(K_\varepsilon) = \int_{\mathbb{S}^{n-1}} \rho_\varepsilon^k (1 + |b_\varepsilon|^2)^{p/2} d\sigma.$$

Substituting F_T into (3.6) and evaluating the radial integral,

$$\mu^p \int_1^\infty t^{n-1-p(\mu+1)} dt = \mu^{p-1},$$

we obtain (3.7).

We refine the preceding function by defining

$$F_*(t, \theta) = t^{-\mu} + \varepsilon \mu Y_\ell(\theta) (t^{-s_\ell} - t^{-\mu}).$$

Since $F_*(1, \theta) = 1$, we extend F_* to K_ε by defining $F_* \equiv 1$ on K_ε . The extended function has trace 1 on ∂K_ε and is an admissible finite-energy test function in the corresponding homogeneous Sobolev space. To approximate F_* by admissible functions in the capacity definition (2.5), fix

$\delta > 0$ and modify F_* near the boundary by setting it equal to 1 for $1 \leq t \leq 1 + \delta$. In the transition region $1 + \delta < t < 1 + 2\delta$, interpolate with respect to t so that the modified function equals 1 at $t = 1 + \delta$ and $F_*(1 + 2\delta, \theta)$ at $t = 1 + 2\delta$. Since

$$F_*(t, \theta) - 1 = O(t - 1), \quad \text{as } t \downarrow 1,$$

uniformly in θ , the interpolation can be chosen with bounded gradient. Its contribution to the energy is therefore $O(\delta)$.

Next, choose a smooth radial cutoff that equals 1 on $\{|x| \leq R\}$ and vanishes on $\{|x| \geq 2R\}$. The energy of the discarded tail and the additional term arising from the gradient of the cutoff are both bounded by

$$O(R^{n-p(\mu+1)}) = O(R^{-\mu}) = o(1), \quad \text{as } R \rightarrow \infty.$$

The truncated Sobolev function is compactly supported and remains identically equal to 1 on a neighborhood of K_ε . Mollifying it at a scale smaller than the width of this neighborhood produces a function in $C_c^\infty(\mathbb{R}^n)$ that is still equal to 1 on K_ε , while its energy converges to that of the truncated function. Let the mollification scale tend to zero. Then $R \rightarrow \infty$, and finally $\delta \rightarrow 0$. Thus we obtain

$$C_p(K_\varepsilon) \leq E_\varepsilon(F_*). \quad (3.8)$$

This approximation argument is the standard passage between the homogeneous Sobolev and smooth definitions of capacity; see [22, Chapter 2].

We next compute the second-order change in energy. For $a > 0$, let

$$u(t) = t^{-\mu}, \quad w_a(t, \theta) = \mu t^{-a} Y_\ell(\theta), \quad T_a = t^{-1} \nabla_{\mathbb{S}} w_a.$$

Define

$$Z = w_{s_\ell} - w_\mu.$$

Then the improved function can be written as

$$F_* = u + \varepsilon Z.$$

Let $c \neq 0$, and decompose

$$z = z_r e_r + z_{\tan} \quad \text{and} \quad z_{\tan} \perp e_r,$$

with $|z| \leq C|c|$. After factoring out $|c|$, the normalized argument remains in a fixed compact set away from the origin for all sufficiently small $|\varepsilon|$. Taylor's theorem then yields the following uniform expansion:

$$\begin{aligned} |c e_r + \varepsilon z|^p &= |c|^p + p|c|^{p-2} c z_r \varepsilon \\ &\quad + \frac{p}{2} |c|^{p-2} ((p-1) z_r^2 + |z_{\tan}|^2) \varepsilon^2 + O(\varepsilon^3 |c|^p). \end{aligned} \quad (3.9)$$

To isolate the second-order angular terms, define

$$\tilde{b}_\varepsilon = b_\varepsilon - \varepsilon \nabla_{\mathbb{S}} Y_\ell = O_{C^1}(\varepsilon^2).$$

The gradient vectors appearing in the integrand of (3.6) can then be written as

$$\begin{aligned} \mathcal{A}_T &= \left(u', \varepsilon T_\mu - u' \tilde{b}_\varepsilon \right), \\ \mathcal{A}_* &= \left(u' + \varepsilon Z_t, \varepsilon T_{s_\ell} - \varepsilon^2 Z_t \nabla_{\mathbb{S}} Y_\ell - u' \tilde{b}_\varepsilon + O(\varepsilon^3 |u'|) \right). \end{aligned}$$

Thus, relative to the radial base vector $(u', 0)$, their respective first-order perturbations are

$$(0, T_\mu) \quad \text{and} \quad (Z_t, T_{s_\ell}).$$

Moreover, the radial factor satisfies

$$\rho_\varepsilon^k = 1 + k\varepsilon Y_\ell + O(\varepsilon^2).$$

The order- ε term in $E_\varepsilon(F_T) - E_\varepsilon(F_*)$ vanishes, since for each fixed t , $Z_t(t, \cdot)$ is a scalar multiple of Y_ℓ , whose spherical mean is zero.

Let

$$W(t) = t^{n-1}|u'(t)|^{p-2}.$$

Applying the vector expansion (3.9) to the exact energy formula (3.6), we obtain the following unreduced second-order expansion:

$$\begin{aligned} E_\varepsilon(F_T) - E_\varepsilon(F_*) &= \frac{p}{2}\varepsilon^2 \int_{\mathbb{S}^{n-1}} \int_1^\infty W(t) (|T_\mu|^2 - |T_{s_\ell}|^2 - (p-1)Z_t^2) dt d\sigma \\ &\quad - pk\varepsilon^2 \int_{\mathbb{S}^{n-1}} \int_1^\infty W(t) u' Y_\ell Z_t dt d\sigma + o(\varepsilon^2). \end{aligned} \quad (3.10)$$

The second integral arises precisely from the interaction between the first-order term $k\varepsilon Y_\ell$ in the expansion of ρ_ε^k and the first-order radial variation of the energy density $|\nabla F_*|^p$. The terms involving the second-order contribution to ρ_ε^k are identical for the two energies and therefore cancel. Moreover, the second-order correction vectors

$$-u'\tilde{b}_\varepsilon \quad \text{and} \quad -\varepsilon^2 Z_t \nabla_{\mathbb{S}} Y_\ell$$

are both tangential and hence orthogonal to the radial base gradient. Their linear pairings with the base gradient therefore vanish. Hence, they produce no additional contribution at order ε^2 .

Define

$$\begin{aligned} Q(w) &= \int_{\mathbb{S}^{n-1}} \int_1^\infty \mu^{p-2} t^{n-1-(\mu+1)(p-2)} \\ &\quad \times [(p-1)w_t^2 + t^{-2}|\nabla_{\mathbb{S}} w|^2] dt d\sigma. \end{aligned} \quad (3.11)$$

Since the weight in (3.11) is W , the difference between the coefficient in (3.10) and $\frac{p}{2}[Q(w_\mu) - Q(w_{s_\ell})]$ equals

$$p(p-1) \int_{\mathbb{S}^{n-1}} \int_1^\infty W(w_\mu)_t Z_t dt d\sigma - pk \int_{\mathbb{S}^{n-1}} \int_1^\infty W u' Y_\ell Z_t dt d\sigma.$$

This vanishes pointwise in the remaining factors since

$$(w_\mu)_t = \mu Y_\ell u', \quad k = (p-1)\mu.$$

Therefore, we have proved

$$E_\varepsilon(F_T) - E_\varepsilon(F_*) = \frac{p}{2}\varepsilon^2 [Q(w_\mu) - Q(w_{s_\ell})] + o(\varepsilon^2). \quad (3.12)$$

The same calculation, with the elementary t -integrals performed, displays the complete second-order coefficient as

$$\begin{aligned} E_\varepsilon(F_T) - E_\varepsilon(F_*) &= \frac{p}{2}\mu^p \varepsilon^2 \int_{\mathbb{S}^{n-1}} Y_\ell^2 d\sigma \\ &\quad \times \left\{ \lambda_\ell \left(\frac{1}{\mu} - \frac{1}{2s_\ell - \mu} \right) - (p-1) \left(\frac{s_\ell^2}{2s_\ell - \mu} - \mu \right) \right\} + o(\varepsilon^2). \end{aligned}$$

Since $t^{\mu-s_\ell} \leq 1$ for $t \geq 1$, the ratios of all perturbation fields to $|u'|$ are uniformly bounded. Thus, for fixed ℓ and sufficiently small $|\varepsilon|$, every line segment joining the base gradient to the corresponding perturbed gradient remains at least $|u'|/2$ away from the origin.

The third derivative of $z \mapsto |z|^p$ on this region is bounded by $C|u'|^{p-3}$. Hence, the pointwise remainder is at most $C|\varepsilon|^3|u'|^p$. Thus, we have

$$\int_1^\infty t^{n-1}|u'|^p dt < \infty.$$

The dominated convergence theorem shows that the remainder in (3.10) is $o(\varepsilon^2)$. The same argument remains valid when $1 < p < 2$.

Using

$$\int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}} Y_\ell|^2 d\sigma = \lambda_\ell \int_{\mathbb{S}^{n-1}} Y_\ell^2 d\sigma,$$

and evaluating the radial integral in (3.11), we obtain

$$Q(w_\tau) = \mu^p \frac{(p-1)\tau^2 + \lambda_\ell}{2\tau - \mu} \int_{\mathbb{S}^{n-1}} Y_\ell^2 d\sigma.$$

Using (3.4), it follows that

$$Q(w_\mu) - Q(w_{s_\ell}) = \mu^{p-1}(p-1)(s_\ell - \mu)^2 \int_{\mathbb{S}^{n-1}} Y_\ell^2 d\sigma. \quad (3.13)$$

Combining (3.8) with $\gamma_{n,p} = \mu^{1-p}$, we obtain

$$S_p(K_\varepsilon) - \gamma_{n,p} C_p(K_\varepsilon) \geq \mu^{1-p} [E_\varepsilon(F_T) - E_\varepsilon(F_*)].$$

The desired estimate (3.5) now follows from (3.12) and (3.13). $\square$

The preceding estimates yield the following counterexample.

Proposition 3.3. *For every $n \geq 2$ and $1 < p < n$, there is a sufficiently high even degree ℓ such that for sufficiently small $\varepsilon \neq 0$,*

$$\Phi_p(K_\varepsilon) > \gamma_{n,p} C_p(K_\varepsilon).$$

Proof. Combining Lemmas 3.1 and 3.2, we obtain

$$\begin{aligned} \Phi_p(K_\varepsilon) - \gamma_{n,p} C_p(K_\varepsilon) &\geq \frac{\varepsilon^2}{2} \left[p(p-1)(s_\ell - \mu)^2 - \frac{n+p}{p} A_\ell^2 \right] \\ &\quad \times \int_{\mathbb{S}^{n-1}} Y_\ell^2 d\sigma + o(\varepsilon^2). \end{aligned} \quad (3.14)$$

It therefore suffices to choose ℓ so that the coefficient in brackets is positive.

If $p = 2q$ is an even integer, then (3.1) immediately yields

$$a_{p,2j} = 0 \quad \text{for every } j > q.$$

If p is not an even integer, the product formula can instead be expressed in terms of Gamma functions as

$$a_{p,2j} = (-1)^j \frac{\Gamma(j-p/2)\Gamma((n+p)/2)}{\Gamma(-p/2)\Gamma(j+(n+p)/2)}.$$

The standard asymptotic formula for ratios of Gamma functions yields

$$|a_{p,2j}| = O(j^{-p-n/2}), \quad A_{2j} = O(j^{2-p-n/2}) = o(1),$$

where the last relation follows from $n + 2p > 4$. By contrast, the definition of s_{2j} gives

$$s_{2j} - \mu = \frac{\sqrt{\mu^2 + 4\lambda_{2j}/(p-1)} - \mu}{2} \sim \frac{2j}{\sqrt{p-1}} \text{ as } j \rightarrow \infty.$$

Therefore, when p is not an even integer, the expression in brackets in (3.14) satisfies

$$p(p-1)(s_{2j} - \mu)^2 - \frac{n+p}{p}A_{2j}^2 = 4pj^2(1 + o(1)).$$

When $p = 2q$ is an even integer, $A_{2j} = 0$ for $j > q$, while the first term in the bracket is strictly positive. Hence, in either case, the bracket is positive for every sufficiently large j . Fix such a j , and let $\ell = 2j$. Since the remainder in (3.14) is $o(\varepsilon^2)$ for this fixed degree, the desired strict inequality holds for every sufficiently small $\varepsilon \neq 0$. $\square$

Proof of (i) in Theorem 1.1. By Proposition 3.3, we obtain part (i) of Theorem 1.1. In particular, the geometric inequality fails even for origin-symmetric convex bodies with C^∞ boundary and everywhere positive Gauss curvature. $\square$

4. PROOF OF (ii) IN THEOREM 1.1

We introduce the standard flat function

$$\vartheta(s) = \begin{cases} \exp(-1/s), & s > 0, \\ 0, & s \leq 0, \end{cases}$$

and define

$$g(s) = \frac{\vartheta(2-s)}{\vartheta(2-s) + \vartheta(s-1)}.$$

Then $g \in C^\infty(\mathbb{R})$, with

$$g \equiv 1 \text{ on } (-\infty, 1], \quad g \equiv 0 \text{ on } [2, \infty),$$

and $g'(s) < 0$ for every $s \in (1, 2)$.

The following proposition transfers the preceding geometric construction to the corresponding functional setting.

Proposition 4.1. *Let $K \in \mathcal{K}_o^n$ be a C^∞ , origin-symmetric convex body satisfying*

$$\Phi_p(K) > \gamma_{n,p} C_p(K).$$

Define

$$f(x) := g(r_K(x)). \tag{4.1}$$

Then f is a nonnegative function in $C_0^\infty(\mathbb{R}^n)$ and satisfies

$$\int_0^\infty \Phi_p(\langle f \rangle_t) dt > \gamma_{n,p} \int_0^\infty C_p(\langle f \rangle_t) dt. \tag{4.2}$$

Proof. The function r_K is smooth on $\mathbb{R}^n \setminus \{0\}$. Since $g \equiv 1$ on $(-\infty, 1]$ and vanishes to infinite order at $s = 2$, the function $f = g \circ r_K$ is smooth at the origin and matches smoothly with the zero function along the boundary of its support. Hence, the function f defined by (4.1) belongs to $C_0^\infty(\mathbb{R}^n)$.

Fix $t \in (0, 1)$. Since g is strictly decreasing on $(1, 2)$, there exists a unique $s = s(t) \in (1, 2)$ such that $t = g(s)$. Applying the computation from the proof of [16, Lemma 8] to this regular level yields

$$\langle f \rangle_t = c(t)K, \quad c(t)^{n-p} = |g'(s)|^{p-1} s^{n-1}. \quad (4.3)$$

Indeed, the level surface $\{f = t\}$ is $s\partial K$, and on this surface,

$$|\nabla f| = |g'(s)| |\nabla r_K|.$$

Moreover, the $(n-1)$ -dimensional surface measure acquires the factor s^{n-1} under dilation. Thus, the Gauss map ν_K satisfies

$$(\nu_K)_* (|\nabla r_K|^{p-1} \mathcal{H}^{n-1} \upharpoonright_{\partial K}) = S_p(K, \cdot).$$

Therefore, the measure appearing in (2.6) is

$$|g'(s)|^{p-1} s^{n-1} dS_p(K, \cdot).$$

Since

$$dS_p(cK, \cdot) = c^{n-p} dS_p(K, \cdot),$$

the choice of $c = c(t)$ in (4.3) gives precisely this measure. Uniqueness for the even L_p -Minkowski problem implies

$$\langle f \rangle_t = c(t)K.$$

Both Φ_p and C_p are homogeneous of degree $n-p$. Hence, for either $\Psi = \Phi_p$ or $\Psi = C_p$, equation (4.3) and the substitution $t = g(s)$ yield

$$\int_0^\infty \Psi(\langle f \rangle_t) dt = A_{g,p} \Psi(K), \quad \text{with } A_{g,p} = \int_1^2 |g'(s)|^p s^{n-1} ds > 0. \quad (4.4)$$

Since $0 \leq f \leq 1$, the relevant level set is empty for $t > 1$, and the endpoint levels have measure zero. Therefore, because $A_{g,p} > 0$, combining (4.4) with

$$\Phi_p(K) > \gamma_{n,p} C_p(K)$$

proves (4.2). $\square$

Proof of (ii) in Theorem 1.1. Part (ii) follows by applying Proposition 4.1 to a convex body constructed in Proposition 3.3. This completes the proof of Theorem 1.1. $\square$

APPENDIX: VERIFICATION OF THE SPECIAL CASE $(n, p) = (3, 2)$

The preceding construction is obtained by a perturbative argument. We conclude by exhibiting an explicit convex body with a quantitative gap, thereby providing an independent verification of the normalizations in the special case $(n, p) = (3, 2)$. For an even integer $q \geq 2$, define

$$B_q^3 := \{x \in \mathbb{R}^3 : x_1^q + x_2^q + x_3^q \leq 1\}.$$

Proposition 4.2. *For an even integer $q \geq 2$,*

$$\Phi_2(B_q^3) = S_2(B_q^3) = \frac{24\pi(q-1)}{q(q+1)\sin(\pi/q)}. \quad (4.5)$$

In particular,

$$\Phi_2(B_{24}^3) > C_2(B_{24}^3).$$

Proof. For a smooth convex body $K \in \mathcal{K}_o^3$, define the symmetric matrix

$$M_K = \int_{\partial K} \frac{\nu_K(x) \otimes \nu_K(x)}{x \cdot \nu_K(x)} d\mathcal{H}^2(x).$$

Note that for $a, b \in \mathbb{R}^3$, the tensor product $a \otimes b$ is the matrix defined by

$$(a \otimes b)_{ij} := a_i b_j, \quad 1 \leq i, j \leq 3.$$

Thus,

$$(M_K)_{ij} = \int_{\partial K} \frac{\nu_{K,i}(x) \nu_{K,j}(x)}{x \cdot \nu_K(x)} d\mathcal{H}^2(x).$$

Since $K \in \mathcal{K}_o^3$, we have $x \cdot \nu_K(x) > 0$ on ∂K . Hence, the matrix M_K is well defined and positive semidefinite. Equation (2.1) yields

$$V_2(K, \bar{u}) = \frac{1}{12} u^\top M_K u.$$

The body B_q^3 is invariant under arbitrary sign changes and permutations of its coordinates. These symmetries imply that

$$M_{B_q^3} = m_q I$$

for some $m_q > 0$. Consequently,

$$V_2(B_q^3, \bar{u}) = \frac{m_q}{12}$$

is independent of $u \in \mathbb{S}^2$. The normalization formulas (2.2) and (2.3) thus give

$$\Phi_2(B_q^3) = S_2(B_q^3).$$

Define the vector field

$$X(x) = (x_1^{q-1}, x_2^{q-1}, x_3^{q-1}).$$

For $x \in \partial B_q^3$, we have

$$\nu(x) = \frac{X(x)}{|X(x)|}, \quad x \cdot X(x) = 1.$$

It follows that

$$\frac{1}{x \cdot \nu(x)} = |X(x)| = X(x) \cdot \nu(x).$$

Therefore, by the divergence theorem and coordinate symmetry,

$$\begin{aligned} S_2(B_q^3) &= \int_{\partial B_q^3} X \cdot \nu d\mathcal{H}^2 = \int_{B_q^3} \operatorname{div} X dx \\ &= 3(q-1) \int_{B_q^3} |x_1|^{q-2} dx. \end{aligned} \tag{4.6}$$

To compute the remaining integral, decompose B_q^3 into its eight orthants and introduce the variables

$$y_i = |x_i|^q, \quad i = 1, 2, 3.$$

The Dirichlet simplex integral then gives

$$\int_{B_q^3} |x_1|^{q-2} dx = \frac{8}{q^3} \frac{\Gamma(1-1/q)\Gamma(1/q)^2}{\Gamma(2+1/q)}.$$

Inserting this expression into (4.6) and applying Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

yields (4.5).

For $q = 24$, it follows that

$$\Phi_2(B_{24}^3) = \frac{23\pi}{25 \sin(\pi/24)} > \frac{552}{25} > 22,$$

where the first strict inequality follows from $\sin x < x$ for $x > 0$. On the other hand,

$$B_{24}^3 \subset [-1, 1]^3 \subset \sqrt{3} B^3.$$

By monotonicity and homogeneity of the 2-capacity, together with $C_2(B^3) = 4\pi$, we obtain

$$C_2(B_{24}^3) \leq C_2(\sqrt{3} B^3) = 4\pi\sqrt{3} < 22.$$

The final strict inequality follows from

$$\pi < \frac{22}{7}, \quad \sqrt{3} < \frac{7}{4}.$$

□

Applying the construction from Proposition 4.1 to $K = B_{24}^3$ gives a fully explicit smooth counterexample to the analytic inequality in $\mathbb{R}^3$ at $p = 2$.

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