

PARTIAL REGULARITY OF HARMONIC MAPS FROM ALEXANDROV SPACES

HUABIN GE, WENSHUAI JIANG, AND HUI-CHUN ZHANG

ABSTRACT. In this paper, we prove the Lipschitz regularity of continuous harmonic maps from an finite dimensional Alexandrov space to a compact smooth Riemannian manifold. This solves a conjecture of F. H. Lin in [36]. The proof extends the argument of Huang-Wang [26].

1. INTRODUCTION

Gromov-Schoen [20] initiated to study harmonic maps into singular spaces by the calculus of variation. A general theory of (variational) harmonic maps between singular spaces was developed by Korevaar-Schoen [33], Jost [30, 31] and Lin [36], independently. The regularity problem is a classical problem in the theory of harmonic maps, which has attracted the attention of many researchers. For the harmonic maps between smooth Riemannian manifolds, many regularity and singularity results have been established (see, for example, [40, 23, 5, 13, 15, 24, 46, 47, 10, 11, 41, 37] and the survey [22] and the book [38]). The regularity of harmonic maps from singular spaces (or manifolds with singular metric) has also been developed extensively, such as [12, 31, 49, 36, 27, 35, 53, 51]).

An Alexandrov space (with curvature bounded from below) is a metric space such that Toponogov comparison theorem of triangles holds locally [7, 6]. Some topological singularity may occur in an Alexandrov space. In this paper, we are interested in the regularity of harmonic maps from an Alexandrov space with curvature bounded from below to a smooth Riemannian manifold. F. H. Lin [36] first established a partial Hölder continuity for energy minimizing maps as follows.

Theorem 1.1 (F. H. Lin [36]). *Let Ω be a bounded open domain in an n -dimensional Alexandrov space with curvature bounded from below by k , and let N be a compact smooth Riemannian manifold. Suppose u is an energy minimizing map, then u is locally Hölder continuous in Ω away from a relatively closed subset of Hausdorff dimension $\leq n - 3$.*

Recall that the theory of regularity for harmonic maps between smooth manifolds includes two main steps: (i) to establish a (partial) Hölder continuity, and (ii) to improve the Hölder continuity to $C^{1,\alpha}$ -regularity for some $\alpha \in (0, 1)$. Considering the regularity of harmonic maps from Alexandrov spaces, based on the partial Hölder regularity of Theorem 1.1, F. H. Lin posted the following conjecture.

Conjecture 1.2 (F. H. Lin [36]). *The Hölder continuity can be improved to the Lipschitz continuity in Theorem 1.1.*

In this paper, we will prove the following Lipschitz regularity.

Theorem 1.3. *Let Ω and N be as in Theorem 1.1. Then any continuous harmonic map (need not to be an energy minimizer) must be locally Lipschitz continuous in Ω . Precisely, there*

exists a constant $\epsilon = \epsilon(n, k, \Omega, N, \sup_N |A|) > 0$ such that the following holds: If $u : \Omega \rightarrow N$ is a harmonic map and a ball $B_{r_0}(x_0) \subset \Omega$, $r_0 \leq 1$, such that u is continuous on $B_{r_0}(x_0)$ and

$$(1.1) \quad \text{osc}_{B_{r_0}(x_0)} u := \sup_{x, y \in B_{r_0}(x_0)} d_N(u(x), u(y)) < \epsilon,$$

then u is Lipschitz continuous on $B_{\frac{r_0}{2}}(x_0)$ (with a Lipschitz constant depending on $n, k, \Omega, r_0, \mu(B_{r_0}(x_0)), \int_{B_{r_0}(x_0)} |\nabla u|^2 d\mu$ and N , and $\sup_N |A|$), where A is the second fundamental form of the isometrically embedding of N into \mathbb{R}^ℓ .

Here and in the sequel of this paper, $\sup_E v$ means always $\text{esssup}_E v$, the essential supremum.

Comparing with the Hölder estimate in Lin [36] and Shi [49], they only used the fact the metric of Alexandrov space is locally L^∞ in the regular point (see Subsection 2.2). However, one can construct example (see Shi [49] or [14]) to show that Hölder estimate is optimal if the coefficient of an elliptic operator is only L^∞ . One cannot expect the Lipschitz estimate for such operator. An example in Chen [12] showed also that the Hölder continuity is optimal if the domain space has no a lower bound of curvature. Therefore, in order to show Theorem 1.3, we have to use more information about Alexandrov space. One important estimate to our proof is the Lipschitz estimate for harmonic function which is a special harmonic map to \mathbb{R} . We will use the Lipschitz estimate (see Section 3, see also [52]) for harmonic function several times in our proof and will use the fact that the smooth target N could be able to embed isometrically into Euclidean space.

As a direct consequence of the combination of Theorem 1.1 and Theorem 1.3, we solve Conjecture 1.2 completely.

Theorem 1.4. *Let Ω, N and u be as in Theorem 1.1. Then u is locally Lipschitz continuous in Ω away from a relatively closed subset of Hausdorff dimension $\leq n - 3$.*

Remark 1.5. In [36] Lin posted a Lipschitz regularity conjecture of harmonic map from an Alexandrov space to a nonpositive curvature metric space. Such conjecture was completely solved by H. C. Zhang-X. P. Zhu [53] by constructing a nonlinear version of Hamilton-Jacobi flow for harmonic map. Our proof of Theorem 1.3 is independent of their techniques and results.

Recalling the case when the domain space of the harmonic maps is smooth, Theorem 1.3 has been proved in [15, 46]. See [8] for an elementary proof in this case. Recently, Huang-Wang [26] provided another new proof in this case, based on Riesz potential estimate and the Green function on \mathbb{R}^n . Our proof of Theorem 1.3 is an extension of Huang-Wang's argument but there is a subtle point. Since it is not known how to get a suitable regularity of Green functions on a general Alexandrov space, then, we will prove a gradient estimate for the Poisson equations with a L^1 -data, via a estimate of heat kernels on Alexandrov spaces, which is given in Sect. 3 (see Proposition 3.2).

In Sect. 2, we will collect some basic concepts and informations of analysis on Alexandrov spaces. In Sect. 4, we will provide some basic facts on harmonic maps on Alexandrov spaces. In the last section, we will give the proof of Theorem 1.3. A key step is established a Morrey-type decay estimate (see Proposition 5.1).

Acknowledgements. H. Ge is partially supported by NSFC 11871094. W. Jiang is partially supported by NSFC 11701507 and the Fundamental Research Funds for the Central

Universities and ARC DECRA. H. C. Zhang is partially supported by NSFC 11521101 and 11571374.

2. PRELIMINARIES

2.1. Alexandrov spaces with curvature bounded below.

Let $(M, |\cdot|)$ be a complete metric space. It is called a *geodesic space* if, for every pair points $p, q \in M$, there exists a point $r \in M$ such that $|pr| = |qr| = |pq|/2$. Fix any $k \in \mathbb{R}$. Given three points p, q, r in a geodesic space M , we can take a triangle $\Delta \bar{p}\bar{q}\bar{r}$ in \mathbb{M}_k^2 such that $|\bar{p}\bar{q}| = |pq|$, $|\bar{q}\bar{r}| = |qr|$ and $|\bar{r}\bar{p}| = |rp|$, where \mathbb{M}_k^2 the simply connected, 2-dimensional space form of constant sectional curvature k . If $k > 0$, we add the assumption $|pq| + |qr| + |rp| < 2\pi/\sqrt{k}$. We let $\tilde{\angle}_k pqr$ denote the angle at the vertex \bar{q} of the triangle $\Delta \bar{p}\bar{q}\bar{r}$, and we call it a *k-comparison angle*.

Definition 2.1. Let $k \in \mathbb{R}$. A geodesic space M is called an *Alexandrov space with curvature bounded below by k* , denoted by $\text{curv} \geq k$, if it satisfies the following properties:

- (i) it is locally compact;
- (ii) for any point $x \in M$, there exists a neighborhood U of x such that the following condition is satisfied: for any two geodesics $\gamma(t) \subset U$ and $\sigma(s) \subset U$ with $\gamma(0) = \sigma(0) := p$, the k -comparison angles $\tilde{\angle}_k \gamma(t)p\sigma(s)$ is non-increasing with respect to each of the variables t and s .

Let M be an Alexandrov space with $\text{curv} \geq k$ for some $k \in \mathbb{R}$. It is well known that the Hausdorff dimension of M is always an integer or $+\infty$ (see, for example, [6, 7, 4]). In the following, the terminology of ‘‘an (n -dimensional) Alexandrov space M ’’ means that M is an Alexandrov space with $\text{curv} \geq k$ for some $k \in \mathbb{R}$ and that its Hausdorff dimension $\dim_{\mathcal{H}} = n$. We denote by $\mu := \mathcal{H}^n$ the n -dimensional Hausdorff measure on M . It holds the corresponding Bishop-Gromov inequality. Moreover, it holds the following local Alfers’ regularity: For any bounded domain Ω in an n -dimensional Alexandrov space with $\text{curv} \geq k$, there exist two positive constants C_1, C_2 , (depending on the diameter of Ω and $\mu(\Omega)$, if $k > 0$, we add to assume that $\text{diam}(\Omega) \leq \pi/(2\sqrt{k})$), such that

$$(2.1) \quad C_1 \leq \frac{\mu(B_r(x))}{r^n} \leq C_2, \quad \forall x \in \Omega, \quad 0 \leq r \leq \text{diam}(\Omega).$$

Indeed, by the Bishop inequality (see [7]), we obtain the upper bound

$$\mu(B_r(x)) \leq \mu(B_r \subset \mathbb{H}^n(k)) \leq C_{n,k,\text{diam}(\Omega)} \cdot r^n$$

and the Bishop-Gromov inequality (see [7]) implies the lower bound

$$\frac{\mu(B_r(x))}{\mu(\Omega)} \geq \frac{\mu(B_r(x))}{\mu(B_{2\cdot\text{diam}(\Omega)}(x))} \geq \frac{\mu(B_r \subset \mathbb{H}^n(k))}{\mu(B_{2\cdot\text{diam}(\Omega)} \subset \mathbb{H}^n(k))} \geq C'_{n,k,\text{diam}(\Omega)} \cdot r^n.$$

On an n -dimensional Alexandrov space M , the angle between any two geodesics $\gamma(t)$ and $\sigma(s)$ with $\gamma(0) = \sigma(0) := p$ is well defined, as the limit

$$\angle \gamma'(0)\sigma'(0) := \lim_{s,t \rightarrow 0} \tilde{\angle}_k \gamma(t)p\sigma(s).$$

We denote by Σ'_p the set of equivalence classes of geodesic $\gamma(t)$ with $\gamma(0) = p$, where $\gamma(t)$ is equivalent to $\sigma(s)$ if $\angle \gamma'(0)\sigma'(0) = 0$. (Σ'_p, \angle) is a metric space, and its completion is called the *space of directions at p* , denoted by Σ_p . It is known (see, for example, [6] or [7]) that

(Σ_p, \angle) is an Alexandrov space with curvature ≥ 1 of dimension $n - 1$. The *tangent cone* at p , T_p , is the Euclidean cone over Σ_p . The “scalar product” is given on T_p by

$$\langle u, v \rangle := \frac{1}{2}(|uo|^2 + |vo|^2 - |uv|^2), \quad \forall u, v \in T_p,$$

where o is the vertex of T_p . The *exponential map* $\exp_p : W_p \subset T_p \rightarrow M$ is defined in the standard way. Generally speaking, the domain W_p may not contain any neighborhood of o . This is one of the technical difficulties in Alexandrov geometry.

Definition 2.2 (Boundary, [7]). The boundary of an Alexandrov space M is defined inductively with respect to dimension. If the dimension of M is one, then M is a complete Riemannian manifold and the *boundary* of M is defined as usual. Suppose that the dimension of M is $n \geq 2$. A point p is a *boundary point* of M if Σ_p has non-empty boundary.

From now on, we always consider Alexandrov spaces without boundary. We refer to the seminar paper [7] or the books [6, 4] for the details.

2.2. Singularity and (almost) Riemannian structure.

Let $k \in \mathbb{R}$ and let M be an n -dimensional Alexandrov space with $\text{curv} \geq k$. For any $\delta > 0$, we denote

$$S_\delta := \{x \in M : \text{vol}(\Sigma_x) \leq (1 - \delta) \cdot \omega_{n-1}\},$$

where ω_{n-1} is the Riemannian volume of the standard $(n - 1)$ -sphere. S_δ is close (see [7]). Each point $p \in S_\delta$ is called a δ -singular point. The set

$$S_M := \cup_{\delta > 0} S_\delta$$

is called *singular set*. A point $p \in M$ is called a *singular point* if $p \in S_M$. Otherwise it is called a *regular point*. Equivalently, a point p is regular if and only if T_p is isometric to \mathbb{R}^n ([7]). Since we always assume that the boundary of M is empty, it is proved in [7] that the Hausdorff dimension of S_M is $\leq n - 2$. We remark that Ostu-Shioya in [42] constructed an Alexandrov space with nonnegative curvature such that its singular set is dense.

Some basic structures of Alexandrov spaces have been known in the following.

Proposition 2.3. *Let $k \in \mathbb{R}$ and let M be an n -dimensional Alexandrov space with $\text{curv} \geq k$. (1) There exists a constant $\delta_{n,k} > 0$ depending only on the dimension n and k such that for each $\delta \in (0, \delta_{n,k})$, the set $M \setminus S_\delta$ forms a Lipschitz manifold ([7]) and has a C^∞ -differentiable structure ([34]).*

(2) *There exists a BV_{loc} -Riemannian metric g on $M \setminus S_\delta$ such that*

- *the metric g is continuous in $M \setminus S_M$ ([42, 44]);*
- *the distance function on $M \setminus S_M$ induced from g coincides with the original one of M ([42]);*
- *the Riemannian measure on $M \setminus S_M$ induced from g coincides with the Hausdorff measure of M ([42]).*

2.3. Sobolev spaces and Laplacian on Alexandrov spaces.

Several different notions of Sobolev spaces on metric spaces have been established, see [9, 2, 34, 48, 33, 21]. They coincide with each other in the setting of Alexandrov spaces.

Let M be an n -dimensional Alexandrov space with $\text{curv} \geq k$ for some $k \in \mathbb{R}$. Let Ω be an open domain in M . We denote by $Lip_{\text{loc}}(\Omega)$ the set of locally Lipschitz continuous functions on Ω , and by $Lip_0(\Omega)$ the set of Lipschitz continuous functions on Ω with compact support in Ω .

For any $1 \leq p \leq +\infty$ and $f \in Lip_{loc}(\Omega)$, its $W^{1,p}(\Omega)$ -norm is defined by

$$\|f\|_{W^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \|\text{Lip}f\|_{L^p(\Omega)},$$

where $\text{Lip}f(x)$ is the *pointwise Lipschitz constant* ([9]) of f at x :

$$\text{Lip}f(x) := \limsup_{y \rightarrow x} \frac{|f(x) - f(y)|}{|xy|}.$$

Sobolev space $W^{1,p}(\Omega)$ is defined by the closure of the set of locally Lipschitz functions f with $\|f\|_{W^{1,p}(\Omega)} < \infty$ under $W^{1,p}(\Omega)$ -norm. The space $W_0^{1,p}(\Omega)$ is defined by the closure of $Lip_0(\Omega)$ under $W^{1,p}(\Omega)$ -norm. We say a function $f \in W_{loc}^{1,p}(\Omega)$ if $f \in W^{1,p}(\Omega')$ for every open subset $\Omega' \subset\subset \Omega$. Here and in the following, “ $\Omega' \subset\subset \Omega$ ” means Ω' is compactly contained in Ω . The space $W^{1,p}(\Omega)$ is reflexible for any $1 < p < \infty$ (see, for example, Theorem 4.48 of [9]). For each $f \in W^{1,p}(\Omega)$, there exists a function $|\nabla f| \in L^p(\Omega)$ such that $\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}$.

Fix a number $\delta \in (0, \delta_{n,k})$ sufficiently small (see Proposition 2.3). Recall that $M^* := M \setminus S_\delta$ is a C^∞ -manifold. Let Ω be an open set and denote $\Omega^* := \Omega \setminus S_\delta = \Omega \cap M^*$. An important fact is the following denseness result given in [34].

Lemma 2.4 (Kuwaie et al. [34]). *Let Ω be bounded open. For each $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, there exists $u_j \in Lip_0(\Omega^*)$ such that $u_j \rightarrow u$ in $W^{1,2}(\Omega)$ and $u_j \xrightarrow{*} u$ in $L^\infty(\Omega)$, as $j \rightarrow \infty$.*

Proof. The convergence $u_j \xrightarrow{W^{1,2}(\Omega)} u$ as $j \rightarrow \infty$ is the assertion of Theorem 1.1 in [34].

If $u \in L^\infty(\Omega)$, from the construction of u_j in [34], we have $\|u_j\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$ for all $j \in \mathbb{N}$. (See Lemma 3.3 in [34], u_j is taken by $\phi_j u$ for some suit cut-off functions). Thus, $\{u_j\}$ is $*$ -weak compact in $L^\infty(\Omega)$. By combining with $u_j \xrightarrow{L^2} u$ as $j \rightarrow \infty$, this yields the second assertion. \square

Definition 2.5 (Distributional Laplacian). The *Laplacian* on Ω is an operator $\Delta (= \Delta_\Omega)$ on $W_{loc}^{1,2}(\Omega)$ defined as the follows. For each function $f \in W_{loc}^{1,2}(\Omega)$, its Laplacian Δf is a linear functional acting on $Lip_0(\Omega)$ given by

$$(2.2) \quad \Delta f(\phi) := - \int_{\Omega} \langle \nabla f, \nabla \phi \rangle d\mu \quad \forall \phi \in Lip_0(\Omega).$$

This Laplacian (on Ω) is linear and satisfies the Chain rule and Leibniz rule (see [42, 34, 16]).

Fix any sufficiently small $\delta > 0$. Thanks to Lemma 2.4, it suffices to take the test function $\phi \in Lip_0(\Omega \setminus S_\delta)$. If $f \in W^{1,2}(\Omega)$, then (2.2) holds for all $\phi \in W_0^{1,2}(\Omega)$.

If, given $f \in W_{loc}^{1,2}(\Omega)$, there exists a function $u_f \in L_{loc}^1(\Omega)$ such that

$$\Delta f(\phi) = \int_{\Omega} u_f \cdot \phi d\mu \quad \forall \phi \in Lip_0(\Omega),$$

then we write as “ $\Delta f = u_f$ in the sense of distributions”. It is similar for $u_f \in L_{loc}^p(\Omega)$ or $W_{loc}^{1,p}(\Omega)$ for any $p \in [1, \infty]$.

A function $f \in W_{loc}^{1,2}(\Omega)$ is call subharmonic if $\Delta f \geq 0$ in the sense of distributions, that is $\int_{\Omega} \langle \nabla f, \nabla \phi \rangle d\mu \leq 0$ for all $0 \leq \phi \in Lip_0(\Omega)$. A basic fact is the maximum principle, which is well-known for experts (see, for example [9, Theorem 7.17] or [19]). For the convenience of readers, we include a proof here.

Lemma 2.6 (Maximum principle). *Let $f \in W^{1,2}(\Omega)$ be a subharmonic function such that $f - g \in W_0^{1,2}(\Omega)$ for some $g \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$. Then*

$$\sup_{\Omega} f \leq \sup_{\Omega} g.$$

Here \sup_{Ω} means esssup_{Ω} .

Proof. Let f_H be the (unique) solution of Dirichlet problem $\Delta f_H = 0$ such that $f_H - f \in W_0^{1,2}(\Omega)$. Thus, $f_H - g \in W_0^{1,2}(\Omega)$. By [9, Theorem 7.8 and Theorem 7.17], we get $f_H \in L^\infty(\Omega)$ and $\sup_{\Omega} f_H \leq \sup_{\Omega} g$. Then, it suffices to show that $\sup_{\Omega} f \leq \sup_{\Omega} f_H$.

Notice that $\Delta(f - f_H) \geq 0$ on Ω and $f - f_H \in W_0^{1,2}(\Omega)$. Then $(f - f_H)^+ \in W_0^{1,2}(\Omega)$. Therefore, we get

$$0 \geq \int_{\Omega} \langle \nabla(f - f_H), \nabla(f - f_H)^+ \rangle d\mu = \int_{\Omega} |\nabla(f - f_H)^+|^2 d\mu.$$

Then $(f - f_H)^+ = 0$ almost everywhere, by Poincaré inequality. That is $f \leq f_H$ almost everywhere in Ω . This yields $\sup_{\Omega} f \leq \sup_{\Omega} f_H$, and finishes the proof. \square

2.4. The heat flows on Alexandrov spaces.

Let M be an n -dimensional Alexandrov space with $\text{curv} \geq k$ for some $k \in \mathbb{R}$. The Dirichlet energy \mathcal{E} is defined by

$$\mathcal{E}(f, g) := \int_M \langle \nabla f, \nabla g \rangle d\mu, \quad \forall f, g \in W^{1,2}(M).$$

This energy \mathcal{E} gives a canonical Dirichlet form on $L^2(M)$ with the domain $D(\mathcal{E}) = W^{1,2}(M)$. It has been shown [34] that the canonical Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is strongly local and that for $f \in D(\mathcal{E})$, the energy measure of f is absolutely continuous w.r.t. μ with the density $|\nabla f|^2$. Moreover, the intrinsic distance $d_{\mathcal{E}}$ induced by \mathcal{E} coincides with the original distance d . 48]). Let $\Delta_{\mathcal{E}}$ and $\{P_t f\}_{t \geq 0}$ be the infinitesimal generator (with the domain $D(\Delta_{\mathcal{E}})$) and the heat flow induced from $(\mathcal{E}, D(\mathcal{E}))$.

It is proved in [50, 34] that there exists a locally Hölder continuous symmetric heat kernel $p_t(x, y)$ of P_t such that

$$(2.3) \quad P_t f(x) = \int_M p_t(x, y) f(y) d\mu, \quad \forall f \in L^2(M).$$

Moreover, the following estimates for heat kernel have been proved in [29].

Lemma 2.7. *Let M be an n -dimensional Alexandrov space with $\text{curv} \geq k$ for some $k \in \mathbb{R}$. There exist two constants $C_1, C_2 > 0$, depending only on k and n , such that*

$$(2.4) \quad \frac{1}{C_1 \cdot \mu(B_{\sqrt{t}}(y))} \exp\left(-\frac{d^2(x, y)}{3t} - C_2 \cdot t\right) \leq p_t(x, y) \\ \leq \frac{C_1}{\mu(B_{\sqrt{t}}(y))} \exp\left(-\frac{d^2(x, y)}{5t} + C_2 \cdot t\right)$$

for all $t > 0$ and all $x, y \in M$, and

$$(2.5) \quad |\nabla p_t(\cdot, y)|(x) \leq \frac{C_1}{\sqrt{t} \cdot \mu(B_{\sqrt{t}}(x))} \exp\left(-\frac{d^2(x, y)}{5t} + C_2 \cdot t\right)$$

for all $t > 0$ and μ -almost all $x, y \in M$.

We need the following result on cut-off functions, (which holds on more general $RCD^*(K, N)$ -spaces, see [39, Lemma 3.1] and [3, 17, 25]).

Lemma 2.8. *Let M be an n -dimensional Alexandrov space with $\text{curv} \geq k$ for some $k \in \mathbb{R}$. Then for every $x_0 \in M$ and $R > 0$ there exists a Lipschitz cut-off function $\chi : M \rightarrow [0, 1]$ satisfying:*

- (i) $\chi = 1$ on $B_{2R/3}(x_0)$ and $\text{supp}(\chi) \subset B_R(x_0)$;
- (ii) $\chi \in D(\Delta_{\mathcal{E}})$ and $\Delta_{\mathcal{E}}\chi \in W^{1,2}(M) \cap L^\infty(M)$, moreover $|\Delta_{\mathcal{E}}\chi| + |\nabla\chi| \leq C(n, k, R)$.

It was shown [16] that the above distributional Laplacian is compatible with the generator $\Delta_{\mathcal{E}}$ of the canonical Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ in the following sense:

$$(2.6) \quad f \in D(\Delta_{\mathcal{E}}) \iff f \in W^{1,2}(M) \text{ and that } \Delta f \text{ is a function in } L^2(M).$$

Moreover, in this case, it holds $\Delta f = \Delta_{\mathcal{E}}f$ in the sense of distributions.

3. GRADIENT ESTIMATES TO POISSON EQUATIONS

We first give a gradient estimate of heat flows as follows.

Lemma 3.1. *Let $n \geq 2$ and $k \in \mathbb{R}$, and let M be an n -dimensional Alexandrov space with $\text{curv} \geq k$. Let $F(t, x) \in L^2(M)$ for any $t \in [0, T]$. Suppose that there exist $\bar{F}(x) \in L^2(M)$ such that*

$$|F(t, x)| \leq \bar{F}(x) \text{ for all } t \in [0, T] \text{ and } \mu\text{-a.e. } x \in M.$$

Let $u(t, x)$ be a solution to the non-homogeneous heat equation

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) = \Delta_{\mathcal{E}} u(t, x) + F(t, x) \\ u(0, x) = u_0(x) \in L^2(M). \end{cases}$$

If both u_0 and $\bar{F}(x)$ are supported in $B_R(x_0)$. Then we have the gradient estimate of $u(t, x)$ as follows.

$$(3.1) \quad |\nabla u(t, x)| \leq \frac{C_{n,k,R}}{t^{\frac{n+1}{2}}} \cdot \int_{B_R(x_0)} |u_0(y)| d\mu(y) + C_{n,k,R} \int_{B_R(x_0)} \frac{\bar{F}(y)}{d^{n-1}(x, y)} d\mu(y)$$

for almost all $x \in B_R(x_0)$ and all $t \in (0, \min\{T, R^2\})$, where $\int_E f d\mu := \frac{1}{\mu(E)} \int_E f d\mu$ for any measurable set E and $f \in L^1(E)$.

Proof. From the Duhamel's principle (see Theorem 3.5 on Page 114 in [43]), the solution $u(t, x)$ is unique and has a representation as follows.

$$(3.2) \quad u(t, x) = \int_M p_t(x, y) u_0(y) d\mu(y) + \int_0^t \int_M p_{t-s}(x, y) F(s, y) d\mu(y) ds.$$

Since both u_0 and \bar{F} are supported in $B_R(x_0)$, we have that for almost all $x \in B_R(x_0)$ and all $t \in (0, T]$

$$(3.3) \quad \begin{aligned} |\nabla u(t, x)| &\leq \int_M |\nabla p_t(x, y)| |u_0(y)| d\mu(y) + \int_0^t \int_X |\nabla p_{t-s}(x, y)| |F(s, y)| d\mu(y) ds \\ &\leq \int_{B_R(x_0)} |\nabla p_t(x, y)| |u_0(y)| d\mu(y) + \int_0^t \int_{B_R(x_0)} |\nabla p_{t-s}(x, y)| \bar{F}(y) d\mu(y) ds \\ &:= I_1 + I_2. \end{aligned}$$

From (2.5) and the Bishop-Gromov inequality we get that for almost all $x \in B_R(x_0)$ and any $\sqrt{t} \leq R$,

$$\begin{aligned} |\nabla p_t(\cdot, y)|(x) &\leq \frac{C_1 \cdot e^{C_2 R}}{\sqrt{t} \cdot \mu(B_{\sqrt{t}}(x))} \cdot \exp\left(-\frac{d^2(x, y)}{5t}\right) \\ &\leq \frac{C_3}{\mu(B_R(x_0))} \cdot t^{-\frac{n+1}{2}} \cdot \exp\left(-\frac{d^2(x, y)}{5t}\right), \end{aligned}$$

where and in the sequel of this proof, all constants C_1, C_2, \dots depend only on n, k and R . Hence, we obtain

$$\begin{aligned} (3.4) \quad I_1 &\leq \frac{C_3}{\mu(B_R(x_0))} \cdot \int_{B_R(x_0)} t^{-\frac{n+1}{2}} \cdot \exp\left(-\frac{d^2(x, y)}{5t}\right) |u_0(y)| d\mu(y) \\ &\leq \frac{C_3}{t^{\frac{n+1}{2}}} \cdot \int_{B_R(x_0)} |u_0(y)| d\mu(y) \end{aligned}$$

and, by taking $\tau = \frac{d^2(x, y)}{t-s}$,

$$\begin{aligned} (3.5) \quad I_2 &\leq \frac{C_3}{\mu(B_R(x_0))} \cdot \int_{B_R(x_0)} d\mu(y) \int_{d^2(x, y)/t}^{\infty} \left(\frac{\tau}{d^2(x, y)}\right)^{\frac{n+1}{2}} \cdot \exp\left(-\frac{\tau}{5}\right) \bar{F}(y) \frac{d^2(x, y) d\tau}{\tau^2} \\ &\leq C_3 \cdot \int_{B_R(x_0)} \frac{\bar{F}(y) d\mu(y)}{d^{n-1}(x, y)} \int_0^{\infty} \tau^{\frac{n-3}{2}} \cdot \exp\left(-\frac{\tau}{5}\right) d\tau \\ &\leq C_4 \cdot \int_{B_R(x_0)} \frac{\bar{F}(y) d\mu(y)}{d^{n-1}(x, y)}, \end{aligned}$$

where we have used $n \geq 2$ and $\int_0^{\infty} \tau^{(n-3)/2} e^{-\tau/5} d\tau \leq C_n$. The desired estimate comes from the combination of (3.3) and (3.4), (3.5). \square

We need the following local gradient estimate for Poisson equations.

Proposition 3.2. *Let $n \geq 2$ and $k \in \mathbb{R}$, and let M be an n -dimensional Alexandrov space with curv $\geq k$. Let $u(x) \in W_{\text{loc}}^{1,2}(B_R(x_0))$ solve the Poisson equation*

$$\Delta u = f \in L_{\text{loc}}^1(B_R(x_0))$$

in the sense of distributions. If $u(x) \in L_{\text{loc}}^\infty(B_R(x_0))$ and if $f \geq 0$, then we have

$$(3.6) \quad |\nabla u(x)| \leq C_{n,k,R} \|u\|_{L^\infty(B_R(x_0))} + C_{n,k,R} \int_{B_R(x_0)} \left(\frac{f}{d^{n-1}(x, y)} + |\nabla u|\right) d\mu(y)$$

for almost all $x \in B_{R/4}(x_0)$.

Proof. (i) We first consider the case of $f \in L_{\text{loc}}^2(B_R(x_0))$.

Let $\phi : M \mapsto [0, 1]$ be a cut-off function such that $\phi = 1$ on $B_{R/3}(x_0)$, $\phi = 0$ out of $B_{2R/3}(x_0)$, and $R|\nabla\phi| + R^2|\Delta\phi| \leq C_{n,k,R}$ (see Lemma 2.8). Then it is clear that $u\phi \in W_0^{1,2}(B_R(x_0)) \subset W^{1,2}(M)$ and

$$g := f\phi + 2\langle \nabla u, \nabla \phi \rangle + u \cdot \Delta \phi \in L^2(M).$$

By (2.6), we obtain that

$$u\phi \in D(\Delta_{\mathcal{E}}) \quad \text{and} \quad \Delta_{\mathcal{E}}(u\phi) = g.$$

By Lemma 3.1, we obtain, by taking $t = R^2$, for almost all $x \in B_{R/4}(x_0)$,

$$(3.7) \quad |\nabla u|(x) = |\nabla(u\phi)|(x) \leq C_1 \int_{B_R(x_0)} |u(y)| + C_1 \int_{B_R(x_0)} \frac{|g(y)|}{d^{n-1}(x, y)} d\mu(y),$$

where the constant C_1 depends only on n, k and R . Noting that $|\nabla\phi| + |\Delta\phi| = 0$ on $B_{R/3}(x_0)$, we have for almost all $x \in B_{R/4}(x_0)$,

$$\begin{aligned} \int_{B_R(x_0)} \frac{|g(y)|}{d^{n-1}(x, y)} d\mu(y) &\leq \int_{B_R(x_0)} \frac{|f|}{d^{n-1}(x, y)} d\mu(y) \\ &\quad + \int_{B_R(x_0) \setminus B_{R/3}(x_0)} \frac{2|\nabla u|/R + |u|/R^2}{d^{n-1}(x, y)} d\mu(y) \\ &\leq \int_{B_R(x_0)} \left(\frac{|f|}{d^{n-1}(x, y)} + C_R(|\nabla u| + |u|) \right) d\mu(y), \end{aligned}$$

where we have used $d(x, y) \geq R/12$ provided $x \in B_{R/4}(x_0)$ and $y \in B_R(x_0) \setminus B_{R/3}(x_0)$. Hence, we get, for almost all $x \in B_{R/4}(x_0)$,

$$(3.8) \quad \begin{aligned} |\nabla u|(x) &\leq C_2 \int_{B_R(x_0)} |u| d\mu + C_2 \int_{B_R(x_0)} \left(\frac{|f|}{d^{n-1}(x, y)} + |\nabla u| \right) d\mu(y) \\ &\leq C_2 \|u\|_{L^\infty(B_R(x_0))} + C_2 \int_{B_R(x_0)} \left(\frac{|f|}{d^{n-1}(x, y)} + |\nabla u| \right) d\mu(y) \end{aligned}$$

where the constant C_2 depends only on n, k and R .

(ii) We consider the case of $0 \leq f \in L^1_{\text{loc}}(B_R(x_0))$. Let $f_j(x) = \min\{f(x), j\}$. We have $0 \leq f_j \in L^\infty \cap L^1(B_{3R/4}(x_0)) \subset L^2(B_{3R/4}(x_0))$ and that $f_j \rightarrow f$ in $L^1(B_{3R/4}(x_0))$, as $j \rightarrow \infty$.

We solve the equation $\Delta u_j = f_j$ on $B_{3R/4}(x_0)$ with $u_j - u \in W_0^{1,2}(B_{3R/4}(x_0))$ (in the sense of distributions). By $\Delta(u - u_j) = f - f_j \geq 0$ in the sense of distributions. The maximum principle (Lemma 2.6) implies that, for almost all $x \in B_{3R/4}(x_0)$,

$$(3.9) \quad \text{esssup}_{B_{3R/4}(x_0)}(u - u_j) \leq 0, \quad \forall j \in \mathbb{N}.$$

Similarly, the combination of the facts $\Delta u_j \geq 0$, $u_j - u \in W_0^{1,2}(B_{3R/4}(x_0))$ and $u \in L^\infty(B_{3R/4}(x_0))$, by the maximum principle (Lemma 2.6) implies that $\sup_{B_{3R/4}(x_0)} u_j \leq \sup_{B_{3R/4}(x_0)} u$, for each $j \in \mathbb{N}$. Then, we obtain, by combining with (3.9),

$$(3.10) \quad \|u_j\|_{L^\infty(B_{3R/4}(x_0))} \leq \|u\|_{L^\infty(B_{3R/4}(x_0))} \quad \forall j \in \mathbb{N}.$$

By using $\Delta(u_j - u) = f_j - f$ in the sense of distributions, we have

$$(3.11) \quad \begin{aligned} \int_{B_{3R/4}(x_0)} |\nabla(u_j - u)|^2 d\mu &\leq \int_{B_{3R/4}(x_0)} |(f_j - f)(u_j - u)| d\mu \\ &\leq \|u_j - u\|_{L^\infty} \cdot \|f_j - f\|_{L^1} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

It follows that $|\nabla u_j| \rightarrow |\nabla u|$ in $L^2(B_{3R/4}(x_0))$ as $j \rightarrow \infty$. In particular, (up to a subsequence,) we have $\lim_{j \rightarrow \infty} |\nabla u_j|(x) = |\nabla u|(x)$ at μ -a.e. $x \in B_{3R/4}(x_0)$. By applying (3.8) to u_j , and facts (3.10) and $0 \leq f_j \leq f$, and then letting $j \rightarrow \infty$, we get

$$|\nabla u|(x) \leq C_2 \|u\|_{L^\infty(B_R(x_0))} + C_2 \int_{B_R(x_0)} \left(\frac{f}{d^{n-1}(x, y)} + |\nabla u| \right) d\mu(y)$$

for almost all $x \in B_{R/4}(x_0)$. The proof is finished. \square

4. HARMONIC MAPS FROM ALEXANDROV SPACES

Let (N, h) be a compact smooth Riemannian manifold. By Nash's imbedding theorem, we can assume that N is isometrically embedded into an Euclidean space \mathbb{R}^ℓ . Let M be an n -dimensional Alexandrov space with $\text{curv} \geq k$ for some $k \in \mathbb{R}$. Fix any open domain $\Omega \subset M$, the Sobolev space $W^{1,2}(\Omega, N)$ is defined by

$$W^{1,2}(\Omega, N) := \left\{ u \in W^{1,2}(\Omega, \mathbb{R}^\ell) \mid u(x) \in N \text{ for } \mu\text{-a.e. } x \in \Omega \right\}$$

and the energy

$$(4.1) \quad E(u) := \int_{\Omega} |\nabla u|^2(x) d\mu(x), \quad |\nabla u|^2 = \sum_{j=1}^{\ell} |\nabla u^j|^2.$$

Definition 4.1. A map $u \in W_{\text{loc}}^{1,2}(\Omega, N)$ is a (weakly) harmonic map, if it is a critical point of $E(\cdot)$ on any subdomain $\Omega' \subset\subset \Omega$. In particular, any energy minimizing map is harmonic.

Lemma 4.2. *Any harmonic map $u : \Omega \rightarrow N$ such that $u \in W^{1,2}(\Omega, N)$, then it satisfies the (weak) harmonic map system:*

$$(4.2) \quad \Delta u = -A(\nabla u, \nabla u)$$

in the sense of distributions, where $A(\cdot, \cdot)$ is the second fundamental form of the embedding $N \subset \mathbb{R}^\ell$. Namely,

$$\int_{\Omega} \langle \nabla u, \nabla \Phi \rangle d\mu = \int_{\Omega} A(\nabla u, \nabla u) \cdot \Phi d\mu, \quad \forall \Phi \in W_0^{1,2}(\Omega, \mathbb{R}^\ell) \cap L^\infty(\Omega, \mathbb{R}^\ell).$$

Moreover, if the image of u is included in a geodesic ball $B_r(Q)$ for some point $Q \in N$ and $r < \frac{\text{inj}(N)}{3}$, ($\text{inj}(N)$ is the injective radius of N), then the function $u_Q(x) := d_N^2(Q, u(x))$ satisfies

$$(4.3) \quad 0 \leq \Delta u_Q \leq C_0 \cdot |\nabla u|^2$$

in the sense of distributions, for some constant $C_0 > 0$ depending on N , but independent of Q .

Proof. (i) Let us first consider the case where Ω is a domain of a smooth manifold with an L^∞ -Riemannian metric. In this case the assertion (4.2) is well-known. If the image $u(\Omega) \subset B_r(Q)$ with $r < \text{inj}(N)/3$, then the function $d_N^2(Q, \cdot)$ is smooth, and by the chain rule of harmonic maps (see [32, Lemma 9.2.2]), we have

$$(4.4) \quad \Delta u_Q = \text{Hess}_u d_N^2(Q, \cdot)(\nabla u, \nabla u)$$

in the sense of distributions, where $\text{Hess}_u d_N^2(Q, \cdot)$ is the Hessian of $d_N^2(Q, \cdot)$ at Q in N . The Hessian comparison theorem asserts

$$0 \leq \text{Hess}_N d_N^2(Q, \cdot) \leq C_1 \cdot I$$

for some constant C_1 depending the bound of $|\text{sec}_N|$ and $\text{inj}(N)$, where the lower bound follows from the fact that $d_N^2(Q, \cdot)$ is convex. Hence, applying to (4.4), we obtain (4.3) in this case.

(ii) Now we consider the general case where Ω is a domain of an Alexandrov space. From Proposition 2.3, we can fix a number $\delta \in (0, \delta_{n,k})$ sufficiently small such that $M^* := M \setminus S_\delta$ is a C^∞ -manifold. Thus, the case (i) asserts that

$$(4.5) \quad \int_{\Omega} \langle \nabla u, \nabla \Phi \rangle d\mu = \int_{\Omega} A(\nabla u, \nabla u) \cdot \Phi d\mu, \quad \forall \Phi \in \text{Lip}_0(\Omega \setminus S_\delta, \mathbb{R}^\ell).$$

By Lemma 2.4, it is clear that the test (vector value) function Φ can be chosen in $W_0^{1,2}(\Omega, \mathbb{R}^\ell) \cap L^\infty(\Omega, \mathbb{R}^\ell)$. This is the assertion (4.2). The estimates (4.3) can be obtained by a similar argument. The proof is finished. \square

5. FROM CONTINUITY TO LIPSCHITZ CONTINUITY

In this section, we will prove Theorem 1.3. Throughout this section, we assume always that Ω is a bounded domain of an n -dimensional Alexandrov space with $\text{curv} \geq k$ for some $k \in \mathbb{R}$, and that (N, h) is compact smooth Riemannian manifold, isometrically embedded into \mathbb{R}^ℓ , and let $u \in W^{1,2}(\Omega, N)$ be a harmonic map.

Remark that the Hölder continuity of energy minimizing maps with small energy was established by Lin [36].

The following Morrey bound for $|\nabla u|$ is the key estimate (see also [26] for a proof in Euclidean domain).

Proposition 5.1. *Assume Ω, N as the above. For any $0 < \alpha < 2$, there exists a constant $\epsilon > 0$ (depending only on α, n, k, Ω and the bound of the second fundamental form $\sup_N |A|$), such that the following holds: If $u : \Omega \rightarrow N$ is a harmonic map (need not to be an energy minimizer) and if a ball $B_{r_0}(x_0) \subset \Omega$ such that*

$$(5.1) \quad \text{osc}_{B_{r_0}(x_0)} u := \sup_{x, y \in B_{r_0}(x_0)} d_N(u(x), u(y)) < \epsilon,$$

then for any $x \in B_{r_0/2}(x_0)$ and any $r \leq r_0/2$, we have

$$(5.2) \quad r^{2-n} \int_{B_r(x)} |\nabla u|^2(y) d\mu(y) \leq Cr^\alpha,$$

where the constant C depends on $\alpha, n, k, \Omega, r_0$ and $\int_{B_{r_0}(x_0)} |\nabla u|^2 d\mu$.

Proof. For any $\epsilon > 0$ to be fixed later when fixing α . For each $r < r_0$, let us solve the Dirichlet problem $\Delta v = 0$ with $u - v \in W_0^{1,2}(B_r(x), \mathbb{R}^\ell)$ on $B_r(x)$. By maximal principle (see Lemma 2.6), we have that

$$(5.3) \quad \sup_{y, z \in B_r(x)} |v(z) - v(y)| \leq \sup_{y, z \in B_r(x)} |u(z) - u(y)| \leq \text{osc}_{B_{r_0}(x_0)} u \leq \epsilon.$$

Thus by noting that $u - v \in W_0^{1,2}(B_r(x), \mathbb{R}^\ell)$, we have

$$(5.4) \quad \sup_{y \in B_r(x)} |(u - v)(y)| \leq \sup_{y, z \in B_r(x)} |(u - v)(z) - (u - v)(y)| \leq 2\epsilon.$$

By using $(v - u) \in W_0^{1,2} \cap L^\infty$ and the equation $\Delta(u - v) + A(u)(\nabla u, \nabla u) = 0$ in the sense of distributions, we have

$$\int_{B_r(x)} \langle \nabla(u - v), \nabla(u - v) \rangle d\mu + \int_{B_r(x)} A(u)(\nabla u, \nabla u)(v - u) d\mu = 0.$$

This implies, by (5.4),

$$(5.5) \quad \int_{B_r(x)} |\nabla(u - v)|^2 \leq 2\epsilon \cdot \sup_N |A| \cdot \int_{B_r(x)} |\nabla u|^2.$$

Recall that the Bochner inequality (see [52]) implies $|\nabla v|^2 \in W_{\text{loc}}^{1,2}(B_r(x))$ and that

$$\Delta |\nabla v|^2 \geq -2k |\nabla v|^2$$

in the sense of distributions on $B_r(x)$. Thus, we have

$$(5.6) \quad \sup_{B_{r/2}(x)} |\nabla v|^2 \leq C_1 \int_{B_r(x)} |\nabla v|^2 d\mu(z),$$

where the constant C_1 depends only on n, k and r_0 . For any $0 < \theta < 1/2$, by (5.5), (5.6) and that v is harmonic on $B_r(x)$, we have

$$(5.7) \quad \int_{B_{\theta r}(x)} |\nabla u|^2 d\mu \leq 2 \int_{B_{\theta r}(x)} |\nabla(u-v)|^2 d\mu + 2 \int_{B_{\theta r}(x)} |\nabla v|^2 d\mu$$

$$(5.8) \quad \leq 2 \int_{B_r(x)} |\nabla(u-v)|^2 d\mu + 2 \int_{B_{\theta r}(x)} |\nabla v|^2 d\mu$$

$$(5.9) \quad \stackrel{(5.5)}{\leq} 4\epsilon \cdot \sup_N |A| \cdot \int_{B_r(x)} |\nabla u|^2 + 2 \int_{B_{\theta r}(x)} |\nabla v|^2 d\mu$$

$$(5.10) \quad \stackrel{(5.6)}{\leq} 4\epsilon \cdot \sup_N |A| \cdot \int_{B_r(x)} |\nabla u|^2 + 2 \cdot C_1 \mu(B_{\theta r}(x)) \int_{B_r(x)} |\nabla v|^2 d\mu$$

$$(5.11) \quad \leq \left(4\epsilon \sup_N |A| + C_2 \theta^n\right) \int_{B_r(x)} |\nabla u|^2 d\mu,$$

where the constant C_2 depends on n, k and Ω , and we have used (2.1) to conclude

$$\frac{\mu(B_{\theta r}(x))}{\mu(B_r(x))} \leq C_{n,k,\Omega} \cdot \theta^n.$$

Multiplying $(\theta r)^{2-n}$ to this inequality (5.11), we get

$$(5.12) \quad (\theta r)^{2-n} \int_{B_{\theta r}(x)} |\nabla u|^2 d\mu \leq \left(4\epsilon \sup_N |A| \cdot \theta^{2-n} + C_2 \theta^2\right) r^{2-n} \int_{B_r(x)} |\nabla u|^2 d\mu.$$

Given $0 < \alpha < 2$, let us choose $\theta_0 = \theta_0(\alpha, C_2) < 1/2$ such that $C_2 \theta_0^2 \leq 1/2 \theta_0^\alpha$. Then fix $\epsilon = \epsilon(n, \theta_0, \alpha, \sup_N |A|)$ such that $4\epsilon \sup_N |A| \theta_0^{2-n} \leq 1/2 \theta_0^\alpha$. Therefore, we arrive at

$$(5.13) \quad (\theta_0 r)^{2-n} \int_{B_{\theta_0 r}(x)} |\nabla u|^2 d\mu \leq \theta_0^\alpha r^{2-n} \int_{B_r(x)} |\nabla u|^2 d\mu$$

for all $r \leq r_0$. By iterating this inequality finitely many times, we have for all $r \leq r_0$ that

$$(5.14) \quad r^{2-n} \int_{B_r(x)} |\nabla u|^2 d\mu \leq r^\alpha \cdot C_{\alpha, \theta_0, r_0} \cdot \int_{B_{\theta_0 r_0}(x)} |\nabla u|^2 d\mu.$$

This is enough to (5.2), since that θ_0 depends only on α, n, k and Ω , and that $B_{\theta_0 r_0}(x) \subset B_{r_0}(x_0)$. \square

In order to dominate the Riesz potentials, we have the following lemma.

Lemma 5.2. *Let $n \geq 2$ and $k \in \mathbb{R}$, and let M be an n -dimensional Alexandrov space with curv $\geq k$. Given any ball $B_{r_0}(x_0) \subset M$ with $r_0 \leq 1$ and any $\beta \in (0, 1)$, there exists a constant $C(n, \beta) > 0$ such that if $g \in L^1_{\text{loc}}(M)$ then*

$$(5.15) \quad \int_{B_{r_0/4}(x)} \frac{|g|(y)}{d^{n-1}(x, y)} d\mu(y) \leq C_{n, \beta} \cdot \sup_{r \leq r_0/2} r^{\beta-n} \int_{B_r(x)} |g|(y) d\mu(y)$$

for μ -a.e. $x \in B_{r_0/2}(x_0)$.

Proof. For any $0 < s < r \leq r_0$, we denote the annulus $A_{s,r}(x) = B_r(x) \setminus \bar{B}_s(x)$. For any $\varepsilon < r_0/4$, we have for all $x \in B_{r_0/2}(x_0)$ that

$$(5.16) \quad \int_{B_{r_0/4}(x)} \frac{|g|(y)}{d^{n-1}(x,y)} d\mu(y) = \int_{B_\varepsilon(x)} \frac{|g|(y)}{d^{n-1}(x,y)} d\mu(y) + \int_{A_{\varepsilon,r_0/4}(x)} \frac{|g|(y)}{d^{n-1}(x,y)} d\mu(y) \\ := I_1 + I_2.$$

It is well-known that

$$(5.17) \quad I_1 \leq C_1 \cdot \varepsilon \cdot M_1 g(x),$$

where $M_1 g(x) := \sup_{0 < r \leq 1} \int_{B_r(x)} |g|(y) d\mu(y)$ is the maximal function of g , and the constant C_1 depends on $n, k, B_{r_0}(x_0)$. Indeed, we have

$$\begin{aligned} \int_{B_\varepsilon(x)} \frac{|g|(y)}{d(x,y)^{n-1}} d\mu(y) &\leq \sum_{2^{-i} \leq \varepsilon} \int_{A_{2^{-i}, 2^{-(i-1)}}(x)} \frac{|g|(y)}{d(x,y)^{n-1}} d\mu(y) \\ &\leq \sum_{2^{-i} \leq \varepsilon} 2^{i(n-1)} \int_{A_{2^{-i}, 2^{-(i-1)}}(x)} |g|(y) d\mu(y) \\ &\leq \sum_{2^{-i} \leq \varepsilon} 2^{i(n-1)} \mu(B_{2^{-(i-1)}}(x)) \int_{B_{2^{-(i-1)}}(x)} |g|(y) d\mu(y) \\ &\leq C_2 \sum_{2^{-i} \leq \varepsilon} 2^{n-i} \int_{B_{2^{-(i-1)}}(x)} |g| d\mu(y) \\ &\leq C_2 \sum_{2^{-i} \leq \varepsilon} 2^{n-i} M_1 g(x) \leq C_2 2^n \cdot 2\varepsilon \cdot M_1 g(x), \end{aligned}$$

where we have used (2.1) and $r_0 \leq 1$ to get

$$\mu(B_{2^{-(i-1)}}(x)) \leq C_2 \cdot 2^{-(i-1)n},$$

for some constant C_2 depends on n, k and $B_{r_0}(x_0)$. This yields (5.17) by taking $C_1 := 2^{n+1} C_2$.

Let us estimate I_2 .

$$\begin{aligned} I_2 &= \sum_{r_0/4 \geq 2^{-i} \geq \varepsilon} \int_{A_{2^{-i}, 2^{-(i-1)}}(x)} \frac{|g|(y)}{d(x,y)^{n-1}} d\mu(y) \\ &\leq \sum_{r_0/4 \geq 2^{-i} \geq \varepsilon} 2^{i(n-1)} \int_{A_{2^{-i}, 2^{-(i-1)}}(x)} |g|(y) d\mu(y) \\ &\leq \sum_{r_0/4 \geq 2^{-i} \geq \varepsilon} 2^{i(n-1)} \int_{B_{2^{-(i-1)}}(x)} |g|(y) d\mu(y) \end{aligned}$$

Hence

$$\begin{aligned}
I_2 &\leq \sum_{r_0/4 \geq 2^{-i} \geq \varepsilon} 2^{i(n-1)} 2^{(i-1)(\beta-n)} 2^{-(i-1)(\beta-n)} \int_{B_{2^{-(i-1)}(x)}} |g|(y) d\mu(y) \\
&\leq \sum_{r_0/4 \geq 2^{-i} \geq \varepsilon} 2^{i(n-1)} 2^{-(-i+1)(-n+\beta)} \cdot \sup_{r \leq r_0/2} r^{\beta-n} \int_{B_r(x)} |g|(y) d\mu(y) \\
&= \sum_{r_0/4 \geq 2^{-i} \geq \varepsilon} 2^{n-\beta-(1-\beta)i} \cdot \sup_{r \leq r_0/2} r^{\beta-n} \int_{B_r(x)} |g|(y) d\mu(y) \\
&\leq C_{n,\beta} \cdot \sup_{r \leq r_0/2} r^{\beta-n} \int_{B_r(x)} |g|(y) d\mu(y)
\end{aligned}$$

for any $\beta \in (0, 1)$. By combining with (5.16) and (5.17), we obtain that for all $\varepsilon < r_0/4$

$$(5.18) \quad \int_{B_{r_0/4}(x)} \frac{|g|(y)}{d^{n-1}(x,y)} d\mu(y) \leq C_1 \cdot \varepsilon \cdot M_1 g(x) + C_{n,\beta} \cdot \sup_{r \leq r_0/2} r^{\beta-n} \int_{B_r(x)} |g|(y) d\mu(y).$$

Since $M_1 g(x) < \infty$ for μ -a.e. $x \in B_{r_0}(x_0)$ by the weakly L^1 -boundedness of the maximal function, we conclude by letting $\varepsilon \rightarrow 0$ that

$$(5.19) \quad \int_{B_{r_0/4}(x)} \frac{|g|(y)}{d^{n-1}(x,y)} d\mu(y) \leq C_{n,\beta} \cdot \sup_{r \leq 1} r^{\beta-n} \int_{B_r(x)} |g|(y) d\mu(y),$$

for μ -a.e. $x \in B_{r_0/2}(x_0)$. The proof is finished. \square

Now we are in the place to prove the main gradient estimate of u .

Theorem 5.3. *Assume Ω, N as the above. Then there exists a constant $\varepsilon \in (0, \frac{\text{inj}(N)}{10})$ (depending only on n, k, Ω and $\sup_N |A|$), such that the following holds: If $u : \Omega \rightarrow N$ is a harmonic map (need not to be an energy minimizer) and if a ball $B_{r_0}(x_0) \subset \Omega$ with $r_0 \leq 1$ such that*

$$\text{osc}_{B_{r_0}(x_0)} u := \sup_{x,y \in B_{r_0}(x_0)} d_N(u(x), u(y)) < \varepsilon,$$

then for any point $Q \in N$ with $d_N(Q, u(B_{r_0}(x_0))) < \frac{\text{inj}(N)}{5}$ and letting $u_Q(x) = d_N^2(Q, u(x))$, we have

$$(5.20) \quad \text{esssup}_{B_{\frac{r_0}{32}}(x_0)} |\nabla u_Q| \leq C.$$

for some constant C depending on $n, k, \Omega, r_0, \mu(B_{r_0}(x_0)), \int_{B_{r_0}(x_0)} |\nabla u|^2 d\mu$ and N , and $\sup_N |A|$.

Proof. Let us fix $\alpha = 3/2$ in Proposition 5.1, we have

$$(5.21) \quad \sup_{r \leq r_0/2} r^{\frac{1}{2}-n} \int_{B_r(x)} |\nabla u|^2(y) d\mu(y) \leq C_1, \quad \forall x \in B_{r_0/2}(x_0),$$

where the constant C_1 depends on n, k, Ω, r_0 and $\int_{B_{r_0}(x_0)} |\nabla u|^2 d\mu$.

From Lemma 5.2, we have (by taking $\beta = 1/2$ and $g = |\nabla u|^2$) that

$$(5.22) \quad \int_{B_{r_0/4}(x)} \frac{|\nabla u|^2(y)}{d^{n-1}(x,y)} d\mu(y) \leq C_{n,1/2} \cdot \sup_{r \leq r_0/2} r^{\frac{1}{2}-n} \int_{B_r(x)} |\nabla u|^2(y) d\mu(y) \stackrel{(5.21)}{\leq} C_2.$$

for μ -a.e. $x \in B_{r_0/2}(x_0)$. In particular, there exists $x_1 \in B_{\frac{r_0}{64}}(x_0)$ such that

$$\int_{B_{r_0/4}(x_1)} \frac{|\nabla u|^2(y)}{d^{n-1}(x_1, y)} d\mu(y) \leq C_2.$$

Take any point $Q \in N$ such that

$$d_N(Q, u(B_{r_0}(x_0))) < \frac{\text{inj}(N)}{5}.$$

We have $u(B_{r_0}(x_0)) \subset B_{\epsilon + \frac{\text{inj}(N)}{5}}(Q) \subset B_{\frac{\text{inj}(N)}{4}}(Q)$. By applying Proposition 3.2 to the estimates (4.3) on $B_{r_0/4}(x_1)$, we obtain that

$$(5.23) \quad \sup_{x \in B_{\frac{r_0}{16}}(x_1)} |\nabla u_Q|(x) \leq C_3 \|u_Q\|_{L^\infty(B_{r_0/4}(x_1))} + \frac{C_4}{\mu(B_{r_0/4}(x_1))} + C_{n,k,r_0} \int_{B_{r_0/4}(x_1)} |\nabla u_Q| d\mu$$

for two positive constants C_3 and C_4 , where C_3 depends on n, k, r_0 , and C_4 depends on $C_2, \sup_N |A|, n, k, r_0$ and the constant C_0 in (4.3). The fact $u(B_{r_0}(x_0)) \subset B_{\frac{\text{inj}(N)}{4}}(Q)$ implies

$$\|u_Q\|_{L^\infty(B_{r_0}(x_0))} \leq \text{inj}^2(N)/16 := C_5.$$

By noticing that

$$|\nabla u_Q| = 2d_N(Q, u(\cdot)) |\nabla d_N(Q, u(\cdot))| \leq 2d_N(Q, u(\cdot)) |\nabla u| \leq 2\sqrt{C_5} |\nabla u|$$

and by combining with the fact $d(x_0, x_1) \leq \frac{r_0}{64}$, and the doubling property, we get from (5.23) that

$$(5.24) \quad \sup_{x \in B_{\frac{r_0}{32}}(x_0)} |\nabla u_Q|(x) \leq C_3 \cdot C_5 + \frac{C_6}{\mu(B_{r_0}(x_0))} + C_7 \int_{B_{r_0}(x_0)} |\nabla u| d\mu \leq C_8,$$

where the constant C_8 depends on $n, k, \Omega, r_0, \mu(B_{r_0}(x_0)), \int_{B_{r_0}(x_0)} |\nabla u|^2 d\mu$ and N , and $\sup_N |A|$. The proof is finished. \square

Now we provide the proof of Theorem 1.3 as follows.

Proof of Theorem 1.3. Let $\epsilon \in (0, \frac{\text{inj}(N)}{10})$ be given in Theorem 5.3. Fix any ball $B_{r_0}(x_0) \subset \Omega$, $r_0 \leq 1$, such that $\text{osc}_{B_{r_0}(x_0)} u < \epsilon$ and that u is continuous on $B_{r_0}(x_0)$. From the gradient estimate of Theorem 5.3, we get that for any $Q \in N$ with $d_N(Q, u(B_{r_0}(x_0))) < \frac{\text{inj}(N)}{5}$,

$$(5.25) \quad \frac{|u_Q(x) - u_Q(y)|}{|xy|} \leq C, \quad \mu\text{-a.e. } x, y \in B_{\frac{r_0}{32}}(x_0),$$

where the constant C is independent of Q (but it may depend on $n, k, \Omega, r_0, \mu(B_{r_0}(x_0)), \int_{B_{r_0}(x_0)} |\nabla u|^2 d\mu$ and N , and $\sup_N |A|$). Since u is continuous in $B_{r_0}(x_0)$, we have u_Q is also continuous in $B_{r_0}(x_0)$. Then, the inequality (5.25) holds for all $x, y \in B_{\frac{r_0}{32}}(x_0)$. That is,

$$(5.26) \quad |u_Q(x) - u_Q(y)| \leq C \cdot |xy|, \quad \forall x, y \in B_{\frac{r_0}{32}}(x_0).$$

Take any $x, y \in B_{\frac{r_0}{32}}(x_0)$. Since $d_N(u(x), u(y)) < \epsilon < \text{inj}(N)/10$, we can extend the geodesic $u(x)u(y)$ to a point Q such that $d_N(Q, u(x)) = \text{inj}(N)/6$ and

$$d_N(u(x), u(y)) = d_N(u(x), Q) - d_N(u(y), Q).$$

By applying (5.26), we obtain

$$(5.27) \quad \begin{aligned} d_N(u(x), u(y)) &= \frac{u_Q(x) - u_Q(y)}{d_N(u(x), Q) + d_N(u(y), Q)} \\ &\leq \frac{C \cdot |xy|}{d_N(u(x), Q) + d_N(u(y), Q)} \leq \frac{6C}{\text{inj}(N)} \cdot |xy|, \end{aligned}$$

where we have used $d_N(u(x), Q) + d_N(u(y), Q) \geq \text{inj}(N)/6$. It asserts that u is Lipschitz continuous on $B_{\frac{r_0}{32}}(x_0)$ with Lipschitz constant $C_1 := 6C/\text{inj}(N)$. The proof is finished. \square

REFERENCES

- [1] L. Ambrosio, N. Gigli, G. Savaré, *Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces*, Rev. Mat. Iberoam., 29 (2013), 969–996.
- [2] L. Ambrosio, N. Gigli, G. Savaré, *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below*, Invent. Math., 195(2) (2014), 289–391.
- [3] L. Ambrosio, A. Mondino, G. Savaré, *On the Bakry-Émery condition, the gradient estimates and the local-to-global property of $RCD^*(K, N)$ metric measure spaces*, J. Geom. Anal., 26(1) (2016), 24–56.
- [4] S. Alexander, V. Kapovitch, A. Petrunin, *Alexandrov geometry*, preprint, available at <https://arxiv.org/abs/1903.08539v1>.
- [5] F. Bethuel, *On the singular set of stationary harmonic maps*, Manuscripta Math. 78 (1993), no. 4, 41–443.
- [6] D. Burago, Y. Burago, S. Ivanov, *A Course in Metric Geometry*, Graduate Studies in Mathematics, vol. 33, AMS (2001).
- [7] Y. Burago, M. Gromov, G. Perelman, *A. D. Alexandrov spaces with curvatures bounded below*, Russian Math. Surveys 47 (1992), 1–58, MR1185284, Zbl 0802.53018.
- [8] S-Y. A. Chang, L. Wang & P. C. Yang, *Regularity of harmonic maps*, Comm. Pure Appl. Math., Vol. LII (1999), 1099–1111.
- [9] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*. Geom. Funct. Anal. 9, (1999), 428–517.
- [10] J. Cheeger, R. Haslhofer, A. Naber, *Quantitative stratification and the regularity of harmonic map flow*. Calc. Var. Partial Differential Equations 53 (2015), no. 1-2, 365–381.
- [11] J. Cheeger, A. Naber, *Quantitative stratification and the regularity of harmonic maps and minimal currents*. Comm. Pure Appl. Math. 66 (2013), no. 6, 965–990.
- [12] J. Chen, *On energy minimizing mappings between and into singular spaces*, Duke Math. J. 79 (1995), 77–99.
- [13] L. C. Evans, *Partial regularity for stationary harmonic maps into spheres*. Arch. Rational Mech. Anal. 116 (1991), no. 2, 101–113.
- [14] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Princeton Univ. Press (1983).
- [15] M. Giaquinta, S. Hildebrandt, *A priori estimates for harmonic mappings*. J. Reine Angew. Math. 336 (1982), 12–164.
- [16] N. Gigli, *On the differential structure of metric measure spaces and applications*, Mem. Amer. Math. Soc. 236 (1113) (2015).
- [17] N. Gigli, S. Mosconi, *The abstract Lewy-Stampacchia inequality and applications*, J. Math. Pures Appl. 104 (2) (2015) 258–275.
- [18] N. Gigli & A. Mondino, *A PDE approach to nonlinear potential theory*, J. Math. Pures Appl. 100 (4) (2013) 505–534.
- [19] A. Grigor’yan, J. Hu, *Heat kernels and Green functions on metric measure spaces*, Canad. J. Math., 66(3) (2014), 641–699.
- [20] M. Gromov, R. Schoen, *Harmonic maps into singular spaces and p -adic superrigidity for lattices in groups of rank one*, Publ. Math. IHES 76 (1992), 165–246.
- [21] P. Hajłasz, P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. 145(688), (2000), x–101.
- [22] R. M. Hardt, *Singularities of harmonic maps*, Bull. Amer. Math. Soc. 34 (1997), 15–34.
- [23] F. Hélein, *Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne* (French) [Regularity of weakly harmonic maps between a surface and a Riemannian manifold], C. R. Acad. Sci. Paris Sér. I Math. 312 (1991), no. 8, 59–596.

- [24] S. Hildebrandt, H. Kaul, K. Widman, *An existence theorem for harmonic mappings of Riemannian manifolds*. Acta Math. 138 (1977), no. 1-2, –16.
- [25] B. Hua, M. Kell & C. Xia, *Harmonic functions on metric measure spaces*, available at <http://arxiv.org/abs/1308.3607>.
- [26] T. Huang, C. Y. Wang, *Notes on the regularity of harmonic map systems*, Proc. Amer. Math. Soc., 138 (2010), 2015–2023.
- [27] W. Ishizuka, C. Y. Wang, *Harmonic maps from manifolds of L^∞ -Riemannian metrics*, Calc. Var. PDE, 32 (2008), 287–405.
- [28] R. Jiang, *Cheeger-harmonic functions in metric measure spaces revisited*, J. Funct. Anal., 266 (2014) 1373–1394.
- [29] R. Jiang, H. Li & H-C. Zhang, *Heat kernel bounds on metric measure spaces and some applications*, Potential Anal. (2016) 44, 601–627.
- [30] J. Jost, *Equilibrium maps between metric spaces*, Calc. Car. PDE 2 (1994), 173–204.
- [31] J. Jost, *Generalized Dirichlet forms and harmonic maps*, Calc. Var. PDE 5, (1997), 1–19.
- [32] J. Jost, *Riemannian Geometry and Geometric Analysis, seventh edition*, ISSN 0172-5939, Universitext, Springer International Publishing AG 2017.
- [33] N. Korevaar, R. Schoen, *Sobolev spaces and harmonic maps for metric space targets*, Comm. Anal. Geom. 1 (1993), 561–659.
- [34] K. Kuwae, Y. Machigashira, T. Shioya, *Sobolev spaces, Laplacian and heat kernel on Alexandrov spaces*, Math. Z. 238(2) (2001), 269–316.
- [35] H. Li, C. Wang, *Harmonic maps on domains with piecewise Lipschitz continuous metrics*, Pac. J. Math., 264(1), (2013), 125–149.
- [36] F. H. Lin, *Analysis on singular spaces*, Collection of papers on geometry, analysis and mathematical physics, 114–126, World Sci. Publ., River Edge, NJ, (1997).
- [37] F. H. Lin, *Gradient estimates and blow-up analysis for stationary harmonic maps*. Ann. of Math. (2) 149 (1999), no. 3, 785-829.
- [38] F. H. Lin, C. Y. Wang, *The analysis of harmonic maps and their heat flows*, World Scientific Publishing Co. Pte. Ltd., 2008.
- [39] A. Mondino and A. Naber, *Structure Theory of Metric-Measure Spaces with Lower Ricci Curvature Bounds*, J. Eur. Math. Soc. (JEMS) 21 (2019), no. 6, 1809-1854.
- [40] C. B. Morrey, *The problem of Plateau on an Riemannian manifold*, Ann. Math., 49. 807–851.
- [41] A. Naber, D. Valtorta, *Rectifiable-Reifenberg and the regularity of stationary and minimizing harmonic maps*. Ann. of Math. (2) 185 (2017), no. 1, 131-227.
- [42] Y. Otsu, T. Shioya, *The Riemannian structure of Alexandrov spaces*, J. Differ. Geom. 39 (1994), 629–658.
- [43] A. Pazy, *Semigroup of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
- [44] G. Perelman, *DC structure on Alexandrov spaces*. Preprint, preliminary version available online at www.math.psu.edu/petrinin/
- [45] A. Petrunin, *Harmonic functions on Alexandrov space and its applications*, ERA Amer. Math. Soc., 9 (2003), 135–141, MR2030174, Zbl 1071.53527.
- [46] R. Schoen, *Analytic aspects of the harmonic map problem*. Seminar on nonlinear partial differential equations (Berkeley, Calif., 1983), 321–358, Math. Sci. Res. Inst. Publ., 2, Springer, New York, 1984
- [47] R. Schoen, K. Uhlenbeck, *A regularity theory for harmonic maps*, J. Differ. Geom. 17, 307–335 (1982).
- [48] N. Shanmugalingam, *Newtonian spaces: An extension of Sobolev spaces to metric measure spaces*. Rev. Mat. Iberoam. 16,(2000), 243–279.
- [49] Y. G. Shi, *A partial regularity result of harmonic maps from manifolds with bounded measurable Riemannian metrics*, Comm. Anal. Geom. 4 (1996), 121–128.
- [50] K. Sturm, *Analysis on local Dirichlet spaces III, The parabolic Harnack inequality*. J. Math. Pure Appl., 75(3), (1996), 273–297.
- [51] H. C. Zhang, X. Zhong, X. P. Zhu, *Quantitative gradient estimates for harmonic maps into singular spaces*, Sci. China Math. (2019). <https://doi.org/10.1007/s11425-018-9493-1>
- [52] H. C. Zhang, X. P. Zhu, *Yau’s gradient estimates on Alexandrov spaces*, J. Differ. Geom., 91(3) (2012), 445–522.
- [53] H. C. Zhang, X. P. Zhu, *Lipschitz continuity of harmonic maps between Alexandrov spaces*, Invent. math. (2018) 211:863–934.

H. GE, SCHOOL OF MATHEMATICS, RENMIN UNIVERSITY OF CHINA, BEIJING, 100872, P.R. CHINA, E-MAIL ADDRESS: HBGE@RUC.EDU.CN

W. JIANG, SCHOOL OF MATHEMATICAL SCIENCES, ZHEJIANG UNIVERSITY, HANGZHOU, 310058, P.R. CHINA, E-MAIL ADDRESS: WSJIANG@ZJU.EDU.CN

SCHOOL OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF SYDNEY, NSW, 2006, AUSTRALIA.

H. C. ZHANG, DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU 510275, P.R. CHINA E-MAIL ADDRESS: ZHANGHC3@MAIL.SYSU.EDU.CN