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Rigidity of manifolds with Bakry–Émery Ricci curvature bounded below

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Abstract Let M be a complete Riemannian manifold with Riemannian volume vol_g and f be a smooth function on M. A sharp upper bound estimate on the first eigenvalue of symmetric diffusion operator $\Delta_f = \Delta - \nabla f \cdot \nabla$ was given by Wu (J Math Anal Appl 361:10–18, 2010) and Wang (Ann Glob Anal Geom 37:393–402, 2010) under a condition that finite dimensional Bakry–Émery Ricci curvature is bounded below, independently. They propounded an open problem is whether there is some rigidity on the estimate. In this note, we will solve this problem to obtain a splitting type theorem, which generalizes Li–Wang's result in Wang (J Differ Geom 58:501–534, 2001, J Differ Geom 62:143–162, 2002). For the case that infinite dimensional Bakry–Emery Ricci curvature of M is bounded below, we do not expect any upper bound estimate on the first eigenvalue of Δ_f without any additional assumption (see the example in Sect. 2). In this case, we will give a sharp upper bound estimate on the first eigenvalue of Δ_f under the additional assuption that $|\nabla f|$ is bounded. We also obtain the rigidity result on this estimate, as another Li–Wang type splitting theorem.

Keywords Bakry–Emery Ricci curvature · Splitting type theorem · Spectrum

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1 Introduction

Let M^n be an n-dimensional complete non-compact Riemannian manifold and $\lambda_1(M^n)$ be the bottom of the spectrum of the Laplacian with respect to the metric of M^n. A classical theorem of Cheng [3] asserts that

$$\lambda_1(M^n) \leqslant \frac{(n-1)^2}{4}$$

provided the Ricci curvature of M^n is bounded from below by $Ric_{M^n} \geqslant -(n-1)$. Li–Wang in [4,5] studied the rigidity of the estimate and proved the following splitting type theorem for optimal case of Cheng's theorem:

Theorem 1.1 (Li–Wang [5]) *Let M^n be a complete n-dimensional Riemannian manifold with $Ric \geqslant -(n-1)$ and $\lambda_1(M^n) = \frac{(n-1)^2}{4}$. Then either:*

(1) *M^n has only one end; or*

(2) *$M^n = \mathbb{R} \times N$ with the warped product metric*

$$ds^2_{M^n} = dt^2 + e^{2t} ds^2_N$$

where N is a compact manifold with nonnegative Ricci curvature; or

(3) *if $n = 3$, $M^3 = \mathbb{R} \times N$ with the warped product metric*

$$ds^2_{M^n} = dt^2 + \cosh^2 t ds^2_N$$

where N is a compact surface with curvature bounded below by -1.

Given a smooth metric measure space $(M, g, e^{-f} d\text{vol}_g)$, where (M, g) is a complete Riemannian manifold with metric g, f is a smooth real-valued function on M, and $d\text{vol}_g$ is the Riemannian volume density on M, the (∞-dimensional) Bakry–Émery Ricci tensor on M is defined by

$$Ric_f(M) = Ric(M) + \text{Hess} f.$$

The Bakry–Émery tensor occurs naturally in many different subjects, such as diffusion processes and Ricci flow, and so on. Let $m \in \mathbb{R}$ with $m > n = \dim(M)$, the m-dimensional Bakry–Émery Ricci tensor is given by

$$Ric_f^m(M) = Ric_f(M) - \frac{\nabla f \otimes \nabla f}{m-n}.$$

It is known that the symmetric diffusion operator $\Delta_f = \Delta - \nabla f \cdot \nabla$ satisfies $CD(m, K)$ condition in the sense of Bakry–Émery provided M has $Ric_f^m(M) \geqslant Kg$ (see for example [8]). On the other hand, from [9,12], suppose that $\int_M e^{-f} d\text{vol}_g = 1$, it is known that the metric measure space $(M, g, e^{-f} d\text{vol}_g)$ satisfies curvature–dimension condition $\mathbf{CD}(m, K)$ in the sense of Sturm–Lott–Villani if and only if $Ric_f^m(M) \geqslant Kg$. As an extension of Ricci curvature, many classical results in Riemannian geometry asserted in terms of Ricci curvature have been extended to the analogous ones on Bakry–Émery Ricci curvature condition. For example, see [8] and [14] for a brief overview of these results.

Let $(M, g, e^{-f} d\text{vol}_g)$ be a smooth metric measure space with m-dimensional Bakry–Émery Ricci curvature bounded below by $-(m-1)$, the sharp upper bound of $\lambda_1(M)$, the bottom of the spectrum of the diffusion operator Δ_f, was obtained in [13] and [15] that

$$\lambda_1(M) \leqslant \frac{(m-1)^2}{4}.$$

They leave an open problem: whether is there some rigidity on this estimate? We will solve this problem and prove the following:

Theorem 1.2 *Let* $(M, g, e^{-f} d\mathrm{vol}_g)$ *be a smooth metric measure space with* $\dim(M) = n \geqslant 3$. *Suppose that* $Ric_f^m(M) \geqslant -(m-1)$ *and* $\lambda_1(M) = \frac{(m-1)^2}{4}$. *Then either:*

(1) *M has only one end; or*

(2) $M = \mathbb{R} \times N$ *with warped product metric*

$$ds_M^2 = dt^2 + e^{-2t} d_N^2$$

and f satisfies $\nabla_{\partial_t} f = m - n$, *where N is a compact manifold with nonnegative Ricci curvature.*

For the case of that smooth metric measure space $(M, g, e^{-f} d\mathrm{vol}_g)$ with ∞-dimensional Bakry–Émery Ricci curvature bounded below, we do not expect any upper bound of $\lambda_1(M)$ without additional assumption. So we will estimate the upper bound of $\lambda_1(M)$ under an additional condition $|\nabla f|$ is bounded. Explicitly, we obtain the following estimate of $\lambda_1(M)$ and its rigidity:

Theorem 1.3 *Let* $(M, g, e^{-f} d\mathrm{vol}_g)$ *be a smooth metric measure space with* $\dim(M) = n \geqslant 3$. *Suppose that* $Ric_f(M) \geqslant -(n-1)$ *and* $|\nabla f| \leqslant A$ *for constant* $A > 0$. *Then we have*

$$\lambda_1(M) \leqslant \frac{\left(n-1+A\right)^2}{4}. \tag{1.1}$$

Furthermore, if the equality holds, then either:

(1) *M has only one end; or*

(2) $M = \mathbb{R} \times N$ *with warped product metric*

$$ds_M^2 = dt^2 + e^{-2t} ds_N^2$$

and $f = At$, *where N is a compact manifold with nonnegative Ricci curvature.*

In Sect. 2, we will provide a simple example to show that λ_1 is unbounded without the assumption that $|\nabla f|$ is bounded.

The two results may be compare with ones in [6], where they consider to extend Theorem 1.1 on a Riemannian manifold with Ricci curvature bounded below and a weighted Poincaré inequality.

After this article was finished, we have learnt Ovidiu Munteanu and JiaPing Wang's related work [10], where they study smooth measure spaces with nonnegative Bakry–Émery Ricci curvature.

2 The sharp upper bound of $\lambda_1(M)$ in infinite dimension case

In this section, we will prove the first assertion of Theorem 1.3 and give an example to show that the assumption $|\nabla f| \leqslant A$ is necessary.

Proposition 2.1 *Let* $(M, g, e^{-f} d\mathrm{vol}_g)$ *be a smooth metric measure space with* $\dim(M) = n$. *Suppose that* $Ric_f(M) \geqslant -(n-1)K$ *and* $|\nabla f| \leqslant A$ *for two constants* $K \geqslant 0$ *and* $A > 0$. *Then we have*

$$\lambda_1 \leqslant \frac{\left((n-1)\sqrt{K} + A\right)^2}{4}. \tag{2.1}$$

Proof Firstly, let us consider the case that $K > 0$.

Let $\mathrm{Vol}_f(B_p(r)) = \int_{B_p(r)} e^{-f} d\mathrm{vol}_g$, the weighted (or f-)volume and $\mathrm{Vol}_K^n(r)$ be the volume of the geodesic ball with radius r in the model space M_K^n, simply connected space form with curvature $-K$. By the volume comparison in [14], we have

$$\mathrm{Vol}_f(B_p(R)) \leqslant \frac{\mathrm{Vol}_f(B_p(1))}{\mathrm{Vol}_K^n(1)} \cdot e^{AR} \mathrm{Vol}_K^n(R) \tag{2.2}$$

for all $R > 1$.

On the other hand, it is well-known that (see for example [11]),

$$\sqrt{\lambda_1(M)} \leqslant \lim_{R\to\infty} \frac{\log \mathrm{Vol}_f(B_p(R))}{2R}. \tag{2.3}$$

Hence, the desired estimate (2.1) follows from (2.2) and (2.3).

If $K = 0$, then $Ric_f \geqslant -(n-1)\epsilon$ for all $\epsilon > 0$. Letting $\epsilon \to 0^+$, we obtain (2.1). Therefore, the proof is completed. □

Next, we will give a family of smooth metric measure spaces $\{(M_j, g_j, e^{-f_j} d\mathrm{vol}_{g_j})\}_{j=1}^{\infty}$ with $\dim(M_j) = n$ and $Ric_{f_j} \geqslant 0$ for all $j \in \mathbb{N}$. But the set $\{\lambda_1(M_j)\}_{j=1}^{\infty}$ is unbounded.

Example Let (M_j, g_j) be $\mathbb{R}^n$ with standard metric, and let $f_j = j \cdot x_1$ for all $j \in \mathbb{N}$. Then

$$\nabla f_j = j \cdot \frac{\partial}{\partial x_1}, \qquad \mathrm{Hess} f_j = 0.$$

Hence the metric measure spaces $(M_j, g_j, e^{-f_j} d\mathrm{vol}_{g_j})$ satisfy $Ric_{f_j} = 0$ for all $j \in \mathbb{N}$.

Let $h_j(x) = \exp(\frac{jx_1}{2})$. By

$$\Delta_{f_j} = \Delta - j \cdot \frac{\partial}{\partial x_1},$$

we have

$$\Delta_{f_j} h_j = \frac{\partial^2}{\partial x_1^2} h_j - j \frac{\partial}{\partial x_1} h_j = -\frac{j^2}{4} h_j, \quad \forall j \in \mathbb{N}.$$

The same proof of Proposition 0.4 in [5] gives that

$$\lambda_1(M_j) \geqslant \frac{j^2}{4}.$$

The above example also shows that the estimate (2.1) is optimal.

3 Rigidity property of $\lambda_1(M)$

The ingredient used in the proof of Theorem 1.1 by Li–Wang is an improved Bochner formula and its metric rigidity. Under the curvature condition $Ric(M) \geqslant -(n-1)$, for any nonconstant harmonic function u, then $|\nabla u|$ satisfies (see [6])

$$|\nabla u|\Delta(|\nabla u|) \geqslant -(n-1)|\nabla u|^2 + \frac{|\nabla(|\nabla u|)|^2}{n-1}.$$

Moreover, if equality holds, then M splits into $\mathbb{R} \times N$ with a warped product metric. Our proof to Theorems 1.2 and 1.3 is basically along Li–Wang's proof of Theorem 1.1. Therefore, the main work of this paper is to extend the above improved Bochner formula and its metric rigidity for smooth metric measure spaces $(M, g, e^{-f}d\mathrm{vol}_g)$ under some suitable Bakry–Émery Ricci curvature conditions.

The following improved Bochner formula with its metric rigidity property is our main tool to prove Theorem 1.3.

Lemma 3.1 *Let $(M, g, e^{-f}d\mathrm{vol}_g)$ be a smooth metric measure space with $\dim(M) \geqslant 2$. Assume that $Ric_f(M) \geqslant -(n-1)$ and $|\nabla f| \leqslant A$ for some constant $A > 0$. Suppose that u is a non-constant solution of $\Delta_f u = 0$, then we have*

$$|\nabla u|\Delta_f(|\nabla u|) \geq \frac{1}{(1+\alpha)(n-1)}|\nabla(|\nabla u|)|^2 - \frac{A^2}{\alpha(n-1)}|\nabla u|^2 - (n-1)|\nabla u|^2 \qquad (3.1)$$

for all $\alpha > 0$. Moreover, if for some $\alpha_0 > 0$, the equality in (3.1) holds, then $\alpha_0 = \frac{A}{(n-1)}$ and $M = \mathbb{R} \times_\eta N$ with the warped product metric

$$ds_M^2 = dt^2 + \exp(-2t)ds_N^2.$$

Proof By the Bochner formula of the Bakry–Émery Ricci tensor (see for example [8]) and $\Delta_f u = 0$, we have

$$|\nabla u| \cdot \Delta_f(|\nabla u|) = |\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 + Ric_f(\nabla u, \nabla u). \qquad (3.2)$$

By choosing an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ such that $|\nabla u|e_1 = \nabla u$, and $e_j u = 0$ for all $j \neq 1$, and using a Yau's trick, we can get

$$|\nabla(|\nabla u|)|^2 = \sum_{j\geqslant 1} u_{1j}^2 \qquad (3.3)$$

and

$$|\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 = \sum_{i\neq 1, j\geqslant 1} u_{ij}^2 \geqslant \sum_{j\geqslant 2} u_{1j}^2 + \frac{1}{n-1}(u_{11} - f_1 u_1)^2, \qquad (3.4)$$

where we used the equation $\Delta_f u = 0$. It is easy to check that

$$(f_1 u_1 - u_{11})^2 \geqslant \frac{u_{11}^2}{1+\alpha} - \frac{(f_1 u_1)^2}{\alpha}, \qquad \forall \alpha > 0.$$

Hence, we have

$$|\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 \geqslant \frac{1}{(n-1)(1+\alpha)} \sum_{j\geqslant 1} u_{1j}^2 - \frac{1}{(n-1)\alpha}(f_1 u_1)^2. \qquad (3.5)$$

The desired estimate (3.1) follows from the combination of (3.2)–(3.5) and $Ric_f \geqslant -(n-1)$.

Now, let us consider the case of equality in (3.1) for some $\alpha_0 > 0$. The equality in (3.1) implies that all the inequalities above become equalities. Then we have

$$\begin{aligned} \nabla f &= f_1 \cdot e_1, \quad f_1 = A; \\ u_{11} &= \frac{1+\alpha_0}{\alpha_0} A u_1, \quad u_{22} = \cdots = u_{nn}; \\ u_{ij} &= 0, \text{ for all } 1 \leqslant i \neq j \leqslant n. \end{aligned}$$

Hence, by $\Delta_f u = 0$, we have

$$\mathrm{Hess} u = \begin{pmatrix} \frac{\alpha_0+1}{\alpha_0} A u_1 & & & \\ & -\frac{A}{(n-1)\alpha_0} u_1 & & \\ & & \ddots & \\ & & & -\frac{A}{(n-1)\alpha_0} u_1 \end{pmatrix}.$$

Let us denote $II = (h_{ij})$ be the second fundamental form of the level set of u and trII be the mean curvature of the level set of u. It is evidently to see that $II = -\frac{A}{(n-1)\alpha_0}(\delta_{ij})$, thus the second fundamental form is a constant multiple of the identity matrix along each level set of u, the splitting of the metric given by the form

$$ds_M^2 = dt^2 + \eta^2(t) ds_N^2$$

with

$$(n-1)\frac{\eta'}{\eta} = \mathrm{tr} II = -\frac{A}{\alpha_0}$$

We have

$$\eta(t) = \exp\left(-\frac{A}{\alpha_0(n-1)} t\right)$$

Then from $\Delta_f u = 0$, we have

$$A u_1 = \Delta u = \mathrm{tr} II \cdot u_1 + u_{11}$$

Let $g = |\nabla u| = u_1$, it becomes

$$\mathrm{tr} II \cdot g + g_1 = A g$$

that is

$$\frac{g_1}{g} = \left(1 + \frac{1}{\alpha_0}\right) A$$

so we have

$$g \Delta_f g = g(g_{11} + \mathrm{tr} II \cdot g_1 - A g_1) = g g_{11} - g_1^2 \equiv 0$$

And by the equality in (3.1)

$$g \Delta_f g = \frac{1}{(1+\alpha_0)(n-1)} g_1^2 - \frac{A^2}{\alpha_0(n-1)} g^2 - (n-1) g^2$$

We must have

$$\alpha_0 = \frac{A}{(n-1)}$$

and

$$\eta(t) = \exp(-t)$$

thus the proof is completed. □

Definition 3.2 An end $E \subset M$ of a smooth metric measure space $(M, g, e^{-f} d\mathrm{vol}_g)$ is said to be f-nonparabolic if there exists a non-constant bounded f-harmonic function on E with finite Dirichlet energy. Otherwise, E is said to be f-parabolic.

The same proof of Lemma 1.2 in [4] gives the following lemma.

Lemma 3.3 *Let $(M, g, e^{-f} d\mathrm{vol}_g)$ be a smooth metric measure space with $\lambda_1(M) > 0$, and let E_1, E_2 be two f-nonparabolic ends in M. Then there exists a non-constant bounded f-harmonic function on M and constant $C > 0$ such that*

$$\int_{B_p(R)} \exp\left(2\sqrt{\lambda_1(M)}r\right)|\nabla u|^2 e^{-f} d\mathrm{vol}_g \leqslant CR$$

for R sufficiently large.

This lemma provides the following upper bound estimate for $\lambda_1(M)$.

Lemma 3.4 *Let $(M, g, e^{-f} d\mathrm{vol})$ be a smooth metric measure space with* $\dim M = n \geqslant 3$. *Assume that $Ric_f \geqslant -(n-1)$ and $|\nabla f| \leq A$ for some $A > 0$. If there exists some $\alpha > 0$ such that*

$$\lambda_1(M) > \max\left\{\left(\frac{A + (n-1)}{2[(n-1)(1+\alpha) - 1]}\right)^2, \rho(\alpha)\right\},$$

then M must have at most one f-nonparabolic end, where

$$\rho(\alpha) = \left(1 - \frac{1}{(1+\alpha)(n-1)}\right)\left((n-1) + \frac{A^2}{\alpha(n-1)}\right).$$

Proof By a contradiction argument, we can assume that M has two f-nonparabolic ends. Let u be a f-harmonic function given in Lemma 3.3. For any $\alpha > 0$, we have

$$\Delta_f g \geqslant -\rho(\alpha) g,$$

where $g = |\nabla u|^b$ and $b = 1 - \frac{1}{(1+\alpha)(n-1)}$.

By using Hölder inequality, we have

$$\int_{B_p(R,2R)} g^2 d\mu \leqslant \left(\int_{B_p(R,2R)} \exp(2r\sqrt{\lambda_1(M)})|\nabla u|^2 d\mu\right)^b \cdot \left(\int_{B_p(R,2R)} \exp(-2r[(n-1)(1+\alpha) - 1]\sqrt{\lambda_1(M)}) d\mu\right)^{1-b},$$

where $d\mu = e^{-f} d\mathrm{vol}_g$ and $B_p(R, 2R) = B_p(2R)\backslash B_p(R)$. Since $Ric_f \geqslant -(n-1)$ and $|\nabla f| \leqslant A$, by volume comparison theorem [14], we have

$$\mathcal{A}(r) \leqslant e^{Ar} \mathcal{A}_{-1}(r),$$

where $\mathcal{A}_{-1}(r)$ is the area of geodesic sphere in simply connected hyperbolic space $\mathbb{H}^n$. Hence, we have

$$\int_{B_p(R,2R)} \exp(-2r[(n-1)(1+\alpha)-1]\sqrt{\lambda_1(M)})d\mu$$
$$\leqslant \int_R^{2R} \exp\Big[\Big(-2[(n-1)(1+\alpha)-1]\sqrt{\lambda_1(M)}+A+(n-1)\Big)r\Big]d\mu.$$

We can assume that

$$\sqrt{\lambda_1(M)} > \frac{A+(n-1)}{2[(n-1)(1+\alpha)-1]}.$$

By combining these inequalities and Lemma 3.3, we have

$$\int_{B_p(R,2R)} g^2 d\mu \leqslant CR \tag{3.6}$$

for R sufficiently large.

We consider ϕ to be a nonnegative compactly supported function on M. Then

$$\int_M |\nabla(g\phi)|^2 d\mu = \int_M |\nabla\phi|^2 g^2 d\mu + \int_M \phi^2 |\nabla g|^2 d\mu + \frac{1}{2}\int_M \langle \nabla(\phi^2), \nabla g^2\rangle d\mu$$
$$= \int_M |\nabla\phi|^2 g^2 d\mu + \rho(\alpha)\int_M \phi^2 g^2 d\mu - \int_M \phi^2 g\big(\Delta_f g + \rho(\alpha) g\big) d\mu$$

for all $\alpha > 0$. By the variational principle of $\lambda_1(M)$ and Lemma 3.1, we have

$$\big(\lambda_1(M)-\rho(\alpha)\big)\int_M \phi^2 g^2 d\mu \leqslant \int_M |\nabla\phi|^2 g^2 d\mu - \int_M \phi^2 g\big(\Delta_f g+\rho(\alpha)g\big)d\mu \leqslant \int_M |\nabla\phi|^2 g^2 d\mu.$$

Now choose ϕ to satisfy $\phi = 1$ on $B_p(R)$, $\phi = 0$ out $B_p(2R)$ and $|\nabla\phi| \leqslant C \cdot R^{-1}$. By the L^2 estimate of g (3.6), letting $R \to \infty$, we have $\lambda_1(M) \leqslant \rho(\alpha)$. This contradicts to the assumption $\lambda_1(M) > \rho(\alpha)$ and completes the proof of the lemma. □

The same argument in the proof of Theorem 2.1 in [7] gives the following proposition.

Proposition 3.5 *Let* $(M, g, e^{-f}d\mathrm{vol})$ *be a smooth metric measure space with* $\dim M = n \geqslant 3$. *Suppose* $h : (0, \infty) \to \mathbb{R}$ *is a function such that*

$$\lim_{r\to\infty} h(r) = 2a > 0.$$

Assume that for any point $p \in M$, *and* $r(x)$ *is the distance function to the point* p, *we have*

$$\Delta_f r(x) \leqslant h(r(x))$$

in the weak sense. Assume also that M *has at least one parabolic end and*

$$\lambda_1(M) = a^2.$$

Then, letting $\gamma(t)$ *be a geodesic ray issuing from a fixed point* p *to infinity of the parabolic end and letting* $\beta(x)$ *be the Buseman function*

$$\beta(x) = \lim_{t\to\infty}(t - r(\gamma(t), x))$$

with respect to γ, *we have*

$$\Delta_f \beta = -2a.$$

Now we are in the position to prove Theorem 1.3.

Proof of Theorem 1.3 The estimate (1.1) has been proved in Sect. 2. Now we consider the case

$$\lambda_1(M) = \frac{(n-1+A)^2}{4}.$$

We first note that $\sqrt{\lambda_1(M)} > \frac{A+(n-1)}{2[(1+\alpha)(n-1)-1]}$ for any $\alpha > 0$ and that, for $\alpha = \frac{A}{n-1}$,

$$\lambda_1(M) - \rho(\alpha) = \frac{1}{4}((n-1)+A)^2 - (n-2+A) = \frac{1}{4}(n-3+A)^2 > 0,$$

Thus by Lemma 3.4, M has at most one f-nonparabolic end.

We can assume that M has two ends at least. Otherwise, we have done. Hence, M has one f-parabolic end at least. Note that the Laplacian comparison theorem of Bakry–Émery Ricci tensor asserts that (see [14])

$$\Delta_f r \leq (n-1)\coth r + A \to (n-1) + A$$

as $r \to \infty$. Then, by Proposition 3.5, we have

$$\Delta_f \beta = -(n-1) - A,$$

where β be the Busemann function with respect to a geodesic ray issuing from a fixed point p to infinity of a parabolic end.

From the elliptic regularity theory, we know that β is smooth. Since $|\nabla\beta| = 1$, M must be $\mathbb{R} \times N$ topologically, where $N = \beta^{-1}(0)$.

The fact $|\nabla\beta| = 1$ implies that $\beta_{1j} = 0$ for all $j \geqslant 1$, for an othorgonal basis $\{e_j\}_{j=1}^n$ with $e_1 = \nabla\beta$, and that, by the Bochner formula

$$0 = \frac{1}{2}\Delta_f |\nabla\beta|^2 = |\nabla^2\beta|^2 + Ric_f(\nabla\beta, \nabla\beta) + \langle \nabla\Delta_f\beta, \nabla\beta\rangle = \sum_{i,j\geqslant 2} \beta_{ij}^2 + Ric_f(e_1, e_1).$$

Then the second fundamental form II and mean curvature of a level set of β satisfy $|II|^2 = |\nabla^2\beta|^2 = -Ric_{11} - f_{11}$ and $H := \mathrm{tr} II = -2a + f_1$, where

$$a = \frac{(n-1)+A}{2}.$$

Note that

$$n-1 \geq -Ric_{11} - f_{11} = |II|^2 \geq \frac{1}{n-1}H^2 = \frac{1}{n-1}\left(4a^2 - 4af_1 + f_1^2\right), \qquad (3.7)$$

we have

$$(4a^2 - 4af_1 + f_1^2) \leq (n-1)^2.$$

Using $2a = n - 1 + A$, we have

$$2(n-1)(A - f_1) + (A - f_1)^2 \leq 0.$$

By the assumption $|\nabla f| \leq A$, we have $f_1 = A$, $f_i = 0$, thus $\nabla f = A\nabla\beta$ and all the inequalities in (3.7) becomes equalities. In particular, we have

$$\begin{cases} \beta_{1j} = 0, & \forall j = 1, \dots, n \\ \beta_{ij} = 0, & i \neq j \\ \beta_{22} = \cdots = \beta_{nn} = -1. \end{cases}$$

Thus M must be $\mathbb{R} \times N$ isometrically, and the metric is given by $ds^2 = dt^2 + \eta^2(t)ds_N^2$ with

$$(n-1)\frac{\eta'}{\eta} = \mathrm{tr}II = -(n-1).$$

Hence $\eta(t) = \exp(-t)$.

Since M has at least two ends, we know N is compact. The Ricci nonnegativity of N follows from Gauss equation directly. □

At last, let us consider the case that a finite-dimensional Bakry–Émery Ricci curvature of M is bounded below and give a proof of Theorem 1.2.

The following improved Bochner formula has been obtain by X.-D. Li in [8] (a similar one can been found in [2]).

Lemma 3.6 *Let* $(M, g, e^{-f}d\mathrm{vol})$ *be a smooth metric space with* $\dim M = n \geqslant 2$. *Assume that* $Ric_f^m \geq -(m-1)$. *Suppose that u is a nonconstant f-harmonic function, that is, u is a solution of* $\triangle_f u = 0$, *then the function* $g = |\nabla u|^{\frac{m-2}{m-1}}$ *must satisfy the following differential inequality:*

$$\Delta_f g \geqslant -(m-2)g \tag{3.8}$$

in the weak sense.

By suing Bakry–Qian's volume comparison theorem [1] and the above improved Bochner formula, the same argument for the proof of Lemma 3.4 gives the following:

Lemma 3.7 *Let* $(M, g, e^{-f}d\mathrm{vol})$ *be a smooth metric measure space with* $\dim M = n \geqslant 3$. *Assume that* $Ric_f \geqslant -(m-1)$. *If*

$$\lambda_1(M) > m-2,$$

then M must have at most one f-nonparabolic end.

Now let us prove Theorem 1.2.

Proof of Theorem 1.2 Note that $m > n \geqslant 3$, hence we have $\frac{(m-1)^2}{4} > m-2$. Assume that M has at least two ends, then, by the same argument in the proof of Theorem 1.2, we have M has only one f-nonparabolic end and

$$\Delta_f \beta = -(m-1),$$

where β is a Busemann function with respect to a geodesic ray issuing from a fixed point p to infinity of a parabolic end.

The fact $|\nabla\beta| = 1$ implies that $\beta_{1j} = 0$ for all $j \geqslant 1$, for an othorgonal basis $\{e_j\}_{j=1}^n$ with $e_1 = \nabla\beta$, and that, by the Bochner formula

$$\begin{aligned} 0 &= \frac{1}{2}\Delta_f|\nabla\beta|^2 = |\nabla^2\beta|^2 + Ric_f(\nabla\beta, \nabla\beta) + \langle\nabla\Delta_f\beta, \nabla\beta\rangle \\ &= \sum_{i,j\geqslant 2}\beta_{ij}^2 + Ric_f^m(e_1, e_1) + \frac{f_1^2}{m-n}. \end{aligned}$$

Then the second fundamental form II and mean curvature of a level set of β satisfy

$$|II|^2 = |\nabla^2\beta|^2 = -\left(Ric_f^m\right)_{11} - \frac{f_1^2}{m-n}$$

and $H := \mathrm{tr}II = -(m-1) + f_1$. By combining with $Ric_f^m \geqslant -(m-1)$ and $H^2 \leqslant (n-1)|II|^2$, we have

$$(-m+1+f_1)^2 \leqslant (n-1)\big(m-1-f_1^2/(m-n)\big).$$

That is,

$$(f_1 - m + n)^2 \leqslant 0.$$

Hence, we obtain $f_1 = m - n$ and all the inequalities above become equalities. In particular, we have $II = \frac{H}{n-1} \cdot (\delta_{ij})_{2\leqslant i,j\leqslant n}$. Note that $H = -(m-1) + f_1 = 1 - n$

$$\begin{cases} \beta_{1j} = 0, & \forall j = 1, \ldots, n \\ \beta_{ij} = 0, & i \neq j \\ \beta_{22} = \cdots = \beta_{nn} = -1. \end{cases}$$

Thus M must be $\mathbb{R} \times N$ isometrically, and the metric is given by $ds^2 = dt^2 + \eta^2(t)ds_N^2$ with

$$(n-1)\frac{\eta'}{\eta} = \mathrm{tr}II = -(n-1).$$

Hence $\eta(t) = \exp(-t)$. □

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References

1. Bakry, D., Qian, Z.: Volume comparison theorems without Jacobi fields. In: Current Trends in Potential Theory, Theta Series in Advanced Mathematics, vol. 4, pp. 115–122 (2005)
2. Bakry, D., Qian, Z.: Some new results on eigenvectors via dimension, diameter, and Ricci curvature. Adv. Math. **155**, 98–153 (2000)
3. Cheng, S.Y.: Eigenvalue comparison theorems and its geometric application. Math. Z. **143**, 289–297 (1975)
4. Li, P., Wang, J.: Complete manifolds with positive spectrum. J. Differ. Geom. **58**, 501–534 (2001)
5. Li, P., Wang, J.: Complete manifolds with positive spectrum, II. J. Differ. Geom. **62**, 143–162 (2002)
6. Li, P., Wang, J.: Weighted Poincaré inequality and rigidity of complete manifolds. Ann. Sci. École Norm. Sup., 4e série **39**, 921–982 (2006)
7. Li, P., Wang, J.: Ends of locally symmetric spaces with maximal bottom spectrum. J. Reine Angew. Math. **632**, 1–35 (2009)
8. Li, X.-D.: Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds. J. Math. Pures Appl. **54**, 1295–1361 (2005)
9. Lott, J., Villani, C.: Weak curvature bounds and functional inequalities. J. Funct. Anal. **245**(1), 311–333 (2007)
10. Munteanu, O., Wang, J.: Smooth metric measure spaces with non-negative curvature. Comm. Anal. Geom. **19**(3), 451–486 (2011)
11. Sturm, K.: Analysis on local Dirichlet spaces, I. J. Reine Angew. Math. **456**, 173–196 (1994)
12. Sturm, K.: On the geometry of metric measure spaces. II. Acta. Math. **196**(1), 133–177 (2006)
13. Wang, L.-F.: The upper bound of the L_μ^2 spectrum. Ann. Glob. Anal. Geom. **37**, 393–402 (2010)
14. Wei, G., Wylie, W.: Comparison geometry for the Bakry–Emery Ricci curvutare. J. Differ. Geom. **83**, 375–405 (2009)
15. Wu, J.-Y.: Upper bounds on the first eigenvalue for a diffusion operator via Bakry-Émery Ricci curvature. J. Math. Anal. Appl. **361**, 10–18 (2010)