

SHARP SPECTRAL GAP AND LI-YAU'S ESTIMATE ON ALEXANDROV SPACES

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ABSTRACT. In the previous work [35], the second and third authors established a Bochner type formula on Alexandrov spaces. The purpose of this paper is to give some applications of the Bochner type formula. Firstly, we extend the sharp lower bound estimates of spectral gap, due to Chen–Wang [9, 10] and Bakry–Qian [6], from smooth Riemannian manifolds to Alexandrov spaces. As an application, we get an Obata type theorem for Alexandrov spaces. Secondly, we obtain (sharp) Li–Yau’s estimate for positive solutions of heat equations on Alexandrov spaces.

1. INTRODUCTION

Let $n \geq 2$ and M a compact n -dimensional Alexandrov space without boundary. It is well known that the first non-zero eigenvalue of the (canonical) Laplacian is given by:

$$\lambda_1(M) := \inf \left\{ \frac{\int_M |\nabla f|^2 d\text{vol}}{\int_M f^2 d\text{vol}} : f \in Lip(M) \setminus \{0\} \text{ and } \int_M f d\text{vol} = 0 \right\},$$

where $Lip(M)$ is the set of Lipschitz functions on M .

When M is a smooth compact Riemannian manifold, the study of the lower bound estimate of first non-zero eigenvalue $\lambda_1(M)$ has a long history, see for example Lichnerowicz [21], Cheeger [7], Li–Yau [20], and so on. For an overview the reader is referred to the introduction of [6], [2, 18] and Chapter 3 in book [31], and references therein.

Let M^n be a compact n -dimensional Riemannian manifold without boundary. Lichnerowicz’s estimate asserts that $\lambda_1(M^n) \geq n$ if Ricci curvature of the manifold M^n is bounded below by $n - 1$. Later Obata [24] proved that the equality holds if and only if the manifold M^n is isometric to \mathbb{S}^n with the standard metric. Zhong–Yang’s estimate [36] asserts that $\lambda_1(M^n) \geq \pi^2/\text{diam}^2(M^n)$ if M^n has nonnegative Ricci curvature. The statement is optimal. In [13], Hang–Wang proved that if the equality holds, then M^n must be isometric to the circle of radius $\text{diam}(M^n)/\pi$. Chen–Wang in [9, 10] and Bakry–Qian in [6] put these two lower bound estimates in a same framework, which is the following comparison theorem:

Theorem 1.1. (Chen–Wang [9, 10], Bakry–Qian [6]) *Let M^n be a compact Riemannian manifold of dimension n (with a convex boundary or without boundary) and $\text{Ric}(M^n) \geq (n - 1)K$. Then the first non-zero (Neumann) eigenvalue satisfies*

$$\lambda_1(M^n) \geq \lambda_1(K, n, d),$$

where d is the diameter of M^n , $\lambda_1(K, n, d)$ denotes the first non-zero Neumann eigenvalue of the following one-dimensional model:

$$v''(x) - (n - 1)T(x)v'(x) = -\lambda v(x) \quad x \in \left(-\frac{d}{2}, \frac{d}{2}\right), \quad v'\left(-\frac{d}{2}\right) = v'\left(\frac{d}{2}\right) = 0$$

and

$$T(x) = \begin{cases} \sqrt{K} \tan(\sqrt{K}x) & \text{if } K \geq 0, \\ -\sqrt{-K} \tanh(\sqrt{-K}x) & \text{if } K < 0. \end{cases}$$

In [27], Petrunin extended the Lichnerowicz's estimate to Alexandrov spaces with curvature ≥ 1 . More generally, in [23], Lott–Villani extended Lichnerowicz's estimate to a metric measure space with $CD(n, (n-1))$ ¹. In particular, Lichnerowicz's estimate holds on an n -dimensional Alexandrov space M with $\text{Ric}(M) \geq n-1$. This was also proved by the second and third named authors in [35] via a different method.

For simplicity, we always assume that the Alexandrov space M has empty boundary. Our first result in this paper is an extension of the above comparison result (Theorem 1.1) on Alexandrov spaces. Explicitly, we will prove the following:

Theorem 1.2. *Let M be a compact n -dimensional Alexandrov space without boundary and $\text{Ric}(M) \geq (n-1)K$. Then its the first non-zero eigenvalue satisfies*

$$\lambda_1(M) \geq \lambda_1(K, n, d),$$

where d is the diameter of M and $\lambda_1(K, n, d)$ as above in Theorem 1.1.

As a consequence, by combining with the maximal diameter theorem in [33], we obtain an Obata type theorem (see [28] for the case of orbifolds).

Corollary 1.3. *Let M be a compact n -dimensional Alexandrov space without boundary and $\text{Ric}(M) \geq (n-1)$. If $\lambda_1(M) = n$, then M is isometric to a spherical suspension over an $(n-1)$ -dimensional Alexandrov space with curvature ≥ 1 .*

There are two different approaches to prove Theorem 1.1. One is a probabilistic way: Chen–Wang [9, 10] used the Kendall–Cranston coupling method to prove Theorem 1.1. This way does not work directly on Alexandrov spaces, since it is not clear how to construct Brownian motions and how to define SDE on Alexandrov spaces. The other is an analytic way, given by Bakry and the first author in [6]. The latter approach consists of three parts. In the first part, by combining Bochner's formula and a smooth maximum principle argument, Kröger in [15] obtained a comparison theorem for the gradient of the eigenfunctions, which was also proved by Bakry–Qian in [6] for general differential operators L with curvature-dimension condition $CD(n, R)$. Secondly, by using the comparison result on the gradient of eigenfunctions and the boundedness of Hessian of eigenfunctions, Bakry–Qian proved a comparison theorem for the maximum of eigenfunctions. In the last part, Bakry–Qian [6] developed a deep analysis on the one-dimensional models to prove Theorem 1.1. For Alexandrov spaces with Ricci curvature bounded below, a Bochner type formula has been established by the second and third authors in [35]. Our proof of Theorem 1.2 is basically along the line of Bakry–Qian's proof in [6]. However, we must overcome the difficulties bringing in due to lacking of a smooth maximum principle and the boundedness of Hessian of eigenfunctions on Alexandrov spaces. To overcome the first difficulty, we will replace the smooth maximum principle argument by a method of upper bound estimate for weak solutions of elliptic equations. To overcome the second difficulty, we will appeal to a mean value inequality of Poisson equations in [35].

After we completed this paper (which was posted on Arxiv in Feb. 2011), we noted that Andrews–Clutterbuck in [1] provided a heat equation proof for the above Theorem 1.1.

¹This is a generalized notion of Ricci curvature bounded below by $n-1$ on metric measure spaces. We refer the reader to a survey [34] for others generalizations of the lower bounds of Ricci curvature on singular spaces, in particular on Alexandrov spaces.

Recently, the sharp estimate has been also extended to Finsler manifolds by Wang-Xia in [32].

The second purpose of this paper is to extend Li-Yau's parabolic estimates from smooth Riemannian manifolds to Alexandrov spaces.

Theorem 1.4. *Let M be a compact n -dimensional Alexandrov space with nonnegative Ricci curvature and $\partial M = \emptyset$. Assume that $u(x, t)$ is a positive solution of heat equation $\frac{\partial}{\partial t}u = \Delta u$ on $M \times [0, \infty)$. Then we have*

$$(1.1) \quad |\nabla \log u|^2 - \frac{\partial}{\partial t} \log u \leq \frac{n}{2t}$$

for any $t > 0$.

Here Δ is the generator of the canonical Dirichlet form on M (see Section 5 for the details). As an application of this estimate, a sharper Harnack inequality of positive solutions of heat equation is obtained (see Corollary 5.2).

When M is a smooth Riemannian manifold, Li-Yau in [22] proved (1.1) by Bochner's formula and smooth maximum principle. In [5], Bakry-Ledoux developed an abstract method to prove (1.1). They used only Bochner's formula. We will use Bakry-Ledoux's method to prove Theorem 1.4. To apply the Bochner type formula in [35], we need to establish necessary regularity for positive solutions of the heat equations.

The paper is organized as follows. In Section 2, we recall some necessary materials for Alexandrov spaces. In Section 3, we will prove the gradient estimate for the first eigenfunction and establish a comparison result for the maximum of eigenfunctions. In Section 4, we will prove Theorem 1.2 and some corollaries of it. Obata type theorem and some explicit lower bound estimates will be given in this section. In the last section, we will consider the heat equations on Alexandrov spaces. Li-Yau's parabolic estimate and a sharper Harnack estimate of positive solutions of heat equations will be obtained in this section.

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2. PRELIMINARIES ON ALEXANDROV SPACES

Let $(X, |\cdot|)$ be a metric space. A rectifiable curve γ connecting two points p, q is called a geodesic if its length is equal to $|pq|$ and it has unit speed. A metric space X is called a geodesic space if every pair points $p, q \in X$ can be connected by *some* geodesic.

Let $k \in \mathbb{R}$ and $l \in \mathbb{N}$. Denote by \mathbb{M}_k^l the simply connected, l -dimensional space form of constant sectional curvature k . The model spaces are \mathbb{M}_k^2 . Given three points p, q, r in a geodesic space X , we can take a comparison triangle $\Delta \bar{p}\bar{q}\bar{r}$ in \mathbb{M}_k^2 such that $|\bar{p}\bar{q}| = |pq|$, $|\bar{q}\bar{r}| = |qr|$ and $|\bar{r}\bar{p}| = |rp|$. If $k > 0$, we add the assumption $|pq| + |qr| + |rp| < 2\pi/\sqrt{k}$. Angles $\tilde{\angle}_k pqr := \angle \bar{p}\bar{q}\bar{r}$ etc. are called comparison angles.

A geodesic space X is called an Alexandrov space (of locally curvature bounded below) if it satisfies the following property:

- (i) it is locally compact;
- (ii) for any point $x \in X$ there exists a neighborhood U_x of x and a real number κ such that, for any four different points p, a, b, c in U_x , we have

$$\tilde{\angle}_\kappa apb + \tilde{\angle}_\kappa bpc + \tilde{\angle}_\kappa cpa \leq 2\pi.$$

The Hausdorff dimension of an Alexandrov space is always an integer. We refer to the seminar paper [4] or the text book [3] for the details.

Let $n \geq 2$ and M an n -dimensional Alexandrov space and Ω be a domain in M . The Sobolev spaces $W^{1,p}(\Omega)$ is well defined (see, for example [16]). We denote by $Lip_0(\Omega)$ the set of Lipschitz continuous functions on Ω with compact support in Ω . Spaces $W_0^{1,p}(\Omega)$ is defined by the closure of $Lip_0(\Omega)$ under $W^{1,p}(\Omega)$ -norm. We say a function $u \in W_{loc}^{1,p}(\Omega)$ if $u \in W^{1,p}(\Omega')$ for every open subset $\Omega' \Subset \Omega$.

Denote by vol the n -dimensional Hausdorff measure on M . The canonical Dirichlet energy (form) $\mathcal{E} : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \rightarrow \mathbb{R}$ is defined by

$$\mathcal{E}(u, v) := \int_{\Omega} \langle \nabla u, \nabla v \rangle d\text{vol} \quad \text{for } u, v \in W_0^{1,2}(\Omega).$$

Given a function $u \in W_{loc}^{1,2}(\Omega)$, a functional \mathcal{L}_u is defined on $Lip_0(\Omega)$ by

$$\mathcal{L}_u(\phi) := - \int_{\Omega} \langle \nabla u, \nabla \phi \rangle d\text{vol}, \quad \forall \phi \in Lip_0(\Omega).$$

Let $f \in L^2(\Omega)$. If $u \in W_{loc}^{1,2}(\Omega)$ such that \mathcal{L}_u is bounded below (or above) in the following sense that

$$\mathcal{L}_u(\phi) \geq \int_{\Omega} f \phi d\text{vol} \quad \left(\text{or } \mathcal{L}_u(\phi) \leq \int_{\Omega} f \phi d\text{vol} \right)$$

for all nonnegative $\phi \in Lip_0(\Omega)$, then the functional \mathcal{L}_u is a signed Radon measure. In this case, u is said to be a sub-solution (super-solution, resp.) of Poisson equation

$$\mathcal{L}_u = f \cdot \text{vol}.$$

A function u is a (weak) solution of Poisson equation $\mathcal{L}_u = f \cdot \text{vol}$ on Ω if it is both a sub-solution and a super-solution of the equation. In particular, a (weak) solution of $\mathcal{L}_u = 0$ is called a harmonic function.

If $f, g \in W^{1,2}(\Omega)$ and \mathcal{L}_g is a signed Radon measure, then

$$(f \mathcal{L}_g)(\phi) = \int_{\Omega} \phi f \mathcal{L}_g(dx) = - \int_{\Omega} \langle f \nabla \phi + \phi \nabla f, \nabla g \rangle d\text{vol}$$

for any $\phi \in Lip_0(\Omega)$. Hence, it is easy to check that if $f, g, fg \in W^{1,2}(\Omega)$ and $\mathcal{L}_f, \mathcal{L}_g, \mathcal{L}_{fg}$ are signed Radon measures, we have

$$\mathcal{L}_{fg} = f \mathcal{L}_g + g \mathcal{L}_f + 2 \langle \nabla f, \nabla g \rangle \cdot \text{vol}$$

and if, in addition, f is bounded, then we have

$$\mathcal{L}_{\Phi(f)} = \Phi'(f) \mathcal{L}_f + \Phi''(f) |\nabla f|^2 \cdot \text{vol}$$

for any $\Phi \in C^2(\mathbb{R})$.

In [33], the second and third authors introduced a notion of ‘‘Ricci curvature has a lower bound R ’’, denoted by $Ric \geq R$. On an n -dimensional Alexandrov space M , the condition $Ric \geq R$ implies that M (equipped with its Hausdorff measure) satisfies Sturm–Lott–Villani’s convexity dimension condition $CD(n, R)$ [30, 23] and Kuwae–Shioya–Ohta’s infinitesimal Bishop–Gromov condition (or measure contraction property) $BG(n, R)$ [17, 25] (see [26] and Appendix in [33]). Of course, an n -dimensional Alexandrov space M with curvature $\geq K$ must have $Ric(M) \geq (n-1)K$.

In [35], the following Bochner type formula was established.

Theorem 2.1. (Theorem 1.2 in [35]) *Let M be an n -dimensional Alexandrov space with Ricci curvature bounded from below by R , and Ω be a bounded domain in M . Let $F(x, s) : \Omega \times [0, +\infty) \rightarrow \mathbb{R}$ be a Lipschitz function and satisfy the following:*

(a) *there exists a zero measure set $\mathcal{N} \subset \Omega$ such that for all $s \geq 0$, the functions $F(\cdot, s)$*

are differentiable at any $x \in \Omega \setminus \mathcal{N}$;

(b) the function $F(x, \cdot)$ is of class C^1 for all $x \in \Omega$ and the function $\frac{\partial F}{\partial s}(x, s)$ is continuous, non-positive on $\Omega \times [0, +\infty)$.

Suppose that u is Lipschitz on Ω and satisfies

$$\mathcal{L}_u = F(x, |\nabla u|^2) \cdot \text{vol}.$$

Then we have $|\nabla u|^2 \in W_{loc}^{1,2}(\Omega)$ and

$$\mathcal{L}_{|\nabla u|^2} \geq \left(\frac{2}{n} F^2(x, |\nabla u|^2) + 2 \langle \nabla u, \nabla F(x, |\nabla u|^2) \rangle + 2R |\nabla u|^2 \right) \cdot \text{vol},$$

provided $|\nabla u|$ is lower semi-continuous at almost all $x \in \Omega$ (That is, there exists a representative of $|\nabla u|$, which is lower semi-continuous at almost all $x \in \Omega$).

For our purpose in this paper, we give the following corollary.

Corollary 2.2. *Let M be an n -dimensional Alexandrov space with Ricci curvature bounded from below by R , and Ω be a bounded domain in M . Let f be a Lipschitz continuous function in Ω and $u \in W_{loc}^{1,2}(\Omega)$ satisfies*

$$\mathcal{L}_u = f \cdot \text{vol}.$$

Suppose that a real function $\Phi(t) \in C^3(\mathbb{R})$ satisfies $\Phi'(t) \neq 0$ and $\Phi''(t) \leq 0$ for all t in the range of u . Then we have $|\nabla \Phi(u)|^2 \in W_{loc}^{1,2}(\Omega)$ and

$$\mathcal{L}_{|\nabla \Phi(u)|^2} \geq \left(\frac{2}{n} \Psi^2 + 2 \langle \nabla \Phi(u), \nabla \Psi \rangle + 2R |\nabla \Phi(u)|^2 \right) \cdot \text{vol},$$

where

$$\Psi(x) := \Phi'(u)f(x) + \Phi''(u)|\nabla u|^2.$$

Proof. Since f is Lipschitz continuous in Ω , in Corollary 5.5 in [35], it is shown that u is locally Lipschitz continuous in Ω , and in Corollary 5.8 in [35], it is shown that $|\nabla u|$ is lower semi-continuous in Ω . Thus, u is differential at almost everywhere in Ω , and $|\nabla \Phi(u)|$ is lower semi-continuous in Ω .

Fix any open set $\Omega' \Subset \Omega$ and define the function $F(x, s) : \Omega' \times [0, +\infty)$ by

$$F(x, s) := \Phi'(u)f(x) + \frac{\Phi''(u)}{[\Phi'(u)]^2} \cdot s.$$

Since both f and u are locally Lipschitz continuous in Ω , we have the function $F(\cdot, s)$ is Lipschitz continuous on $\Omega' \times [0, +\infty)$. Note that $\Phi''(t) \leq 0$, it is easy check that $F(x, s)$ satisfies the conditions (a) and (b) in Theorem 1.2.

From $\mathcal{L}_u = f \cdot \text{vol}$, we get

$$\begin{aligned} \mathcal{L}_{\Phi(u)} &= \Phi'(u)\mathcal{L}_u + \Phi''(u)|\nabla u|^2 \cdot \text{vol} \\ &= \Psi(x) \cdot \text{vol} = F(x, |\nabla \Phi(u)|^2) \cdot \text{vol}. \end{aligned}$$

Now, we can apply Theorem 2.1 to conclude the desired result in this corollary. \square

The same trick as in the proof of Theorem 6 in [6] gives an improvement of the Bochner inequality as following:

Corollary 2.3. *Let M , u and f be as above in Corollary 2.2, and let $\Omega \subset M$ be an open set. Assume $|\nabla u| \geq c > 0$ a.e. on Ω for some constant c . Then we have the following improved Bochner formula*

$$(2.1) \quad \mathcal{L}_{|\nabla u|^2} \geq \left(\frac{2}{l} f^2 + 2 \langle \nabla u, \nabla f \rangle + 2R |\nabla u|^2 + \frac{2l}{l-1} \left(\frac{f}{l} - \frac{\langle \nabla u, \nabla |\nabla u|^2 \rangle}{2|\nabla u|^2} \right)^2 \right) \cdot \text{vol}$$

on Ω , for all real number $l \geq n$.

The following mean value inequality was also obtained in [35].

Proposition 2.4. (Corollary 4.5 in [35]) *Let M be an n -dimensional Alexandrov space with Ricci curvature bounded from below by R and let Ω be a domain in M . If u is continuous and satisfies that $\mathcal{L}_u \leq c_1 \cdot \text{vol}$ on Ω and $u \geq 0$, then, for any $p \in \Omega$, there exists a constant $c_2 = c_2(n, \Omega, p, c_1)$ such that*

$$\frac{1}{\text{vol}(B_o(r) \subset T_p^{R/(n-1)})} \int_{B_p(r)} u d\text{vol} \leq u(p) + c_2 r^2$$

for any sufficiently small r with $B_p(r) \Subset \Omega$, where $T_p^{R/(n-1)}$ is the $\frac{R}{n-1}$ -cone over Σ_p , the space of directions (see [3] p. 354).

3. COMPARISON THEOREMS ON GRADIENT AND MAXIMUM OF EIGENFUNCTIONS

Let M be a compact n -dimensional Alexandrov space without boundary and $\text{Ric}(M) \geq R := (n-1)K$. Let λ_1 be the non-zero first eigenvalue and f be a first eigenfunction on M . That is, f is a minimizer of

$$\lambda_1 := \inf \left\{ \frac{\int_M |\nabla \phi|^2 d\text{vol}}{\int_M \phi^2 d\text{vol}} : \phi \in \text{Lip}(M) \setminus \{0\} \quad \text{and} \quad \int_M \phi d\text{vol} = 0 \right\}.$$

It is easy to check that \mathcal{L}_f is a measure and satisfies

$$(3.1) \quad \mathcal{L}_f = -\lambda_1 f \cdot \text{vol}.$$

We set $G = |\nabla f|^2$ in this section. The following regularity result is necessary for us.

Lemma 3.1. *G is lower semi-continuous on M and lies in $W^{1,2}(M)$.*

Proof. It was proved that f is Lipschitz continuous on M in [27] (see also Theorem 4.3 in [12] or Corollary 5.5 in [35]). Now by applying Theorem 2.1 to the equation (3.1), we may deduce that $G \in W^{1,2}(M)$ and

$$\mathcal{L}_G \geq (-2\lambda_1 + 2R)G \cdot \text{vol}.$$

If $2\lambda_1 - 2R \leq 0$, then $\mathcal{L}_G \geq 0$. This concludes that G has a lower semi-continuous representation in $W^{1,2}(M)$ (see Theorem 5.1 in [14]). If $\mu := 2\lambda_1 - 2R > 0$, we consider the function $g = e^{\sqrt{\mu}t}G$ on $M \times \mathbb{R}$ with directly product metric and obtain $\mathcal{L}_g \geq 0$. Hence g has also a lower semi-continuous representation, and therefore G is lower semi-continuous. \square

Let us recall the one-dimensional model operators $L_{R,l}$ in [6]. Given $R \in \mathbb{R}$ and $l > 1$, the one-dimensional models $L_{R,l}$ are defined as follows: setting $K = R/(l-1)$,

- (1) If $R > 0$, $L_{R,l}$ defined on $(-\pi/2\sqrt{K}, \pi/2\sqrt{K})$ by

$$L_{R,l}v(x) = v''(x) - (l-1)\sqrt{K} \tan(\sqrt{K}x)v'(x);$$

- (2) If $R < 0$, $L_{R,l}$ defined on $(-\infty, \infty)$ by

$$L_{R,l}v(x) = v''(x) + (l-1)\sqrt{-K} \tanh(\sqrt{-K}x)v'(x);$$

and

- (3) If $R = 0$, $L_{R,l}$ defined on $(-\infty, \infty)$ by

$$L_{R,l}v(x) = v''(x).$$

We refer the readers to [6] for the properties of $L_{R,l}$.

The first purpose of this section is to show the following comparison result on the gradients of eigenfunctions, which is an extension of Kröger's comparison result in [15]. In smooth case, the proof of this result in [15, 6] relies on smooth maximum principle. For (singular) Alexandrov spaces we need to use a method of upper bound estimate for weak solutions of elliptic equations.

Theorem 3.2. *Let $l \in \mathbb{R}$ and $l \geq n$. Suppose $\lambda_1 > \max\{0, \frac{lR}{l-1}\}$. Let v be a Neumann eigenfunction of $L_{R,l}$ with respect to the same eigenvalue λ_1 on some interval. If $[\min f, \max f] \subset [\min v, \max v]$, then*

$$G := |\nabla f|^2 \leq (v' \circ v^{-1})^2(f).$$

Proof. Without loss of generality, we may assume that

$$[\min f, \max f] \subset (\min v, \max v).$$

Denote by $T(x)$ the function such that

$$L_{R,l}(v) = v'' - Tv'.$$

As in Corollary 3 in Section 4 of [6], we can choose a smooth bounded function h_1 on $[\min f, \max f]$ such that

$$h_1' < \min\{Q_1(h_1), Q_2(h_1)\},$$

where Q_1, Q_2 are given by following

$$\begin{aligned} Q_1(h_1) &:= -(h_1 - T)\left(h_1 - \frac{2l}{l-1}T + \frac{2\lambda_1 v}{v'}\right), \\ Q_2(h_1) &:= -h_1\left(\frac{l-2}{2(l-1)}h_1 - T + \frac{\lambda_1 v}{v'}\right). \end{aligned}$$

We can then take a smooth function g on $[\min f, \max f]$ such that $g \leq 0$ and $g' = -\frac{h_1}{v'} \circ v^{-1}$.

Now define a function F on M by

$$\psi(f)F = G - \phi(f),$$

where

$$\psi(f) := e^{-g(f)} \quad \text{and} \quad \phi(f) := (v' \circ v^{-1})^2(f).$$

It suffices to show $F \leq 0$ on M .

Let us argue by contradiction. Suppose there exists a positive small number ϵ_0 such that the set $\{x \in M : F(x) \geq \epsilon_0\}$ has positive measure.

Consider the set $\Omega = \{x \in M : F(x) > \frac{\epsilon_0}{2}\}$. By Lemma 3.1 and the continuity of f , we know that F is lower semi-continuous on M , hence, Ω is an open subset in M . Without loss of generality, we may assume that Ω is connected. Since $\psi(f) \geq 1$ and $\phi(f) \geq 0$, we have $G > \frac{\epsilon_0}{2}$ on Ω .

In the calculation below, we write only Φ instead of $\Phi(f)$ for any function Φ on \mathbb{R} .

By applying (2.1) to $\mathcal{L}_f = -\lambda_1 f$, we have

$$(3.2) \quad \mathcal{L}_G \geq \left(-2\lambda_1 G + \frac{2\lambda_1^2}{l}f^2 + 2RG + \frac{2l}{l-1}\left(\frac{\lambda_1 f}{l} + \frac{\langle \nabla f, \nabla G \rangle}{2G}\right)^2\right) \text{vol}.$$

Noticing that $\langle \nabla f, \nabla G \rangle = \psi'FG + \phi'G + \psi \langle \nabla f, \nabla F \rangle$, we get

$$\begin{aligned} \left(\frac{\lambda_1 f}{l} + \frac{\langle \nabla f, \nabla G \rangle}{2G}\right)^2 &= \left(\frac{\lambda_1 f}{l} + \frac{\phi'}{2}\right)^2 + \frac{\psi'^2 F^2}{4} + \frac{\psi^2 (\langle \nabla f, \nabla F \rangle)^2}{4G^2} \\ &\quad + \left(\frac{\lambda_1 f}{l} + \frac{\phi'}{2}\right) \cdot \left(F\psi' + \frac{\psi \langle \nabla f, \nabla F \rangle}{G}\right) + \frac{F\psi'\psi}{2G} \langle \nabla f, \nabla F \rangle. \end{aligned}$$

Now $\phi(f)$ the following (see pp. 133 in [6])

$$\phi(-2\lambda_1 - \phi'' + 2R) + \lambda_1 f \phi' + \frac{2}{l} \lambda_1^2 f^2 + \frac{2l}{l-1} \left(\frac{\lambda_1 f}{l} + \frac{\phi'}{2} \right)^2 = 0.$$

Putting these equations to $\mathcal{L}_{\psi F} = \mathcal{L}_G - \mathcal{L}_\phi$, we have

$$\mathcal{L}_F \geq \mathcal{A} \cdot \text{vol},$$

where

$$\begin{aligned} \mathcal{A} = & \frac{l}{2(l-1)} \frac{\psi'^2}{\psi} F^2 - \frac{\psi''}{\psi} FG + \frac{2l}{l-1} \langle \nabla f, \nabla F \rangle \left(\frac{1}{G} \left(\frac{\lambda_1 f}{l} + \frac{\phi'}{2} \right) + \frac{F\psi'}{2G} \right) \\ & + \frac{1}{\psi} \left(\lambda_1 \psi' f + \frac{2l\psi'}{l-1} \left(\frac{\lambda_1 f}{l} + \frac{\phi'}{2} \right) + (-2\lambda_1 + 2R - \phi'') \right) F - 2 \frac{\psi'}{\psi} \langle \nabla f, \nabla F \rangle. \end{aligned}$$

Then by substituting $G = \psi F + \phi$ and $\psi = e^{-g}$ into the above expression, we obtain the following inequality

$$(3.3) \quad \mathcal{L}_F \geq \left(\psi(f) T_1 \cdot F^2 + T_2 \cdot F + T_3 \langle \nabla f, \nabla F \rangle \right) \text{vol},$$

where

$$v'^2 T_1 = Q_2(h_1) - h'_1, \quad T_2 = Q_1(h_1) - h'_1$$

and

$$T_3 = \frac{2l}{l-1} \left(-\frac{g'}{2} + \frac{1}{2G} \left(\frac{2\lambda_1 f}{l} + \phi' + \phi g' \right) \right) + 2g'.$$

Note that both T_1 and T_2 are positive, and both T_3 and $|\nabla f|$ are bounded on Ω . It follows from (3.3) that

$$(3.4) \quad \mathcal{L}_F \geq -c |\nabla F| \cdot \text{vol}$$

on Ω for some constant c .

Recall that we have assumed that the set $\{x \in M : F(x) \geq \epsilon_0\}$ has positive measure. To get the desired contradiction, we only need to show

$$(3.5) \quad \sup_{\Omega} F \leq \frac{\epsilon_0}{2}.$$

Take any constant k to satisfy $\epsilon_0/2 \leq k < \sup_{\Omega} F$, and set $\phi_k = (F - k)^+$. (If no such k exists, we are done.) By the definition of domain Ω , we have $\phi_k \in W_0^{1,2}(\Omega)$. From (3.4), we have

$$\begin{aligned} \int_{\Omega} \langle \nabla F, \nabla \phi_k \rangle d\text{vol} &= - \int_{\Omega} \phi_k d\mathcal{L}_F \leq c \int_{\Omega} \phi_k |\nabla F| d\text{vol} \\ &\leq c \left(\int_{\Omega_k} |\nabla F|^2 d\text{vol} \right)^{1/2} \left(\int_{\Omega_k} \phi_k^2 d\text{vol} \right)^{1/2}, \end{aligned}$$

where $\Omega_k = \text{supp} |\nabla \phi_k| \subset \text{supp} \phi_k \subset \Omega$. Here we have used the fact that $|\nabla \phi_k| = |\nabla F|$ in $\text{supp} \phi_k$. Hence, we have

$$(3.6) \quad \int_{\Omega_k} |\nabla \phi_k|^2 d\text{vol} \leq c^2 \int_{\Omega_k} \phi_k^2 d\text{vol}.$$

Since Ω is bounded (by that M is compact) and $\text{Ric}(M) \geq R$, we have the following Sobolev inequality on Ω (see, for example [16] or [34]): there exists $\nu > 2$ and $C_S = C_S(n, \nu, \Omega) > 0$ such that

$$(3.7) \quad C_S \left(\int_{\Omega} |\psi|^{\nu} d\text{vol} \right)^{2/\nu} \leq \int_{\Omega} |\nabla \psi|^2 d\text{vol}, \quad \forall \psi \in W_0^{1,2}(\Omega).$$

By combining with (3.6), we get

$$\begin{aligned} \|\phi_k\|_{L^2(\Omega_k)}^2 &\leq \|\phi_k\|_{L^\nu(\Omega_k)}^2 \cdot (\text{vol}(\Omega_k))^{1-2/\nu} \\ &\leq C_S^{-1} \int_{\Omega} |\nabla \phi_k|^2 d\text{vol} \cdot (\text{vol}(\Omega_k))^{1-2/\nu} \\ &\leq c^2 C_S^{-1} \cdot (\text{vol}(\Omega_k))^{1-2/\nu} \cdot \|\phi_k\|_{L^2(\Omega_k)}^2. \end{aligned}$$

Thus we deduce that

$$\text{vol}(\Omega_k) \geq C$$

for some constant $C = C(c, n, \nu, C_S) > 0$, which is independent of k . Noting that $\Omega_k \subset \text{supp}|\nabla F| \cap \{F \geq k\}$ and letting k tend to $\sup_{\Omega} F$, we have

$$\text{vol}(\text{supp}|\nabla F| \cap \{F = \sup_{\Omega} F\}) \geq C.$$

This is impossible, since $|\nabla F| = 0$ a.e. in $\{F = \sup_{\Omega} F\}$ (see Proposition 2.22 in [8]). Hence the desired (3.5) is proved. Therefore, we have completed the proof of Theorem 3.2. \square

Given $R, l \in \mathbb{R}$ with $l \geq n$ and $\lambda_1 > \max\{\frac{lR}{l-1}, 0\}$, let $v_{R,l}$ be the solution of the equation

$$L_{R,l}v = -\lambda_1 v$$

with initial value $v(a) = -1$ and $v'(a) = 0$, where

$$a = \begin{cases} -\frac{\pi}{2\sqrt{R/(l-1)}} & \text{if } R > 0, \\ 0 & \text{if } R \leq 0. \end{cases}$$

We denote

$$b = \inf\{x > a : v'_{R,l}(x) = 0\}$$

and

$$m_{R,l} = v_{R,l}(b).$$

The second purpose of this section is to show the following comparison result on the maximum of eigenfunctions.

Theorem 3.3. *Let M be n -dimensional Alexandrov space without boundary and $\text{Ric}(M) \geq R$. Suppose that M has the first eigenvalue λ_1 and a corresponding eigenfunction f . Suppose $\lambda_1 > \max\{0, \frac{nR}{n-1}\}$ and $\min f = -1$, $\max f \leq 1$. Then we have*

$$\max f \geq m_{R,n}.$$

In smooth case, the proof of this result in [6] relies on the fact that the Hessian of f is bounded. Since we are not sure the existence of the Hessian for an eigenfunction on Alexandrov spaces, we have to give an alternative argument.

Proof of Theorem 3.3. Let us argue by contradiction. Suppose $\max f < m_{R,n}$.

Since $m_{R,l}$ is continuous on l , we can find some real number $l > n$ such that

$$\max f \leq m_{R,l} \quad \text{and} \quad \lambda_1 > \max\{0, \frac{lR}{l-1}\}.$$

Denote $v = v_{R,l}$. Recall from (the same proof of) Proposition 5 of [6] that the ratio

$$R(s) = -\frac{\int_M f \mathbf{1}_{\{f \leq v(s)\}} d\text{vol}}{\rho(s)v'(s)}$$

is increasing on $[a, v^{-1}(0)]$ and decreasing on $[v^{-1}(0), b]$, where the function ρ is

$$\rho(s) := \begin{cases} \cos^{l-1}(\sqrt{K}s) & \text{if } K = R/(l-1) > 0; \\ s^{l-1} & \text{if } K = R/(l-1) = 0; \\ \sinh^{l-1}(\sqrt{-K}s) & \text{if } K = R/(l-1) < 0. \end{cases}$$

It follows that for any $s \in [a, v^{-1}(-1/2)]$, we have

$$(3.8) \quad \text{vol}(\{f \leq v(s)\}) \leq -2 \int_M f 1_{\{f \leq v(s)\}} d\text{vol} \leq 2C\rho(s)v'(s),$$

where $C = R(v^{-1}(0))$.

Take $p \in M$ with $f(p) = -1$. By

$$f - f(p) \geq 0, \quad \text{and} \quad \mathcal{L}_{f-f(p)} = -\lambda_1 f \cdot \text{vol} \leq \lambda_1 \cdot \text{vol}.$$

The mean value inequality, Proposition 2.4, implies that there exists a constant C_1 such that

$$\frac{1}{\text{vol}(B_o(r) \subset T_p^{R/(n-1)})} \int_{B_p(r)} (f - f(p)) d\text{vol} \leq C_1 r^2$$

for any sufficiently small $r > 0$. Let $A(r) = \{f - f(p) > 2C_1 r^2\} \cap B_p(r)$. Then

$$\frac{\text{vol}(A(r))}{\text{vol}(B_p(r))} \leq \frac{\int_{B_p(r)} (f - f(p)) d\text{vol}}{2C_1 r^2 \text{vol}(B_p(r))} \leq \frac{\text{vol}(B_o(r) \subset T_p^{R/(n-1)})}{2\text{vol}(B_p(r))} \leq \frac{2}{3}$$

for any sufficiently small $r > 0$. Here we have used the fact

$$\lim_{r \rightarrow 0^+} \frac{\text{vol}(B_o(r) \subset T_p^{R/(n-1)})}{\text{vol}(B_p(r))} = 1.$$

Hence

$$\begin{aligned} \frac{1}{3} \text{vol}(B_p(r)) &\leq \text{vol}(B_p(r) \setminus A(r)) \leq \text{vol}(\{f \leq f(p) + 2C_1 r^2\}) \\ &= \text{vol}(\{f \leq -1 + 2C_1 r^2\}) \end{aligned}$$

for any sufficiently small $r > 0$. By combining this with (3.8), we have

$$(3.9) \quad \text{vol}(B_p(r)) \leq 6C\rho(s) \cdot v'(s)$$

for any sufficiently small $r > 0$, where

$$s = v^{-1}(-1 + 2C_1 r^2) = v^{-1}(v(a) + 2C_1 r^2).$$

Rewriting $L_{R,l}v = -\lambda_1 v$ as $(\rho v')' = -\lambda_1 \rho v$ and noting that $v'(a) = 0$, we get

$$(\rho v')(s) = -\lambda_1 \int_a^s \rho v dt \quad \text{and} \quad \frac{v'(s) - v'(a)}{s - a} = -\lambda_1 \frac{\int_a^s \rho v dt}{(s - a)\rho(s)}.$$

By applying L'Hospital's rule, we have

$$v''(a) = -\lambda_1 \lim_{s \rightarrow a} \frac{\int_a^s \rho v dt}{(s - a)\rho(s)} = -\lambda_1 \lim_{s \rightarrow a} \frac{v(s)}{1 + (s - a)\rho'(s)/\rho(s)}.$$

Noting that

$$v(a) = -1 \quad \text{and} \quad \lim_{s \rightarrow a} (s - a) \frac{\rho'(s)}{\rho(s)} = l - 1,$$

we get $v''(a) = \lambda_1/l$. Hence there exists two constants C_2 and C_3 such that

$$0 < C_2 \leq v'' \leq C_3 < \infty$$

in a neighborhood of a . The combination of $v'(a) = 0$ and $v''(s) \geq C_2$ implies that

$$(3.10) \quad v(s) - v(a) \geq \frac{C_2}{2}(s - a)^2$$

for s sufficiently near a . On the other hand, the combination of $v'(a) = 0$ and $0 < v''(s) \leq C_3$ implies that

$$0 \leq v'(s) \leq C_3(s - a)$$

for s sufficiently near a . Note that, by the definition of function ρ ,

$$\lim_{s \rightarrow a^+} \frac{\rho^{\frac{1}{l-1}}(s)}{s - a} = \begin{cases} \sqrt{K}, & \text{if } K = R/(l-1) > 0; \\ 1, & \text{if } K = R/(l-1) = 0; \\ \sqrt{-K}, & \text{if } K = R/(l-1) < 0. \end{cases}$$

Thus, we have

$$\rho(s) \leq C_3'(s - a)^{l-1}$$

for s sufficiently near a and for some constant C_3' . By combining with $0 \leq v'(s) \leq C_3(s - a)$, we have

$$(3.11) \quad \rho(s)v'(s) \leq C_3 \cdot C_3' \cdot (s - a)^l := C_4(s - a)^l$$

for s sufficiently near a .

The combination of (3.9), (3.11) and (3.10) implies that

$$\text{vol}(B_p(r)) \leq 6C \cdot C_4(s - a)^l \leq 6C \cdot C_4 \cdot \left(\frac{2}{C_2}(v(s) - v(a)) \right)^{l/2}$$

for s sufficiently near a . Noting that $v(s) - v(a) = 2C_1r^2$, we have

$$(3.12) \quad \text{vol}(B_p(r)) \leq 6C \cdot C_4 \cdot \left(\frac{4C_1}{C_2}r^2 \right)^{l/2} := C_5r^l$$

for any sufficiently small r .

Fix $r_0 > 0$. By Bishop–Gromov volume comparison, we have

$$\frac{\text{vol}(B_p(r))}{\mathcal{H}^n(B^K(r))} \geq \frac{\text{vol}(B_p(r_0))}{\mathcal{H}^n(B^K(r_0))}$$

for any $0 < r < r_0$, where $\mathcal{H}^n(B^K(r))$ is the volume of a geodesic ball with radius r in n -dimensional simply connected space form with sectional curvature K . Thus, there exists a constant C_6 such that

$$(3.13) \quad \text{vol}(B_p(r)) \geq \frac{\text{vol}(B_p(r_0))}{\mathcal{H}^n(B^K(r_0))} \cdot \mathcal{H}^n(B^K(r)) \geq C_6r^n$$

for any sufficiently small r .

The combination of (3.12) and (3.13) implies that $C_5 \cdot r^{l-n} \geq C_6$ holds for any sufficiently small r . Hence, we get $l \leq n$. This contradicts to the assumption $l > n$. Therefore, the proof of Theorem 3.3 is finished. \square

4. COMPARISON THEOREMS ON THE FIRST EIGENVALUE AND ITS APPLICATIONS

In this section, we will prove Theorem 1.2 in Introduction and its corollaries.

Proof of Theorem 1.2. Without loss of generality, we may assume $K \in \{-1, 0, 1\}$. Let λ_1 and f be the first non-zero eigenvalue and a corresponding eigenfunction with $\min f = -1$ and $\max f \leq 1$.

By Lichnerowicz's estimate, we have $\lambda_1 \geq n$, if $K = 1$. Now fix any $R < (n-1)K$, we have

$$\lambda_1 > \max\left\{\frac{nR}{n-1}, 0\right\}.$$

Then, by using the above Theorem 3.3 and Corollary 1 and 2 in Section 3 of [6], we can find an interval $[a, b]$ such that the one dimensional model operator $L_{R,n}$ has the first Neumann eigenvalue λ_1 and a corresponding eigenfunction v with $\min v = -1$, $\max v = \max f$. Applying Theorem 13 in Section 7 of [6], we have

$$(4.1) \quad \lambda_1 \geq \lambda_1(R/(n-1), n, b-a),$$

where $\lambda_1(R/(n-1), n, b-a)$ is the first non-zero Neumann eigenvalue of $L_{R,n}$ on the symmetric interval $(-\frac{b-a}{2}, \frac{b-a}{2})$.

By Theorem 3.2, we have

$$|\nabla(v^{-1} \circ f)| \leq 1.$$

The canonical Dirichlet form \mathcal{E} induces a pseudo-metric

$$d_{\mathcal{E}}(x, y) := \sup\{u(x) - u(y) : u \in W^{1,2}(M) \cap C(M) \text{ and } |\nabla u| \leq 1 \text{ a.e.}\}.$$

Since f is Lipschitz continuous and $|\nabla(v^{-1} \circ f)| \leq 1$, we have

$$b-a = v^{-1}(\max f) - v^{-1}(\min f) \leq \max_{x,y \in M} d_{\mathcal{E}}(x, y).$$

On the other hand, Kuwae–Machigashira–Shioya in [16] proved that the induced pseudo-metric $d_{\mathcal{E}}(x, y)$ is equal to the origin metric $d(x, y)$. Then $\max_{x,y \in M} d_{\mathcal{E}}(x, y)$ is equal to d , the diameter of M . By combining this with (4.1), we have

$$\lambda_1 \geq \lambda_1(R/(n-1), n, d).$$

Therefore, Theorem 1.2 follows from the combination of this and the arbitrariness of R . \square

In the rest of this section, we will apply Theorem 1.2 to conclude some explicit lower bounds for λ_1 .

The same computation as in [10] gives the following explicit lower bounds for $\lambda_1(M)$:

Corollary 4.1. (Chen–Wang [10]) *Let M be a compact $n(\geq 2)$ -dimensional Alexandrov space without boundary and $\text{Ric}(M) \geq (n-1)K$. Then its first non-zero eigenvalue $\lambda_1(M)$ satisfies:*

(1) if $K = 1$, then

$$\lambda_1(M) \geq \frac{n}{1 - \cos^n(d/2)} \quad \text{and} \quad \lambda_1(M) \geq \frac{\pi^2}{d^2} + (n-1) \cdot \max\left\{\frac{\pi}{4n}, 1 - \frac{2}{\pi}\right\};$$

(2) if $K = -1$, then

$$\lambda_1(M) \geq \frac{\pi^2}{d^2} \cosh^{1-n}\left(\frac{d}{2}\right) \cdot \sqrt{1 + \frac{2(n-1)d^2}{\pi^4}} \quad \text{and} \quad \lambda_1(M) \geq \frac{\pi^2}{d^2} - (n-1)\left(\frac{\pi}{2} - 1\right);$$

(3) if $K = 0$, then $\lambda_1(M) \geq \frac{\pi^2}{d^2}$, (Zhong–Yang's estimate [36])
where d is the diameter of M .

A direct computation shows that the first eigenvalue of any n -dimensional spherical suspension is exactly n . On the other hand, by combining Corollary 4.1 (1) and the maximal diameter theorem for Alexandrov space in [34], we conclude the following Obata type theorem:

Corollary 4.2. *Let M be a compact n -dimensional Alexandrov space without boundary and $\text{Ric}(M) \geq (n-1)$. Then $\lambda_1(M) = n$ if and only if M is isometric to a spherical suspension over an $(n-1)$ -dimensional Alexandrov space with curvature ≥ 1 .*

In the end of this section, we give some explicit lower bounds of $\lambda_1(M)$.

Corollary 4.3. *Let M be a compact $n(\geq 2)$ -dimensional Alexandrov space without boundary and $\text{Ric}(M) \geq (n-1)K$. Then its first non-zero eigenvalue $\lambda_1(M)$ satisfies*

$$\lambda_1(M) \geq 4s(1-s)\frac{\pi^2}{d^2} + s(n-1)K$$

for all $s \in (0, 1)$, where d is the diameter of M .

Proof. When $K > 0$, we can assume $d < \pi/\sqrt{K}$. Otherwise,

$$\lambda_1(M) = nK = K + (n-1)K \geq 4s(1-s)\frac{\pi^2}{d^2} + s(n-1)K$$

for all $s \in (0, 1)$.

Denote by

$$D = \frac{d}{2}, \quad f = v' \quad \text{and} \quad F = -(n-1)T,$$

where v and T are the one variable functions in Theorem 1.1. Clearly, we have

$$-f'' = F'f + Ff' + \lambda_1(K, n, d) \cdot f$$

and $f(\pm D) = 0$, $f(x) > 0$ on $x \in (-D, D)$.

For any $a > 1$, by multiplying f^{a-1} and integrating over $(-D, D)$, we get

$$(4.2) \quad -\int_{-D}^D f^{a-1} f'' dx = \int_{-D}^D (\lambda_1(K, n, d) + F') f^a dx + \int_{-D}^D F f^{a-1} f' dx.$$

Next, by $f(\pm D) = 0$, we have

$$-\int_{-D}^D f^{a-1} f'' dx = (a-1) \int_{-D}^D f^{a-2} f'^2 dx = \frac{4(a-1)}{a^2} \int_{-D}^D [(f^{a/2})']^2 dx.$$

On the other hand, by $f(\pm D) = 0$ again, we have

$$\begin{aligned} \int_{-D}^D F f^{a-1} f' dx &= -\int_{-D}^D f (F' f^{a-1} + (a-1) F f^{a-2} f') dx \\ &= -\int_{-D}^D F' f^a dx - (a-1) \int_{-D}^D F f^{a-1} f' dx. \end{aligned}$$

Hence, we have

$$\int_{-D}^D F f^{a-1} f' dx = -\frac{1}{a} \int_{-D}^D F' f^a dx.$$

Putting these equations to (4.2), we have

$$\frac{4(a-1)}{a^2} \int_{-D}^D [(f^{a/2})']^2 dx = \int_{-D}^D (\lambda_1(K, n, d) + (1 - \frac{1}{a})F') f^a dx.$$

Letting $s = 1 - \frac{1}{a} \in (0, 1)$, we get

$$\begin{aligned} 4s(1-s) \int_{-D}^D [(f^{a/2})']^2 dx &= \int_{-D}^D (\lambda_1(K, n, d) + sF') f^a dx \\ &\leq (\lambda_1(K, n, d) + s \max_{x \in (-D, D)} F') \int_{-D}^D f^a dx. \end{aligned}$$

Since $f^{a/2}(\pm D) = 0$, by Wirtinger's inequality, we have

$$4s(1-s) \left(\frac{\pi}{2D} \right)^2 \leq \lambda_1(K, n, d) + s \max_{x \in (-D, D)} F'.$$

Note also that

$$\max_{x \in (-D, D)} F' = -(n-1) \min_{x \in (-D, D)} T' = -(n-1)K.$$

Therefore, by applying Theorem 1.2, we get the desired estimate. \square

Remark 4.4. (1) If let $s = \frac{1}{2}$, we get

$$\lambda_1(M) \geq \frac{\pi^2}{d^2} + \frac{1}{2}(n-1)K.$$

This improves Chen–Wang's result in both $K > 0$ and $K < 0$. It also improves Ling's recent results in [19].

(2) If $K > 0$, Peter Li conjectures that $\lambda_1(M) \geq \pi^2/d^2 + (n-1)K$. Corollary 4.3 implies that $\lambda_1(M) \geq \frac{3}{4}(\pi^2/d^2 + (n-1)K)$.

(3) If $n \leq 5$ and $K > 0$, by choosing some suitable constant s , we have

$$\lambda_1(M) \geq \frac{\pi^2}{d^2} + \frac{1}{2}(n-1)K + \frac{(n-1)^2 K^2 d^2}{16\pi^2}.$$

5. LI–YAU'S PARABOLIC ESTIMATES

In this section, we consider heat equations on Alexandrov spaces.

Let M be an n -dimensional compact Alexandrov space without boundary and let $\mathcal{E} : W^{1,2}(M) \times W^{1,2}(M) \rightarrow \mathbb{R}$ be the canonical Dirichlet energy. Associated with the Dirichlet form $(\mathcal{E}, W^{1,2}(M))$, there exists an infinitesimal generator Δ which acts on a dense subspace $\mathbf{D}(\Delta)$ of $W^{1,2}(M)$, defined by

$$\int_M g \Delta f d\text{vol} = -\mathcal{E}(f, g), \quad \forall f \in \mathbf{D}(\Delta) \text{ and } g \in W^{1,2}(M).$$

By the definition, it is easy to check that $f \in \mathbf{D}(\Delta)$ implies $\mathcal{L}f = \Delta f \cdot \text{vol}$.

By the general theory of analytic semigroups (see for example [11]), the operator Δ generates an analytic semigroup $(T_t)_{t \geq 0}$ on $L^2(M)$. For any $f \in L^2(M)$, $u(x, t) := T_t f(x)$ solves the (linear) heat equation

$$\frac{\partial}{\partial t} u(x, t) = \Delta u(x, t)$$

with initial value $u(x, 0) = f(x)$ in the sense that

- (1) $T_t f \rightarrow f$ in $L^2(M)$, as $t \rightarrow 0$;
- (2) $T_t f \in \mathbf{D}(\Delta)$ and $\frac{\partial}{\partial t} T_t f = \Delta T_t f$ for all $t > 0$.

Moreover, $T_t f$ satisfies the following properties (see, for example [11])

- (3) $T_t f \in D((\Delta)^m)$, for all $t > 0$ and all $m \in \mathbb{N}$. If $f \in D((\Delta)^m)$, then

$$(\Delta)^m T_t f = T_t (\Delta)^m f, \quad \forall t \geq 0.$$

The existence of the heat kernel was proved in [16]. More precisely, there exists a unique, measurable, nonnegative, and locally Hölder continuous function $p_t(x, y)$ on $(0, \infty) \times M \times M$ satisfying the following properties (i)–(iii):

(i) For any $f \in L^2(M)$, $x \in M$ and $t > 0$,

$$(5.1) \quad T_t f(x) = \int_M p_t(x, y) f(y) d\text{vol}(y);$$

(ii) For any $s, t > 0$ and $x, y \in M$, we have

$$\begin{aligned} p_t(x, y) &= p_t(y, x) > 0, \\ p_{t+s}(x, y) &= \int_M p_t(x, z) p_s(z, y) d\text{vol}(z), \\ \int_M p_t(x, y) d\text{vol}(y) &= 1, \end{aligned}$$

the last equality follows from the fact that M is compact;

(iii) Denote by $0 < \lambda_1 \leq \lambda_2 \leq \dots$ all the non-zero eigenvalues of Δ with multiplicity and by $\{\phi_j\}_{j=1}^\infty$ the sequence of associated eigenfunctions which is a complete orthonormal basis of $W^{1,2}(M)$ and $\|\phi_j\|_{L^2(M)} = 1$ for all $j \in \mathbb{N}$. Then we have

$$(5.2) \quad p_t(x, y) = \frac{1}{\text{vol}(M)} + \sum_{j=1}^{\infty} e^{-\lambda_j t} \phi_j(x) \phi_j(y)$$

for all $t > 0$ and $x, y \in M$. Moreover, Cheng–Li proved that Sobolev inequality (3.7) implies that

$$\lambda_j \geq C \cdot (j+1)^{\frac{\nu-2}{\nu}}$$

for some constant $C = C(n, \nu, M, C_S) > 0$ (see, for example, Section 3.5 in book [31]).

Lemma 5.1. *Let M be an n -dimensional Alexandrov space with $\partial M = \emptyset$. Then for any $f \in L^2(M)$, $t > 0$ and $m \in \mathbb{N}$, $(\Delta)^m T_t f$ is Lipschitz continuous in M .*

Proof. For any $m \in \mathbb{N}$, we have $T_t f \in D((\Delta)^m)$ for all $t > 0$. This concludes that $(\Delta)^m T_t f$ is a solution of equation

$$\frac{\partial}{\partial t} u(x, t) = \Delta u(x, t).$$

On the other hand, the same proof of the locally Lipschitz continuity of Dirichlet heat kernel on a bounded domain (Theorem 5.14 in [34]) proves that $p_t(\cdot, y)$ is Lipschitz continuous on M , for any $y \in M$. From (5.1), we get that any solution of equation $\frac{\partial}{\partial t} u(x, t) = \Delta u(x, t)$ is Lipschitz continuous on M (or see Theorem 4.4 in [12]). Therefore, the proof of the lemma is completed. \square

Next, we adapt Bakry–Ledoux’s method in [5] to prove Theorem 1.4.

Proof of Theorem 1.4. Fix any small $\epsilon > 0$ and set $f = u(\cdot, \epsilon)$. Since u is positive and continuous on M , we know that f is bounded by a positive constant from below. Without loss of generality, we may assume that $f > 1$.

The combination of equation (5.1) and

$$\int_M p_t(x, y) d\text{vol}(y) = 1$$

implies $T_t f > 1$ for all $t > 0$. By the definition of $\mathbf{D}(\Delta)$, (5.1) and $f \geq 1$, $T_t f \geq 1$, we have $\log f, \log T_t f \in \mathbf{D}(\Delta)$ for all $t > 0$.

Fix $t > 0$ and, as in [5], let us consider the function

$$\psi(s) = T_s \left(T_{t-s} f \cdot |\nabla \log T_{t-s} f|^2 \right), \quad 0 \leq s \leq t.$$

Setting $g_s = \log T_{t-s} f$, by $\frac{\partial}{\partial s} T_{t-s} f = -\Delta T_{t-s} f$, we have

$$\begin{aligned} \psi(s, x) &= \int_M p_s(x, y) \cdot \exp g_s(y) \cdot |\nabla g_s(y)|^2 dy, \\ \exp g_s \cdot \frac{\partial}{\partial s} g_s &= -\Delta \exp g_s \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial s} \psi(s, x) &= \int_M \frac{\partial}{\partial s} p_s(x, y) \cdot \exp g_s(y) \cdot |\nabla g_s(y)|^2 dy \\ &\quad + \int_M p_s(x, y) \cdot \frac{\partial}{\partial s} g_s(y) \cdot \exp g_s(y) \cdot |\nabla g_s(y)|^2 dy \\ (5.3) \quad &\quad + \int_M p_s(x, y) \cdot \exp g_s(y) \cdot 2 \left\langle \frac{\partial}{\partial s} \nabla g_s(y), \nabla g_s(y) \right\rangle dy \\ &= \int_M \Delta p_s(x, y) \cdot \exp g_s(y) \cdot |\nabla g_s(y)|^2 dy \\ &\quad - T_s \left(\Delta \exp g_s \cdot |\nabla g_s|^2 \right) - T_s \left(\exp g_s \cdot 2 \langle \nabla(\Delta g_s + |\nabla g_s|^2), \nabla g_s \rangle \right). \end{aligned}$$

By the definition of Δ and the functional \mathcal{L} , and $\exp g \in \mathbf{D}(\Delta)$, we have

$$\begin{aligned} &\int_M \Delta p_s(x, y) \cdot \exp g_s(y) \cdot |\nabla g_s(y)|^2 dy \\ &= \int_M p_s(x, y) d\mathcal{L}_{(\exp g_s(y) \cdot |\nabla g_s(y)|^2)} \\ &= T_s \left(\Delta \exp g_s \cdot |\nabla g_s|^2 \right) + 2T_s \left(\langle \nabla \exp g_s, \nabla |\nabla g_s|^2 \rangle \right) \\ &\quad + \int_M p_s(x, y) \exp g_s(y) d\mathcal{L}_{|\nabla g_s|^2}. \end{aligned}$$

From Lemma 5.1 and $T_{t-s} f \geq 1$, we get that $\Delta T_{t-s} f$ is Lipschitz continuous on M , for all $0 < s < t$. Note that $g_s \in \mathbf{D}(\Delta)$. Now, because M has nonnegative Ricci curvature, we can apply Corollary 2.2 (Bochner type formula) to equation

$$\mathcal{L}_{T_{t-s} f} = \Delta T_{t-s} f \cdot \text{vol}$$

and function $\Phi(t) = \log t$ to conclude that

$$(5.4) \quad \mathcal{L}_{|\nabla g_s|^2} \geq \left(\frac{2(\Delta g_s)^2}{n} + 2 \langle \nabla g_s, \nabla \Delta g_s \rangle \right) \cdot \text{vol}.$$

Putting these above equations and the nonnegativity of $p_t(x, y)$ to (5.3), we have

$$(5.5) \quad \frac{\partial}{\partial s} \psi(s, x) \geq \frac{2}{n} T_s \left(\exp g_s \cdot (\Delta g_s)^2 \right).$$

The rest of the proof follows exactly from the corresponding argument in [5].

From (5.5), we have $\psi(0) \leq \psi(t)$, i.e.,

$$T_t f \cdot |\nabla \log T_t f|^2 \leq T_t (f |\nabla \log f|^2), \quad \forall t > 0.$$

Since $T_t f \in \mathbf{D}(\Delta)$ for all $t > 0$, the above inequality implies

$$(5.6) \quad T_t f \cdot \Delta(\log T_t f) \geq T_t (f \Delta(\log f)), \quad \forall t > 0.$$

By applying

$$\Delta g_s = \frac{\Delta T_{t-s}f}{T_{t-s}f} - |\nabla g_s|^2$$

and Cauchy–Schwarz inequality, we have

$$\begin{aligned} T_s(\exp g_s(\Delta g_s)^2) &= T_s\left(T_{t-s}f\left(\frac{\Delta T_{t-s}f}{T_{t-s}f} - |\nabla g_s|^2\right)^2\right) \\ &= T_s\left(\frac{(\Delta T_{t-s}f - T_{t-s}f|\nabla g_s|^2)^2}{T_{t-s}f}\right) \\ &\geq \left(T_s(\Delta T_{t-s}f - T_{t-s}f|\nabla g_s|^2)\right)^2 / T_s(T_{t-s}f) \\ &= (\Delta T_t f - \psi(s))^2 / T_t f. \end{aligned}$$

Putting this into the equation (5.5), we get

$$(\psi(s) - \Delta T_t f)' \geq \frac{2}{nT_t f}(\psi(s) - \Delta T_t f)^2.$$

This implies

$$(5.7) \quad \varphi(s_2) - \varphi(s_1) \geq \frac{2}{nT_t f}(s_2 - s_1) \cdot \varphi(s_2) \cdot \varphi(s_1),$$

for all $0 \leq s_1 < s_2 \leq t$, where

$$\varphi(s) = \psi(s) - \Delta T_t f.$$

In particular, we have

$$\varphi(t) - \varphi(0) \geq \frac{2t}{nT_t f}\varphi(t)\varphi(0).$$

That is,

$$(5.8) \quad -\varphi(0) \geq \frac{2t}{nT_t f}\varphi(t)\varphi(0) - \varphi(t) = -\varphi(t) \cdot \left(1 - \frac{2t}{nT_t f}\varphi(0)\right).$$

Note that

$$\varphi(0) = -T_t f \Delta(\log T_t f), \quad \varphi(t) = -T_t(f \Delta(\log f)).$$

Hence, by the equation (5.8), we have

$$(5.9) \quad T_t f \cdot \Delta(\log T_t f) \geq T_t(f \Delta(\log f)) \left(1 + \frac{2t}{n} \Delta(\log T_t f)\right)$$

for all $t > 0$.

We now claim that

$$(5.10) \quad 1 + \frac{2t}{n} \Delta(\log T_t f) \geq 0, \quad \forall t > 0.$$

Fix any $t > 0$. Indeed, if $\Delta(\log T_t f) \geq 0$, we are done. Then we may assume that $\Delta(\log T_t f) < 0$. Since $T_t f \geq 1 > 0$, the equation (5.6) implies

$$T_t(f \Delta(\log f)) < 0.$$

Now the equation (5.9) shows that

$$1 + \frac{2t}{n} \Delta(\log T_t f) \geq \frac{T_t f \cdot \Delta(\log T_t f)}{T_t(f \Delta(\log f))} \geq 0.$$

This proves the equation (5.10).

Rewriting the equation (5.10), we get

$$|\nabla \log T_t f|^2 - \frac{\partial}{\partial t} \log T_t f \leq \frac{n}{2t} \quad \forall t > 0.$$

Noting that $T_t f(x) = u(x, t + \epsilon)$, we get

$$|\nabla \log u|^2 - \frac{\partial}{\partial t} \log u \leq \frac{n}{2(t - \epsilon)} \quad \forall t > \epsilon.$$

The desired inequality (1.1) follows from the arbitrariness of ϵ . Therefore, the proof of Theorem 1.4 is completed. \square

Since $u(x, t)$ is continuous in $M \times (0, \infty)$, a direct application of Theorem 1.4, as in [22], gives the following parabolic Harnack inequality.

Corollary 5.2. *Let M be a compact n -dimensional Alexandrov space with nonnegative Ricci curvature and $\partial M = \emptyset$. Assume that $u(x, t)$ is a positive solution of heat equation $\frac{\partial}{\partial t} u = \Delta u$ on $M \times [0, \infty)$. Then we have*

$$u(x_1, t_1) \leq u(x_2, t_2) \left(\frac{t_2}{t_1} \right)^{\frac{n}{2}} \exp \left(\frac{|x_1 x_2|^2}{4(t_2 - t_1)} \right)$$

for all $x_1, x_2 \in M$ and $0 < t_1 < t_2 < \infty$.

The Harnack inequality is sharper than that in [29].

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