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Harmonic Maps Between Alexandrov Spaces

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Abstract In this paper, we shall discuss the existence, uniqueness and regularity of harmonic maps from an Alexandrov space into a geodesic space with curvature $\leqslant 1$ in the sense of Alexandrov.

Keywords Harmonic maps · Alexandrov spaces · Positive upper curvature bounds · Global regularity

Mathematics Subject Classification 58E20 · 53C43

1 Introduction

After a remarkable work [13], the theory of harmonic maps into or between singular spaces has been studied extensively. In [17], Korevaar and Schoen introduced a concept of energy and Sobolev maps for a metric space target, and developed a satisfactory existence and regularity theory for the Dirichlet problem of energy minimizers whenever the target space has non-positive curvature in the sense of Alexandrov. See also [6–8,15,16,20] for related results. In general, in absence of conditions on the curvature of the target, one does not have either existence or regularity of the minimizers.

Let $(X, |\cdot,\cdot|)$, (Y, d) be two metric spaces and let Ω be a bounded domain (connected open subset) of X. μ is a Radon measure on X. Given $p \geqslant 1$, $\varepsilon > 0$ and a Borel measurable map $u : \Omega \to Y$, the approximating energy functional $E^u_{p,\varepsilon}$ of u is given as follows. For each compactly supported continuous function $\varphi \in C_c(\Omega)$, we set

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$$E^u_{p,\varepsilon}(\varphi) := C_{n,p} \int_\Omega \varphi(x)\mathrm{d}\mu(x) \int_{B_\varepsilon(x)\cap\Omega} \frac{d^p(u(x), u(y))}{\varepsilon^{n+p}}\mathrm{d}\mu(y),$$

where $C_{n,p}$ is a normalized constant. The p-th energy functional of u is defined by

$$E^u_p(\varphi) := \limsup_{\varepsilon\to 0} E^u_{p,\varepsilon}(\varphi), \quad \forall\varphi \in C_c(\Omega).$$

We say that $u \in W^{1,p}(\Omega, Y)$ if $u \in L^p(\Omega, Y)$ and it has finite p-energy

$$\sup_{\varphi\in C_c(\Omega),\, 0\leqslant\varphi\leqslant 1} E^u_p(\varphi) < \infty.$$

When Ω is a domain of a smooth manifold, and (Y, d) is an arbitrary metric space, Korevaar–Schoen [17] proved that, for any $\varphi \in C_c(\Omega)$, the limit $\lim_{\varepsilon\to 0} E^u_{p,\varepsilon}(\varphi)$ exists. This is extended to the case where Ω is a domain of a Lipschitz manifold [12], or a domain of a polyhedra [8], or Ω is a domain of an Alexandrov space with curvature bounded below [19].

Given a domain Ω of an Alexandrov space with curvature bounded below and a metric space (Y, d). A map $u : \Omega \mapsto Y$ is called a harmonic map if it is a local energy minimizer of E^u_2. Our purpose in this paper is to study the Dirichlet problem of harmonic maps, including the existence and regularity, from a bounded domain of an Alexandrov space with curvature bounded below into a complete geodesic space of curvature $\leqslant 1$ in the sense of Alexandrov.

When the target (Y, d) is a non-positively curved space, by utilizing the convexity of distance function $d(\cdot, \cdot)$ on the product space $Y \times Y$, the Dirichlet problem of harmonic maps has been solved for a large variety of domains by [6,8,10,16,17,20], including a domain of an Alexandrov space.

Consider the case when (Y, d) has curvature $\leqslant 1$ in the sense of Alexandrov. If Ω is a smooth domain of a Riemannian manifold, the Dirichlet problem of harmonic maps from Ω to Y has been solved by Serbinowski [26] for the case where the images of the maps are contained in a geodesic ball of Y with radius $< \pi/2$. In [17,26], the essential tool is a concept of directional energy, which is a generalization of the directional derivative of functions, defined by a C^1-vector field on Ω. In fact, the energy of a Sobolev map in [17] is able to be represented as an integration of its directional energy. J. Eells and B. Fuglede [8,10,11] extended this method to the case where Ω is a domain of a Riemannian polyhedra.

It is well known that the set of singular points might be dense in a general Alexandrov space with curvature bounded from below [21]. When Ω is a domain of an Alexandrov space, since the C^1 (even Lipschitz continuous) vector fields do not make sense, it seems difficult to employ directly the method of directional energy in the setting of Alexandrov spaces. We need more arguments to conclude the following existence result when the domain is in an Alexandrov space. For a subset $A \subset X$ and a map $u : A \to Y$, we denote by $u(A) := \{u(x) : x \in A\}$.

Theorem 1.1 *Let X be an Alexandrov space with curvature $\geqslant k$, $\Omega \subset X$ be a bounded domain, and let Y be a complete geodesic space with curvature $\leqslant 1$ (in the sense of*

Alexandrov). Given a ball $B_\rho(Q) \subset Y$ with $\rho < \pi/2$, and $\varphi \in W^{1,2}(\Omega, Y)$ with $\varphi(\Omega) \subset B_\rho(Q)$, we write

$$W^{1,2}_\varphi(\Omega, B_\rho(Q)) := \{v \in W^{1,2}(\Omega, Y) : d(v, \varphi) \in W^{1,2}_0(\Omega), v(\Omega) \subset B_\rho(Q)\}.$$

Then $W^{1,2}_\varphi(\Omega, B_\rho(Q))$ has a unique element u of least 2-energy. It is called the harmonic map on Ω which agrees φ on $\partial\Omega$.

Next, we will consider the regularity of harmonic maps given in the above Theorem 1.1. See, for example, [6,8,9,16,17,29] for the relational results for the case that the target has non-positive curvature. Let $u : \Omega \to Y$ be a harmonic map from a smooth Riemannian domain Ω to a metric space with curvature $\leqslant 1$, it is proved in [26] that u is locally Lipschitz continuous and globally Hölder continuous. J. Eells and B. Fuglede [8–11] proved the global Hölder continuity for harmonic maps from a domain with regular boundary of a Riemannian polyhedra to a metric space with curvature $\leqslant 1$.

Here, we shall prove the global Hölder continuity for harmonic maps from a domain of an Alexandrov space to a metric space with curvature $\leqslant 1$. Let Ω be a domain of an Alexandrov space X. It is said to satisfy the *measure density condition*, if there exists a constant $C > 0$ such that

$$\mu(B_r(x) \cap \Omega) \geqslant C\mu(B_r(x)) \tag{1.2}$$

for all $x \in \overline{\Omega}$ and all $0 < r < \min\{1, \mathrm{Diam}(\Omega)\}$. Here $\mathrm{Diam}(\Omega) := \sup_{x,y\in\Omega} |xy|$.

Theorem 1.3 *Let X be an Alexandrov space with curvature $\geqslant k$ and let $\Omega \subset X$ be a bounded domain such that both domains Ω and $X\backslash\overline{\Omega}$ satisfy the measure density condition, and let Y be a complete geodesic space with curvature $\leqslant 1$ (in the sense of Alexandrov). Assume that $w \in W^{1,2}(X, Y)$ is Hölder continuous on $\overline{\Omega}$ and the image $w(X)$ is contained in a geodesic ball $B_\rho(Q) \subset Y$ with radius $\rho < \pi/2$. Suppose that u is the harmonic map on Ω which agrees w on $\partial\Omega$. Then u is Hölder continuous on $\overline{\Omega}$.*

Remark 1.4 (1) If $\Omega \subset M$ is a domain of a smooth manifold with Lipschitz continuous boundary $\partial\Omega$, then both Ω and $M\backslash\overline{\Omega}$ satisfy the measure density condition.

(2) Let p be any point in an Alexandrov space X with curvature bounded below. Then there exists a number $\varepsilon_p > 0$ such that, for any $r \in (0, \varepsilon_p)$, both domains $B_r(p)$ and $X\backslash\overline{B_r(p)}$ satisfy the measure density condition (see [28, from line −4 on p. 472 to line 3 on p. 473].

Recall that every harmonic map $u : \Omega \subset X \to Y$ must be locally Lipschitz continuous in one of the following cases:

(a) X is an Alexandrov space with curvature bounded from below, and Y is a non-positively curved metric space (see [29]);
(b) X is a smooth manifold, Y is a metric space with curvature $\leqslant 1$, and the image $u(\Omega)$ is contained in a geodesic ball with radius $< \pi/2$ (see [26]).

An interesting problem is to improve the Hölder continuity in Theorem 1.3 to the Lipschitz continuity. A similar problem is asked in [20].

Organization of the paper. In Sect. 2, we will provide some necessary materials on Alexandrov spaces, Sobolev spaces for maps, and prove some necessary properties. In particular, we will prove that the energy measure of a $W^{1,p}$-map ($p > 1$) must be absolutely continuous with respect to the Hausdorff measure. In Sect. 3, we will discuss the existence of harmonic maps and prove Theorem 1.1. The regularity of harmonic maps is included in the last section.

2 Preliminaries

2.1 Alexandrov Spaces

Let $k \in \mathbb{R}$ and $l \in \mathbb{N}$. Denote by $\mathbb{M}^l_k$ the simply connected, l-dimensional space form of constant sectional curvature k. The space $\mathbb{M}^2_k$ is called k-plane.

Let $(X, |\cdot,\cdot|)$ be a complete metric space. A rectifiable curve γ connecting two points p, q is called a *geodesic* if its length is equal to $|pq|$ and it has unit speed. A metric space X is called a *geodesic space* if, for every pair of points $p, q \in X$, there exists some geodesic connecting them.

Fix any $k \in \mathbb{R}$. Given three points p, q, r in a geodesic space X, we can take a triangle $\triangle\bar{p}\bar{q}\bar{r}$ in k-plane $\mathbb{M}^2_k$ such that $|\bar{p}\bar{q}| = |pq|$, $|\bar{q}\bar{r}| = |qr|$ and $|\bar{r}\bar{p}| = |rp|$. If $k > 0$, we add the assumption that $|pq| + |qr| + |rp| < 2\pi/\sqrt{k}$. The triangle $\triangle\bar{p}\bar{q}\bar{r} \subset \mathbb{M}^2_k$ is unique up to a rigid motion. We call it *comparison triangle*. We let $\widetilde{\angle}_k pqr$ denote the angle at the vertex $\bar{q}$ of the triangle $\triangle\bar{p}\bar{q}\bar{r}$, and we call it a *$k$-comparison angle*.

Definition 2.1 Let $k \in \mathbb{R}$. A geodesic space X is called an *Alexandrov space with curvature* $\geqslant k$ if it satisfies the following properties:

(1) it is locally compact;
(2) for any point $x \in X$, there exists a neighborhood U of x such that the following condition is satisfied: for any two geodesics $\gamma(t) \subset U$ and $\sigma(s) \subset U$ with $\gamma(0) = \sigma(0) := p$, the k-comparison angle

$$\widetilde{\angle}_\kappa \gamma(t) p \sigma(s)$$

is non-increasing with respect to each of the variables t and s.

It is well known that the Hausdorff dimension of an Alexandrov space with curvature $\geqslant k$, for some constant $k \in \mathbb{R}$, is always an integer (see, for example, [4] or [1]). Let X be an n-dimensional Alexandrov space with curvature $\geqslant k$. We denote by μ the n-dimensional Hausdorff measure on X. The Bishop inequality and the Bishop–Gromov inequality are satisfied on X, i.e., for every $x \in X$, the ratio

$$\frac{\mu(B_r(x))}{V^k_r} \leqslant 1$$

and

$$\frac{\mu(B_r(x))}{V_r^k}$$

is non-increasing in r, where V_r^k is the volume of a ball of radius r in the space form $\mathbb{M}_k^n$, cf. [1]. In particular, the doubling condition is satisfied on X.

Let $p \in X$, given two geodesics $\gamma(t)$ and $\sigma(s)$ with $\gamma(0) = \sigma(0) = p$, the angle

$$\angle\gamma'(0)\sigma'(0) := \lim_{s,t\to 0} \widetilde{\angle}_\kappa \gamma(t)p\sigma(s)$$

is well defined. We denote by Σ_p' the set of equivalence classes of geodesic $\gamma(t)$ with $\gamma(0) = p$, where $\gamma(t)$ is equivalent to $\sigma(s)$ if $\angle\gamma'(0)\sigma'(0) = 0$. The completion of metric space $(\Sigma_p', \angle)$ is called the space of directions at p, denoted by Σ_p. The tangent cone at p, T_p, is the Euclidean cone over Σ_p. For two tangent vectors $u, v \in T_p$, their "scalar product" is defined by (see Sect. 1 in [23])

$$\langle u, v\rangle := \frac{1}{2}(|u|^2 + |v|^2 - |uv|^2).$$

For any $\delta > 0$, we denote

$$X^\delta := \big\{x \in X :\ \mathrm{vol}(\Sigma_x) > (1-\delta)\cdot \mathrm{vol}(\mathbb{S}^{n-1})\big\},$$

where $\mathbb{S}^{n-1}$ is the standard $(n-1)$-sphere, and vol denotes the $(n-1)$-dimensional Hausdorff measure. X^δ an open set (see [1]). The set $S_\delta := X\backslash X^\delta$ is called the δ-*singular set*. Each point $p \in S_\delta$ is called a δ-*singular point*. The set

$$S_X := \bigcup_{\delta>0} S_\delta$$

is called *singular set*. A point $p \in X$ is called a *singular point* if $p \in S_X$. Otherwise it is called a *regular point*. Equivalently, a point p is regular if and only if T_p is isometric to $\mathbb{R}^n$ ([1]). It is proved in [1] that the Hausdorff dimension of S_X is $\leqslant n-1$. We remark that the singular set S_X might be dense in X ([21]).

Some basic structures of Alexandrov spaces are in the following.

Proposition 2.2 *Let $k \in \mathbb{R}$ and let X be an n-dim Alexandrov space with curvature $\geqslant k$.*

(1) *There exists a constant $\delta_{n,k} > 0$ depending only on the dimension n and k such that for each $\delta \in (0, \delta_{n,k})$, the set X^δ forms a Lipschitz manifold* ([1]) *and has a C^∞-differentiable structure* ([18]).
(2) *There exists a BV_{loc}-Riemannian metric g on X^δ such that*
   - *the metric g is continuous in $X\backslash S_X$* ([21,22])*;*
   - *the distance function on $X\backslash S_X$ induced from g coincides with the original one of X* ([21])*;*

- *the Riemannian measure on $X \backslash S_X$ induced from g coincides with the Hausdorff measure of X* ([21]).

Definition 2.3 ([1]) The boundary of an Alexandrov space X is defined inductively with respect to dimension. If the dimension of X is one, then X is a complete Riemannian manifold and the *boundary* of X is defined as usual. Suppose that the dimension of X is $n \geqslant 2$. A point p is a *boundary point* of X if Σ_p has non-empty boundary.

From now on, we always consider Alexandrov spaces without boundary.

2.2 Sobolev Spaces and Laplacian

Several different notions of Sobolev spaces have been established on metric spaces; see [5,14,17–19,27].[1] They coincide with each other on Alexandrov spaces.

From now on, we assume that X is an n-dim Alexandrov space with curvature $\geqslant k$, $\Omega \subset X$ is a domain of X. It is well known that the metric measure space $(X, |\cdot,\cdot|, \mu)$ supports a local (weak) Poincaré inequality.

We denote by $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ the set of locally Lipschitz continuous functions on Ω, and by $\mathrm{Lip}_c(\Omega)$ the set of Lipschitz continuous functions on Ω with compact support.

For a continuous function $u : X \to \mathbb{R}$, define

$$\mathrm{Lip}u(x) = \limsup_{y \to x} \frac{|u(y) - u(x)|}{|yx|}.$$

Definition 2.4 For any $1 \leqslant p \leqslant +\infty$ and $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, its $W^{1,p}(\Omega)$-norm is defined by

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|\mathrm{Lip}u\|_{L^p(\Omega)}.$$

Sobolev spaces $W^{1,p}(\Omega)$ are defined by the closure of the set

$$\{u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega) : \|u\|_{W^{1,p}(\Omega)} < +\infty\},$$

under $W^{1,p}(\Omega)$-norm. Spaces $W_0^{1,p}(\Omega)$ are defined by the closure of $\mathrm{Lip}_c(\Omega)$ under $W^{1,p}(\Omega)$-norm. We denote by $W_c^{1,p}(\Omega) = \{f \in W_0^{1,p}(\Omega) : f \text{ has compact support}\}$. We say that a function $f \in W_{\mathrm{loc}}^{1,p}(\Omega)$ if $f \in W^{1,p}(\Omega')$ for every open subset $\Omega' \subset\subset \Omega$.

Cheeger [5, Theorem 4.48] proved that $W^{1,p}(\Omega)$ is reflexive when $1 < p < \infty$. We denote by ∇u the weak gradient of $u \in W^{1,p}(\Omega)$.

[1] In [5,14,27], Sobolev spaces are defined on metric measure spaces supporting a doubling condition and a Poincaré inequality.

We recall the chain derivation property of $W^{1,2}(\Omega)$. It is formulated as follows: For $f, g \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and $\Phi : \mathbb{R} \to \mathbb{R}$ which is C^1 on the range of f, then $\Phi(f)$ belongs to $W^{1,2}(\Omega)$ and

$$\langle \nabla \Phi(f), \nabla g \rangle = \Phi'(f) \cdot \langle \nabla f, \nabla g \rangle \quad \mu\text{-a.e.}$$

Given $f \in W^{1,2}_{\rm loc}(\Omega)$, the Laplacian of f is defined as a functional on $\mathrm{Lip}_c(\Omega)$ by

$$\mathscr{L}_f(\varphi) := -\int_\Omega \langle \nabla f, \nabla \varphi \rangle \, \mathrm{d}\mu, \quad \forall\, \varphi \in \mathrm{Lip}_c(\Omega).$$

Given $h \in L^1_{\rm loc}(\Omega)$. A function $f \in W^{1,2}_{\rm loc}(\Omega)$ is said to satisfy the inequality

$$\mathscr{L}_f \leqslant h \mathrm{d}\mu$$

in the sense of distribution if the inequality

$$\mathscr{L}_f(\varphi) \leqslant \int_\Omega h\varphi \mathrm{d}\mu$$

holds for all $0 \leqslant \varphi \in \mathrm{Lip}_c(\Omega)$. In this case, the functional $\mathscr{L}_f$ is a signed Radon measure.

Remark 2.5 Moreover, the measure $\mathscr{L}_f$ can be extended to a functional on $\varphi \in L^\infty(\Omega) \cap W^{1,2}_c(\Omega)$.

Indeed, let us fix arbitrarily a compact set $K \subset \Omega$. Take a function $\ell_K \in \mathrm{Lip}_c(\Omega)$ with $\ell_K \equiv 1$ on K. Given any function $\varphi \in \mathrm{Lip}_c(K)$, we have

$$0 \leqslant \|\varphi\|_{L^\infty} \cdot \ell_K \pm \varphi \in \mathrm{Lip}_c(\Omega),$$

and then

$$\begin{aligned} \mathscr{L}_f\Big(\|\varphi\|_{L^\infty} \cdot \ell_K \pm \varphi\Big) &\leqslant \int_\Omega h \cdot \Big(\|\varphi\|_{L^\infty}\ell_K \pm \varphi\Big)\mathrm{d}\mu \\ &\leqslant \|\varphi\|_{L^\infty} \cdot \int_{\mathrm{supp}\ell_K} |h|(\ell_K + 1)\mathrm{d}\mu. \end{aligned}$$

This follows

$$\pm \mathscr{L}_f(\varphi) \leqslant \|\varphi\|_{L^\infty} \cdot \big(2\|h\|_{L^1(\mathrm{supp}\ell_K)} + |\mathscr{L}_f(\ell_K)|\big) := C \cdot \|\varphi\|_{L^\infty}.$$

Thus, since $\mathrm{supp}\varphi \subset K$, we have

$$\left|\int_K \langle \nabla f, \nabla \varphi \rangle \mathrm{d}\mu\right| = |\mathscr{L}_f(\varphi)| \leqslant C \cdot \|\varphi\|_{L^\infty}.$$

At last, any function $\phi \in W^{1,2}_c(\Omega) \cap L^\infty(\Omega)$ can be $W^{1,2}(\Omega)$-approximated by a sequence of functions $\varphi_j \in \mathrm{Lip}_c(\Omega)$ with $\|\varphi_j\|_{L^\infty} \leqslant \|\phi\|_{L^\infty}$ (see, for example, [5, Theorem 4.24]).

2.3 Energy Functional and Sobolev Spaces into Metric Space

From now on, we assume that Ω is a domain of an Alexandrov space and that (Y, d) is a complete metric space. Fix any $p \in [1, \infty)$. Fix a point $P \in Y$. A Borel measurable map $u : \Omega \to Y$ is said to be in the space $L^p(\Omega, Y, P)$ if it has separable range and

$$\int_\Omega d^p\big(u(x), P\big)\mathrm{d}\mu(x) < \infty.$$

If $\mu(\Omega) < \infty$, the space $L^p(\Omega, Y, P)$ does not depend on the choice of the point $P \in Y$. In this case, we denote the space by $L^p(\Omega, Y)$. The space $L^p(\Omega, Y)$ is a metric space under the distance

$$d_p(f, g) = \left(\int_\Omega d^p(f(x), g(x))\mathrm{d}\mu(x)\right)^{\frac{1}{p}}.$$

The space $(L^p(\Omega, Y), d_p)$ forms a complete metric space.

Definition 2.6 For $u \in L^p(\Omega, Y)$, the approximating energy $E^u_{p,\varepsilon}$ is defined by

$$E^u_{p,\varepsilon}(\varphi) = \int_\Omega \varphi(x) e^u_{p,\varepsilon}\mathrm{d}\mu(x),$$

for any $\varphi \in C_c(\Omega)$, where the approximating density of u is defined by

$$e^u_{p,\varepsilon}(x) = \frac{n+p}{c_{n,p}\varepsilon^n}\int_{B_\varepsilon(x)\cap\Omega} \frac{d^p(u(x), u(y))}{\varepsilon^p}\mathrm{d}\mu(y)$$

where the constant $c_{n,p} = \int_{\mathbb{S}^{n-1}} |x^1|^p \sigma(\mathrm{d}x)$, and σ is the canonical volume on $\mathbb{S}^{n-1}$.

Given any $\varphi \in C_c(\Omega)$, it is easy to check that, for any sufficiently small $\varepsilon > 0$, the approximating energy $E^u_{p,\varepsilon}(\varphi)$ coincides (up to a constant) with the one defined by Kuwae and Shioya in [19], that is,

$$\widetilde{E}^u_{p,\varepsilon}(\varphi) := \frac{n}{2\omega_{n-1}\varepsilon^n}\int_\Omega \varphi(x)\int_{B_\varepsilon(x)\cap\Omega} \frac{d^p(u(x), u(y))}{\varepsilon^p} \cdot I_{Q(\Omega)}(x, y)\mathrm{d}\mu(y)\mathrm{d}\mu(x),$$

where

$$Q(\Omega) := \big\{(x, y) \in \Omega \times \Omega : \ |xy| < |\gamma_{xy}, \partial\Omega|, \ \ \forall \text{geodesic } \gamma_{xy} \ \text{ from } x \text{ to } y\big\},$$

and ω_{n-1} is the volume of $\mathbb{S}^{n-1}$. It is proved in [19] that, for each $\varphi \in C_c(\Omega)$, the limit

$$E_p^u(\varphi) := \lim_{\varepsilon \to 0^+} E_{p,\varepsilon}^u(\varphi)$$

exists. We call $E_p^u(\varphi)$ the *energy functional* of u, defined on $C_c(\Omega)$.

For $u \in L^p(\Omega, Y)$, the p-energy of u is defined by

$$E_p^u(\Omega) := \sup_{\phi \in C_c(\Omega), 0 \leqslant \varphi \leqslant 1} E_p^u(\varphi).$$

Definition 2.7 We define the Sobolev space from Ω to Y by

$$W^{1,p}(\Omega, Y) = \left\{ u \in L^p(\Omega, Y) : E_p^u(\Omega) < \infty \right\}.$$

Notice that the space of Lipschitz maps $\mathrm{Lip}(\Omega, Y) \subset W^{1,p}(\Omega, Y)$.

Proposition 2.8 ([19]) *Let* $1 < p < \infty$ *and* $u \in W^{1,p}(\Omega, Y)$. *Then the following assertions (1)–(5) hold.*

(1) (Contraction property, [19, Lemma 3.3]) *Consider another complete metric space* (Z, d_Z) *and a Lipschitz map* $\psi : Y \to Z$, *we have* $\psi \circ u \in W^{1,p}(\Omega, Z)$ *and*

$$E_p^{\psi \circ u}(\varphi) \leqslant |\psi|_{Lip}^p \cdot E_p^u(\varphi)$$

for any $0 \leqslant \varphi \in C_c(\Omega)$, *where*

$$|\psi|_{Lip} := \sup_{y, y' \in Y,\ y \neq y'} \frac{d_Z(\psi(y), \psi(y'))}{d(y, y')}.$$

In particular, for any point $Q \in Y$, *we have* $d\big(Q, u(\cdot)\big) \in W^{1,p}(\Omega, \mathbb{R})$ *and*

$$E_p^{d(Q, u(\cdot))}(\varphi) \leqslant E_p^u(\varphi)$$

for any $0 \leqslant \varphi \in C_c(\Omega)$.

(2) (Lower semi-continuity, [19, Theorem 3.2]) *For any sequence* $u_j \to u$ *in* $L^p(\Omega, Y)$ *as* $j \to \infty$, *we have*

$$E_p^u(\varphi) \leqslant \liminf_{j \to \infty} E_p^{u_j}(\varphi)$$

for any $0 \leqslant \varphi \in C_c(\Omega)$.

(3) (Energy measure, [19, Theorem 4.1 and Proposition 5.1]) *There exists a finite Borel measure, denoted by* E_p^u *again, on* Ω, *which is called the* energy measure *of* u, *such that for any* $0 \leqslant \varphi \in C_c(\Omega)$,

$$E_p^u(\varphi) = \int_\Omega \varphi(x) \mathrm{d}E_p^u(x).$$

Furthermore, the measure is strongly local. That is, for any nonempty open subset $O \subset \Omega$, we have $u|_O \in W^{1,p}(O, Y)$, and moreover, if u is a constant map almost everywhere on O, then $E^u_p(O) = 0$.

(4) (Weak Poincaré inequality, [19, Theorem 4.2(ii)]) *For any open set $O = B_R(Q)$ with $B_{6R}(Q) \subset\subset \Omega$, there exists a positive constant $C = C(n, k, R)$ such that the following holds: for any $z \in O$ and any $0 < r < R/2$, we have*

$$\int_{B_r(z)}\int_{B_r(z)} d^p\big(u(x), u(y)\big)\mathrm{d}\mu(x)\mathrm{d}\mu(y) \leqslant Cr^{n+2} \cdot \int_{B_{6r}(z)} \mathrm{d}E^u_p(x),$$

where the constant C given in [19, p. 61] *depends only on the constants R, ϑ, and Θ in Definition 2.1 for WMCPBG condition in* [19]*. In particular, for the case of Alexandrov spaces as shown in the proof of Theorem 2.1 in* [19]*, one can choose $R > 0$ arbitrarily, $\vartheta = 1$ and*

$$\Theta = \sup_{0<r<R} \frac{\mathrm{vol}(B_r(o) \subset \mathbb{M}^n_k)}{\mathrm{vol}(B_r(o) \subset \mathbb{R}^n)} = C(n, k, R).$$

(5) (Equivalence for $Y = \mathbb{R}$, [19, Theorem 6.2]) *If $Y = \mathbb{R}$, the above Sobolev space $W^{1,p}(\Omega, \mathbb{R})$ is equivalent to the Sobolev space $W^{1,p}(\Omega)$ given in Definition 2.4. Precisely, for any $u \in W^{1,p}(\Omega, \mathbb{R})$, the energy measure E^u_p is absolutely continuous with respect to μ and*

$$\frac{\mathrm{d}E^u_p}{\mathrm{d}\mu} = |\nabla u|^p.$$

For a subset $A \subset X$, denote by

$$\mathrm{Diam}(A) := \sup_{x,y\in A} |xy|.$$

Lemma 2.9 *Let $\Omega \subset X$ be a bounded domain. Then there exist positive constants C_1, C_2 such that*

$$C_2 \leqslant \frac{\mu(B_r(x))}{r^n} \leqslant C_1$$

for any ball $B_r(x) \subset \Omega$.

Proof Set $D := \mathrm{Diam}(\Omega)$. Using the Bishop inequality, we have, for any ball $B_r(x) \subset \Omega$,

$$\mu(B_r(x)) \leqslant V^k_r \leqslant \sup_{0<r\leqslant D} \frac{V^k_r}{r^n} \cdot r^n := C_1(n, D) \cdot r^n,$$

where V^k_r is the volume of a ball of radius r in the space form $\mathbb{M}^n_k$.

For any ball $B_r(x) \subset \Omega$, by using the Bishop–Gromov inequality, we have, for all $0 < r \leqslant D$,

$$\frac{\mu(B_r(x))}{V_r^k} \geqslant \frac{\mu(B_D(x))}{V_D^k}.$$

Note that $\Omega \subset B_D(x)$, thus we have

$$\frac{\mu(B_r(x))}{r^n} \geqslant \frac{\mu(\Omega)}{V_D^k} \cdot \frac{V_r^k}{r^n} \geqslant \inf_{0<r\leqslant D} \frac{V_r^k}{r^n} \cdot \frac{\mu(\Omega)}{V_D^k} := C_2(n, \Omega).$$

Now the proof is complete. □

To simplify, we use the following notation throughout this paper. Given a μ-measurable function f and a μ-measurable subset A, we denote by

$$\fint_A f \mathrm{d}\mu := \frac{1}{\mu(A)} \int_A f \mathrm{d}\mu.$$

2.4 Absolute Continuity of the Energy Measures

We continue to assume that Ω is a domain of an Alexandrov space and that (Y, d) is a complete metric space.

In [24], Reshetnyak proposed a similar approach to define the Sobolev spaces for maps into a metric space. Given $1 \leqslant p < \infty$, by the definition in [24], a map $u \in L^p(\Omega, Y)$ belongs to Reshetnyak–Sobolev space, denoted by $R^{1,p}(\Omega, Y)$, if for every Lipschitz function $\psi : Y \to \mathbb{R}$ the composition $\psi \circ u \in W^{1,p}(\Omega)$, and there is a function $w \in L^p(\Omega)$ (independent of ψ) such that, for every Lipschitz function $\psi : Y \to \mathbb{R}$, the inequality holds

$$|\nabla(\psi \circ u)|(x) \leqslant |\psi|_{Lip} \cdot w(x) \qquad \mu\text{−a.e. } x \in \Omega.$$

Lemma 2.10 ([14]) *Let* $1 < p < \infty$*, we have* $W^{1,p}(\Omega, Y) \subset R^{1,p}(\Omega, Y)$.

Proof The lemma is due to Heinonen–Koskela–Shanmugalingam–Tyson [14] essentially. For completeness, we include a proof here.

Let $u \in W^{1,p}(\Omega, Y)$. Given any Lipschitz function $\psi : Y \to \mathbb{R}$, we have the composition $\psi \circ u \in W^{1,p}(\Omega)$, because of Proposition 2.8(1) and (5). According to Proposition 2.8(1), we know that the energy measure of u and $\psi \circ u$ satisfies

$$E_p^{\psi \circ u} \leqslant |\psi|_{Lip}^p \cdot E_p^u. \tag{2.11}$$

Consider the Lebesgue decomposition of E_p^u with respect to μ on Ω,

$$E_p^u = |\nabla u|_p \cdot \mu + (E_p^u)^s,$$

where $|\nabla u|_p \in L^1(\Omega)$, because $E_p^u(\Omega) < \infty$. By Proposition 2.8(5), the measure $E_p^{\psi\circ u}$ is absolutely continuous w.r.t. μ, and has density $|\nabla(\psi \circ u)|^p$. From (2.11), we have

$$|\nabla(\psi \circ u)|^p \cdot \mu \leqslant |\psi|_{Lip}^p \cdot |\nabla u|_p \cdot \mu + |\psi|_{Lip}^p \cdot (E_p^u)^s.$$

The singular part $(E_p^u)^s$ is supported in a μ-zero measure set. Then we have

$$|\nabla(\psi \circ u)|(x) \leqslant |\psi|_{Lip} \cdot |\nabla u|_p^{1/p}(x) \qquad \mu\text{−a.e. } x \in \Omega.$$

Therefore, $u \in R^{1,p}(\Omega, Y)$. □

Remark 2.12 If Ω is a bounded domain of a smooth manifold, it is shown in [14,25] that $W^{1,p}(\Omega, Y) = R^{1,p}(\Omega, Y)$. It is interesting to clarify whether the assertion is still valid when Ω is a bounded domain of an Alexandrov space.

Proposition 2.13 (Absolute continuity for the energy measures) *Let* $1 < p < \infty$. *For each* $u \in W^{1,p}(\Omega, Y)$*, the energy measure* E_p^u *is absolutely continuous w.r.t the volume* μ*, i.e., there exists* $|\nabla u|_p \in L^1(\Omega)$ *such that*

$$\frac{\mathrm{d}E_p^u}{\mathrm{d}\mu} = |\nabla u|_p.$$

Proof By [14, Corollary 3.21] that $R^{1,p}(\Omega, Y) = N^{1,p}(\Omega, Y) \subset N^{1,p}(X, V := \ell^\infty(Y))$, where $N^{1,p}(\Omega, Y)$ is the Newtonian–Sobolev space in [14, Definition 3.9] by the upper gradient. From Lemma 2.10, we have $u \in N^{1,p}(X, V)$. By [14, Proposition 4.6], we have

$$d(u(x), u(y)) \leqslant C(n, \Omega)(Mg(x) + Mg(y)) \cdot |xy|,$$

for μ-a.e. $x, y \in \Omega$, where $Mg(x)$ is the maximal function of g defined by

$$Mg(x) = \sup_{r>0} ⨍_{B_r(x)} g(y)\mathrm{d}\mu(y),$$

and $g \in L^p(\Omega)$ is a weak upper gradient in the sense of [14, Definition 3.9]. Denote by $h = Mg$ for convenience. Without loss of generality, for any $0 \leqslant \varphi \in \mathrm{Lip}_c(\Omega)$, we have

$$\begin{aligned}\frac{c_{n,p}}{n+p} E_{p,\varepsilon}^u(\varphi) &= \int_\Omega \varphi(x)\mathrm{d}\mu(x) \int_{B_\varepsilon(x)} \frac{d^p(u(x), u(y))}{\varepsilon^{n+p}}\mathrm{d}\mu(y) \\ &\leqslant C(n, p, \Omega) \int_\Omega \varphi(x)\mathrm{d}\mu(x) \int_{B_\varepsilon(x)} \frac{h^p(x) + h^p(y)}{\varepsilon^n}\mathrm{d}\mu(y) \qquad (2.14)\end{aligned}$$

$$\leqslant C(n,p,\Omega)\int_\Omega \varphi(x)h^p(x)\mathrm{d}\mu(x) + C(n,p,\Omega)\int_\Omega \varphi(x)\mathrm{d}\mu(x)\int_{B_\varepsilon(x)}\frac{h^p(y)}{\varepsilon^n}\mathrm{d}\mu(y),$$

where we have used that $\mu(B_\varepsilon(x)) \leqslant C_2\varepsilon^n$. Set

$$|\varphi|_{Lip} = \sup_{x,y\in\Omega, x\neq y}\frac{|\varphi(y)-\varphi(x)|}{d(x,y)}.$$

Define a function $I_\varepsilon(x,y)$ on $\Omega\times\Omega$ by

$$I_\varepsilon(x,y) = \begin{cases} 1, & \text{if } |xy| < \varepsilon, \\ 0, & \text{if } |xy| \geqslant \varepsilon. \end{cases}$$

Note that $I_\varepsilon(x,y) = I_\varepsilon(y,x)$. On the other hand,

$$\begin{aligned}
&\int_\Omega \varphi(x)\mathrm{d}\mu(x)\int_{B_\varepsilon(x)}\frac{h^p(y)}{\varepsilon^n}\mathrm{d}\mu(y)\\
&= \int_{\Omega\times\Omega}\varphi(x)\frac{h^p(y)}{\varepsilon^n}I_\varepsilon(x,y)\mathrm{d}\mu(y)\mathrm{d}\mu(x)\\
&\leqslant \int_{\Omega\times\Omega}(\varphi(y)+|\varphi|_{Lip}\cdot|xy|)\frac{h^p(y)}{\varepsilon^n}I_\varepsilon(x,y)\mathrm{d}\mu(y)\mathrm{d}\mu(x)\\
&= \int_{\Omega\times\Omega}(\varphi(y)+|\varphi|_{Lip}\cdot|xy|)\frac{h^p(y)}{\varepsilon^n}I_\varepsilon(x,y)\mathrm{d}\mu(x)\mathrm{d}\mu(y)\\
&= \int_\Omega h^p(y)\mathrm{d}\mu(y)\int_{B_\varepsilon(y)}\frac{\varphi(y)+|\varphi|_{Lip}\cdot|xy|}{\varepsilon^n}\mathrm{d}\mu(x)\\
&\leqslant \int_\Omega h^p(y)\mathrm{d}\mu(y)\int_{B_\varepsilon(y)}\frac{\varphi(y)+|\varphi|_{Lip}\cdot\varepsilon}{\varepsilon^n}\mathrm{d}\mu(x)\\
&\leqslant C\int_\Omega(\varphi(y)+|\varphi|_{Lip}\cdot\varepsilon)h^p(y)\mathrm{d}\mu(y)\\
&= C\int_\Omega\varphi(y)h^p(y)\mathrm{d}\mu(y) + C\varepsilon\int_\Omega|\varphi|_{Lip}\cdot h^p(y)\mathrm{d}\mu(y).
\end{aligned} \tag{2.15}$$

Note that by the classical Hardy–Littlewood estimate, $h = Mg \in L^p(\Omega)$. Then, by combining (2.14) and (2.15), and taking $\varepsilon \to 0$, we obtain that

$$E_p^u(\varphi) \leqslant C\int_\Omega \varphi(x)h^p(x)\mathrm{d}\mu(x).$$

Hence, the proof is complete. □

Recall that a result about the point-wise convergence of approximating density is given in [29, Corollary 4.6].

Lemma 2.16 ([29]) *Let* $1 < p < \infty$ *and* $u \in W^{1,p}(\Omega, Y)$. *Then, for any sequence of positive numbers* $\{\epsilon_i\}$ *converging to* 0*, there is a subsequence* $\{\varepsilon_i\} \subset \{\epsilon_i\}$ *such that*

$$\lim_{i\to\infty} e^u_{p,\varepsilon_i}(x) = |\nabla u|_p(x) \qquad \mu\text{−a.e. } x \in \Omega.$$

Combining the absolute continuity of energy measure and the above point-wise convergence result, we can obtain a locally L^1-convergence of the approximating density.

Proposition 2.17 *Let* $1 < p < \infty$ *and* $u \in W^{1,p}(\Omega, Y)$. *Then we have*

$$e^u_{p,\varepsilon} \to |\nabla u|_p \quad \textit{in } L^1_{\text{loc}}(\Omega),$$

as $\varepsilon \to 0$.

Proof It suffices to show that for any ball $B \subset\subset \Omega$ we have $e^u_{p,\varepsilon} \to |\nabla u|_p$ in $L^1(B)$. We argue it by contradiction. Suppose that there exists some ball $B \subset\subset \Omega$, such that

$$e^u_{p,\epsilon} \nrightarrow |\nabla u|_p \quad \text{in } L^1(B),$$

as $\varepsilon \to 0$. Then there exist $\delta_0 > 0$ and a sequence of positive numbers $\{\epsilon_i\}$, going to 0, such that

$$\int_B |e^u_{p,\epsilon_i} - |\nabla u|_p| \geqslant \delta_0, \tag{2.18}$$

for all large enough $i \geqslant 1$.

Notice that $\mu(\partial B) = 0$. By Proposition 2.13, we have $E^u_p(\partial B) = 0$. Recall that $E^u_{p,\epsilon_i} = e^u_{p,\epsilon_i} \mathrm{d}\mu \rightharpoonup E^u_p = |\nabla u|_p \mathrm{d}\mu$ weakly as Radon measures on Ω. Thus, we have

$$\lim_{i\to\infty} E^u_{p,\epsilon_i}(B) = E^u_p(B).$$

That is,

$$\lim_{i\to\infty} \int_B e^u_{p,\epsilon_i} \mathrm{d}\mu = \int_B |\nabla u|_p \mathrm{d}\mu. \tag{2.19}$$

By Lemma 2.16, there is a subsequence $\{\varepsilon_i\} \subset \{\epsilon_i\}$ such that

$$\lim_{i\to\infty} e^u_{p,\varepsilon_i}(x) = |\nabla u|_p \qquad \mu\text{−a.e. } x \in \Omega. \tag{2.20}$$

Thus, by combination of (2.19) and (2.20) implies that

$$e^u_{p,\varepsilon_i} \to |\nabla u|_p \quad \text{in } L^1(B).$$

This contradicts with (2.18). Hence, the proof is complete. □

3 Harmonic Maps into Space of Curvature Bounded from Above

We continue to assume that Ω is a domain of an Alexandrov space and that (Y, d) is a complete metric space. In this section, we will prove the existence of harmonic maps, Theorem 1.1.

3.1 Some Representations of Energy Functionals

We will give some technical lemmas about the representation of energy functionals.

Lemma 3.1 *For any fixed $\alpha > 0$ and $u \in W^{1,2}(\Omega, Y)$, we have for any $v \in W^{1,2}(\Omega, Y)$ and $\varphi \in C_c(\Omega)$*

$$\frac{c_{n,2}}{n+2}E_2^v(\varphi) = \lim_{\varepsilon\to 0}\int_\Omega \varphi(x)\mathrm{d}\mu(x)\int_{\{y\in B_\varepsilon(x):d(u(x),u(y))<\alpha\}}\frac{d^2(v(x),v(y))}{\varepsilon^{n+2}}\mathrm{d}\mu(y). \tag{3.2}$$

Proof **Step 1**. We firstly show that (3.2) holds in the case where Ω is a bounded of the Euclidean space $\mathbb{R}^n$.

Let $\omega \in \mathbb{S}^{n-1}$ be an unit vector. The corresponding directional energy ${}^\omega E_2^v$ of v is defined by (see [17, Theorem 1.8.1])

$${}^\omega E_2^v(\varphi) := \lim_{\varepsilon\to 0}\int_\Omega \frac{d^2(v(x),v(x+\varepsilon\omega))}{\varepsilon^2}\varphi(x)\mathrm{d}\mu(x).$$

According to [26, Corollary 1.5], we have

$$\frac{d(v(x),v(x+\varepsilon\omega))}{\varepsilon} \to |v_*(\omega)|(x) \quad \text{in } L^2_{\text{loc}}(\Omega), \tag{3.3}$$

as $\varepsilon \to 0$, for some $|v_*(\omega)|(x) \in L^2(\Omega)$.

We denote by $\Omega_\varepsilon := \{x \in \Omega : |x, \partial\Omega| > \varepsilon\}$ and $A_u^\varepsilon = \{x \in \Omega_\varepsilon : d(u(x), u(x + \varepsilon\omega)) \geqslant \alpha\}$.

Sublemma 3.4 *For any $\varphi \in C_c(\Omega)$, we have*

$${}^\omega E_2^v(\varphi) = \lim_{\varepsilon\to 0}\int_{\Omega\setminus A_u^\varepsilon} \frac{d^2(v(x),v(x+\varepsilon\omega))}{\varepsilon^2}\varphi(x)\mathrm{d}\mu(x). \tag{3.5}$$

Proof of Sublemma By definition, we have

$$\begin{aligned}{}^\omega E_2^v(\varphi) &= \lim_{\varepsilon\to 0}\int_{\Omega\setminus A_u^\varepsilon} \frac{d^2(v(x),v(x+\varepsilon\omega))}{\varepsilon^2}\varphi(x)\mathrm{d}\mu(x)\\ &\quad + \lim_{\varepsilon\to 0}\int_{A_u^\varepsilon} \frac{d^2(v(x),v(x+\varepsilon\omega))}{\varepsilon^2}\varphi(x)\mathrm{d}\mu(x).\end{aligned}$$

However, we have

$$\left|\int_{A_u^\varepsilon}\frac{d^2(v(x),v(x+\varepsilon\omega))}{\varepsilon^2}\varphi(x)\mathrm{d}\mu(x)\right|\leqslant|\varphi|_\infty\int_{A_u^\varepsilon\cap\mathrm{supp}\varphi}\frac{d^2(v(x),v(x+\varepsilon\omega))}{\varepsilon^2}\mathrm{d}\mu(x)$$

converges to 0, because of (3.3) and that $\mu(A_u^\varepsilon\cap\mathrm{supp}\varphi)\to 0$. Hence, we have proved this sublemma. □

We now continue the proof of Lemma 3.1. Fix any $\varphi\in C_c(\Omega)$. Without loss of generality, we may assume that $\varphi\geqslant 0$. Define a function $I_u^\varepsilon(x,\omega)$ on $\Omega_\varepsilon\times\mathbb{S}^{n-1}$ by

$$I_u^\varepsilon(x,\omega)=\begin{cases}1, & \text{if } d(u(x),u(x+\varepsilon\omega))<\alpha,\\ 0, & \text{if } d(u(x),u(x+\varepsilon\omega))\geqslant\alpha.\end{cases}$$

It follows from (3.5) that (by [17, Theorem 1.8.1])

$$\begin{aligned}c_{n,2}\cdot E_2^v(\varphi)&=\int_{\mathbb{S}^{n-1}}{}^{\omega}E_2^v(\varphi)\mathrm{d}\sigma(\omega)\\&=\int_{\mathbb{S}^{n-1}}\mathrm{d}\sigma(\omega)\lim_{\varepsilon\to0}\int_\Omega\frac{d^2(v(x),v(x+\varepsilon\omega))}{\varepsilon^2}I_u^\varepsilon(x,\omega)\varphi(x)\mathrm{d}\mu(x).\end{aligned}\tag{3.6}$$

By [17, Lemma 1.8.2], we have

$$\int_\Omega\frac{d^2(v(x),v(x+\varepsilon\omega))}{\varepsilon^2}\varphi(x)\mathrm{d}\mu(x)\leqslant C|\varphi|_\infty E_2^v(\Omega),\tag{3.7}$$

for some constant $C>0$, independent of ω. Hence, by (3.6), (3.7) and the bounded convergence theorem, we have

$$c_{n,2}\cdot E_2^v(\varphi)=\lim_{\varepsilon\to0}\int_{\mathbb{S}^{n-1}}\mathrm{d}\sigma(\omega)\int_\Omega\frac{d^2(v(x),v(x+\varepsilon\omega))}{\varepsilon^2}I_u^\varepsilon(x,\omega)\varphi(x)\mathrm{d}\mu(x).\tag{3.8}$$

Applying Fubini's theorem to (3.8), we obtain that

$$c_{n,2}\cdot E_2^v(\varphi)=\lim_{\varepsilon\to0}\int_\Omega\varphi(x)\mathrm{d}\mu(x)\int_{\mathbb{S}^{n-1}}\frac{d^2(v(x),v(x+\varepsilon\omega))}{\varepsilon^2}I_u^\varepsilon(x,\omega)\mathrm{d}\sigma(\omega).\tag{3.9}$$

Let $S(x,\varepsilon)$ be the sphere centered at x with radius ε, i.e., $S(x,\varepsilon):=\{y:|xy|=\varepsilon\}$, and $\mathrm{d}\sigma_{x,\varepsilon}$ be the area volume on $S(x,\varepsilon)$. Define $K_{u,x}^\varepsilon(y)$ be the characteristic function of $\{y\in S(x,\varepsilon):d(u(x),u(y))<\alpha\}$. We obtain from (3.9) that

$$c_{n,2}\cdot E_2^v(\varphi)=\lim_{\varepsilon\to0}\int_\Omega\varphi(x)\mathrm{d}\mu(x)\int_{S(x,\varepsilon)}\frac{d^2(v(x),v(y))}{\varepsilon^{n+1}}K_{u,x}^\varepsilon(y)\mathrm{d}\sigma_{x,\varepsilon}(y).\tag{3.10}$$

Divding both sides of (3.10) by $n+2$, we have

$$\frac{c_{n,2}}{n+2}E_2^v(\varphi) = \int_0^1 \lambda^{n+1}d\lambda \lim_{\varepsilon\to 0}\int_\Omega \varphi(x)\mathrm{d}\mu(x)\int_{S(x,\lambda\varepsilon)}\frac{d^2(v(x),v(y))}{(\lambda\varepsilon)^{n+1}}K_{u,x}^{\lambda\varepsilon}(y)\mathrm{d}\sigma_{x,\lambda\varepsilon}(y). \tag{3.11}$$

According to [17, Theorem 1.5.1], there exists $\varepsilon_0 > 0$, such that

$$\int_\Omega \varphi(x)\mathrm{d}\mu(x)\int_{S(x,\delta)}\frac{d^2(v(x),v(y))}{\delta^{n+1}}\mathrm{d}\sigma_{x,\delta}(y) \leqslant 1 + E_2^v(\varphi)$$

for any $0<\delta<\varepsilon_0$. Thus, by applying the bounded convergence theorem to (3.11), we have

$$\begin{aligned}\frac{c_{n,2}}{n+2}E_2^v(\varphi) &= \lim_{\varepsilon\to 0}\int_0^1 \lambda^{n+1}d\lambda\int_\Omega \varphi(x)\mathrm{d}\mu(x)\int_{S(x,\lambda\varepsilon)}\frac{d^2(v(x),v(y))}{(\lambda\varepsilon)^{n+1}}K_{u,x}^{\lambda\varepsilon}(y)\mathrm{d}\sigma_{x,\lambda\varepsilon}(y)\\ &= \lim_{\varepsilon\to 0}\int_0^1 d\lambda\int_\Omega \varphi(x)\mathrm{d}\mu(x)\int_{S(x,\lambda\varepsilon)}\frac{d^2(v(x),v(y))}{\varepsilon^{n+1}}K_{u,x}^{\lambda\varepsilon}(y)\mathrm{d}\sigma_{x,\lambda\varepsilon}(y)\\ &= \lim_{\varepsilon\to 0}\int_\Omega \varphi(x)\mathrm{d}\mu(x)\int_0^1 d\lambda\int_{S(x,\lambda\varepsilon)}\frac{d^2(v(x),v(y))}{\varepsilon^{n+1}}K_{u,x}^{\lambda\varepsilon}(y)\mathrm{d}\sigma_{x,\lambda\varepsilon}(y).\end{aligned} \tag{3.12}$$

Note that $\mathrm{d}\mu = \varepsilon\mathrm{d}\lambda\mathrm{d}\sigma_{x,\lambda\varepsilon}$. Thus, we obtain from (3.12) that

$$\frac{c_{n,2}}{n+2}E_2^v(\varphi) = \lim_{\varepsilon\to 0}\int_\Omega \varphi(x)\mathrm{d}\mu(x)\int_{\{y\in B_\varepsilon(x):d(u(x),u(y))<\alpha\}}\frac{d^2(v(x),v(y))}{\varepsilon^{n+2}}\mathrm{d}\mu(y).$$

Equivalently,

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi(x)\mathrm{d}\mu(x)\int_{\{y\in B_\varepsilon(x):d(u(x),u(y))\geqslant\alpha\}}\frac{d^2(v(x),v(y))}{\varepsilon^{n+2}}\mathrm{d}\mu(y) = 0. \tag{3.13}$$

Hence, we have proved this lemma for the case where the domain $\Omega \subset \mathbb{R}^n$.

Step 2. We secondly show that (3.2) holds in the case where Ω is a bounded domain of a Lipschitz manifold with an L^∞-Riemannian metric. Equivalently, we want to show

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi(x)\mathrm{d}\mu(x)\int_{\{y\in B_\varepsilon(x):d(u(x),u(y))\geqslant\alpha\}}\frac{d^2(v(x),v(y))}{\varepsilon^{n+2}}\mathrm{d}\mu(y) = 0. \tag{3.14}$$

By compactness of $\overline{\Omega}$ and the partition of unit, we may assume that Ω is contained in some local chart (U, ψ). Notice that ψ is bi-Lipschitz, i.e.,

$$m \cdot |xy| \leqslant |\psi(x) - \psi(y)|_{\mathbb{R}^n} \leqslant M \cdot |xy|$$

for some constants $m, M > 0$. Denote by $B_r^e(x)$ the ball centered at $x \in \mathbb{R}^n$ with radius r, and by $g = \det(g_{ij})_{n\times n}$. Without loss of generality, for any $0 \leqslant \varphi \in C_c(\Omega)$,

$$\begin{aligned}
&\int_\Omega \varphi(x)\mathrm{d}\mu(x) \int_{\{y\in B_\varepsilon(x): d(u(x),u(y))\geqslant\alpha\}} \frac{d^2(v(x), v(y))}{\varepsilon^{n+2}}\mathrm{d}\mu(y)\\
&\quad\leqslant \int_{\psi(\Omega)} \varphi\circ\psi^{-1}(x')\sqrt{g(x')}\mathrm{d}x'\\
&\qquad \int_{\{y'\in B^e_{M\varepsilon}(x'): d(u\circ\psi^{-1}(x'),\, u\circ\psi^{-1}(y'))\geqslant\alpha\}} \frac{d^2(v\circ\psi^{-1}(x'), v\circ\psi^{-1}(y'))}{\varepsilon^{n+2}}\sqrt{g(y')}\mathrm{d}y'\\
&\quad= M^{n+2}\int_{\psi(\Omega)} \varphi\circ\psi^{-1}(x')\sqrt{g(x')}\mathrm{d}x'\\
&\qquad \int_{\{y'\in B^e_{M\varepsilon}(x'): d(u\circ\psi^{-1}(x'),\, u\circ\psi^{-1}(y'))\geqslant\alpha\}} \frac{d^2(v\circ\psi^{-1}(x'), v\circ\psi^{-1}(y'))}{(M\varepsilon)^{n+2}}\sqrt{g(y')}\mathrm{d}y'\\
&\quad\to 0, \quad \text{as } \varepsilon\to 0 \quad (\text{by } (3.13)).
\end{aligned}$$

Hence, (3.14) holds true.

Step 3. At last, we shall finish the lemma by proving (3.2) in the case where Ω is a bounded domain of an Alexandrov space.

Fix $\delta \in (0, \delta_{n,k})$, where $\delta_{n,k}$ is defined in Proposition 2.2. Let $\Omega^\delta = \{x \in \Omega : \mathrm{vol}(\Sigma_x) > (1-\delta)\mathrm{vol}(\mathbb{S}^{n-1})\}$. Ω^δ is an open subset of Ω. Let $K^\delta = \Omega\backslash\Omega^\delta$. Then $\mu(K^\delta) = 0$. Furthermore, by Proposition 2.2, Ω^δ forms a Lipschitz manifold.

Hence we have for each $\varphi \in C_c(\Omega^\delta)$,

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi(x)\mathrm{d}\mu(x)\int_{\{y\in B_\varepsilon(x):d(u(x),u(y))\geqslant\alpha\}} \frac{d^2(v(x), v(y))}{\varepsilon^{n+2}}\mathrm{d}\mu(y) = 0. \tag{3.15}$$

We now prove that (3.15) still holds for each $\varphi \in C_c(\Omega)$. Fix any $\varphi \in C_c(\Omega)$. Without loss of generality, we may assume that $\varphi \geqslant 0$. For any $\gamma > 0$ sufficiently small, we can choose $f \in C_c(\Omega^\delta)$, such that $0 \leqslant f \leqslant \varphi$ and

$$E_2^v(\varphi) - E_2^v(f) \leqslant \kappa_2(\gamma), \tag{3.16}$$

where $\kappa_2(\gamma)$ is a positive function with

$$\lim_{\gamma\to 0}\kappa_2(\gamma) = 0. \tag{3.17}$$

This can be easily done by the following arguments. Recall that $K^\delta = \Omega\backslash\Omega^\delta$ and $\mu(K^\delta) = 0$. Let $K^\delta_\gamma = \{x \in \Omega : |x, K^\delta| < \gamma\}$ and $h : \Omega \mapsto [0, 1]$ be a continuous function such that $h\big|_{\Omega\setminus K^\delta_\gamma} = 1$ and $h\big|_{K^\delta_{\gamma/2}} = 0$. Choose $f = h\circ\varphi$, then by Proposition 2.13, we have

$$E_2^v(\varphi) - E_2^v(f) \leqslant \int_{K^\delta_\gamma} \varphi|\nabla v|_2 \mathrm{d}\mu := \kappa_2(\gamma).$$

Hence, (3.16) holds true. We compute

$$\begin{aligned} &\int_\Omega \varphi(x)\mathrm{d}\mu(x) \int_{\{y\in B_\varepsilon(x): d(u(x),u(y))\geqslant\alpha\}} \frac{d^2(v(x),v(y))}{\varepsilon^{n+2}}\mathrm{d}\mu(y) \\ &\quad\leqslant \int_\Omega \varphi(x) - f(x)\mathrm{d}\mu(x)\int_{B_\varepsilon(x)} \frac{d^2(v(x),v(y))}{\varepsilon^{n+2}}\mathrm{d}\mu(y) \\ &\qquad + \int_\Omega f(x)\mathrm{d}\mu(x)\int_{\{y\in B_\varepsilon(x): d(u(x),u(y))\geqslant\alpha\}} \frac{d^2(v(x),v(y))}{\varepsilon^{n+2}}\mathrm{d}\mu(y) \\ &\quad := I_1 + I_2. \end{aligned} \tag{3.18}$$

It follows from (3.16) that

$$\lim_{\varepsilon\to 0} I_1 \leqslant \kappa_2(\gamma). \tag{3.19}$$

We obtain from (3.15) that

$$\lim_{\varepsilon\to 0} I_2 = 0. \tag{3.20}$$

Hence, the combination of (3.17)–(3.20) implies that

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi(x)\mathrm{d}\mu(x) \int_{\{y\in B_\varepsilon(x): d(u(x),u(y))\geqslant\alpha\}} \frac{d^2(v(x),v(y))}{\varepsilon^{n+2}}\mathrm{d}\mu(y) = 0.$$

Equivalently, (3.2) holds true. □

Corollary 3.21 *Fix any $\alpha,\beta > 0$ and $u, v \in W^{1,2}(\Omega, Y)$. Let $I_\varepsilon(x) := \{y \in B_\varepsilon(x) : d(u(x),u(y)) \leqslant \alpha$ and $d(v(x),v(y)) \leqslant \beta\}$. Then we have, for any $w \in W^{1,2}(\Omega, Y)$ and any $\varphi \in C_c(\Omega)$,*

$$E_2^w(\varphi) = \int_\Omega |\nabla w|_2\varphi\mathrm{d}\mu = \frac{n+2}{c_{n,2}}\lim_{\varepsilon\to 0}\int_\Omega \varphi(x)\mathrm{d}\mu(x)\int_{I_\varepsilon(x)} \frac{d^2(w(x),w(y))}{\varepsilon^{n+2}}\mathrm{d}\mu(y).$$

Corollary 3.22 *Fix any $\alpha,\beta > 0$ and $u, v \in W^{1,2}(\Omega, Y)$. Let $I_\varepsilon(x) := \{y \in B_\varepsilon(x) : d(u(x),u(y)) \leqslant \alpha$ and $d(v(x),v(y)) \leqslant \beta\}$. Then, for any $w \in W^{1,2}(\Omega, Y)$ and any $\varphi \in L^\infty(\Omega)$ with compact support*

$$E_2^w(\varphi) = \int_\Omega |\nabla w|_2\varphi\mathrm{d}\mu = \frac{n+2}{c_{n,2}}\lim_{\varepsilon\to 0}\int_\Omega \varphi(x)\mathrm{d}\mu(x)\int_{I_\varepsilon(x)} \frac{d^2(w(x),w(y))}{\varepsilon^{n+2}}\mathrm{d}\mu(y)$$

Proof By Proposition 2.17, we obtain that

$$e^{w}_{2,\varepsilon} \to |\nabla w|_2 \quad \text{in } L^1_{\mathrm{loc}}(\Omega).$$

Thus, for any $0 \leqslant \varphi \in L^\infty(\Omega)$ with compact support, we have

$$\int_\Omega |\nabla w|_2 \varphi \mathrm{d}\mu = \frac{n+2}{c_{n,2}} \lim_{\varepsilon\to 0} \int_\Omega \varphi(x)\mathrm{d}\mu(x) \int_{B_\varepsilon(x)} \frac{d^2(w(x), w(y))}{\varepsilon^{n+2}} \mathrm{d}\mu(y) \quad (3.23)$$

On the other hand, choose $f \in C_c(\Omega)$ such that $\varphi \leqslant f$. Then, by Corollary 3.21, we have

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \int_\Omega \varphi(x)\mathrm{d}\mu(x) \int_{B_\varepsilon(x)\setminus I_\varepsilon(x)} \frac{d^2(w(x), w(y))}{\varepsilon^{n+2}} \mathrm{d}\mu(y) \\ &\quad \leqslant \limsup_{\varepsilon\to 0} \int_\Omega f(x)\mathrm{d}\mu(x) \int_{B_\varepsilon(x)\setminus I_\varepsilon(x)} \frac{d^2(w(x), w(y))}{\varepsilon^{n+2}} \mathrm{d}\mu(y) = 0. \end{aligned} \quad (3.24)$$

Thus, this corollary follows from the combination of (3.23) and (3.24). □

Corollary 3.25 *For any $\delta > 0$ and $u \in W^{1,2}(\Omega, Y)$ with $\mathrm{Diam}(u(\Omega)) < \infty$, we have*

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi(x)\mathrm{d}\mu(x) \int_{B_\varepsilon(x)} \frac{d^{2+\delta}(u(x), u(y))}{\varepsilon^{n+2}} \mathrm{d}\mu(y) = 0$$

for any $\varphi \in C_c(\Omega)$.

Proof Let $D = \mathrm{Diam}(u(\Omega))$. Fix any $\alpha > 0$. Without loss of generality, for any $0 \leqslant \varphi \in C_c(\Omega)$,

$$\begin{aligned} &\int_\Omega \varphi(x)\mathrm{d}\mu(x) \int_{B_\varepsilon(x)} \frac{d^{2+\delta}(u(x), u(y))}{\varepsilon^{n+2}} \mathrm{d}\mu(y) \\ &\quad = \int_\Omega \varphi(x)\mathrm{d}\mu(x) \int_{\{y\in B_\varepsilon(x): d(u(x),u(y))\leqslant\alpha\}} \frac{d^{2+\delta}(u(x), u(y))}{\varepsilon^{n+2}} \mathrm{d}\mu(y) \\ &\qquad + \int_\Omega \varphi(x)\mathrm{d}\mu(x) \int_{\{y\in B_\varepsilon(x): d(u(x),u(y))>\alpha\}} \frac{d^{2+\delta}(u(x), u(y))}{\varepsilon^{n+2}} \mathrm{d}\mu(y) \\ &\quad \leqslant \alpha^\delta \int_\Omega \varphi(x)\mathrm{d}\mu(x) \int_{B_\varepsilon(x)} \frac{d^2(u(x), u(y))}{\varepsilon^{n+2}} \mathrm{d}\mu(y) \\ &\qquad + D^\delta \int_\Omega \varphi(x)\mathrm{d}\mu(x) \int_{\{y\in B_\varepsilon(x): d(u(x),u(y))\geqslant\alpha\}} \frac{d^2(u(x), u(y))}{\varepsilon^{n+2}} \mathrm{d}\mu(y) \\ &\quad := I_1 + I_2. \end{aligned}$$

One has

$$\lim_{\varepsilon\to 0} I_1 = \alpha^\delta E^u_2(\varphi)$$

By Lemma 3.1, we obtain that

$$\lim_{\varepsilon\to 0} I_2 = 0.$$

Thus, we have

$$\limsup_{\varepsilon\to 0}\int_\Omega \varphi(x)\mathrm{d}\mu(x)\int_{B_\varepsilon(x)}\frac{d^{2+\delta}(u(x),u(y))}{\varepsilon^{n+2}}\mathrm{d}\mu(y)\leqslant \alpha^\delta E_2^u(\varphi).$$

By taking $\alpha\to 0$, we conclude this corollary. □

Corollary 3.26 *Let $h:[0,\beta)\to[0,\infty)$ be a continuous function which is differentiable at 0 with $h(0)=0$, $h'(0)=1$, and satisfies $h(t)\leqslant ct$ for some constant $c>0$ and all $t\in(0,\beta)$. Fix any $\alpha>0$ and $u\in W^{1,2}(\Omega,Y)$, we have for any $v\in W^{1,2}(\Omega,Y)$ and $\varphi\in C_c(\Omega)$,*

$$\frac{c_{n,2}}{n+2}E_2^v(\varphi)=\lim_{\varepsilon\to 0}\int_\Omega\varphi(x)\mathrm{d}\mu(x)\int_{\{y\in B_\varepsilon(x):d(u(x),u(y))<\alpha\}}\frac{h^2(d(v(x),v(y)))}{\varepsilon^{n+2}}\mathrm{d}\mu(y).$$

In particular, we have

$$\frac{c_{n,2}}{n+2}E_2^v(\varphi)=\lim_{\varepsilon\to 0}\int_\Omega\varphi(x)\mathrm{d}\mu(x)\int_{\{y\in B_\varepsilon(x):d(u(x),u(y))<\alpha\}}\frac{\sin^2(d(v(x),v(y)))}{\varepsilon^{n+2}}\mathrm{d}\mu(y).$$

Proof For any given $\gamma\in(0,1)$, choose $\delta>0$ such that

$$|h(t)-t|\leqslant\gamma t\quad\text{when } 0\leqslant t<\delta. \tag{3.27}$$

Fix any $\varphi\in C_c(\Omega)$. Without loss of generality, we may assume that $\varphi\geqslant 0$. Then we have,

$$\begin{aligned}
&\int_\Omega\varphi(x)\mathrm{d}\mu(x)\int_{\{y\in B_\varepsilon(x):d(u(x),u(y))<\alpha\}}\frac{h^2(d(v(x),v(y)))}{\varepsilon^{n+2}}\mathrm{d}\mu(y)\\
&\quad=\int_\Omega\varphi(x)\mathrm{d}\mu(x)\int_{\{y\in B_\varepsilon(x):d(u(x),u(y))<\alpha\text{ and }d(v(x),v(y))<\delta\}}\frac{h^2(d(v(x),v(y)))}{\varepsilon^{n+2}}\mathrm{d}\mu(y)\\
&\qquad+\int_\Omega\varphi(x)\mathrm{d}\mu(x)\int_{\{y\in B_\varepsilon(x):d(u(x),u(y))<\alpha\text{ and }d(v(x),v(y))\geqslant\delta\}}\frac{h^2(d(v(x),v(y)))}{\varepsilon^{n+2}}\mathrm{d}\mu(y)\\
&\quad:=I_1+I_2.
\end{aligned}$$

By Corollary 3.21 and (3.27), we have

$$(1-\gamma)^2\frac{c_{n,2}}{n+2}E_2^v(\varphi)\leqslant\liminf_{\varepsilon\to 0}I_1\leqslant\limsup_{\varepsilon\to 0}I_1\leqslant\frac{c_{n,2}}{n+2}(1+\gamma)^2E_2^v(\varphi).$$

Recall that $h(t)\leqslant ct$ for some constant $c>0$. Applying Lemma 3.1, we have

$$\lim_{\varepsilon\to 0}I_2=0.$$

Finally, by taking $\gamma \to 0$, we conclude this corollary. □

3.2 The Existence of Harmonic Maps to a Space with Curvature Bounded Above

Let us begin from an introduction of the space with curvature $\leqslant 1$.

Definition 3.28 A geodesic space (Y, d) is said to be with curvature $\leqslant 1$ (in the sense of Alexandrov), if the following comparison property holds: Given any triple of points $P, Q, R \in Y$ with $d(P, Q) + d(Q, R) + d(R, Q) < 2\pi$, and a point $S \in$ geodesic QR with

$$d(Q, S) = d(R, S) = \frac{1}{2}d(Q, R),$$

there exists a comparison triangle $\triangle\bar{P}\bar{Q}\bar{R}$ in $\mathbb{S}^2$ and a point $\bar{S} \in$ geodesic $\bar{Q}\bar{R}$ with

$$d_{\mathbb{S}^2}(\bar{Q}, \bar{S}) = d_{\mathbb{S}^2}(\bar{R}, \bar{S}) = \frac{1}{2}d_{\mathbb{S}^2}(\bar{Q}, \bar{R})$$

such that

$$d(P, S) \leqslant d_{\mathbb{S}^2}(\bar{P}, \bar{S}).$$

As a simple consequence of Definition 3.28, we have, for any pair of points $P, Q \in Y$ with $d(P, Q) < \pi$, there exists a unique geodesic joining them.

Definition 3.29 Given any pair of points $P, Q \in Y$ with $d(P, Q) < \pi$ and $\lambda \in [0, 1]$, one defines $(1 - \lambda)P + \lambda Q$ to be a point on a unique geodesic joining P and Q that is a fraction λ away from P, that is

$$d((1 - \lambda)P + \lambda Q, P) = \lambda d(P, Q).$$

From now on, we always assume that Y is a complete geodesic space with curvature $\leqslant 1$.

Definition 3.30 We say that a quadruple of points $P, Q, R, S \in Y$ determines a quadrilateral $\square PQRS$ in Y if $d(P, Q) + d(Q, R) + d(R, S) + d(S, P) < 2\pi$.

Definition 3.31 We say that a triple of points $P, Q, R \in Y$ determines a triangle $\triangle PQR$ in Y if $d(P, Q) + d(Q, R) + d(R, P) < 2\pi$.

Lemma 3.32 ([26], Estimate I) *Let $\square PQRS$ be a quadrilateral in Y. Let $Q_{1/2} = \frac{Q+R}{2}$ and $P_{1/2} = \frac{P+S}{2}$. Then*

$$\begin{aligned}2d^2(P_{1/2}, Q_{1/2}) \leqslant &\frac{1}{\cos^2(d(P,S)/2)}(d^2(P,Q) + d^2(R,S)) \\ &- \frac{1}{2\cos^2(d(P,S)/2)}(d(Q,R) - d(P,S))^2 \\ &+ \mathrm{Cub}(d(P,Q), d(R,S), |d(Q,R) - d(P,S)|).\end{aligned}$$

Here $\mathrm{Cub}(\cdot,\cdot,\cdot)$ *represents terms that are cubic in the indicated variables.*

Lemma 3.33 ([26], Estimate II) *Let $\triangle PQS$ be a triangle in Y. For a pair of numbers $0 \le \eta, \eta' \le 1$, define $P_{\eta'} = (1-\eta')P + \eta' Q$ and $S_\eta = (1-\eta)S + \eta Q$. Then*

$$\begin{aligned}d^2(P_{\eta'}, S_\eta) \leqslant &\frac{\sin^2((1-\eta)d(Q,S))}{\sin^2(d(Q,S))}(d^2(P,S) - (d(Q,P) - d(Q,S))^2) \\ &+ ((1-\eta')d(Q,P) - (1-\eta)d(Q,S))^2 + \mathrm{Cub}(|\eta' - \eta|, d(P,S)).\end{aligned}$$

Here $\mathrm{Cub}(\cdot,\cdot)$ *represents terms that are cubic in the indicated variables.*

Given a ball $B_\rho(Q) \subset Y$ with $\rho < \pi/2$, we denote by

$$W^{1,2}(\Omega, B_\rho(Q)) = \{u \in W^{1,2}(\Omega, Y) : u(\Omega) \subset B_\rho(Q)\}$$

and

$$W^{1,2}_\varphi(\Omega, B_\rho(Q)) = \{u \in W^{1,2}(\Omega, B_\rho(Q)) : d(u,\varphi) \in W^{1,2}_0(\Omega))\}$$

for $\varphi \in W^{1,2}(\Omega, B_\rho(Q))$.

Given $u_0, u_1 \in W^{1,2}(\Omega, B_\rho(Q))$. For any $x, y \in \Omega$, the sum

$$d(u_0(x), u_0(y)) + d(u_0(y), u_1(y)) + d(u_1(y), u_1(x)) + d(u_1(x), u_0(x))$$

may exceed 2π, the four points $u_0(x), u_0(y), u_1(y), u_1(x)$ do not determine a quadrilateral in this case. Thanks to Corollary 3.22, when computing the energies, we only need to consider the points x, y such that

$$d(u_0(x), u_0(y)) \leqslant \pi - 2\rho$$

and

$$d(u_1(x), u_1(y)) \leqslant \pi - 2\rho.$$

In this case, we have

$$d(u_0(x), u_0(y)) + d(u_0(y), u_1(y)) + d(u_1(y), u_1(x)) + d(u_1(x), u_0(x)) < 2\pi.$$

Hence, the four points $u_0(x), u_0(y), u_1(y), u_1(x)$ determine a quadrilateral in this case.

A convexity of energy functional from a smooth Riemannian domain has been established by Serbinowski in [26]. In the following, we will extend the convexity to a domain of an Alexandrov space.

Theorem 3.34 (Convexity of the energy functional) *Given a ball $B_\rho(Q) \subset Y$ with $\rho < \pi/2$, and $u_0, u_1 \in W^{1,2}(\Omega, B_\rho(Q))$. Let $u_{1/2}(x) = \frac{1}{2}(u_0(x) + u_1(x))$, $\lambda(x) = d(u_0(x), u_1(x))$, $R(x) = d(u_{1/2}(x), Q)$, $w(x) = (1-\eta(x))u_{1/2}(x) + \eta(x)Q$, where $\eta : \Omega \to [0, 1]$ is defined by*

$$\begin{cases} \frac{\sin((1-\eta)R)}{\sin R} = \cos\frac{\lambda}{2} & \text{when } R > 0 \\ 1-\eta = \cos\frac{\lambda}{2} & \text{when } R = 0. \end{cases} \tag{3.35}$$

Then we have, for any $0 \leqslant \phi \in C_c(\Omega)$,

$$\cos^8\rho \int_\Omega \left|\nabla \frac{\tan(\lambda/2)}{\cos R}\right|^2 \phi \mathrm{d}\mu \leqslant \frac{1}{2}E_2^{u_0}(\phi) + \frac{1}{2}E_2^{u_1}(\phi) - E_2^{w}(\phi).$$

In particular,

$$\cos^8\rho \int_\Omega \left|\nabla \frac{\tan(\lambda/2)}{\cos R}\right|^2 \mathrm{d}\mu \leqslant \frac{1}{2}E_2^{u_0}(\Omega) + \frac{1}{2}E_2^{u_1}(\Omega) - E_2^{w}(\Omega).$$

Proof For any $x, y \in \Omega$ with $d(u_0(x), u_0(y)) \leqslant \pi - 2\rho$ and $d(u_1(x), u_1(y)) \leqslant \pi - 2\rho$, by applying Lemma 3.32 to the quadrilateral $\Box u_0(x)u_0(y)u_1(y)u_1(x)$, we get

$$\begin{aligned} d^2(u_{1/2}(x), u_{1/2}(y)) \leqslant & \frac{1}{2\cos^2(\lambda(x)/2)}(d^2(u_0(x), u_0(y)) + d^2(u_1(x), u_1(y))) \\ & - \frac{1}{4\cos^2(\lambda(x)/2)}(\lambda(y) - \lambda(x))^2 \\ & + \mathrm{Cub}(d(u_0(x), u_0(y)), d(u_1(x), u_1(y)), |\lambda(y) - \lambda(x)|). \end{aligned} \tag{3.36}$$

Applying Lemma 3.33 to the triangle $\triangle Qu_{1/2}(x)u_{1/2}(y)$, we have

$$\begin{aligned} d^2(w(y), w(x)) \leqslant & \frac{\sin^2((1-\eta(x))R(x))}{\sin^2 R(x)} d^2(u_{1/2}(x), u_{1/2}(y)) \\ & - \frac{\sin^2((1-\eta(x))R(x))}{\sin^2 R(x)}(R(y) - R(x))^2 \\ & + ((1-\eta(y))R(y) - (1-\eta(x))R(x))^2 \\ & + \mathrm{Cub}(|\eta(y) - \eta(x)|, |R(y) - R(x)|). \end{aligned} \tag{3.37}$$

Recall (3.35), the definition of η. The combination of (3.36) and (3.37) implies that

$$\begin{aligned} & -d^2(w(y),w(x)) + \frac{d^2(u_0(x),u_0(y))}{2} + \frac{d^2(u_1(x),u_1(y))}{2} \\ & \quad \geqslant \frac{(\lambda(y)-\lambda(x))^2}{4} + \cos^2\left(\frac{\lambda(x)}{2}\right)(R(y)-R(x))^2 \\ & \quad - ((1-\eta(y))R(y) - (1-\eta(x))R(x))^2 \\ & \quad + \mathrm{Cub}(|\eta(y)-\eta(x)|, |R(y)-R(x)|) \\ & \quad + \mathrm{Cub}(d(u_0(x),u_0(y)), d(u_1(x),u_1(y)), |\lambda(y)-\lambda(x)|). \end{aligned} \tag{3.38}$$

For $\varepsilon > 0$ sufficiently small, we let $I_\varepsilon(x) := \{y \in B_\varepsilon(x) : d(u_0(x),u_0(y)) \leqslant \pi - 2\rho \text{ and } d(u_1(x),u_1(y)) \leqslant \pi - 2\rho\}$. For any $0 \leqslant \phi \in C_c(\Omega)$, we obtain from (3.38) that

$$\begin{aligned} & -\int_\Omega \phi(x)\mathrm{d}\mu(x)\int_{I_\varepsilon(x)} \frac{d^2(w(y),w(x))}{\varepsilon^{n+2}}\mathrm{d}\mu(y) \\ & \quad + \frac{1}{2}\int_\Omega \phi(x)\mathrm{d}\mu(x)\int_{I_\varepsilon(x)} \frac{d^2(u_0(x),u_0(y))}{\varepsilon^{n+2}}\mathrm{d}\mu(y) \\ & \quad + \frac{1}{2}\int_\Omega \phi(x)\mathrm{d}\mu(x)\int_{I_\varepsilon(x)} \frac{d^2(u_1(x),u_1(y))}{\varepsilon^{n+2}}\mathrm{d}\mu(y) \\ & \quad \geqslant \frac{1}{4}\int_\Omega \phi(x)\mathrm{d}\mu(x)\int_{I_\varepsilon(x)} \frac{(\lambda(y)-\lambda(x))^2}{\varepsilon^{n+2}}\mathrm{d}\mu(y) \\ & \quad + \int_\Omega \phi(x)\cos^2\left(\frac{\lambda(x)}{2}\right)\mathrm{d}\mu(x)\int_{I_\varepsilon(x)} \frac{(R(y)-R(x))^2}{\varepsilon^{n+2}}\mathrm{d}\mu(y) \\ & \quad - \int_\Omega \phi(x)\mathrm{d}\mu(x)\int_{I_\varepsilon(x)} \frac{((1-\eta(y))R(y)-(1-\eta(x))R(x))^2}{\varepsilon^{n+2}}\mathrm{d}\mu(y) \\ & \quad + \int_\Omega \phi(x)\mathrm{d}\mu(x)\int_{I_\varepsilon(x)} \frac{\mathrm{Cub}(|\eta(y)-\eta(x)|, |R(y)-R(x)|)}{\varepsilon^{n+2}}\mathrm{d}\mu(y) \\ & \quad + \int_\Omega \phi(x)\mathrm{d}\mu(x)\int_{I_\varepsilon(x)} \frac{\mathrm{Cub}(d(u_0(x),u_0(y)), d(u_1(x),u_1(y)), |\lambda(y)-\lambda(x)|)}{\varepsilon^{n+2}}\mathrm{d}\mu(y). \end{aligned}$$

Note that $\phi(x)\cos^2\left(\frac{\lambda(x)}{2}\right) \in L^\infty(\Omega)$ has compact support. Applying Corollary 3.25 and Corollary 3.22 to the above inequality, we have

$$-E_2^w(\phi) + \frac{1}{2}E_2^{u_0}(\phi) + \frac{1}{2}E_2^{u_1}(\phi) \geqslant \frac{1}{4}E_2^\lambda(\phi) + E_2^R\left(\cos^2\left(\frac{\lambda}{2}\right)\phi\right) - E_2^{(1-\eta)R}(\phi),$$

Recall that the energy measures are absolutely continuous w.r.t μ. Hence, we obtain that

$$
\begin{aligned}
&-E_2^w(\phi) + \frac{1}{2}E_2^{u_0}(\phi) + \frac{1}{2}E_2^{u_1}(\phi) \\
&\quad \geqslant \int_\Omega \left(\frac{1}{4}|\nabla\lambda|^2 + \cos^2\left(\frac{\lambda}{2}\right)|\nabla R|^2 - |\nabla((1-\eta)R)|^2\right)\phi \mathrm{d}\mu \\
&\quad = \int_\Omega \left(\frac{1}{4}|\nabla\lambda|^2 + \cos^2\left(\frac{\lambda}{2}\right)|\nabla R|^2 - \frac{|\nabla(\cos\frac{\lambda}{2}\sin R)|^2}{1-\cos^2\frac{\lambda}{2}\sin^2 R}\right)\phi \mathrm{d}\mu \\
&\quad = \int_\Omega \frac{\cos^4 R\cos^4\frac{\lambda}{2}}{1-\sin^2 R\cos^2\frac{\lambda}{2}}\left|\nabla\frac{\tan\frac{\lambda}{2}}{\cos R}\right|^2 \phi \mathrm{d}\mu \\
&\quad \geqslant \cos^8\rho \int_\Omega \left|\nabla\frac{\tan\frac{\lambda}{2}}{\cos R}\right|^2 \phi \mathrm{d}\mu
\end{aligned}
$$

Thus, by letting $\phi \nearrow 1$, we have

$$
\cos^8\rho \int_\Omega \left|\nabla\frac{\tan(\lambda/2)}{\cos R}\right|^2 \mathrm{d}\mu(x) \leqslant \frac{1}{2}E_2^{u_0}(\Omega) + \frac{1}{2}E_2^{u_1}(\Omega) - E_2^w(\Omega).
$$

□

Now, we are in position to prove the existence result Theorem 1.1.

Proof of Theorem 1.1 Denote by

$$
E^0 = \inf\{E_2^v(\Omega) : v \in W_\varphi^{1,2}(\Omega, B_\rho(Q))\}.
$$

Choose a sequence $\{u_n\} \subset W_\varphi^{1,2}(\Omega, B_\rho(Q))$ such that $E_2^{u_n}(\Omega) \downarrow E^0$. Let $\lambda_{n,m} = d(u_n, u_m)$, $R_{n,m} = d\left(\frac{u_n+u_m}{2}, Q\right)$, $w_{n,m}(x) = (1-\eta_{n,m}(x))(\frac{u_n+u_m}{2})(x) + \eta_{n,m}(x)Q$, where $\eta_{n,m} : \Omega \to [0,1]$ is defined by

$$
\begin{cases}
\frac{\sin((1-\eta_{n,m})R_{n,m})}{\sin R_{n,m}} = \cos\frac{\lambda_{n,m}}{2} & \text{when } R_{n,m} > 0 \\
1 - \eta_{n,m} = \cos\frac{\lambda_{n,m}}{2} & \text{when } R_{n,m} = 0.
\end{cases}
$$

Notice that $w_{n,m} \in W_\varphi^{1,2}(\Omega, B_\rho(Q))$ and hence $E^0 \leqslant E_2^{w_{n,m}}(\Omega)$. Thus, we have

$$
\frac{1}{2}E_2^{u_n}(\Omega) + \frac{1}{2}E_2^{u_m}(\Omega) - E_2^{w_{n,m}}(\Omega) \leqslant \frac{1}{2}E_2^{u_n}(\Omega) + \frac{1}{2}E_2^{u_m}(\Omega) - E^0.
$$

By Poincaré's inequality and Theorem 3.34, we obtain that

$$
\lim_{n,m\to 0}\int_\Omega \lambda_{n,m}^2(x)\mathrm{d}\mu(x) \leqslant C \lim_{n,m\to 0}\int_\Omega \left|\nabla\frac{\tan(\lambda_{n,m}/2)}{\cos R_{n,m}}\right|^2 \mathrm{d}\mu = 0.
$$

Hence, $\{u_n\}$ is a Cauchy sequence in $L^2(\Omega, Y)$. By completeness of $L^2(\Omega, Y)$, $\{u_n\}$ converges to some $u \in L^2(\Omega, Y)$, in the sense of $L^2(\Omega, Y)$. By semi-continuity of the energy functional, $E_2^u(\Omega) \leqslant E^0$ and $u \in W^{1,2}(\Omega, B_\rho(Q))$. Notice that $d(u_n, \varphi) \in W_0^{1,2}(\Omega)$, and $d(u_n, \varphi) \to d(u, \varphi)$ in $L^2(\Omega)$, so $d(u, \varphi) \in W_0^{1,2}(\Omega)$. Hence, we have $u \in W_\varphi^{1,2}(\Omega, B_\rho(Q))$ and $E_2^u(\Omega) = E^0$. The uniqueness follows directly from Theorem 3.34. □

4 Global Hölder Regularity for Harmonic Maps

Let Ω be a bounded domain of an Alexandrov space and let (Y, d_Y) be a geodesic space with curvature $\leqslant 1$. Suppose that $u : \Omega \to Y$ is a harmonic map such that the image $u(\Omega)$ is contained in a ball $B_\rho(Q)$ with radius $\rho < \pi/2$.

4.1 Interior Hölder Regularity

Proposition 4.1 *Assume that $u : \Omega \to B_\rho(Q) \subset Y$ is a harmonic map with $0 < \rho < \pi/4$. Let $R(x) := d(u(x), P)$ for $P \in B_\rho(Q)$ fixed. Then we have*

(1) *$\mathscr{L}_{\cos R} \leqslant -\cos R \cdot |\nabla u|_2 \mathrm{d}\mu$ in the sense of distribution;*
(2) *if $\rho < \pi/4$, then $\mathscr{L}_R \geqslant 0$ in the sense of distribution;*
(3) *if $\rho < \pi/4$, then $\mathscr{L}_{R^2} \geqslant 2\cos(2\rho) \cdot |\nabla u|_2 \mathrm{d}\mu$ in the sense of distribution.*

Proof (1) Fix any $0 \leqslant \phi \in \mathrm{Lip}_c(\Omega)$. Without loss of generality, we may assume that $|\phi|_\infty \leqslant 1$. Let $w = (1-\phi)u + \phi P$. Applying Lemma 3.33 to the triangle $\triangle P u(x) u(y)$, we have

$$\begin{aligned} d^2(w(y), w(x)) \leqslant & \frac{\sin^2((1-\phi(x))R(x))}{\sin^2 R(x)} d^2(u(x), u(y)) \\ & - \frac{\sin^2((1-\phi(x))R(x))}{\sin^2 R(x)} (R(y) - R(x))^2 \\ & + ((1-\phi(y))R(y) - (1-\phi(x))R(x))^2 \\ & + \mathrm{Cub}(|\phi(y) - \phi(x)|, d(u(y), u(x))). \end{aligned} \tag{4.2}$$

Fix any $0 \leqslant \eta \in C_c(\Omega)$. For $\varepsilon > 0$ sufficiently small, we obtain from (4.2) that

$$\begin{aligned} & \int_\Omega \eta(x) \int_{B_\varepsilon(x)} \frac{d^2(w(y), w(x))}{\varepsilon^{n+2}} \mathrm{d}\mu(y) \\ & \leqslant \int_\Omega \eta(x) \frac{\sin^2((1-\phi(x))R(x))}{\sin^2 R(x)} \int_{B_\varepsilon(x)} \frac{d^2(u(x), u(y))}{\varepsilon^{n+2}} \mathrm{d}\mu(y) \\ & \quad - \int_\Omega \eta(x) \frac{\sin^2((1-\phi(x))R(x))}{\sin^2 R(x)} \int_{B_\varepsilon(x)} \frac{(R(y) - R(x))^2}{\varepsilon^{n+2}} \mathrm{d}\mu(y) \\ & \quad + \int_\Omega \eta(x) \int_{B_\varepsilon(x)} \frac{((1-\phi(y))R(y) - (1-\phi(x))R(x))^2}{\varepsilon^{n+2}} \mathrm{d}\mu(y) \\ & \quad + \int_\Omega \eta(x) \int_{B_\varepsilon(x)} \frac{\mathrm{Cub}(|\phi(y) - \phi(x)|, d(u(y), u(x)))}{\varepsilon^{n+2}} \mathrm{d}\mu(y). \end{aligned} \tag{4.3}$$

Note that $\eta(x)\frac{\sin^2((1-\phi(x))R(x))}{\sin^2 R(x)} \in L^\infty(\Omega)$ has compact support. Applying Corollary 3.25 and Proposition 2.17 to (4.3), we obtain that

$$E_2^w(\eta) \leq \int_\Omega \eta \frac{\sin^2((1-\phi)R)}{\sin^2 R}|\nabla u|_2 \mathrm{d}\mu - \int_\Omega \eta \frac{\sin^2((1-\phi)R)}{\sin^2 R}|\nabla R|^2 \mathrm{d}\mu \\ + \int_\Omega \eta |\nabla((1-\phi)R)|^2 \mathrm{d}\mu.$$

Thus, by letting $\eta \nearrow 1$, we get

$$E_2^w(\Omega) \leq \int_\Omega \frac{\sin^2((1-\phi)R)}{\sin^2 R}|\nabla u|_2 \mathrm{d}\mu - \int_\Omega \frac{\sin^2((1-\phi)R)}{\sin^2 R}|\nabla R|^2 \mathrm{d}\mu \\ + \int_\Omega |\nabla((1-\phi)R)|^2 \mathrm{d}\mu.$$

Recall that u is a harmonic map, then $E_2^w(\Omega) \geq E_2^u(\Omega)$. Thus, we have

$$E_2^u(\Omega) \leq \int_\Omega \frac{\sin^2((1-\phi)R)}{\sin^2 R}|\nabla u|_2 \mathrm{d}\mu - \int_\Omega \frac{\sin^2((1-\phi)R)}{\sin^2 R}|\nabla R|^2 \mathrm{d}\mu \\ + \int_\Omega |\nabla((1-\phi)R)|^2 \mathrm{d}\mu,$$

and hence

$$0 \leq \int_\Omega \frac{\sin^2((1-\phi)R) - \sin^2 R}{\sin^2 R}|\nabla u|_2 \mathrm{d}\mu - \int_\Omega \frac{\sin^2((1-\phi)R)}{\sin^2 R}|\nabla R|^2 \mathrm{d}\mu \\ + \int_\Omega |\nabla R - R\nabla\phi - \phi\nabla R|^2 \mathrm{d}\mu. \tag{4.4}$$

For any $t \in (0, 1)$, by replacing ϕ with $t\phi$ in (4.4), and dividing by t, we get

$$0 \leq \int_\Omega \frac{\sin^2((1-t\phi)R) - \sin^2 R}{t \sin^2 R}|\nabla u|_2 \mathrm{d}\mu - \int_\Omega \frac{\sin^2((1-t\phi)R)}{t\sin^2 R}|\nabla R|^2 \mathrm{d}\mu \\ + \int_\Omega \frac{|\nabla R - tR\nabla\phi - t\phi\nabla R|^2}{t} \mathrm{d}\mu.$$

Letting $t \searrow 0$, we obtain that

$$0 \leqslant -2\int_\Omega \phi R \cot R |\nabla u|_2 \mathrm{d}\mu + 2\int_\Omega (-R\langle\nabla\phi, \nabla R\rangle - \phi|\nabla R|^2) \mathrm{d}\mu \\ + 2\int_\Omega \phi R |\nabla R|^2 \cot R \mathrm{d}\mu,$$

i.e.,

$$0 \leq \int_\Omega \langle \nabla \left(\phi \frac{R}{\sin R}\right), \nabla \cos R\rangle - \int_\Omega \phi \frac{R}{\sin R} \cos R \cdot |\nabla u|_2 \mathrm{d}\mu. \tag{4.5}$$

Note that $\frac{2\rho}{\sin 2\rho} \geqslant \frac{R}{\sin R} \geqslant 1$, and hence $\phi/(R/\sin R) \in L^\infty(\Omega) \cap W_c^{1,2}(\Omega)$. From Remark 2.5, we know that (4.5) continues to hold for any nonnegative function in $L^\infty(\Omega) \cap W_c^{1,2}(\Omega)$. Thus, by replacing ϕ with $\phi/(R/\sin R)$ in (4.5), we have

$$\mathscr{L}_{\cos R}(\phi) \leqslant -\int_\Omega \phi \cos R \cdot |\nabla u|_2 \mathrm{d}\mu \tag{4.6}$$

for any $0 \leqslant \phi \in \mathrm{Lip}_c(\Omega)$. This proves the assertion (1).

(2) Fix any $\varepsilon > 0$ sufficiently small. Note that $0 \leqslant R \leqslant 2\rho < \pi$. Let $R_\varepsilon := \arccos(\cos R - \varepsilon)$. Hence

$$\mathscr{L}_{\cos R} = \mathscr{L}_{\cos R_\varepsilon}. \tag{4.7}$$

For any $0 \leqslant \phi \in \mathrm{Lip}_c(\Omega)$, by (4.6) and (4.7), we have

$$\begin{aligned}
0 &\geqslant -\int_\Omega \langle \nabla \cos R_\varepsilon, \nabla\phi\rangle \,\mathrm{d}\mu + \int_\Omega \cos R \cdot |\nabla u|_2 \phi \mathrm{d}\mu \\
&= \int_\Omega \langle \sin R_\varepsilon \nabla R_\varepsilon, \nabla\phi\rangle \,\mathrm{d}\mu + \int_\Omega \cos R \cdot |\nabla u|_2 \phi \mathrm{d}\mu \\
&= \int_\Omega \langle \nabla R_\varepsilon, \sin R_\varepsilon \nabla\phi\rangle \,\mathrm{d}\mu + \int_\Omega \cos R \cdot |\nabla u|_2 \phi \mathrm{d}\mu \\
&= \int_\Omega \langle \nabla R_\varepsilon, \nabla(\phi \sin R_\varepsilon) - \phi \nabla \sin R_\varepsilon\rangle \,\mathrm{d}\mu + \int_\Omega \cos R \cdot |\nabla u|_2 \phi \mathrm{d}\mu \\
&= \int_\Omega \langle \nabla R_\varepsilon, \nabla(\phi \sin R_\varepsilon)\rangle \,\mathrm{d}\mu - \int_\Omega \langle \phi \cos R_\varepsilon \nabla R_\varepsilon, \nabla R_\varepsilon\rangle \,\mathrm{d}\mu \\
&\quad + \int_\Omega \cos R \cdot |\nabla u|_2 \phi \mathrm{d}\mu \\
&= -\mathscr{L}_{R_\varepsilon}(\phi \sin R_\varepsilon) - \int_\Omega \langle \phi \cos R_\varepsilon \nabla R_\varepsilon, \nabla R_\varepsilon\rangle \,\mathrm{d}\mu + \int_\Omega \cos R \cdot |\nabla u|_2 \phi \mathrm{d}\mu
\end{aligned}$$

Hence, we get

$$\mathscr{L}_{R_\varepsilon}(\phi \sin R_\varepsilon) \geqslant \int_\Omega \phi \cos R \cdot |\nabla u|_2 \mathrm{d}\mu - \int_\Omega \phi \cos R_\varepsilon |\nabla R_\varepsilon|^2 \mathrm{d}\mu. \tag{4.8}$$

From Remark 2.5, we know that (4.8) continues to hold for any nonnegative function in $L^\infty(\Omega) \cap W_c^{1,2}(\Omega)$.

On the other hand, it is easy to check that

$$0 < \kappa(\varepsilon) \leqslant \sin R_\varepsilon \leqslant 1$$

for some positive function $\kappa(\varepsilon)$ with $\lim_{\varepsilon\to 0} \kappa(\varepsilon) = 0$. Hence, $\phi/\sin R_\varepsilon \in L^\infty(\Omega) \cap W_c^{1,2}(\Omega)$. Thus, by replacing ϕ with $\phi/\sin R_\varepsilon$ in (4.8), we obtain that

$$\mathscr{L}_{R_\varepsilon}(\phi) \geqslant \int_\Omega \phi \frac{\cos R}{\sin R_\varepsilon} |\nabla u|_2 \mathrm{d}\mu - \int_\Omega \phi \cot R_\varepsilon \cdot |\nabla R_\varepsilon|^2 \mathrm{d}\mu \tag{4.9}$$

for any $0 \leqslant \phi \in \mathrm{Lip}_c(\Omega)$. If $\rho < \pi/4$, then we have $0 \leqslant R < \pi/2$ and hence $\cos R > 0$. Note that $|\nabla u|_2 \geqslant |\nabla R|^2$ and $\nabla R_\varepsilon = \frac{\sin R}{\sin R_\varepsilon}\nabla R$. Thus, we obtain from (4.9) that

$$\begin{aligned}
\mathscr{L}_{R_\varepsilon}(\phi) &\geqslant \int_\Omega \phi \frac{\cos R}{\sin R_\varepsilon}|\nabla u|_2 \mathrm{d}\mu - \int_\Omega \phi \cot R_\varepsilon \cdot |\nabla R_\varepsilon|^2 \mathrm{d}\mu \\
&\geqslant \int_\Omega \phi \frac{\cos R}{\sin R_\varepsilon}|\nabla R|^2 \mathrm{d}\mu - \int_\Omega \phi \cot R_\varepsilon \cdot \frac{\sin^2 R}{\sin^2 R_\varepsilon}|\nabla R|^2 \mathrm{d}\mu \\
&= \int_\Omega \phi \frac{\cos R \sin^2 R_\varepsilon - \cos R_\varepsilon \sin^2 R}{\sin^3 R_\varepsilon}|\nabla R|^2 \mathrm{d}\mu \\
&= \int_\Omega \phi \frac{\varepsilon(1 - \varepsilon \cos R + \cos^2 R)}{\sin^3 R_\varepsilon}|\nabla R|^2 \mathrm{d}\mu \\
&\geqslant 0.
\end{aligned}$$

On the other hand, it is easy to check that R_ε is uniformly bounded in $W^{1,2}(\Omega)$ and

$$R_\varepsilon \to R \quad \text{in } L^2(\Omega) \text{ as } \varepsilon \to 0.$$

Recall that $W^{1,2}(\Omega)$ is reflexive. Hence, by taking a subsequence of R_ε, we conclude that

$$\mathscr{L}_R \geqslant 0.$$

This is the assertion (2).

(3) Fix any $\varepsilon > 0$ sufficiently small. Let $R_\varepsilon := \arccos(\cos R - \varepsilon)$. For any $0 \leqslant \phi \in \mathrm{Lip}_c(\Omega)$, we have

$$\begin{aligned}
\mathscr{L}_{R_\varepsilon^2}(\phi) &= -\int_\Omega \left\langle \nabla R_\varepsilon^2, \nabla\phi \right\rangle \mathrm{d}\mu \\
&= -\int_\Omega \langle 2R_\varepsilon \nabla R_\varepsilon, \nabla\phi\rangle \, \mathrm{d}\mu \\
&= -\int_\Omega \langle \nabla R_\varepsilon, 2R_\varepsilon \nabla\phi\rangle \, \mathrm{d}\mu \\
&= -\int_\Omega \langle \nabla R_\varepsilon, \nabla(2R_\varepsilon\phi) - \phi\nabla(2R_\varepsilon)\rangle \, \mathrm{d}\mu \\
&= \mathscr{L}_{R_\varepsilon}(2R_\varepsilon\phi) + 2\int_\Omega \phi|\nabla R_\varepsilon|^2 \mathrm{d}\mu.
\end{aligned} \tag{4.10}$$

We obtain from (4.9) and (4.10) that

$$\begin{aligned}
\mathscr{L}_{R_\varepsilon^2}(\phi) &\geqslant 2\int_\Omega \phi R_\varepsilon \frac{\cos R}{\sin R_\varepsilon}|\nabla u|_2 \mathrm{d}\mu - 2\int_\Omega \phi R_\varepsilon \cot R_\varepsilon |\nabla R_\varepsilon|^2 \mathrm{d}\mu \\
&\quad + 2\int_\Omega \phi|\nabla R_\varepsilon|^2 \mathrm{d}\mu \\
&= 2\int_\Omega \phi R_\varepsilon \frac{\cos R}{\sin R_\varepsilon}|\nabla u|_2 \mathrm{d}\mu + 2\int_\Omega \phi(1 - R_\varepsilon \cot R_\varepsilon)|\nabla R_\varepsilon|^2 \mathrm{d}\mu.
\end{aligned}$$

If $\rho < \pi/4$, then $R < 2\rho < \pi/2$ and hence $R_\varepsilon < \pi/2$. Notice that $1 - R_\varepsilon \cot R_\varepsilon \geqslant 0$. Thus, we have

$$\mathscr{L}_{R_\varepsilon^2}(\phi) \geqslant 2\int_\Omega \phi R_\varepsilon \frac{\cos R}{\sin R_\varepsilon} |\nabla u|_2 \mathrm{d}\mu.$$

Note that $R_\varepsilon \geqslant \sin R_\varepsilon$ and $\cos R \geqslant \cos(2\rho)$. Hence,

$$\mathscr{L}_{R_\varepsilon^2} \geqslant 2\cos(2\rho) \cdot |\nabla u|_2 \mathrm{d}\mu.$$

On the other hand, it is easy to check that R_ε^2 is uniformly bounded in $W^{1,2}(\Omega)$ and

$$R_\varepsilon^2 \to R^2 \quad \text{in } L^2(\Omega) \text{ as } \varepsilon \to 0.$$

Recall that $W^{1,2}(\Omega)$ is reflexive. Hence, by taking a subsequence of R_ε^2, we conclude that

$$\mathscr{L}_{R^2} \geqslant 2\cos(2\rho) \cdot |\nabla u|_2 \mathrm{d}\mu. \tag{4.11}$$

This is the assertion (3), and hence the proof is finished. □

Along the same arguments as in the proof of [11, Theorem 2], the inequality (4.11) (as same as [11, Eq. (5.2)]) implies the following interior Hölder estimate:

Theorem 4.12 *Let X be an (n-dim) Alexandrov space with curvature $\geqslant k$, $\Omega \subset X$ be a bounded domain, and let Y be a complete geodesic space with curvature $\leqslant 1$. Suppose that $u : \Omega \to \overline{B}_\rho(Q) \subset Y$ is a harmonic map with $\rho < \pi/2$. Then u is locally Hölder continuous on Ω.*

4.2 Boundary Hölder Regularity

We will prove the boundary Hölder regularity by constructing a barrier via harmonic functions. Let us recall the concept of the relative capacity.

Definition 4.13 ([2]) Let $B \subset X$ be a ball and $E \subset B$. We define

$$\mathrm{Cap}_p(E, 2B) := \inf_u \int_{2B} |\nabla u|^p \mathrm{d}\mu,$$

where the infimum is taken over all $u \in W_0^{1,p}(2B)$ such that $u \geqslant 1$ on E.

To obtain the boundary regularity of the harmonic maps, we need the following theorem for harmonic functions:

Theorem 4.14 ([2], Theorem 2.12) *Let X be a doubling metric measure space admitting a weak (1,p)-Poincaré inequality for some $p < 2$, and let $\Omega \subset X$ be a bounded domain. Then there exists a constant $C > 0$ such that if $w \in W^{1,2}(X)$ is Hölder continuous at $x_0 \in \partial\Omega$,*

$$\liminf_{\theta \to 0} \frac{1}{|\log \theta|} \int_\theta^1 \exp\left(-C\gamma(p,\theta)^{2/(2-p)}\right) \frac{\mathrm{d}\theta}{\theta} > 0, \tag{4.15}$$

and $u \in W^{1,2}(X)$ *is a harmonic function on* Ω *with boundary data* w*, then* u *is Hölder continuous at* x_0*. Here*

$$\gamma(p,\theta) := \frac{\theta^{-p}\mu(B_\theta(x_0))}{\mathrm{Cap}_p(B_\theta(x_0)\backslash\Omega, B_{2\theta}(x_0))}. \tag{4.16}$$

Remark 4.17 From [2, Remark 2.15], the inequality (4.15) in Theorem 4.14 is satisfied if the measure density condition is satisfied for the domain $X\backslash\overline{\Omega}$ at x_0, namely, there exists a constant $C > 0$ such that

$$\mu\big(B_r(x_0)\backslash\Omega\big) \geqslant C \cdot \mu\big(B_r(x_0)\big)$$

for all $0 < r < \min\{1, \mathrm{Diam}(X\backslash\overline{\Omega})\}$.

Consider the modified distance function on $\overline{B}_\rho(Q) \subset Y$ with $\rho < \pi/2$ as follows

$$f(x) := \frac{1-\cos(d(P,x))}{\cos(d(Q,x))}, \tag{4.18}$$

here $P \in \overline{B}_\rho(Q)$ is fixed. Note that f is Lipschitz continuous on $\overline{B}_\rho(Q)$.

Definition 4.19 We say that a function f is λ-convex on a domain $E \subset Y$, if $f \circ \gamma$ is λ-convex for all geodesics $\gamma \subset E$.

Lemma 4.20 ([4], Proposition 9.2.18) *Let* (Z, d_Z) *be a complete geodesic space. A continuous function* $f : Z \to \mathbb{R}$ *is* λ*-convex if and only if, for any* $x, y \in Z$ *and* z *a midpoint (*z *lies on some geodesic* γ *connecting* x *and* y*, and satisfies* $d_Z(z,x) = d_Z(z,y) = d_Z(x,y)/2$*) of* x *and* y*,*

$$f(z) \leqslant \frac{f(x)+f(y)}{2} - \frac{\lambda}{4}d_Z^2(x,y).$$

Lemma 4.21 f *is* λ*-convex on* $\overline{B}_\rho(Q)$*, for some positive constant* λ*.*

Proof For any geodesic $xy \subset \overline{B}_\rho(Q)$, choose $x', y' \in \overline{B}_\rho(Q') \subset \mathbb{S}^2$ such that

$$d(Q,x) = d_{\mathbb{S}^2}(Q',x'), \quad d(x,y) = d_{\mathbb{S}^2}(x',y'), \quad d(Q,y) = d_{\mathbb{S}^2}(Q',y'),$$

then choose $P' \in \mathbb{S}^2$ such that P' and Q' are on the same side of the geodesic $x'y'$ and

$$d(P,x) = d_{\mathbb{S}^2}(P',x'), \quad d(P,y) = d_{\mathbb{S}^2}(P',y').$$

Let $z = \frac{x+y}{2}$ and $z' = \frac{x'+y'}{2}$. Define a function g on $\overline{B}_\rho(Q') \subset \mathbb{S}^2$ by

$$g(x) := \frac{1-\cos(d_{\mathbb{S}^2}(P',x))}{\cos(d_{\mathbb{S}^2}(Q',x))}.$$

According to [30, Lemma 3], the function g is λ-convex for some $\lambda > 0$. By curvature comparison, we have

$$d(P, z) \leqslant d_{\mathbb{S}^2}(P', z') \quad \text{and} \quad \mathrm{d}(Q, z) \leqslant d_{\mathbb{S}^2}(Q', z').$$

Hence, $f(z) \leqslant g(z')$. By Lemma 4.20, we obtain that

$$\begin{aligned} f(z) \leqslant g(z') &\leqslant \frac{g(x') + g(y')}{2} - \frac{\lambda}{4} d^2_{\mathbb{S}^2}(x', y') \\ &= \frac{f(x) + f(y)}{2} - \frac{\lambda}{4} d^2(x, y). \end{aligned}$$

By Lemma 4.20 again, we conclude that f is λ-convex. □

Lemma 4.22 *Let f be defined in (4.18), and let $u : \Omega \to \overline{B}_\rho(Q) \subset Y$ $(0 < \rho < \pi/2)$ be a harmonic map. Then*

$$\mathscr{L}_{f \circ u} \geqslant 0$$

in the sense of distribution.

Proof Fix any $\varepsilon \in (0, 1)$. Denote by $R_p = \cos d(u(x), P)$, $R_q = \cos d(u(x), Q)$ and

$$f_\varepsilon = \frac{1 + \varepsilon - R_p}{R_q}.$$

Thus, we have

$$1 + \varepsilon - R_p = R_q \cdot f_\varepsilon. \tag{4.23}$$

Then we obtain from (4.23) that

$$\langle \nabla R_p, \nabla R_q \rangle = -\langle \nabla (R_q f_\varepsilon), \nabla R_q \rangle = -|\nabla R_q|^2 f_\varepsilon - R_q \langle \nabla f_\varepsilon, \nabla R_q \rangle,$$

and hence

$$\langle \nabla f_\varepsilon, \nabla R_q \rangle = \frac{1}{R_q} \left(-\langle \nabla R_p, \nabla R_q \rangle - |\nabla R_q|^2 f_\varepsilon \right). \tag{4.24}$$

For any $0 \leqslant \phi \in \mathrm{Lip}_c(\Omega)$, by (4.23), we have

$$\begin{aligned}
-\mathscr{L}_{R_p}(\phi) &= \mathscr{L}_{R_q f_\varepsilon}(\phi) \\
&= -\int_\Omega \langle \nabla\phi, \nabla(R_q f_\varepsilon)\rangle \mathrm{d}\mu \\
&= -\int_\Omega \langle \nabla\phi, R_q \nabla f_\varepsilon + f_\varepsilon \nabla R_q\rangle \mathrm{d}\mu \\
&= -\int_\Omega \langle \nabla f_\varepsilon, R_q \nabla\phi\rangle \mathrm{d}\mu - \int_\Omega \langle f_\varepsilon \nabla\phi, \nabla R_q\rangle \mathrm{d}\mu \\
&= -\int_\Omega \langle \nabla f_\varepsilon, \nabla(\phi R_q) - \phi\nabla R_q\rangle \mathrm{d}\mu - \int_\Omega \langle \nabla R_q, \nabla(f_\varepsilon\phi) - \phi\nabla f_\varepsilon\rangle \mathrm{d}\mu \\
&= 2\int_\Omega \langle \phi\nabla f_\varepsilon, \nabla R_q\rangle \mathrm{d}\mu + \mathscr{L}_{f_\varepsilon}(\phi R_q) + \mathscr{L}_{R_q}(f_\varepsilon\phi).
\end{aligned} \tag{4.25}$$

The combination of (4.24) and (4.25) implies that

$$-\mathscr{L}_{R_p}(\phi) = 2\int_\Omega \phi \frac{-|\nabla R_q|^2 f_\varepsilon - \langle \nabla R_p, \nabla R_p\rangle}{R_q} \mathrm{d}\mu + \mathscr{L}_{R_q}(f_\varepsilon\phi) + \mathscr{L}_{f_\varepsilon}(\phi R_q). \tag{4.26}$$

Note that $f_\varepsilon > 0$, by Young's inequality, we have

$$-\langle \nabla R_p, \nabla R_q\rangle \leqslant |\nabla R_q|^2 f_\varepsilon + \frac{|\nabla R_p|^2}{4 f_\varepsilon}.$$

Hence, we obtain from the above inequality and (4.26) that

$$\begin{aligned}
\mathscr{L}_{f_\varepsilon}(\phi R_q) &= -\mathscr{L}_{R_p}(\phi) + 2\int_\Omega \phi \frac{|\nabla R_q|^2 f_\varepsilon + \langle \nabla R_p, \nabla R_p\rangle}{R_q} \mathrm{d}\mu - \mathscr{L}_{R_q}(f_\varepsilon\phi) \\
&\geqslant -\mathscr{L}_{R_p}(\phi) - \int_\Omega \frac{|\nabla R_p|^2}{2R_q f_\varepsilon} \phi \mathrm{d}\mu - \mathscr{L}_{R_q}(f_\varepsilon\phi)
\end{aligned}$$

By the above inequality and Proposition 4.1(1), we have

$$\begin{aligned}
\mathscr{L}_{f_\varepsilon}(\phi R_q) &\geqslant \int_\Omega R_q f_\varepsilon |\nabla u|_2 \phi \mathrm{d}\mu + \int_\Omega R_p |\nabla u|_2 \phi \mathrm{d}\mu - \int_\Omega \frac{|\nabla R_p|^2}{2R_q f_\varepsilon} \phi \mathrm{d}\mu \\
&= \int_\Omega (1+\varepsilon-R_p)|\nabla u|_2 \phi \mathrm{d}\mu + \int_\Omega R_p |\nabla u|_2 \phi \mathrm{d}\mu - \int_\Omega \frac{|\nabla R_p|^2}{2R_q f_\varepsilon} \phi \mathrm{d}\mu \\
&= \int_\Omega (1+\varepsilon)|\nabla u|_2 \phi \mathrm{d}\mu - \int_\Omega \frac{|\nabla R_p|^2}{2(1+\varepsilon-R_p)} \phi \mathrm{d}\mu
\end{aligned} \tag{4.27}$$

Notice that $|\nabla R_p|^2 = \sin^2 d(u(x), P) \cdot |\nabla d(u(x), P)|^2 = (1 - R_p^2)|\nabla d(u(x), P)|^2$, thus we obtain from (4.27) that

$$\begin{aligned}\mathscr{L}_{f_\varepsilon}(\phi R_q) &\geqslant \int_\Omega (1+\varepsilon)|\nabla u|_2 \phi \mathrm{d}\mu - \int_\Omega \frac{(1-R_p^2)|\nabla d(u(x), P)|^2}{2(1+\varepsilon - R_p)} \phi \mathrm{d}\mu \\ &\geqslant \int_\Omega (1+\varepsilon)|\nabla u|_2 \phi \mathrm{d}\mu - \int_\Omega \frac{(1-R_p^2)|\nabla d(u(x), P)|^2}{2(1 - R_p)} \phi \mathrm{d}\mu \\ &= \int_\Omega (1+\varepsilon)|\nabla u|_2 \phi \mathrm{d}\mu - \int_\Omega \frac{(1+R_p)|\nabla d(u(x), P)|^2}{2} \phi \mathrm{d}\mu\end{aligned}$$

Note that $\frac{1+R_p}{2} < 1+\varepsilon$ and $|\nabla u|_2 \geqslant |\nabla d(u(x), P)|^2$. Hence, we conclude that

$$\mathscr{L}_{f_\varepsilon}(\phi R_q) \geqslant 0. \tag{4.28}$$

From Remark 2.5, we know that (4.28) continues to hold for any nonnegative function in $L^\infty(\Omega) \cap W_c^{1,2}(\Omega)$.

Note that $1 \geqslant R_q \geqslant \cos\rho > 0$, hence $0 \leqslant \phi/R_q \in W_c^{1,2}(\Omega) \cap L^\infty(\Omega)$. Thus, by replacing ϕ with ϕ/R_q in (4.28), we have

$$\mathscr{L}_{f_\varepsilon}(\phi) \geqslant 0$$

for any $0 \leqslant \phi \in W_c^{1,2}(\Omega) \cap L^\infty(\Omega)$. Finally, it is easy to check that

$$f_\varepsilon \to f \circ u \quad \text{as } \varepsilon \to 0, \text{ in } W^{1,2}(\Omega).$$

Thus, we conclude that

$$\mathscr{L}_{f\circ u} \geqslant 0.$$

This completes the proof. □

Theorem 4.29 *Let X be an (n-dim) Alexandrov space with curvature $\geqslant k$, and let $\Omega \subset X$ be a bounded domain satisfying the measure density condition. Let Y be a complete geodesic space with curvature $\leqslant 1$ (in the sense of Alexandrov). Given a ball $B_\rho(Q) \subset Y$ with radius $0 < \rho < \pi/2$. Then there exists a constant $C_0 > 0$ such that if $w \in W^{1,2}(X, Y)$ with $w(X) \subset B_\rho(Q)$ is Hölder continuous at $x_0 \in \partial\Omega$,*

$$\liminf_{\theta \to 0} \frac{1}{|\log\theta|} \int_\theta^1 \exp\left(-C_0 \gamma(p, \theta)^{2/(2-p)}\right) \frac{\mathrm{d}\theta}{\theta} > 0 \tag{4.30}$$

holds for some $1 < p < 2$, and $u \in W^{1,2}(\Omega, B_\rho(Q))$ is a harmonic map on Ω which agrees w on $\partial\Omega$, then u is Hölder continuous at x_0. Here $\gamma(p, \theta)$ is defined in (4.16).

Proof Let f be the modified distance function given in (4.18) with $P = w(x_0)$. Let $g \in W^{1,2}(\Omega)$ be the solution of the Dirichlet problem

$$\begin{cases} \{\mathscr{L}_g & = 0 \quad \text{on } \Omega \\ g - f \circ w & \in W_0^{1,2}(\Omega). \end{cases}$$

Because that Ω satisfies the measure density condition, according to [3, Proposition 5.1], there exists a $W^{1,2}(X)$-extension of the function $(g - f \circ w)$ by taking value $= 0$ outside of Ω. Note that $f \circ w \in W^{1,2}(X)$, by Proposition 2.8(1), one has $g \in W^{1,2}(X)$. Using Theorem 4.14 and the assumption (4.30), we have g is Hölder continuous at x_0.

By Lemma 4.22, we have

$$\mathscr{L}_{f\circ u - g} \geqslant 0,$$

then by applying the weak maximum principle (see, for example, [5, Theorem 7.17]), we get

$$f \circ u - g \leqslant 0 \quad \text{a.e. } x \in \Omega.$$

Note that $f(w(x_0)) = 0$, $g - f \circ w \in W_0^{1,2}(\Omega)$ and that both g and $f \circ w$ are Hölder continuous at x_0, we have $g(x_0) = 0$. Hence, we have

$$f(u(x)) - f(w(x_0)) \leqslant g(x) - g(x_0) \quad \text{a.e. } x \in \Omega.$$

Choose a geodesic γ connecting $w(x_0)$ and $u(x)$, by the λ-convexity of f for some $\lambda > 0$, Lemma 4.21, we have, for any $x \in \Omega$,

$$f(u(x)) - f(w(x_0)) \geqslant \frac{d^+}{dt}\bigg|_{t=0} (f \circ \gamma(t)) + \frac{\lambda}{2} d^2(u(x), u(x_0)) \geqslant \frac{\lambda}{2} d^2(u(x), u(x_0)),$$

where we have used $\frac{d^+}{dt}|_{t=0}(f \circ \gamma(t)) \geqslant 0$, since $\gamma(0) = w(x_0)$ is a minimum point of f. Thus, we have

$$d^2(u(x), w(x_0)) \leqslant \frac{2}{\lambda}(g(x) - g(x_0)) \qquad \text{a.e. } x \in \Omega.$$

By using the triangle inequality, we get

$$d^2\big(u(x), w(x)\big) \leqslant 2d^2\big(w(x), w(x_0)\big) + \frac{4}{\lambda}(g(x) - g(x_0)) \qquad \text{a.e. } x \in \Omega. \quad (4.31)$$

By using that Ω satisfies the measure density condition again, according to [3, Proposition 5.1], the zero-extension of $d\big(u(x), w(x)\big)$ is in $W^{1,2}(X)$. Since $w(X)$ is contained in $B_\rho(Q)$ with $\rho < \pi/2$, we have $f(w(x)) \geqslant 0$. Hence, by $g - f \circ w \geqslant 0$, we have $g(x) \geqslant 0$. Thus, the right-hand side of (4.31) is nonnegative. It follows, for a fixed constant $\varepsilon > 0$, that

$$d^2\big(u(x), w(x)\big) \leqslant 2d^2\big(w(x), w(x_0)\big) + \frac{4}{\lambda}(g(x) - g(x_0)) \qquad \text{a.e. } x \in B_\varepsilon(x_0).$$

Since both w and g are Hölder continuous at x_0, we get

$$\operatorname*{esssup}_{B_\delta(x_0)} d^2\big(u(x), w(x)\big) \leqslant C(\delta^\alpha + \delta^\beta)$$

for some $C > 0$ and $0 < \alpha, \beta < 1$ and all $0 < \delta < \varepsilon$. After a redefinition on a set of measure zero, $d\big(u(x), w(x)\big)$ is Hölder continuous at x_0.

At last, by using the facts that both w and $d\big(u(x), w(x)\big)$ are Hölder continuous at x_0 and the triangle inequality, we have u is Hölder continuous at x_0. Now the proof is finished. □

Theorem 1.3 is a consequence of the combination of Theorem 4.12, Remark 4.17 and Theorem 4.29.

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