SHARP SPECTRAL GAPS ON METRIC MEASURE SPACES

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Abstract. In this paper, we extend the sharp lower bounds of spectal gap, due to Chen-Wang [12, 13], Bakry-Qian [6] and Andrews-Clutterbuck [5], from smooth Riemannian manifolds to general metric measure spaces with Riemannian curvature-dimension condition $RCD^*(K,N)$.

1. INTRODUCTION

Let (X,d,m) be a compact metric measure space. Given a Lipschitz function $f:X\to\mathbb{R}$, its point-wise Lipschitz constant $\mathrm{Lip}f(x)$ is defined as

$$\mathrm{Lip}f(x):=\limsup_{y\to x}\frac{|f(y)-f(x)|}{d(x,y)}.$$

In this paper, we are concerned with the spectral gap

$$\lambda_1(X):=\inf\Big\{\frac{\int_X(\mathrm{Lip}f)^2\mathrm{d}m}{\int_X f^2\mathrm{d}m}:\ f\in Lip(X)\backslash\{0\}\quad\text{and}\quad\int_X f\mathrm{d}m=0\Big\},\tag{1.1}$$

where $Lip(X)$ is the space of Lipschitz functions on X.

When M is a compact smooth Riemannian manifold without boundary (or with a convex boundary ∂M), the study of the lower bounds of the first eigenvalue λ_1 of the Laplace-Beltrami operator Δ has a long history. See for example, Lichnerowicz [27], Cheeger [10], Li-Yau [26], and so on. For an overview the reader is referred to the introduction of [7, 6, 25] and Chapter 3 in book [35], and references therein. In particular the following comparison theorem for λ_1 has been established by Chen-Wang [12, 13], Bakry-Qian [6] and Andrews-Clutterbuck [5] independently, via three different methods.

Theorem 1.1 (Chen-Wang [12, 13], Bakry-Qian [6], Andrews-Clutterbuck [5])**.** *Let M be an N-dimensional compact Riemannian manifold without boundary (or with a convex boundary). Suppose that the Ricci curvature $Ric(M)\geqslant K$ and that the diameter $\leqslant d$. Let λ_1 be the first (non-zero) eigenvalue (with Neumann boundary condition if the boundary is not empty). Then*

$$\lambda_1(M)\geqslant\hat{\lambda}(K,N,d)$$

where $\hat{\lambda}(K,N,d)$ denotes the first non-zero Neumann eigenvalue of the following one-dimensional model:

$$v''(x)-(N-1)T(x)v'(x)=-\lambda v(x)\qquad x\in(-\frac{d}{2},\frac{d}{2}),\qquad v'(-\frac{d}{2})=v'(\frac{d}{2})=0$$

and

$$T(x)=\begin{cases}\sqrt{\frac{K}{N-1}}\tan(\sqrt{\frac{K}{N-1}}x) & \text{if } K\geqslant 0,\\ \sqrt{\frac{-K}{N-1}}\tanh(\sqrt{\frac{-K}{N-1}}x) & \text{if } K<0.\end{cases}$$

This comparison Theorem 1.1 implies the classical Lichnerowicz estimate [27] for $K = n-1$ and also Zhong-Yang's estimate [42] for $K = 0$. Some lower bounds of the spectral gaps have been extended to singular spaces. In [36], Shioya discussed spectral gaps in Riemannian orbifolds. In [31], Petrunin proved the Linchnerowiz estimate for Alexandrov spaces with curvature $\geqslant 1$ in the sense of Alexandrov. Recently, Theorem 1.1 has been extended to Alexandrov spaces in [32] using a notion of generalized lower Ricci curvature bounds in [41], and by Wang-Xia [40] to Finsler manifolds.

In the last few years, several notions of "the generalized Ricci curvature bounded below" on general metric spaces have been introduced. Sturm [38, 39] and Lott-Villani [28], independently, introduced a so-called *curvature-dimension condition*, denoted by CD, on metric measure spaces via optimal transportation. A refinement of this notion is given in Ambrosio-Gigli-Savaré [3], which is called *Riemannian curvature-dimension condition*, denoted by RCD^*. Recently, in two remarkable works, Ambrosio-Gigli-Savaré [1] and Erbar-Kuwada-Sturm [16], they proved the equivalence of the Riemannian curvature-dimension condition and of the Bochner formular of Bakry-Émery via an abstract Γ_2-calculus, denoted by BE. Notice that in the case where M is a (compact) Riemannian manifold. Given two numbers $K \in \mathbb{R}$ and $N \geqslant 1$, M satisfying the Riemannian curvature-dimension condition $RCD^*(K,N)$ is equivalent to that the Ricci curvature $Ric(M) \geqslant K$ and the dimension $dim \leqslant N$.

We will consider the spectral gap on metric measure spaces under a suitable Riemannian curvature-dimension condition. Lott-Villani [29] and Erbar-Kuwada-Sturm [16] extended Linchnerowicz's estimate to metric measure spaces with $CD(K,N)$ or $RCD^*(K,N)$ for $K > 0$ and $1 \leqslant N < \infty$.

In this paper, we will extend Theorem 1.1 to general metric measure spaces. Precisely, we have the following theorem.

Theorem 1.2. *Let $K \in \mathbb{R}$, $1 \leqslant N < \infty$ and $d > 0$. Let (X, d, m) be a compact metric measure space satisfying the Riemannian curvature-dimension condition $RCD^*(K,N)$ and the diameter $\leqslant d$. Then the spectral gap $\lambda_1(X)$ has the following lower bound*

$$\lambda_1(X) \geqslant \hat{\lambda}(K, N, d), \tag{1.2}$$

where $\hat{\lambda}(K,N,d)$ is given in Theorem 1.1.

Our proof of Theorem 1.2 relies on the self-improvement of regularity under the Riemannian curvature-dimension condition (Theorem 2.6) and a version of maximum principle, which is similar to the classical maximum principle for C^2-functions on manifolds (see Proposition 3.1 and Remark 3.2).

Remark 1.3. (1) When $N > 1$ and $K = N-1$, the above Theorem 1.2 implies that

$$\lambda_1(X) \geqslant \frac{N}{1 - \cos^N(d/2)}.$$

In particular, this gives that if $\lambda_1(X) = N$, then $d = \pi$. The combination of this and the maximal diameter theorem in [22] implies an Obata-type rigidity theorem for general metric measure spaces, which is also proved in [23] by Ketterer, independently.

(2) Very recently Cavalletti and Mondino [8, 9] use a different method to establish a further generalization of this result. They prove the same sharp spectral gap estimates (and some other sharp isoperimetric and functional inequalities) for non-branching CD-spaces.

2. Preliminaries

In this section, we recall some basic notions and the calculus on metric measure spaces. For our purpose in this paper, we will focus only on the case of compact spaces. Let (X,d) be a compact metric space, and let m be a Radon measure with $\mathrm{supp}(m)=X$.

2.1. **Riemannian curvature-dimension condition** $RCD^*(K,N)$.

Let (X,d,m) be a compact metric space. The Cheeger energy is given in [2] from the relaxation in $L^2(X,m)$ of the point-wise Lipschitz constant of Lipschitz functions. That is, given a function $f\in L^2(X,m)$, the Cheeger energy of f is defined [2] by

$$\mathrm{Ch}(f):=\inf\Big\{\liminf_{j\to\infty}\frac{1}{2}\int_X(\mathrm{Lip}f_j)^2dm\Big\},$$

where the infimum is taken over all sequences of Lipschitz functions $\{f_j\}$ converging to f in $L^2(X,m)$. If $\mathrm{Ch}(f)<\infty$, then there is a (unique) so-called minimal relaxed gradient $|Df|_w$ such that

$$\mathrm{Ch}(f)=\frac{1}{2}\int_X|Df|_w^2\mathrm{d}m.$$

The domain of Ch in $L^2(X,m)$, D(Ch), is a Banach space with norm $\sqrt{\|f\|^2_{L^2}+\||Df|_w\|^2_{L^2}}$.

Definition 2.1. ([2]) A metric measure space (X,d,m) is called *infinitesimally Hilbertian* if the associated Cheeger energy Ch is a quadratic form.

Let (X,d,m) be an infinitesimally Hilbertian space. It is proved in [3] that the scalar product

$$\Gamma(f,g):=\lim_{\epsilon\to0^+}\frac{|D(f+\epsilon g)|_w^2-|Df|_w^2}{2\epsilon}\qquad f,g\in \mathrm{D(Ch)}$$

exists in $L^1(X,m)$. In the following we denote by $\mathbb{V}$ the Hilbert space D(Ch) with the scalar product

$$(f,g)_{\mathbb{V}}:=\int_X\big(fg+\Gamma(f,g)\big)\mathrm{d}m.$$

The quadratic form Ch canonically induces a symmetric, regular, strongly local Dirichlet form $(\mathrm{Ch},\mathbb{V})$. The regular property of $(\mathrm{Ch},\mathbb{V})$ comes from that X is always assumed to be compact. Moreover, for any $f,g\in\mathbb{V}$, $\Gamma(f,g)$ provides an explicit expression of the *Carré du champ* of the Dirichlet form $(\mathrm{Ch},\mathbb{V})$. The associated energy measure of f is absolutely continuous with respect to m with density $\Gamma(f)=|Df|_w^2$.

Denote by $(H_t)_{t>0}$ and Δ the associated Markov semigroup in $L^2(X,m)$ and its generator respectively. Since X is compact, according to [33], the $RCD^*(K,N)$ condition implies that (X,d,m) supports a global Poincaré inequality. Moreover, the operator $(-\Delta)^{-1}$ is a compact operator. Then the spectral theorem gives that the $\lambda_1(X)$ in (1.1) is the first non-zero eigenvalue of $-\Delta$. (See, for example, [15].)

We adopt the notations given in [4]:

$$D_{\mathbb{V}}(\Delta):=\big\{f\in\mathbb{V}:\Delta f\in\mathbb{V}\big\}$$

and, for every $p\in[1,\infty]$,

$$D_{L^p}(\Delta):=\big\{f\in\mathbb{V}\cap L^p(X,m):\Delta f\in L^2\cap L^p(X,m)\big\}.$$

Definition 2.2. ([4, 16]) Let $K \in \mathbb{R}$ and $N \geqslant 1$. An infinitesimally Hilbertian space (X, d, m) is said to satisfy the *condition* $BE(K, N)$ if the associated Dirichlet form $(\mathrm{Ch}, \mathbb{V})$ satisfies

$$\int_X \Big(\frac{1}{2}\Gamma(f)\Delta\phi - \Gamma(f, \Delta f)\phi\Big)\mathrm{d}m \geqslant K \int_X \Gamma(f)\phi\mathrm{d}m + \frac{1}{N}\int_X (\Delta f)^2\phi\mathrm{d}m$$

for all $f \in D_{\mathbb{V}}(\Delta)$ and all nonnegative $\phi \in D_{L^\infty}(\Delta)$.

According to [4, 16], the Riemannian curvature-dimension condition $RCD^*(K, N)$ is equivalent to the corresponding Bakry-Émery condition $BE(K, N)$ with a slight regularity. We shall use the following definition for $RCD^*(K, N)$ (Notice that X is always assumed to be compact in the paper).

Definition 2.3. ([4, 16]), Let $K \in \mathbb{R}$ and $N \geqslant 1$. A compact, infinitesimally Hilbertian geodesic space (X, d, m) is said to satisfy the $RCD^*(K, N)$-*condition* (or metric $BE(K, N)$ condition) if it satisfies $BE(K, N)$ and that every $f \in \mathbb{V}$ with $\|\Gamma(f)\|_{L^\infty} \leqslant 1$ has a 1-Lipschitz representative.

Recall that a (locally) compact metric (X, d) is a geodesic space if the distance between any two points in X can be realized as the length of some curve connecting them. Notice that if (X, d, m) satisfies $RCD^*(K, N)$ condition then $d = d_{\mathrm{Ch}}$, where d_{Ch} is the induced metric by the Dirichlet form $(\mathrm{Ch}, \mathbb{V})$. For any $f \in \mathbb{V}$ with $\Gamma(f) \in L^\infty(X, m)$, we always identify f with its Lipschitz representative. Moreover, $H_t f$, $H_t(|\nabla f|_w^2)$ and $\Delta H_t f$ have continuous representatives (see Proposition 4.4 of [16]).

2.2. **The self-improvement of regularity on $RCD^*(K, N)$-spaces.**

Let $K \in \mathbb{R}$ and $1 \leqslant N < \infty$, and let (X, d, m) be a compact metric measure space satisfying the $RCD^*(K, N)$ condition.

Let us recall an extension of the generator Δ of $(\mathrm{Ch}, \mathbb{V})$, which is introduced in [4, 34]. Denote by $\mathbb{V}'$ the set of continuous linear functionals $\ell : \mathbb{V} \to \mathbb{R}$, and $\mathbb{V}'_+$ denotes the set of positive linear functionals $\ell \in \mathbb{V}'$ such that $\ell(\varphi) \geqslant 0$ for all $\varphi \in \mathbb{V}$ with $\varphi \geqslant 0$ m-a.e. in X. An important characterization of functionals in $\mathbb{V}'_+$ is that, for each $\ell \in \mathbb{V}'_+$ there exists a unique corresponding Radon measure μ_ℓ on X such that

$$\ell(\varphi) = \int_X \tilde{\varphi}\mathrm{d}\mu_\ell \quad \forall \varphi \in \mathbb{V},$$

where $\tilde{\varphi}$ is a quasi continuous representative of φ. Denote by

$$\mathbb{M}_\infty := \Big\{ f \in \mathbb{V} \cap L^\infty(X, m) : \exists\ \mu \ \text{ such that } \ -\mathcal{E}(f, \varphi) = \int_X \tilde{\varphi}\mathrm{d}\mu \quad \forall \varphi \in \mathbb{V}\Big\},$$

where $\mu = \mu_+ - \mu_-$ with $\mu_+, \mu_- \in \mathbb{V}'_+$. When a function $f \in \mathbb{M}_\infty$, we set $\Delta^* f := \mu$, and denote its Lebesgue decomposition w.r.t m as $\Delta^* f = \Delta^{ab} f \cdot m + \Delta^s f$. It is clear that if $f \in D(\Delta) \cap L^\infty(X, m)$ then $f \in \mathbb{M}_\infty$ and $\Delta^* f = \Delta f \cdot m$.

Lemma 2.4. *Let $K \in \mathbb{R}$ and $N \geqslant 1$, and let (X, d, m) be a compact metric measure space satisfying $RCD^*(K, N)$ condition.*
(i) (Chain rule, [34, Lemma 3.2]) *If $g \in D(\Delta) \cap Lip(X)$ and $\phi \in C^2(\mathbb{R})$ with $\phi(0) = 0$, then we have*

$$\phi \circ g \in D(\Delta) \cap Lip(X) \qquad \text{and} \qquad \Delta(\phi \circ g) = \phi' \circ g \cdot \Delta g + \phi'' \circ g \cdot \Gamma(g);$$

(ii) (Leibniz rule, [34, Corollary 2.7]) *If $g_1 \in \mathbb{M}_\infty$ and $g_2 \in D(\Delta) \cap Lip(X)$, then we have*

$$g_1 \cdot g_2 \in \mathbb{M}_\infty \qquad \text{and} \qquad \Delta^*(g_1 \cdot g_2) = g_2 \cdot \Delta^* g_1 + g_1 \cdot \Delta g_2 \cdot m + 2\Gamma(g_1, g_2) \cdot m.$$

Remark 2.5. We can take $\phi \in C^2(\mathbb{R})$ without the restriction $\phi(0) = 0$ in the Chain rule. This comes from the fact that $1 \in D(\Delta)$ and $\Delta 1 = 0$, because X is assumed to be compact.

The following self-improvement of regularity is given in Lemma 3.2 of [34]. (See also Theorem 2.7 of [17]).

Theorem 2.6. ([34, 17]) *Let $K \in \mathbb{R}$ and $1 \leqslant N < \infty$, and let (X, d, m) be a compact metric measure space satisfying $RCD^*(K,N)$ condition. If $f \in D_{\mathbb{V}}(\Delta) \cap Lip(X)$, then we have $\Gamma(f) \in \mathbb{M}_\infty$ and*

$$\frac{1}{2}\Delta^*\Gamma(f) - \Gamma(f, \Delta f) \cdot m \geqslant K\Gamma(f) \cdot m + \frac{1}{N}(\Delta f)^2 \cdot m. \tag{2.1}$$

A crucial fact, which is implied by the above inequality, is that the singular part of $\Delta^*\Gamma(f)$ has a correct sign: $\Delta^s\Gamma(f)$ is non-negative.

Using the same trick as in the proof of Bakry-Qian [6, Thm 6] and [34, Thm 3.4], one can prove the following Corollary of Theorem 2.6 (see [21, Lemma 2.3] for a detailed proof):

Corollary 2.7. *Let $K \in \mathbb{R}$ and $1 \leqslant N < \infty$, and let (X, d, m) be a compact metric measure space satisfying $RCD^*(K,N)$ condition. If $f \in D_{\mathbb{V}}(\Delta) \cap Lip(X)$, then $\Delta^s f \geqslant 0$ and the following holds m-a.e. on $\{x \in X : \Gamma(f)(x) \neq 0\}$,*

$$\left(\frac{1}{2}\Delta^{ab}\Gamma(f) - \Gamma(f, \Delta f) - K\Gamma(f) - \frac{1}{N}(\Delta f)^2\right) \geqslant \frac{N}{N-1}\left(\frac{\Delta f}{N} - \frac{\Gamma(f, \Gamma(f))}{2\Gamma(f)}\right)^2. \tag{2.2}$$

For $\kappa \in \mathbb{R}$ and $\theta \geqslant 0$ we denote the function

$$\mathfrak{s}_\kappa(\theta) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\theta), & \kappa > 0, \\ \theta, & \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}\theta), & \kappa < 0. \end{cases}$$

Proposition 2.8 (Bishop-Gromov inequality, [18, 39]). *For each $x_0 \in X$ and $0 < r < R \leqslant \pi\sqrt{(N-1)/(K \vee 0)}$, we have*

$$\frac{m(B_r(x_0))}{m(B_R(x_0))} \geqslant \frac{\int_0^r \mathfrak{s}_{\frac{K}{N-1}}(t)^{N-1}\mathrm{d}t}{\int_0^R \mathfrak{s}_{\frac{K}{N-1}}(t)^{N-1}\mathrm{d}t}. \tag{2.3}$$

Proof. By Corollary of 1.5 in [18], (X, d, m) satisfies $MCP(K,N)$ condition. The desired Bishop-Gromov inequality (2.3) holds on $MCP(K,N)$-spaces by Remark 5.3 of [39]. □

We need also the following mean value inequality in [30]. See also Lemma 2.1 of [14].

Lemma 2.9. [30, Lemma 3.4] *Let $f \in D(\Delta)$ be a non-negative, continuous function with $\Delta f \leqslant c_0$ m-a.e.. Then there exists a constant $C(K, N, diamX)$ such that the following holds:*

$$\fint_{B_r(x)} f \mathrm{d}m \leqslant C(f(x) + c_0 r^2). \tag{2.4}$$

At last, we need the following Sobolev inequality, whose proof is similar to that of Theorem 13.1 of [19]. For the reader's convenience, we include a proof here.

Lemma 2.10. *Let $E \subset X$ be an m-measurable subset with $m(E) > 0$. Then there exist constants $\nu > 2$ and $\widetilde{C_S}$ which depend only on K, N, X and E, such that for any $f \in \mathbb{V}$ with $f = 0$ m-a.e. in E, the following Sobolev inequality holds:*

$$\|f\|_{L^\nu(X)} \leqslant \widetilde{C_S}\left(\int_X \Gamma(f)\mathrm{d}m\right)^{\frac{1}{2}}. \tag{2.5}$$

Proof. The above Bishop-Gromov inequality (2.3) implies the doubling property; and by Theorem 2.1 of [33], a Poincaré inequality holds. These two ingredients imply the following Sobolev inequality by Theorem 9.7 of [19]: there exist constants $\nu > 2$ and $C_S > 0$, depending on K, N and $\mathrm{diam}X$, such that for all $f \in \mathbb{V}$,

$$\left(\fint_X |f - \fint_X f|^\nu\right)^{\frac{1}{\nu}} \leqslant C_S \left(\fint_X \Gamma(f)\mathrm{d}m\right)^{\frac{1}{2}}, \tag{2.6}$$

where $\fint_X \Gamma(f) := \frac{1}{m(X)} \int_X \Gamma(f)\mathrm{d}m$.

Note that $\fint_X f$ is a constant and that $f = 0$ on E, thus we have $\|f - \fint_X f\|_{L^\nu(E)} = (m(E))^{\frac{1}{\nu}} \cdot |\fint_X f|$ and

$$\begin{aligned} \|\fint_X f\|_{L^\nu(X)} &= m(X)^{\frac{1}{\nu}} |\fint_X f| \\ &= \left(\frac{m(X)}{m(E)}\right)^{\frac{1}{\nu}} \|f - \fint_X f\|_{L^\nu(E)} \\ &\leqslant \left(\frac{m(X)}{m(E)}\right)^{\frac{1}{\nu}} \|f - \fint_X f\|_{L^\nu(X)}. \end{aligned}$$

Then, by Minkowski inequality, we have

$$\begin{aligned} \|f\|_{L^\nu(X)} &\leqslant \|f - \fint_X f\|_{L^\nu(X)} + \|\fint_X f\|_{L^\nu(X)} \\ &\leqslant \left[1 + (\tfrac{m(X)}{m(E)})^{\frac{1}{\nu}}\right] \|f - \fint_X f\|_{L^\nu(X)} \\ &\overset{(2.6)}{\leqslant} \left[1 + (\tfrac{m(X)}{m(E)})^{\frac{1}{\nu}}\right] \cdot C_S m(X)^{\frac{1}{\nu} - \frac{1}{2}} \left(\int_X \Gamma(f)\mathrm{d}m\right)^{\frac{1}{2}}. \end{aligned}$$

Let $\widetilde{C_S} = C_S m(X)^{\frac{1}{\nu} - \frac{1}{2}} \left[1 + \frac{m(X)}{m(E)})^{\frac{1}{\nu}}\right]$, thus we have completed the proof. □

3. Eigenvalue estimate for $RCD^*(K,N)$-spaces

Let $K \in \mathbb{R}$ and $1 \leqslant N < \infty$, and let (X,d,m) be a compact $RCD^*(K,N)$-space. We need a version of maximum principle on X as follows.

Proposition 3.1. *Let $u \in \mathbb{M}_\infty$ and let $\varepsilon_0 > 0$. If the measure $\Delta^* u$ satisfies that the singular part $\Delta^s u \geqslant 0$ on X and that the absolutely continuous part*

$$\Delta^{ab} u \geqslant C_1 \cdot u - C_2 \cdot \sqrt{\Gamma(u)} \quad m\text{−a.e. on} \quad \{x : u(x) \geqslant \varepsilon_0\} \tag{3.1}$$

holds for some positive constants C_1 and C_2 (they may depend on ε_0). Then $u \leqslant \varepsilon_0$ m-a.e. on X.

Remark 3.2. If X is a smooth Riemannian manifold, and if u is a C^2-function, then the Proposition 3.1 is a corollary of the classical maximum principle. In fact, if the assertion is false in this case, we assume that u achieves its maximum at point p, where $u(p) > \epsilon_0$. By using the maximum principle on C^2-functions, we have

$$\Delta u(p) \leqslant 0 \quad \text{and} \quad \Gamma(u)(p) = 0.$$

Hence, by (3.1), we have $u(p) \leqslant 0$. This contradicts to $u(p) > \varepsilon_0$.

In the setting of metric measure spaces, we need a new argument.

Proof of Proposition 3.1. Since $u \in L^\infty(X,m)$, we have $\sup_X u < \infty$, where $\sup_X u = \inf\{l : (u-l)_+ = 0, m\text{−}a.e. \text{ in } X\}$.

Let us argue by contradiction. Suppose that $\varepsilon_0 < \sup_X u$.

Take any constant $k \in [\varepsilon_0, \sup_X u)$ and set $\phi_k = (u-k)^+$. Then $\phi_k \in \mathbb{V}$. Since the singular part $\Delta^s u \geqslant 0$, we have

$$\begin{aligned} -\int_X \Gamma(u, \phi_k)\,\mathrm{d}m \quad &= \int_X \widetilde{\phi_k} d\Delta^* u \\ &\geqslant \int_X \widetilde{\phi_k} \Delta^{ab} u \,\mathrm{d}m \\ &= \int_{\{x: u(x) \geqslant k\}} \phi_k \Delta^{ab} u \,\mathrm{d}m \\ &\overset{(3.1)}{\geqslant} -C_2 \int_{X_k} \phi_k \sqrt{\Gamma(u)} \mathrm{d}m \\ &\geqslant -C_2 \left(\int_{X_k} {\phi_k}^2\right)^{\frac{1}{2}} \left(\int_{X_k} \Gamma(u)\right)^{\frac{1}{2}}, \end{aligned}$$

where $X_k := \{x : \Gamma(u) \neq 0\} \cap \{x : u(x) > k\}$.

By the truncation property in [37] and $\Gamma(u, \phi_k) = \Gamma(u) = \Gamma(\phi_k)$ m-a.e. in X_k, we have

$$\int_X \Gamma(u, \phi_k) = \int_{X_k} \Gamma(u, \phi_k) = \int_{X_k} \Gamma(u) = \int_{X_k} \Gamma(\phi_k).$$

The combination of the above two equations implies that

$$\int_{X_k} \Gamma(\phi_k) \leqslant C_2^2 \int_{X_k} {\phi_k}^2. \tag{3.2}$$

Now we claim that there exists a constant $k_0 \in [\varepsilon_0, \sup_X u)$ such that

$$m(\{x : u(x) < k_0\}) > 0. \tag{3.3}$$

Suppose that (3.3) fails for any $k \in [\varepsilon_0, \sup_X u)$. That is, $m(\{x : u(x) < k\}) = 0$ for any $k \in [\varepsilon_0, \sup_X u)$. Letting k tend to $\sup_X u$, we get $m(\{x : u(x) < \sup_X u\}) = 0$. Thus $u = \sup_X u$ m-a.e. in X. Now, we have $\Delta^* u = 0$ and $\Gamma(u) = 0$ m-a.e. in X. This contradicts (3.1) and proves the claim.

Fix such a constant $k_0 \in [\varepsilon_0, \sup_X u)$ such that (3.3) holds. Denote $E = \{x : u(x) < k_0\}$. For all $k \in (k_0, \sup_X u)$, we have $\phi_k = 0$ m-a.e. in E. By applying Lemma 2.10, we conclude that

$$\|\phi_k\|_{L^\nu(X)} \leqslant \widetilde{C_S} \left(\int_X \Gamma(\phi_k) \mathrm{d}m\right)^{\frac{1}{2}}, \quad \forall k \in (k_0, \sup_X u). \tag{3.4}$$

We shall show that $m(X_k) > 0$ for all $k \in (k_0, \sup_X u)$. Fix any $k \in (k_0, \sup_X u)$, the set $\{x : u(x) > k\}$ has positive measure, because $k < \sup_X u$. Hence, $\|\phi_k\|_{L^\nu(X)} > 0$. By using (3.4), we get $m(\{x : \Gamma(\phi_k) \neq 0\}) > 0$. Note that

$$\Gamma(\phi_k) = \begin{cases} \Gamma(u) & m\text{−}a.e. \text{ in } \{x : u(x) > k\} \\ 0 & m\text{−}a.e. \text{ in } \{x : u(x) \leqslant k\}, \end{cases}$$

we have $\{x : \Gamma(\phi_k) \neq 0\} \subset X_k$ up to a zero measure set. Thus, we get $m(X_k) \geqslant m(\{x : \Gamma(\phi_k) \neq 0\}) > 0$.

On the other hand, we have

$$\begin{aligned} \|\phi_k\|_{L^2(X_k)} &\leqslant \|\phi_k\|_{L^\nu(X_k)} \cdot (m(X_k))^{1/2-1/\nu} \leqslant \|\phi_k\|_{L^\nu(X)} \cdot (m(X_k))^{1/2-1/\nu} \\ &\leqslant \widetilde{C_S} \cdot \Big(\int_{X_k} \Gamma(\phi_k) \mathrm{d}m\Big)^{1/2} (m(X_k))^{1/2-1/\nu} \\ &\overset{(3.2)}{\leqslant} \widetilde{C_S} \cdot C_2 \cdot \|\phi_k\|_{L^2(X_k)} (m(X_k))^{1/2-1/\nu}, \end{aligned}$$

where we have used that $\{x : \Gamma(\phi_k) \neq 0\} \subset X_k$ up to a zero measure set again. Note that $m(X_k) > 0$, hence $\|\phi_k\|_{L^2(X_k)} \neq 0$, for all $k \in (k_0, \sup_X u)$, there is a constant $C > 0$, such that $m(X_k) > C$ for all $k_0 \leqslant k < \sup_X u$. Recall that $X_k = \{x : \Gamma(u) \neq 0\} \cap \{x : u(x) > k\}$, by letting $k \to \sup_X u$, we have

$$m(\{x : \Gamma(u) \neq 0\} \cap \{u = \sup_X u\}) \geqslant C.$$

This contradicts the fact that $\Gamma(u) = 0$ a.e. in $\{u = \sup_X u\}$ (see Proposition 2.22 of [11]), and proves the proposition. □

Let us recall the one-dimensional model operators $L_{R,l}$ in [6]. Given $R \in \mathbb{R}$ and $l > 1$, the one-dimensional models $L_{R,l}$ are defined as follows: let $L = R/(l-1)$,

(1) If $R > 0$, $L_{R,l}$ defined on $(-\pi/2\sqrt{L}, \pi/2\sqrt{L})$ by

$$L_{R,l}v(x) = v''(x) - (l-1)\sqrt{L}\tan(\sqrt{L}x)v'(x);$$

(2) If $R < 0$, $L_{R,l}$ defined on $(-\infty, \infty)$ by

$$L_{R,l}v(x) = v''(x) - (l-1)\sqrt{-L}\tanh(\sqrt{-L}x)v'(x);$$

(3) If $R = 0$, $L_{R,l}$ defined on $(-\infty, \infty)$ by

$$L_{R,l}v(x) = v''(x).$$

Next we will apply Corollary 2.7 to eigenfunctions and prove the following comparison theorem on the gradient of the eigenfunctions, which is an extension of Kröger's comparison result in [24].

Theorem 3.3. *Let (X, d, m) be a compact $RCD^*(K, N)$-space, and let λ_1 be the first eigenvalue on X. Let $l \in \mathbb{R}$ and $l \geqslant N$, and let f be an eigenfunction with respect to λ_1. Suppose $\lambda_1 > \max\left\{0, \frac{lK}{l-1}\right\}$. Let v be a Neumann eigenfunction of $L_{K,l}$ with respect to the same eigenvalue λ_1 on some interval. If $[\min f, \max f] \subset [\min v, \max v]$, then*

$$\Gamma(f) \leqslant (v' \circ v^{-1})^2(f) \ \ m\text{-a.e.}.$$

Proof. Without loss of generality, we may assume that $[\min f, \max f] \subset (\min v, \max v)$.

Denote by $T(x)$ the function such that

$$L_{K,l}(v) = v'' - Tv.$$

As in Corollary 3 in section 4 of [6], we can choose a smooth bounded function h_1 on $[\min f, \max f]$ such that

$$h_1' < \min\{Q_1(h_1), Q_2(h_1)\},$$

where Q_1, Q_2 are given by following

$$Q_1(h_1) := -(h_1 - T)\left(h_1 - \frac{2l}{l-1}T + \frac{2\lambda_1 v}{v'}\right),$$

$$Q_2(h_1) := -h_1\left(\frac{l-2}{2(l-1)}h_1 - T + \frac{\lambda_1 v}{v'}\right).$$

We can then take a smooth function g on $[\min f, \max f]$, $g \leqslant 0$ and $g' = -\frac{h_1}{v'} \circ v^{-1}$.

According to [3, Theorem 6.5] (see also [20, Theorem 1.1]), we have that f is Lipschitz continuous. Notice that $\Delta f = -\lambda_1 f \in \mathbb{V}$. Hence $f \in D_{\mathbb{V}}(\Delta) \cap Lip(X)$.

Now define a function F on X by

$$\psi(f)F = \Gamma(f) - \phi(f),$$

where $\psi(f) := e^{-g(f)}$ and $\phi(f) := (v' \circ v^{-1})^2(f)$. Since $f \in D_{\mathbb{V}}(\Delta) \cap Lip(X)$, by Theorem 2.6, we have $\Gamma(f) \in \mathbb{M}_\infty$. According to Lemma 2.4 and Remark 2.5, we have $\psi(f), \phi(f) \in D(\Delta) \cap Lip(X)$ and $F \in \mathbb{M}_\infty$. Moreover

$$\Delta^* F = \frac{1}{\psi}\Delta^*\Gamma + \frac{1}{\psi}\big(- 2\Gamma(\psi, F) - \Delta\psi F - \Delta\phi\big) \cdot m,$$

where and in the sequel, we denote by $\Gamma = \Gamma(f)$ and $\phi = \phi(f), \psi = \psi(f)$. By using Theorem 2.6 again, we have $\Delta^s F \geqslant 0$ on X and

$$\Delta^{ab} F = \frac{1}{\psi}\Big(\Delta^{ab}\Gamma - 2\Gamma(\psi, F) - \Delta\psi F - \Delta\phi\Big) \quad m-a.e. \text{ in } X.$$

Since $l \geqslant N$, the (X, d, m) satisfies also $RCD^*(K, l)$ condition. Applying inequality (2.2) to f and using $\Delta f = -\lambda_1 f$, we have, for m-a.e. $x \in \{x : \Gamma(x) > 0\}$,

$$\Delta^{ab}\Gamma \geqslant -2\lambda_1\Gamma + \frac{2{\lambda_1}^2}{l} f^2 + 2K\Gamma + \frac{2l}{l-1}\left(\frac{\lambda_1 f}{l} + \frac{\Gamma(f,\Gamma)}{2\Gamma}\right)^2.$$

Fix arbitrarily a constant $\epsilon_0 > 0$. We want to show $F \leqslant \epsilon_0$ m-a.e. in X.

Since $F \leqslant e^g \cdot \Gamma \leqslant \Gamma$, we have $\{x : F(x) \geqslant \epsilon_0\} \subset \{x : \Gamma(x) > 0\}$. Following the argument from line 29 on page 1182 to line 10 on page 1183 of [32], we get:

$$\Delta^{ab} F \geqslant \psi T_1 \cdot F^2 + T_2 \cdot F + T_3\Gamma(f, F) \quad m-a.e. \text{ on } \{x : F(x) \geqslant \epsilon_0\}, \tag{3.5}$$

where

$$v'^2 T_1 = Q_2(h_1) - h_1', \quad T_2 = Q_1(h_1) - h_1',$$

and

$$T_3 = \frac{2l}{l-1}\left(-\frac{g'}{2} + \frac{1}{2\Gamma}\left(\frac{2\lambda_1 f}{l} + \phi' + \phi g'\right)\right) + 2g'.$$

Note that both T_1 and T_2 are positive, Γ is bounded on X and T_3 is bounded on $\{x : F(x) \geqslant \epsilon_0\}$. It follows from (3.5) that

$$\Delta^{ab} F \geqslant c_1 \cdot F - c_2 \cdot \sqrt{\Gamma(F)} \quad a.e. \text{ on } \{x : F(x) \geqslant \epsilon_0\} \tag{3.6}$$

for some constant $c_2 > 0$ and $c_1 = \min_{s\in[\min f, \max f]} T_2(s) > 0$. By combining with $\Delta^s F \geqslant 0$ on X and Proposition 3.1, we conclude that $F \leqslant \epsilon_0$ m-a.e. in X.

At last, by the arbitrariness of ϵ_0, we have $F \leqslant 0$ m-a.e. in X. This completes the proof of Theorem 3.3. □

Let $v_{R,l}$ be the solution of the equation

$$L_{R,l} v = -\lambda_1 v$$

with initial value $v(a) = -1$ and $v'(a) = 0$, where

$$a = \begin{cases} -\frac{\pi}{2\sqrt{R/(l-1)}} & \text{if } R > 0, \\ 0 & \text{if } R \leqslant 0. \end{cases}$$

We denote

$$b = \inf\{x > a : \ v'_{R,l}(x) = 0\}$$

and

$$m_{R,l} = v_{R,l}(b)$$

Note that $v_{R,l}$ is non-decreasing on $[a, b]$.

Next we show the following comparison theorem on the maximum of eigenfunctions.

Theorem 3.4. *Let (X,d,m) be a compact $RCD^*(K,N)$-space, and let f be an eigenfunction with respect to the first eigenvalue λ_1 on X. Suppose* $\min f=-1, \max f\leqslant 1$. *Then we have*
$$\max f\geqslant m_{K,N}.$$

Proof. We argue by contradiction. Suppose $\max f<m_{K,N}$. Since $m_{K,l}$ is continuous on l, we can find some real number $l>N$ such that
$$\max f\leqslant m_{K,l} \text{ and } \lambda_1>\max\{0,\frac{lK}{l-1}\}.$$

Then following the proof of Proposition 5 in [6], we obtain that the ratio
$$R(s)=-\frac{\int_X f1_{\{f\leqslant v(s)\}}\mathrm{d}m}{\rho(s)v'(s)}$$
is increasing on $[a,v^{-1}(0)]$ and decreasing on $[v^{-1}(0),b]$, where the function ρ is
$$\rho(s)\ :=\begin{cases}\cos^{l-1}(\sqrt{L}s) & \text{if } L>0\\ s^{l-1} & \text{if } L=0\\ \sinh^{l-1}(\sqrt{-L}s) & \text{if } L<0\end{cases}$$
and $L=K/(l-1)$. It follows that for any $s\in[a,v^{-1}(-1/2)]$, since $v(s)\leqslant-\frac{1}{2}$, we have
$$m(\{f\leqslant v(s)\})\leqslant-2\int_X f1_{\{f\leqslant v(s)\}}\mathrm{d}m\leqslant 2C\rho(s)v'(s), \tag{3.7}$$
where $C=R(v^{-1}(0))$.

Take $p\in X$ with $f(p)=-1$. By
$$f-f(p)\geqslant 0,\qquad \text{and } \Delta(f-f(p))=-\lambda_1 f\leqslant\lambda_1$$
The mean value inequality (2.4) implies that
$$⨍_{B_r(p)}(f-f(p))\mathrm{d}m\leqslant C\lambda_1 r^2$$
for all $r>0$ such that $B_r(x)\subset X$. Denote $C_1=C\lambda_1$. Let $A(r)=\{f-f(p)>2C_1r^2\}\cap B_p(r)$. Then
$$\frac{m(A(r))}{m(B_p(r))}\leqslant\frac{⨍_{B_r(p)}(f-f(p))\mathrm{d}m}{2C_1r^2m(B_r(p))}\leqslant\frac{1}{2}.$$
Hence
$$\begin{aligned}\tfrac{1}{2}m(B_r(p)) &\leqslant m(B_r(p)\backslash A(r))\\ &\leqslant m(\{f-f(p)\leqslant 2C_1r^2\})\\ &=m(\{f\leqslant -1+2C_1r^2\}).\end{aligned}$$
By using (3.7) and following the argument from line 1 on Page 1186 to line 3 on page 1187 of [32], one can get that there exists a constant $C_2>0$ such that
$$m(B_p(r))\leqslant C_2r^l$$
for all sufficiently small $r>0$.

Fix $r_0>0$. By Bishop-Gromov inequality (2.3), we have
$$m(B_p(r))\geqslant\frac{m(B_p(r_0))}{\int_0^{r_0}\mathfrak{s}_{\frac{K}{N-1}}(t)^{N-1}\mathrm{d}t}\int_0^r\mathfrak{s}_{\frac{K}{N-1}}(t)^{N-1}\mathrm{d}t\geqslant C_3r^N$$

for any $0 < r < r_0$. The combination of the above two inequalities implies that $C_2 r^{l-N} \geqslant C_3$ holds for any sufficiently small r. Hence, we have $l \leqslant N$, which contradicts to the assumption $l > N$. Therefore, the Proof of Theorem 3.4 is finished. $\square$

Now we are in the position to prove the main result—Theorem 1.2.

Proof of Theorem 1.2. Let λ_1 and f denote respectively the first non-zero eigenvalue and a corresponding eigenfunction with $\min f = -1$ and $\max f \leqslant 1$. By Theorem 4.22 of [16], we have $\lambda_1 \geqslant NK/(N-1)$ if $K > 0$ and $N > 1$. Now fix any $R < K$, we have

$$\lambda_1 > \max\left\{\frac{NR}{N-1}, 0\right\}.$$

Then we may use the results of Section 6 and Section 3 of [6], we can find an interval $[a, b]$ such that the one-dimensional model operator $L_{R,N}$ has the first Neumann eigenvalue λ_1 and a corresponding eigenfunction v with $v(a) = \min v = -1$ and $v(b) = \max v = \max f$. By Theorem 13 in Section 7 of [6], we have

$$\lambda_1 \geqslant \hat{\lambda}(R, N, b-a), \tag{3.8}$$

where $\hat{\lambda}(R, N, b-a)$ is the first non-zero Neumann eigenvalue of $L_{R,N}$ on the symmetric interval $(-\frac{b-a}{2}, \frac{b-a}{2})$. Note that f is continuous, we take two points x and y in X such that $f(x) = -1$ and $f(y) = \max f$. Let $g = v^{-1} \circ f$, then $g(x) = a$, $g(y) = b$ and, by Theorem 3.3, $\Gamma(g) \leqslant 1$ m-a.e. in X. Hence, we have

$$b - a = g(y) - g(x) \leqslant d(x, y) \leqslant \max_{z_1, z_2 \in X} d(z_1, z_2) := d,$$

where d is the diameter of X. Together with (3.8) and the fact that the function $\hat{\lambda}(R, N, s)$ decreases with s, we conclude

$$\lambda_1 \geqslant \hat{\lambda}(R, N, d).$$

By the arbitrariness of R, we finally prove the theorem. $\square$

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