

DIV-CURL CHARACTERIZATION OF SURFACE ORIENTATION

ZUOQIANG SHI AND BO ZHU

ABSTRACT. Consistent orientation of surface normals is a global problem in computational geometry. Orientation is intrinsically global: local geometry determines a normal line, but not which of its two directions is outward. We study the orientation problem at the continuum level for a smooth closed surface $\Gamma = \partial\Omega$. Starting from the Newtonian single-layer potential, we consider its averaged boundary divergence and curl traces, denoted by B and C . The classical scalar Gauss constraint $B\varphi = 1/2$ alone does not determine a free vector density. We show that adding the curl condition $C\varphi = 0$ makes the system complete. More precisely, for every $s \in \mathbb{R}$,

$$\|\varphi\|_{H^s(\Gamma; \mathbb{R}^3)} \leq C_{\Gamma, s} (\|B\varphi\|_{H^s(\Gamma)}^2 + \|C\varphi\|_{H^s(\Gamma; \mathbb{R}^3)}^2)^{1/2}.$$

$P = (B, C)$ is therefore injective with closed range, and $P\varphi = (1/2, 0)$ has the unique solution $\varphi = n_\Omega$. The Clifford Cauchy involution $T^2 = I/4$ is classical. The new result proved here is the same-order lower bound for $P = (B, C)$; its problem-specific proof step is the factorization $TM = JP$, which identifies B and C with the scalar and bivector parts of the same Cauchy singular integral. This factorization also yields a compatibility condition for general boundary data. If the samples are distributed according to a positive density $d\mu = \rho \, dS$, the weighted system recovers $q = n_\Omega/\rho$ uniquely, so normalization removes the unknown sampling density and returns the outward orientation. Regularized point-cloud experiments illustrate the continuum result.

1. INTRODUCTION

Normal orientation is global geometric information. Local fitting methods can estimate a tangent plane, and hence a normal *line*, from nearby samples, but local geometry alone cannot determine which of its two directions is outward. This distinction matters in downstream tasks. For example, Poisson surface reconstruction assumes consistently oriented point normals and uses the resulting global vector field as input [17]. The orientation problem is therefore not merely one of local differential estimation; it is an inside–outside problem for the sampled surface.

The computational literature treats orientation as a global problem. The classical reconstruction of Hoppe et al. combines local tangent-plane estimation with graph-based propagation of orientation [14]. Later work replaced greedy propagation by global graph energies or patch-level propagation, improving consistency on large or geometrically difficult point sets [30, 26]. Fu, Zheng, and Yu, for example, solve a least-squares sign problem on a Delaunay graph constructed from an implicit shell [9]. These methods separate local normal-line estimation from the global choice of signs.

Another class of methods couples orientation to an ambient implicit or volumetric field. Iterative Poisson surface reconstruction removes the oriented-input requirement by alternating reconstruction and normal updates [15], while Poisson formulations with isovalue constraints optimize normals together with an implicit function [33]. Generalized winding numbers provide an inside–outside representation for point-set geometry [2]; later orientation methods regularize the winding-number field or impose consistency between that field and its gradient [34, 19]. Stochastic formulations supply a further global inside–outside objective [16], including implementations designed for GPU scale and robustness [35].

Potential-theoretic Gauss formulations use closely related operators. Parametric Gauss Reconstruction treats oriented surface elements as unknowns in the classical Gauss formula [20]. Two recent extensions enrich the resulting linear information by modifying the ambient potential problem: convection-augmented Gauss reconstruction uses a family of convection-modified fundamental solutions to generate additional equations [23], whereas anisotropic Gauss reconstruction varies the elliptic coefficients to introduce additional constraints and obtain a stable linear system [22]. These developments expose an issue already

2020 *Mathematics Subject Classification.* Primary 65R20; Secondary 65D18, 30G35, 42B20, 58J32.

Key words and phrases. surface orientation, normal orientation, point cloud, div–curl system, boundary integral operator, Cauchy–Clifford operator, singular integral, sampling measure.

present in the classical Gauss formulation: a single scalar potential identity does not determine a free vector density. Our completion is different. We retain the Newtonian kernel and append the curl trace of the same vector single-layer potential. Completeness is therefore a property of one continuum boundary operator and requires neither auxiliary convection directions nor anisotropic coefficients.

Other methods also impose global constraints. Gotsman and Hormann derive sparse linear least-squares constraints from Stokes' theorem for precomputed normal lines [11]. For large, non-watertight scenes, divide-and-conquer methods explicitly optimize relative orientations between spatial blocks [18]. Wavelet-based reconstruction has also combined an implicit indicator representation with divergence-free constraints for sparse unoriented point clouds [21]. These works provide several global mechanisms—graph propagation and optimization, Poisson or implicit reconstruction, winding-number and stochastic inside–outside models, Stokes or Delaunay-graph formulations, and enriched Gauss systems—with increasing emphasis on noise, sparsity, scale, and incomplete geometry.

Here we address a complementary question. Instead of starting from a discrete sign variable, a modified family of potentials, or a volumetric reconstruction, we ask whether a canonical *linear* boundary operator can characterize the outward orientation of a smooth closed surface when its unknown is an unrestricted vector density, and whether that characterization is quantitatively stable. This formulation separates the continuum problem from local normal-line estimation and discrete sign propagation. The answer is affirmative once the scalar Gauss equation is supplemented by the curl trace of the same vector single-layer potential: the scalar block is noninjective on free vector fields, whereas the joint div–curl operator admits a global lower bound. The proof is independent of any graph or discretization and instead uses the Cauchy singular integral for the three-dimensional Clifford algebra.

Let $\Omega \subset \mathbb{R}^3$ be bounded with smooth boundary Γ , and let n_Ω denote the outward unit normal. Suppose that the samples are distributed according to a positive surface measure

$$(1) \quad d\mu = \rho \, dS, \quad 0 < \rho_{\min} \leq \rho \leq \rho_{\max} < \infty.$$

The density ρ need not be known. The natural linear unknown is the scaled normal

$$(2) \quad q_* = \frac{n_\Omega}{\rho}.$$

Its magnitude absorbs the sampling density, whereas its direction is the desired orientation:

$$(3) \quad \frac{q_*}{\|q_*\|} = n_\Omega.$$

Equivalently, $q_* \, d\mu = n_\Omega \, dS$ is the vector area measure.

With the Newtonian kernel

$$(4) \quad \Phi(z) = \frac{1}{4\pi|z|}, \quad -\Delta\Phi = \delta_0,$$

define, for a vector density q integrated against μ ,

$$(5) \quad (B_\mu q)(x) = \text{p. v.} \int_\Gamma \nabla_x \Phi(x-y) \cdot q(y) \, d\mu(y),$$

$$(6) \quad (C_\mu q)(x) = \text{p. v.} \int_\Gamma \nabla_x \Phi(x-y) \times q(y) \, d\mu(y).$$

Our geometric characterization is

$$(7) \quad B_\mu q = \frac{1}{2}, \quad C_\mu q = 0,$$

with the unique solution $q = q_*$. The system contains neither n_Ω nor ρ explicitly. It depends only on the support Γ , the sampling measure μ , and the translation-invariant Newtonian kernel. The outward unit normal is obtained by solving the linear system and normalizing the result.

The proof separates a classical analytic input from the problem-specific structural step. With the normal multiplier $M\varphi = -n_\Omega\varphi$ and the grade isometry $J(b, c) = -b + *c$, the boundary Cauchy singular integral T satisfies

$$(8) \quad TM = JP, \quad T^2 = \frac{1}{4}I.$$

The identity $T^2 = I/4$ and the Sobolev boundedness of T are classical consequences of Calderón–Cauchy theory. The step specific to the present problem is the exact factorization $TM = JP$: it identifies $B\varphi$ and $C\varphi$ as the scalar and bivector components of a single Cauchy singular integral. Such grade decompositions belong to the standard Clifford framework. The new assertion is that this particular joint boundary operator has a same-order lower bound. Combining the factorization with the classical involution transfers the invertibility of T to a same-order Sobolev lower bound for P . This is the main new estimate; injectivity, closed range, exact orientation recovery, and the compatibility and least-squares formulas follow from it. The same-order form is stronger than uniqueness: it gives a bounded inverse on the range without loss of Sobolev derivatives. The normal appears only in the auxiliary map M used in the proof, not in the definition of P or in the orientation equation. The classical analytic input comes from boundary singular-integral and Clifford/Dirac theory [5, 24, 25, 27, 31], together with standard pseudodifferential mapping theory [6, 12, 29].

The main results are summarized as follows.

- (i) **Same-order stability.** Theorem 3.1 proves, for every $s \in \mathbb{R}$,

$$\|\varphi\|_{H^s(\Gamma; \mathbb{R}^3)} \leq C_{\Gamma, s} (\|B\varphi\|_{H^s}^2 + \|C\varphi\|_{H^s}^2)^{1/2}.$$

Hence $P = (B, C)$ is injective and has closed range. The estimate has the same Sobolev order as the data and follows by combining the problem-specific factorization $TM = JP$ with the classical identity $T^2 = I/4$.

- (ii) **Range and compatibility.** Proposition 3.2 shows that $\text{Ran } P$ is a proper closed subspace of the data space, gives an explicit compatibility condition through the Cauchy operator, and provides an inverse formula on compatible data. The system is stable and overdetermined.
- (iii) **Exact recovery of orientation.** Corollary 3.4 proves that

$$B\varphi = \frac{1}{2}, \quad C\varphi = 0$$

has the unique solution $\varphi = n_\Omega$. Under nonuniform sampling, $P_\mu = PM_\rho$ therefore gives the unique weighted solution $q = n_\Omega/\rho$, whose normalization is n_Ω . The result applies to the complete smooth boundary of a bounded domain without introducing a topology-dependent nullspace, and in particular covers disconnected boundary components and cavities.

- (iv) **Least-squares recovery for incompatible data.** Theorem 3.5 proves existence and uniqueness of the least-squares solution for arbitrary prescribed boundary data, identifies it with the pseudoinverse $P_s^\dagger$, and shows that the residual is exactly the component orthogonal to $\text{Ran } P$. The reconstruction is Lipschitz stable with norm $m_{\Gamma, s}^{-1}$.

The results are at the continuum level. Our contribution is a uniqueness and stability theorem for a canonical boundary operator, together with an explanation of why the scalar Gauss constraint becomes complete after its curl companion is included. A structure-preserving discretization and its convergence theory remain separate problems.

Section 2 defines the Newtonian single-layer potential and the averaged div–curl boundary operators, establishes their jump relations and Sobolev boundedness, and derives the data generated by n_Ω . Section 3 proves the Clifford factorization, stability estimate, range characterization, orientation recovery, and least-squares inverse. Section 4 presents the point-cloud experiments. The appendices collect the potential-theoretic and Clifford-algebra facts used in the proofs.

2. BOUNDARY DIV–CURL OPERATORS AND THE ORIENTATION PROBLEM

2.1. Geometry and the Newtonian single-layer potential. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^∞ boundary

$$\Gamma = \partial\Omega = \bigsqcup_{j=1}^M \Sigma_j,$$

where each Σ_j is a compact connected embedded surface. The components may have arbitrary finite genus and may occur as the boundaries of nested cavities. The domain determines a canonical orientation of Γ , namely its outward unit normal n_Ω . Our aim is to characterize n_Ω as the unique solution of a boundary integral system with prescribed divergence and curl data.

Sobolev spaces on the disconnected surface are interpreted componentwise; for example,

$$H^s(\Gamma) = \bigoplus_{j=1}^M H^s(\Sigma_j).$$

The smoothness assumption allows arbitrary Sobolev order $s \in \mathbb{R}$: n_Ω is smooth, and the boundary singular integral operators used below act boundedly on every H^s . For a boundary of finite regularity, both the trace statements and the admissible range of s require corresponding restrictions.

The construction starts from the potential whose first derivatives determine the boundary divergence and curl values.

Let

$$\Phi(z) = \frac{1}{4\pi|z|}, \quad -\Delta\Phi = \delta_0,$$

be the Newtonian kernel in $\mathbb{R}^3$. For a smooth vector density $\varphi \in C^\infty(\Gamma; \mathbb{R}^3)$, define its componentwise single-layer potential by

$$(9) \quad (A\varphi)(x) = \int_{\Gamma} \Phi(x-y)\varphi(y) dS_y, \quad x \in \mathbb{R}^3 \setminus \Gamma.$$

Here x is the observation point and $y \in \Gamma$ is the integration variable. The kernel has an integrable singularity on the two-dimensional surface, so the potential itself has a well-defined continuous boundary value.

Lemma 2.1 (Continuity of the Newtonian single-layer potential). *The function $A\varphi$ extends continuously from $\mathbb{R}^3 \setminus \Gamma$ to all of $\mathbb{R}^3$. In particular, its two traces on Γ agree and are given by*

$$(A\varphi)(x) = \int_{\Gamma} \Phi(x-y)\varphi(y) dS_y, \quad x \in \Gamma.$$

The proof is given in Appendix A.

2.2. Averaged div–curl traces and Sobolev boundedness. Although $A\varphi$ is continuous across Γ , its first derivatives generally have different interior and exterior traces. We record these derivatives through the scalar divergence and the vector curl. For $x \notin \Gamma$, differentiation under the integral sign gives

$$(10) \quad \operatorname{div}(A\varphi)(x) = \int_{\Gamma} \nabla_x \Phi(x-y) \cdot \varphi(y) dS_y,$$

$$(11) \quad \operatorname{curl}(A\varphi)(x) = \int_{\Gamma} \nabla_x \Phi(x-y) \times \varphi(y) dS_y.$$

These identities motivate the scalar boundary operator B and the vector-valued boundary operator C . There is, however, a stronger singularity at the boundary: whereas $\Phi(x-y) = O(|x-y|^{-1})$ is locally integrable on the surface, $\nabla_x \Phi(x-y) = O(|x-y|^{-2})$ is not absolutely integrable there. Consequently, the boundary values of (10)–(11) must be interpreted through cancellation.

Fix $x \in \Gamma$. Since $n_\Omega(x)$ points out of Ω , the points $x - tn_\Omega(x)$ lie inside Ω and the points $x + tn_\Omega(x)$ lie outside $\overline{\Omega}$ for all sufficiently small $t > 0$. The candidate interior and exterior divergence traces are therefore

$$\begin{aligned} (\operatorname{div} A\varphi)_{\text{int}}(x) &= \lim_{t \downarrow 0} \operatorname{div}(A\varphi)(x - tn_\Omega(x)), \\ (\operatorname{div} A\varphi)_{\text{ext}}(x) &= \lim_{t \downarrow 0} \operatorname{div}(A\varphi)(x + tn_\Omega(x)). \end{aligned}$$

The corresponding curl traces are

$$\begin{aligned} (\operatorname{curl} A\varphi)_{\text{int}}(x) &= \lim_{t \downarrow 0} \operatorname{curl}(A\varphi)(x - tn_\Omega(x)), \\ (\operatorname{curl} A\varphi)_{\text{ext}}(x) &= \lim_{t \downarrow 0} \operatorname{curl}(A\varphi)(x + tn_\Omega(x)). \end{aligned}$$

The terms *interior* and *exterior* specify the side from which x is approached. These formulas define the intended limits but do not assume their existence. The next lemma proves existence. The classical jump theorem in fact gives nontangential limits, which is stronger than the normal approach used here [25, Theorem 6.11, pp. 202–205].

Lemma 2.2 (Jump relations for the div–curl traces). *For $\varphi \in C^\infty(\Gamma; \mathbb{R}^3)$, the four traces defined above exist at every $x \in \Gamma$. With the principal value understood as*

$$\text{p. v.} \int_{\Gamma} K(x, y) dS_y := \lim_{\varepsilon \downarrow 0} \int_{\Gamma \setminus B_\varepsilon(x)} K(x, y) dS_y,$$

define

$$(12) \quad (B\varphi)(x) = \text{p. v.} \int_{\Gamma} \nabla_x \Phi(x - y) \cdot \varphi(y) dS_y,$$

$$(13) \quad (C\varphi)(x) = \text{p. v.} \int_{\Gamma} \nabla_x \Phi(x - y) \times \varphi(y) dS_y.$$

Then

$$(14) \quad (\text{div } A\varphi)_{\text{int}} = B\varphi + \frac{1}{2}n_\Omega \cdot \varphi, \quad (\text{div } A\varphi)_{\text{ext}} = B\varphi - \frac{1}{2}n_\Omega \cdot \varphi,$$

$$(15) \quad (\text{curl } A\varphi)_{\text{int}} = C\varphi + \frac{1}{2}n_\Omega \times \varphi, \quad (\text{curl } A\varphi)_{\text{ext}} = C\varphi - \frac{1}{2}n_\Omega \times \varphi.$$

In particular, $B\varphi$ and $C\varphi$ are the arithmetic means of the corresponding interior and exterior traces:

$$(16) \quad B\varphi = \frac{1}{2}((\text{div } A\varphi)_{\text{int}} + (\text{div } A\varphi)_{\text{ext}}), \quad C\varphi = \frac{1}{2}((\text{curl } A\varphi)_{\text{int}} + (\text{curl } A\varphi)_{\text{ext}}).$$

The proof is given in Appendix A.

Lemma 2.3 (Smooth boundary values and the cross-boundary jump). *Let $\varphi \in C^\infty(\Gamma; \mathbb{R}^3)$. Then the common trace*

$$S\varphi := A\varphi|_\Gamma$$

belongs to $C^\infty(\Gamma; \mathbb{R}^3)$. The one-sided gradient traces are also smooth matrix-valued functions on Γ and satisfy

$$(17) \quad \gamma_{\text{int}}(\nabla A\varphi) - \gamma_{\text{ext}}(\nabla A\varphi) = n_\Omega \otimes \varphi, \quad (n_\Omega \otimes \varphi)_{ij} = (n_\Omega)_i \varphi_j.$$

Consequently, $A\varphi$ is smooth along Γ but, at every point $x \in \Gamma$ for which $\varphi(x) \neq 0$, it is not C^1 across Γ .

The proof is given in Appendix A.

The operators B and C are the averaged boundary divergence and curl of the potential. Neither operator changes Sobolev order.

Proposition 2.4 (Sobolev boundedness of B and C). *For every $s \in \mathbb{R}$, the operators initially defined on smooth densities extend uniquely to bounded maps*

$$\begin{aligned} B &: H^s(\Gamma; \mathbb{R}^3) \longrightarrow H^s(\Gamma), \\ C &: H^s(\Gamma; \mathbb{R}^3) \longrightarrow H^s(\Gamma; \mathbb{R}^3). \end{aligned}$$

Equivalently, there is a constant $M_{\Gamma, s} > 0$ such that

$$(18) \quad \|B\varphi\|_{H^s(\Gamma)} + \|C\varphi\|_{H^s(\Gamma; \mathbb{R}^3)} \leq M_{\Gamma, s} \|\varphi\|_{H^s(\Gamma; \mathbb{R}^3)}$$

for every $\varphi \in H^s(\Gamma; \mathbb{R}^3)$.

The proof is given in Appendix A.

2.3. Normal boundary values and the orientation equation. For $s \in \mathbb{R}$, write

$$Y^s = H^s(\Gamma) \times H^s(\Gamma; \mathbb{R}^3), \quad P\varphi = (B\varphi, C\varphi).$$

Proposition 2.4 therefore gives a bounded map

$$P : H^s(\Gamma; \mathbb{R}^3) \longrightarrow Y^s.$$

The formulas defining B , C , and P involve the Euclidean kernel, surface measure, and the unknown density φ , but no chosen normal field. The outward normal enters only in the analysis of the solution. Its boundary divergence and curl determine the right-hand side of the orientation equation.

Lemma 2.5 (Div–curl values generated by the outward normal). *For every $s \in \mathbb{R}$, the outward unit normal satisfies $n_\Omega \in H^s(\Gamma; \mathbb{R}^3)$ and*

$$(19) \quad Bn_\Omega = \frac{1}{2}, \quad Cn_\Omega = 0; \quad \text{equivalently,} \quad Pn_\Omega = \left(\frac{1}{2}, 0\right).$$

Here $1/2$ denotes the constant scalar function on every component of Γ .

Proof. Smoothness of Γ implies that n_Ω is smooth, and hence $n_\Omega \in H^s(\Gamma; \mathbb{R}^3)$ for every $s \in \mathbb{R}$. Define

$$q(z) = \int_\Omega \Phi(z - y) dy.$$

Here $z \in \mathbb{R}^3 \setminus \Gamma$ is the fixed observation point, while y is always the integration variable: $y \in \Gamma$ in a surface integral and $y \in \Omega$ in a volume integral. Boundary traces will be obtained only afterward by letting $z \rightarrow x$ for a fixed $x \in \Gamma$.

For the surface-to-volume conversion, let e_j be the j -th coordinate vector. If $z \notin \overline{\Omega}$, the vector field $y \mapsto \Phi(z - y)e_j$ is smooth on $\overline{\Omega}$, and the divergence theorem gives

$$(20) \quad \begin{aligned} \int_\Gamma \Phi(z - y)(n_\Omega)_j(y) dS_y &= \int_\Gamma (\Phi(z - y)e_j) \cdot n_\Omega(y) dS_y \\ &= \int_\Omega \operatorname{div}_y (\Phi(z - y)e_j) dy \\ &= \int_\Omega \partial_{y_j} \Phi(z - y) dy. \end{aligned}$$

If $z \in \Omega$, apply the same argument to $\Omega \setminus \overline{B_\varepsilon(z)}$. The additional boundary integral over $\partial B_\varepsilon(z)$ vanishes, because $\Phi(z - y) = (4\pi\varepsilon)^{-1}$ is constant on the sphere and the integral of its unit normal is zero:

$$\int_{\partial B_\varepsilon(z)} \Phi(z - y)v_j(y) dS_y = \frac{1}{4\pi\varepsilon} \int_{\partial B_\varepsilon(z)} v_j(y) dS_y = 0.$$

Since $|\nabla_y \Phi(z - y)| = O(|z - y|^{-2})$ is locally integrable in three dimensions, letting $\varepsilon \downarrow 0$ proves (20) also for $z \in \Omega$.

Collecting the three component identities gives

$$\int_\Gamma \Phi(z - y)n_\Omega(y) dS_y = \int_\Omega \nabla_y \Phi(z - y) dy.$$

Finally, $\nabla_y \Phi(z - y) = -\nabla_z \Phi(z - y)$; hence

$$\begin{aligned} (An_\Omega)(z) &= \int_\Omega \nabla_y \Phi(z - y) dy \\ &= - \int_\Omega \nabla_z \Phi(z - y) dy = -\nabla_z q(z), \quad z \notin \Gamma. \end{aligned}$$

Consequently,

$$\operatorname{curl}(An_\Omega) = 0 \quad \text{in } \mathbb{R}^3 \setminus \Gamma.$$

Since $-\Delta \Phi = \delta_0$, convolution with the characteristic function of Ω gives, for $z \notin \Gamma$,

$$\operatorname{div}(An_\Omega) = -\Delta q = (-\Delta \Phi) * \chi_\Omega = \chi_\Omega = \begin{cases} 1, & z \in \Omega, \\ 0, & z \in \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases}$$

Now fix $x \in \Gamma$ and let $z \rightarrow x$ first from Ω and then from $\mathbb{R}^3 \setminus \overline{\Omega}$. The preceding identities give

$$(\operatorname{div} An_\Omega)_{\text{int}}(x) = 1, \quad (\operatorname{div} An_\Omega)_{\text{ext}}(x) = 0,$$

and

$$(\operatorname{curl} An_\Omega)_{\text{int}}(x) = (\operatorname{curl} An_\Omega)_{\text{ext}}(x) = 0.$$

Substitution into the averaged-trace formulas (16) yields

$$Cn_\Omega = \frac{1}{2}(0 + 0) = 0, \quad Bn_\Omega = \frac{1}{2}(1 + 0) = \frac{1}{2}.$$

This proves (19). □

The inverse problem is to recover n_Ω from the boundary values in (19):

$$(21) \quad \text{find } \varphi \in H^s(\Gamma; \mathbb{R}^3) \quad \text{such that} \quad P\varphi = \left(\frac{1}{2}, 0\right).$$

The lemma proves existence. The a priori estimate below proves that no second density can have the same averaged divergence and curl values.

3. STABILITY, RANGE, AND EXACT RECOVERY

For $s \in \mathbb{R}$, set

$$X^s = H^s(\Gamma; \mathbb{R}^3), \quad Y^s = H^s(\Gamma) \times H^s(\Gamma; \mathbb{R}^3), \quad P = (B, C) : X^s \rightarrow Y^s.$$

Section 2 identified the outward normal as a solution of the orientation equation. We prove uniqueness and determine the range of the div–curl operator

$$P = (B, C) : X^s \longrightarrow Y^s.$$

These are distinct questions: the a priori estimate makes P injective with closed range, whereas the range calculation shows that P is not surjective. Exact recovery applies to compatible right-hand sides; arbitrary right-hand sides are treated by least squares in Subsection 3.3.

The proof uses one classical fact and one new identification. The classical fact is that the boundary Clifford Cauchy operator T is bounded on every Sobolev space and satisfies $T^2 = I/4$. The new identification is the exact factorization

$$TM = JP,$$

where M encodes a vector density as an even Clifford field and J encodes its div–curl data. This factorization is the step that converts the classical Cauchy involution into the lower bound for P .

3.1. A priori estimate for P .

Theorem 3.1 (A priori estimate). *For every $s \in \mathbb{R}$, there is a constant $C_{\Gamma,s} > 0$ such that*

$$(22) \quad \|\varphi\|_{H^s(\Gamma; \mathbb{R}^3)} \leq C_{\Gamma,s} \left(\|B\varphi\|_{H^s(\Gamma)}^2 + \|C\varphi\|_{H^s(\Gamma; \mathbb{R}^3)}^2 \right)^{1/2}$$

for every $\varphi \in H^s(\Gamma; \mathbb{R}^3)$.

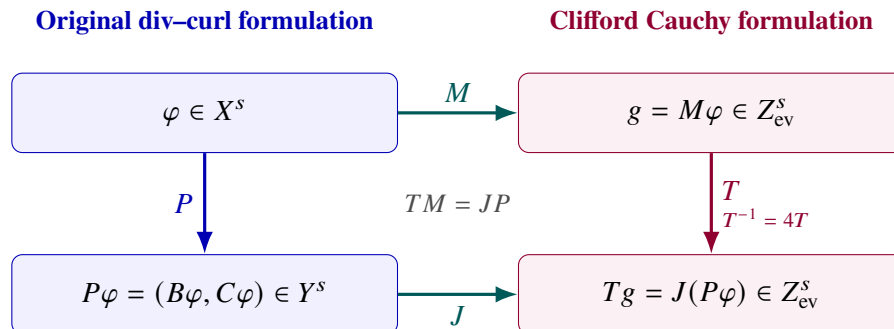
Proof. Set

$$X^s = H^s(\Gamma; \mathbb{R}^3), \quad Y^s = H^s(\Gamma) \times H^s(\Gamma; \mathbb{R}^3), \quad Z^s = H^s(\Gamma; \text{Cl}_3),$$

where Cl_3 is the real Clifford algebra of $\mathbb{R}^3$, and let

$$Z_{\text{ev}}^s := H^s(\Gamma; \Lambda^0 \mathbb{R}^3 \oplus \Lambda^2 \mathbb{R}^3) \subset Z^s$$

be the even-grade subspace. The proof compares the original operator $P : X^s \rightarrow Y^s$ with the Clifford Cauchy singular integral $T : Z^s \rightarrow Z^s$. The maps M and J , defined in Step 1 below, convert the unknown φ and the pair $(B\varphi, C\varphi)$ into Clifford-valued fields. The comparison is summarized by the commutative diagram below.



The left vertical arrow is the operator whose stability is to be proved. The upper horizontal arrow sends the vector density φ to the even Clifford field $M\varphi$, whereas the lower horizontal arrow combines the scalar field $B\varphi$ and the vector field $C\varphi$ into the even Clifford field $J(B\varphi, C\varphi)$. Commutativity means that the two routes from the upper-left corner to the lower-right corner agree:

$$T(M\varphi) = J(P\varphi).$$

The proof establishes that M has a bounded inverse on its range, J is an isometry, and $T^{-1} = 4T$. It then proves the factorization $TM = JP$, which reduces the estimate for P to the bounded invertibility of T .

The normal n_Ω appears only in the auxiliary map M used in this proof. Neither the definition of P nor the orientation equation requires a previously chosen normal field.

Step 1: Clifford conventions and the horizontal maps. The algebraic and Sobolev details are proved in Appendix B.2. Recall the orthogonal decomposition

$$\text{Cl}_3 = \Lambda^0 \mathbb{R}^3 \oplus \Lambda^1 \mathbb{R}^3 \oplus \Lambda^2 \mathbb{R}^3 \oplus \Lambda^3 \mathbb{R}^3.$$

For vectors $v, w \in \mathbb{R}^3$, write

$$vw = -v \cdot w + v \wedge w, \quad *(v \wedge w) = v \times w.$$

The Clifford-valued Sobolev norm is taken componentwise with respect to this decomposition.

The upper horizontal map. For $\varphi \in X^s$, define

$$(M\varphi)(x) := -n_\Omega(x)\varphi(x).$$

The order matters:

$$\begin{aligned} -n_\Omega \varphi &= n_\Omega \cdot \varphi - n_\Omega \wedge \varphi, \\ -\varphi n_\Omega &= n_\Omega \cdot \varphi + n_\Omega \wedge \varphi. \end{aligned}$$

Reversing the factors changes the sign of the bivector component. Proposition B.1 proves that $M\varphi$ is even, that M is an L^2 isometry, and that for every $s \in \mathbb{R}$,

$$\begin{aligned} M : X^s &\longrightarrow Z_{\text{ev}}^s && \text{is bounded and injective,} \\ n_\Omega(M\varphi) &= \varphi, && \|\varphi\|_{X^s} \leq C_{M,s} \|M\varphi\|_{Z^s}. \end{aligned}$$

The lower horizontal map. Define

$$J : Y^s \longrightarrow Z_{\text{ev}}^s, \quad J(b, c) = -b + *c.$$

Here $-b$ is scalar and $*c$ is a bivector. These grades are orthogonal, and the Hodge star is an isometry. Hence

$$\begin{aligned} \|J(b, c)\|_{Z^s}^2 &= \|-b\|_{H^s(\Gamma)}^2 + \|\ast c\|_{H^s(\Gamma; \Lambda^2 \mathbb{R}^3)}^2 \\ &= \|b\|_{H^s(\Gamma)}^2 + \|c\|_{H^s(\Gamma; \mathbb{R}^3)}^2 = \|(b, c)\|_{Y^s}^2. \end{aligned}$$

Thus J is an isometry.

Step 2: the classical Clifford Cauchy involution. The Clifford algebra and Cauchy formulas used below are collected in Subsections B.3 and B.4. Set

$$\Omega_{\text{ext}} := \mathbb{R}^3 \setminus \overline{\Omega}.$$

Definition of the boundary operator. Let e_1, e_2, e_3 be the standard Clifford generators and define

$$\mathcal{D} = \sum_{j=1}^3 e_j \partial_j.$$

The relations $e_i e_j + e_j e_i = -2\delta_{ij}$ give

$$\begin{aligned} \mathcal{D}^2 &= \sum_{j=1}^3 e_j^2 \partial_j^2 + \sum_{1 \leq i < j \leq 3} (e_i e_j + e_j e_i) \partial_i \partial_j \\ &= -\sum_{j=1}^3 \partial_j^2 = -\Delta. \end{aligned}$$

For $\Phi(z) = (4\pi|z|)^{-1}$, set

$$E(z) = \mathcal{D}\Phi(z) = \nabla\Phi(z) = -\frac{z}{4\pi|z|^3}.$$

Since $-\Delta\Phi = \delta_0$,

$$\mathcal{D}E = \mathcal{D}^2\Phi = (-\Delta)\Phi = \delta_0.$$

Thus E is the fundamental solution of $\mathcal{D}$ for the present sign convention. Define

$$(23) \quad T : C^\infty(\Gamma; \text{Cl}_3) \longrightarrow C^\infty(\Gamma; \text{Cl}_3),$$

$$(Tg)(x) = \text{p. v.} \int_{\Gamma} E(x-y)n_{\Omega}(y)g(y) dS_y, \quad x \in \Gamma.$$

The factors in (23) must remain in the displayed order because Clifford multiplication is not commutative. The kernel has the critical singularity $|x-y|^{-2}$ on the two-dimensional surface. Proposition B.2 proves that the principal value exists and defines a continuous map from $C^\infty(\Gamma; \text{Cl}_3)$ to itself. As in Lemma 2.3, this is intrinsic smoothness on Γ ; it does not assert that the associated Cauchy field is smooth across Γ .

Claim. For every $s \in \mathbb{R}$, T extends uniquely to a bounded operator

$$(24) \quad T : Z^s = H^s(\Gamma; \text{Cl}_3) \longrightarrow Z^s, \quad T^2 = \frac{1}{4}I.$$

Consequently, T is boundedly invertible and $T^{-1} = 4T$.

Proof of the claim: the Cauchy transform and its trace. First take $g \in C^\infty(\Gamma; \text{Cl}_3)$ and define

$$C : C^\infty(\Gamma; \text{Cl}_3) \longrightarrow C^\infty(\mathbb{R}^3 \setminus \Gamma; \text{Cl}_3)$$

by the off-surface integral

$$(Cg)(x) = \int_{\Gamma} E(x-y)n_{\Omega}(y)g(y) dS_y, \quad x \in \mathbb{R}^3 \setminus \Gamma.$$

A Clifford-valued field u on an open set U is called *left monogenic* if $\mathcal{D}u = 0$ in U . Denote the space of such fields by

$$\mathcal{M}(U; \text{Cl}_3) := \{u \in C^\infty(U; \text{Cl}_3) : \mathcal{D}u = 0\}.$$

If $x \notin \Gamma$, then $\text{dist}(x, \Gamma) > 0$. The kernel is therefore smooth near x , differentiation under the integral is valid, and $\mathcal{D}_x E(x-y) = 0$ for $x \neq y$. Hence

$$\mathcal{D}(Cg)(x) = \int_{\Gamma} \mathcal{D}_x E(x-y)n_{\Omega}(y)g(y) dS_y = 0.$$

More explicitly,

$$\mathcal{D}(Cg) = 0 \quad \text{in } \Omega, \quad \mathcal{D}(Cg) = 0 \quad \text{in } \Omega_{\text{ext}}.$$

Equivalently,

$$C : C^\infty(\Gamma; \text{Cl}_3) \longrightarrow \mathcal{M}(\Omega; \text{Cl}_3) \oplus \mathcal{M}(\Omega_{\text{ext}}; \text{Cl}_3).$$

Hence Cg is monogenic separately in the interior and exterior; the two fields generally have different traces on Γ .

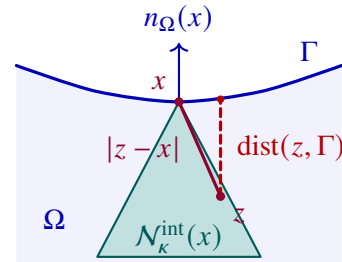
The definition of the interior trace and its geometric interpretation are displayed side by side.

Fix an aperture $\kappa > 1$. For $x \in \Gamma$, define

$$\mathcal{N}_{\kappa}^{\text{int}}(x) := \{z \in \Omega : |z-x| < \kappa \text{dist}(z, \Gamma)\}.$$

Here $\text{dist}(z, \Gamma)$ is the shortest Euclidean distance from z to the boundary. The inequality prevents z from approaching x tangentially: z must remain in an interior cone with vertex x . For a field $u : \Omega \rightarrow \text{Cl}_3$, its interior nontangential trace is defined at those $x \in \Gamma$ for which the limit on the right-hand side exists:

$$(\gamma_{\text{int}}u)(x) := \lim_{\substack{z \rightarrow x \\ z \in \mathcal{N}_{\kappa}^{\text{int}}(x)}} u(z).$$



Two-dimensional cross-section; the teal region is the admissible nontangential approach region.

Let $\mathcal{T}_k^{\text{int}}(\Omega; \text{Cl}_3)$ denote the class of fields $u : \Omega \rightarrow \text{Cl}_3$ for which this limit exists at every $x \in \Gamma$. Thus

$$\gamma_{\text{int}} : \mathcal{T}_k^{\text{int}}(\Omega; \text{Cl}_3) \longrightarrow \text{Map}(\Gamma, \text{Cl}_3).$$

No continuity or Sobolev regularity of the boundary field is part of this definition; such regularity follows from the trace theorem being applied.

Because n_Ω is the outward normal, $x - tn_\Omega(x) \in \Omega$ for all sufficiently small $t > 0$, and this normal path is nontangential. Hence

$$(\gamma_{\text{int}}u)(x) = \lim_{t \downarrow 0} u(x - tn_\Omega(x))$$

whenever the nontangential trace exists.

Plemelj trace and Cauchy reproduction. Two formulas are needed, with different domains. The Plemelj–Sokhotski formula takes a density on Γ to a boundary trace. The Cauchy reproduction formula reconstructs a monogenic field in Ω from its trace, subject to nontangential L^2 bounds.

1. Interior Plemelj–Sokhotski trace formula. Let $g \in C^\infty(\Gamma; \text{Cl}_3)$. The successive objects are

$$\boxed{\begin{array}{ccccc} g & \xrightarrow{C} & (Cg)|_\Omega & \xrightarrow{\gamma_{\text{int}}} & \gamma_{\text{int}}(Cg) \\ \in C^\infty(\Gamma; \text{Cl}_3) & & \in \mathcal{M}(\Omega; \text{Cl}_3) & & \in C^\infty(\Gamma; \text{Cl}_3) \end{array}}.$$

The field $(Cg)|_\Omega$ is monogenic. Its interior nontangential trace exists, belongs to $C^\infty(\Gamma; \text{Cl}_3)$, and satisfies

$$(25) \quad \gamma_{\text{int}}(Cg) = \left(-\frac{1}{2}I + T \right) g.$$

Here Tg is the principal-value integral in (23), while $-\frac{1}{2}g$ is the local jump term associated with approach from Ω . Thus

$$\gamma_{\text{int}}C : C^\infty(\Gamma; \text{Cl}_3) \longrightarrow C^\infty(\Gamma; \text{Cl}_3), \quad g \longmapsto \left(-\frac{1}{2}I + T \right) g.$$

This is the Clifford Plemelj–Sokhotski formula. It is stated in [7, formula (36), p. 8]; a general Dirac-system version is [27, formula (2.3.23), p. 679]. Those references write the sign according to their kernel and symbol conventions; translating to $E = \nabla\Phi$ and the outward normal gives (25).¹

2. Cauchy reproduction formula. Recall that $\mathcal{M}(\Omega; \text{Cl}_3)$ is the space of smooth fields in Ω satisfying $\mathcal{D}u = 0$. Monogenicity alone is not the full hypothesis of the reproduction theorem. For $u \in \mathcal{M}(\Omega; \text{Cl}_3)$, define its interior nontangential maximal function by

$$N_\kappa u(x) := \sup_{z \in \mathcal{N}_\kappa^{\text{int}}(x)} |u(z)|, \quad x \in \Gamma.$$

Define the Hardy–Dirac class

$$\mathcal{H}_\mathcal{D}^2(\Omega) := \{u \in \mathcal{M}(\Omega; \text{Cl}_3) : N_\kappa u \in L^2(\Gamma), \quad \gamma_{\text{int}}u \text{ exists a.e. and belongs to } L^2(\Gamma; \text{Cl}_3)\}.$$

This is the $p = 2$ case of the Hardy space in [27, formula (3.0.1), p. 681]. Every monogenic $u \in C^\infty(\overline{\Omega}; \text{Cl}_3)$ belongs to this class, but an arbitrary element of $\mathcal{M}(\Omega; \text{Cl}_3)$ need not. For $u \in \mathcal{H}_\mathcal{D}^2(\Omega)$, the relevant maps are summarized by

$$\boxed{\begin{array}{ccccc} u & \xrightarrow{\gamma_{\text{int}}} & \gamma_{\text{int}}u & \xrightarrow{-C} & -C(\gamma_{\text{int}}u)|_\Omega = u \\ \in \mathcal{H}_\mathcal{D}^2(\Omega) & & \in L^2(\Gamma; \text{Cl}_3) & & \in \mathcal{H}_\mathcal{D}^2(\Omega) \end{array}}.$$

The Cauchy transform reproduces the original field:

$$(26) \quad -C(\gamma_{\text{int}}u)(x) = u(x), \quad x \in \Omega.$$

Equation (26) is a quoted representation theorem, not a consequence of monogenicity alone and not a boundary-limit formula. In the notation of [27, formula (3.0.5), p. 681], it is $u = C_D(u|_{\partial\Omega})$. That identity

¹The sign can also be checked directly on the unit sphere. At its center $x = 0$, the outward normal is $n(y) = y$ for $|y| = 1$, and hence

$$(C1)(0) = \int_{S^2} E(-y)y \, dS_y = \int_{S^2} \frac{y^2}{4\pi} \, dS_y = -1.$$

Here $y^2 = -|y|^2 = -1$ in Cl_3 . Thus, with our kernel convention, the Cauchy transform reproduces the constant field with a minus sign in the interior.

follows there from the Cauchy–Pompeiu formula [27, Theorem 2.2.1, p. 674]; its volume term vanishes because $\mathcal{D}u = 0$. With $E = \nabla\Phi$ and the outward normal, their C_D is $-C$ in our notation, which gives (26).

The field used in the projection argument below does satisfy the extra hypotheses. Indeed, if $g \in C^\infty(\Gamma; \text{Cl}_3)$ and $u := -(Cg)|_\Omega$, then the nontangential maximal estimate [27, formula (2.3.22), p. 679], together with (25), gives $N_\kappa u \in L^2(\Gamma)$ and $\gamma_{\text{int}} u \in C^\infty(\Gamma; \text{Cl}_3)$. Hence $u \in \mathcal{H}_D^2(\Omega)$, so (26) applies to this u .

The projection identity. By (25), the interior Cauchy boundary operator is

$$\Pi_{\text{int}} := -\gamma_{\text{int}} C = \frac{1}{2}I - T$$

and defines a map

$$\Pi_{\text{int}} : C^\infty(\Gamma; \text{Cl}_3) \longrightarrow C^\infty(\Gamma; \text{Cl}_3).$$

It remains to prove that this map is a projection.

Fix $g \in C^\infty(\Gamma; \text{Cl}_3)$ and set

$$u := -(Cg)|_\Omega.$$

The mapping property of C shows that $u \in \mathcal{M}(\Omega; \text{Cl}_3)$. Since g is smooth, the Plemelj formula (25) also shows that $\gamma_{\text{int}}(Cg)$ exists and belongs to $C^\infty(\Gamma; \text{Cl}_3)$. Consequently, the interior trace of u is well defined and

$$\gamma_{\text{int}} u = -\gamma_{\text{int}} Cg = \Pi_{\text{int}} g.$$

Thus $\Pi_{\text{int}} g$ is precisely the boundary trace of the interior monogenic Cauchy field u .

The preceding verification of the nontangential hypotheses showed that $u \in \mathcal{H}_D^2(\Omega)$. The Cauchy reproduction formula (26) therefore gives

$$-C(\Pi_{\text{int}} g) = -C(\gamma_{\text{int}} u) = u \quad \text{in } \Omega.$$

Because $\Pi_{\text{int}} g \in C^\infty(\Gamma; \text{Cl}_3)$, the Plemelj formula applied to the new density $\Pi_{\text{int}} g$ guarantees that the interior trace on the left exists. Taking the interior trace of the last identity is therefore legitimate, and gives

$$\begin{aligned} \Pi_{\text{int}}^2 g &= -\gamma_{\text{int}} C(\Pi_{\text{int}} g) \\ &= \gamma_{\text{int}} u \\ &= \Pi_{\text{int}} g. \end{aligned}$$

Hence $\Pi_{\text{int}}^2 = \Pi_{\text{int}}$ on $C^\infty(\Gamma; \text{Cl}_3)$: the operator maps a boundary density to the trace of its interior Cauchy field and acts as the identity on every trace it produces.

It remains to translate this projection identity into an identity for T . Substituting $\Pi_{\text{int}} = \frac{1}{2}I - T$ and expanding the square gives

$$\begin{aligned} \left(\frac{1}{2}I - T\right)^2 &= \frac{1}{2}I - T, \\ \frac{1}{4}I - T + T^2 &= \frac{1}{2}I - T, \end{aligned}$$

and cancellation of the common term $-T$ yields

$$T^2 = \frac{1}{4}I \quad \text{on } C^\infty(\Gamma; \text{Cl}_3).$$

The two formulas give the implication

$$\text{Plemelj trace formula} + \text{Cauchy reproduction} \implies \Pi_{\text{int}}^2 = \Pi_{\text{int}} \implies T^2 = \frac{1}{4}I.$$

Extension to Z^s and inversion. So far, $T^2 = \frac{1}{4}I$ has been proved only on smooth boundary fields. Fix $s \in \mathbb{R}$. Both the operator and the identity extend to

$$Z^s = H^s(\Gamma; \text{Cl}_3).$$

Bounded extension. On a compact C^∞ boundary, the Calderón projector Π_{int} is a classical pseudodifferential operator of order zero; see [12, Theorem 11.8, pp. 314–315]. Since

$$T = \frac{1}{2}I - \Pi_{\text{int}},$$

T is also an order-zero pseudodifferential operator. The Sobolev mapping theorem for such operators [12, Theorem 8.5, pp. 206–207] therefore gives a bounded map

$$T : Z^s \longrightarrow Z^s.$$

More precisely, the operator initially defined on $C^\infty(\Gamma; \text{Cl}_3)$ extends uniquely to Z^s , because $C^\infty(\Gamma; \text{Cl}_3)$ is dense in Z^s . This extension is again denoted by T .

Extension of the quadratic identity. Let $g \in Z^s$ and choose $g_k \in C^\infty(\Gamma; \text{Cl}_3)$ such that $g_k \rightarrow g$ in Z^s . Boundedness of T first gives

$$Tg_k \longrightarrow Tg \quad \text{in } Z^s,$$

and a second application of T gives

$$T^2g_k = T(Tg_k) \longrightarrow T(Tg) = T^2g \quad \text{in } Z^s.$$

For every k , the smooth quadratic identity gives $T^2g_k = \frac{1}{4}g_k$. Hence

$$T^2g = \lim_{k \rightarrow \infty} T^2g_k = \lim_{k \rightarrow \infty} \frac{1}{4}g_k = \frac{1}{4}g \quad \text{in } Z^s.$$

Thus the identity $T^2 = \frac{1}{4}I$ in (24) holds on all of Z^s .

Inversion and norm estimate. For every $g \in Z^s$,

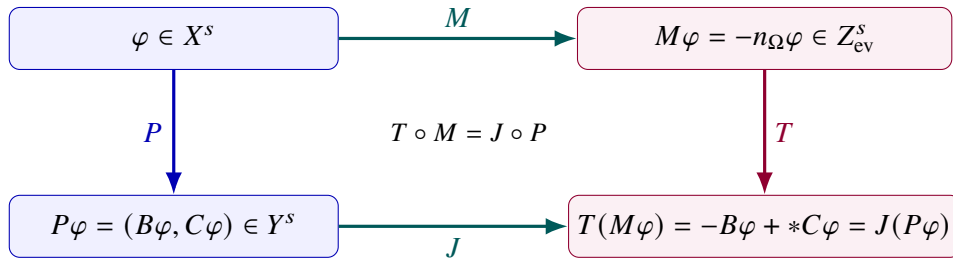
$$(4T)Tg = 4T^2g = g, \quad T(4T)g = 4T^2g = g.$$

Therefore T is boundedly invertible on Z^s and $T^{-1} = 4T$. Applying the boundedness of T to Tg gives

$$\|g\|_{Z^s} = 4\|T(Tg)\|_{Z^s} \leq 4\|T\|_{Z^s \rightarrow Z^s}\|Tg\|_{Z^s}.$$

This proves the claim.

Step 3: the div–curl factorization specific to P . The two routes from φ to an even Clifford-valued boundary field agree. This is the new structural step in the argument. The lower-right box records the identity proved below.



The required identity is

$$(27) \quad T(M\varphi) = J(P\varphi) = -B\varphi + *C\varphi.$$

The identity for smooth densities. First let $\varphi \in C^\infty(\Gamma; \mathbb{R}^3)$ and fix $x \in \Gamma$. For $\varepsilon > 0$, truncate each integral to $\Gamma \setminus B_\varepsilon(x)$. The pointwise inverse identity from Step 1 gives

$$n_\Omega(y)M\varphi(y) = \varphi(y),$$

and hence

$$\int_{\Gamma \setminus B_\varepsilon(x)} E(x-y)n_\Omega(y)M\varphi(y) dS_y = \int_{\Gamma \setminus B_\varepsilon(x)} E(x-y)\varphi(y) dS_y.$$

For $y \neq x$, both $E(x-y)$ and $\varphi(y)$ are vectors. By the Clifford product convention in Step 1,

$$E(x-y)\varphi(y) = -E(x-y) \cdot \varphi(y) + E(x-y) \wedge \varphi(y).$$

Hence the integrand has only scalar and bivector components; no vector or trivector term can occur. Let $\langle q \rangle_k$ denote the grade- k component of $q \in \text{Cl}_3$. Since the grade projections are linear maps on the

finite-dimensional algebra Cl_3 , they commute with integration and with every convergent principal-value limit. Taking the scalar and bivector components gives

$$\begin{aligned}\langle T(M\varphi)(x) \rangle_0 &= -\text{p. v.} \int_{\Gamma} E(x-y) \cdot \varphi(y) dS_y = -B\varphi(x), \\ \langle T(M\varphi)(x) \rangle_2 &= \text{p. v.} \int_{\Gamma} E(x-y) \wedge \varphi(y) dS_y.\end{aligned}$$

For the bivector component, set

$$Q_\varepsilon \varphi(x) := \int_{\Gamma \setminus B_\varepsilon(x)} E(x-y) \wedge \varphi(y) dS_y.$$

The Hodge star is a fixed linear map on Cl_3 . The convention $*(a \wedge b) = a \times b$ therefore gives

$$*Q_\varepsilon \varphi(x) = \int_{\Gamma \setminus B_\varepsilon(x)} E(x-y) \times \varphi(y) dS_y \longrightarrow C\varphi(x) \quad (\varepsilon \downarrow 0).$$

Since $*^2 = I$ on grades one and two, this is equivalent to

$$Q_\varepsilon \varphi(x) \longrightarrow *C\varphi(x).$$

Thus the bivector principal value exists and equals $*C\varphi(x)$. The scalar principal value is $-B\varphi(x)$ by Proposition 2.4. Combining the two grade components gives

$$T(M\varphi)(x) = -B\varphi(x) + *C\varphi(x) = J(B\varphi(x), C\varphi(x)) = J(P\varphi)(x).$$

Passage to Sobolev spaces. Let $\varphi \in X^s$. Choose $\varphi_k \in C^\infty(\Gamma; \mathbb{R}^3)$ with $\varphi_k \rightarrow \varphi$ in X^s . The boundedness of M and T , proved in Steps 1 and 2, implies

$$M\varphi_k \longrightarrow M\varphi \quad \text{in } Z^s, \quad T(M\varphi_k) \longrightarrow T(M\varphi) \quad \text{in } Z^s.$$

On the other route, Proposition 2.4 and the boundedness of J imply

$$P\varphi_k \longrightarrow P\varphi \quad \text{in } Y^s, \quad J(P\varphi_k) \longrightarrow J(P\varphi) \quad \text{in } Z^s.$$

Since $T(M\varphi_k) = J(P\varphi_k)$ for every k , passage to the limit proves (27) for every $\varphi \in X^s$. The isometry of J then gives the exact norm identity

$$\|T(M\varphi)\|_{Z^s}^2 = \|J(P\varphi)\|_{Z^s}^2 = \|P\varphi\|_{Y^s}^2 = \|B\varphi\|_{H^s(\Gamma)}^2 + \|C\varphi\|_{H^s(\Gamma; \mathbb{R}^3)}^2.$$

Step 4: completion of the a priori estimate. For $\varphi \in X^s$, the bounded inverse of M , the identity $M\varphi = 4T(T(M\varphi))$, and the norm identity from Step 3 give

$$\begin{aligned}\|\varphi\|_{X^s} &\leq C_{M,s} \|M\varphi\|_{Z^s} \\ &= 4C_{M,s} \|T(T(M\varphi))\|_{Z^s} \\ &\leq 4C_{M,s} \|T\|_{Z^s \rightarrow Z^s} \|T(M\varphi)\|_{Z^s} \\ &= 4C_{M,s} \|T\|_{Z^s \rightarrow Z^s} \left(\|B\varphi\|_{H^s}^2 + \|C\varphi\|_{H^s}^2 \right)^{1/2}.\end{aligned}$$

Thus (22) holds with the explicit admissible constant

$$C_{\Gamma,s} := 4C_{M,s} \|T\|_{Z^s \rightarrow Z^s}.$$

This completes the proof of the a priori estimate. $\square$

3.2. Range, compatibility, and exact recovery. Theorem 3.1 shows that P is injective with closed range; we characterize that range. Writing an even Clifford field uniquely as $a + *q$, set

$$\mathcal{R}_M^s := M(X^s) = \{a + *q \in Z_{\text{ev}}^s : q \cdot n_\Omega = 0\}.$$

The factorization $TM = JP$ then gives the sequence

$$X^s \xrightarrow{M} \mathcal{R}_M^s \xrightarrow{T} T(\mathcal{R}_M^s) \xrightarrow{J^{-1}} \text{Ran } P.$$

A value $y \in Y^s$ is called *compatible* if $y \in \text{Ran } P$. For $g \in \mathcal{R}_M^s$, the inverse of M is the left Clifford product $M^{-1}g = n_\Omega g$, which is vector-valued.

Proposition 3.2 (Range, compatibility, and recovery). *For every $s \in \mathbb{R}$,*

$$\text{Ran } P = J^{-1}T(\mathcal{R}_M^s), \quad \text{Ran } P \subseteq Y^s, \quad \text{Ran } P \text{ is closed.}$$

For every $y \in Y^s$,

$$y \in \text{Ran } P \iff 4T(Jy) \in \mathcal{R}_M^s.$$

For $y \in \text{Ran } P$,

$$P^{-1}y = n_\Omega(4T(Jy)).$$

Proof. Step 1: the range of M . Proposition B.1 gives $M(X^s) = \mathcal{R}_M^s$. Define

$$L_s : Z_{\text{ev}}^s \longrightarrow H^s(\Gamma), \quad L_s(a + *q) = q \cdot n_\Omega.$$

Multiplication by the smooth field n_Ω is bounded on H^s , so L_s is bounded and

$$\mathcal{R}_M^s = \ker L_s.$$

Hence $\mathcal{R}_M^s$ is closed. It is proper because

$$L_s(*n_\Omega) = |n_\Omega|^2 = 1.$$

Step 2: the action of T on even fields. For vectors $E(x - y)$ and $n_\Omega(y)$, the product $E(x - y)n_\Omega(y)$ has only scalar and bivector components. Left multiplication by this even element maps even Clifford fields to even Clifford fields. Thus Tg is even whenever g is a smooth even field; boundedness together with density then gives

$$T(Z_{\text{ev}}^s) \subset Z_{\text{ev}}^s.$$

Since $T^2 = \frac{1}{4}I$, the restriction

$$T : Z_{\text{ev}}^s \longrightarrow Z_{\text{ev}}^s$$

is an isomorphism with inverse $4T$.

Step 3: the range of P . The map $J : Y^s \rightarrow Z_{\text{ev}}^s$ is an isometric isomorphism. Using $TM = JP$ and Step 1 gives

$$J(\text{Ran } P) = T(M(X^s)) = T(\mathcal{R}_M^s).$$

Applying J^{-1} proves the range formula. Since T and J are isomorphisms and $\mathcal{R}_M^s$ is closed and proper, $\text{Ran } P$ is also closed and proper.

Step 4: compatibility and recovery. For $\varphi \in X^s$ and $y \in Y^s$,

$$\begin{aligned} P\varphi = y &\iff TM\varphi = Jy \\ &\iff M\varphi = 4T(Jy). \end{aligned}$$

The last equation is solvable precisely when $4T(Jy) \in M(X^s) = \mathcal{R}_M^s$. If this condition holds, put

$$g := 4T(Jy), \quad \varphi := n_\Omega g.$$

Because $g \in \mathcal{R}_M^s$, Proposition B.1 shows that φ is vector-valued. Moreover,

$$M\varphi = -n_\Omega(n_\Omega g) = -(n_\Omega^2)g = g,$$

and therefore

$$TM\varphi = Tg = 4T^2(Jy) = Jy.$$

Since J is injective, $P\varphi = y$. Uniqueness follows from Theorem 3.1. $\square$

Remark 3.3 (Fredholm status). The operator $P : X^s \rightarrow Y^s$ is not Fredholm. It is injective and has closed range, but its cokernel is infinite-dimensional. Indeed, L_s is surjective because

$$L_s(*f n_\Omega) = f \quad (f \in H^s(\Gamma)),$$

and the isomorphisms T and J give

$$Y^s / \text{Ran } P \cong Z_{\text{ev}}^s / T(\mathcal{R}_M^s) \cong Z_{\text{ev}}^s / \mathcal{R}_M^s \cong H^s(\Gamma).$$

Thus $P : X^s \rightarrow Y^s$ is upper semi-Fredholm, while $P : X^s \rightarrow \text{Ran } P$ is an isomorphism.

The orientation datum $(1/2, 0)$ is compatible by Lemma 2.5. Proposition 3.2 and the a priori estimate give exact recovery of the outward normal.

Corollary 3.4 (Unique orientation). *For every $s \in \mathbb{R}$, the system*

$$(28) \quad B\varphi = \frac{1}{2}, \quad C\varphi = 0$$

has exactly one solution in $H^s(\Gamma; \mathbb{R}^3)$, namely $\varphi = n_\Omega$.

Proof. Existence follows from (19). If φ_1 and φ_2 solve (28), then $\psi = \varphi_1 - \varphi_2$ satisfies $B\psi = C\psi = 0$. Theorem 3.1 gives $\|\psi\|_{H^s} = 0$. Hence $\varphi_1 = \varphi_2$, so the solution above is unique. $\square$

3.3. Least-squares recovery for incompatible values. Proposition 3.2 shows that the compatible values form the proper closed subspace

$$\mathcal{R}_P^s := P(X^s) \subseteq Y^s.$$

An arbitrary prescribed pair $y \in Y^s$ need not belong to the range of P ; equivalently, the equation $P\varphi = y$ need not be solvable. Least-squares recovery replaces y by its orthogonal projection onto $\mathcal{R}_P^s$ and then applies the inverse of P on that subspace.

Fix $s \in \mathbb{R}$ and the Hilbert norms on X^s and Y^s . Let

$$\Pi_s : Y^s \longrightarrow \mathcal{R}_P^s$$

be the orthogonal projection, and let $P_s^* : Y^s \rightarrow X^s$ be the Hilbert adjoint of P . The lower modulus

$$(29) \quad m_{\Gamma,s} := \inf_{\varphi \in X^s \setminus \{0\}} \frac{\|P\varphi\|_{Y^s}}{\|\varphi\|_{X^s}} > 0$$

is positive by Theorem 3.1.

3.3.1. Projection, normal equation, and least-squares inverse.

Theorem 3.5 (Least-squares projection and inverse). *For every $y \in Y^s$, the functional*

$$(30) \quad \mathcal{F}_y(\varphi) := \frac{1}{2} \|P\varphi - y\|_{Y^s}^2, \quad \varphi \in X^s,$$

has a unique minimizer $\varphi_{\text{ls}}(y)$. Define the least-squares inverse

$$(31) \quad P_s^\dagger := (P_s^* P)^{-1} P_s^* : Y^s \longrightarrow X^s.$$

Then

$$(32) \quad \varphi_{\text{ls}}(y) = P_s^\dagger y, \quad P\varphi_{\text{ls}}(y) = \Pi_s y,$$

$$(33) \quad r_s(y) := y - P\varphi_{\text{ls}}(y) = (I - \Pi_s)y, \quad r_s(y) \in (\mathcal{R}_P^s)^\perp = \ker P_s^*.$$

Equivalently, the minimizer is the unique solution of the normal equation

$$(34) \quad P_s^* P\varphi = P_s^* y.$$

The two projection identities are

$$(35) \quad P_s^\dagger P = I_{X^s}, \quad P P_s^\dagger = \Pi_s,$$

and the norm of the least-squares inverse is sharp:

$$(36) \quad \|P_s^\dagger\|_{Y^s \rightarrow X^s} = m_{\Gamma,s}^{-1}.$$

Consequently,

$$(37) \quad \begin{aligned} \|\varphi_{\text{ls}}(y_1) - \varphi_{\text{ls}}(y_2)\|_{X^s} &\leq m_{\Gamma,s}^{-1} \|\Pi_s(y_1 - y_2)\|_{Y^s} \\ &\leq m_{\Gamma,s}^{-1} \|y_1 - y_2\|_{Y^s}, \end{aligned}$$

and

$$(38) \quad \min_{\varphi \in X^s} \mathcal{F}_y(\varphi) = \frac{1}{2} \|r_s(y)\|_{Y^s}^2 = \frac{1}{2} \text{dist}_{Y^s}(y, \mathcal{R}_P^s)^2.$$

Proof. Step 1: orthogonal projection and the minimizer. The subspace $\mathcal{R}_P^s$ is closed by Proposition 3.2. The projection theorem gives the unique decomposition

$$(39) \quad y = \Pi_s y + (I - \Pi_s)y, \quad \Pi_s y \in \mathcal{R}_P^s, \quad (I - \Pi_s)y \perp \mathcal{R}_P^s.$$

For every $\varphi \in X^s$, the two terms in

$$P\varphi - y = (P\varphi - \Pi_s y) - (I - \Pi_s)y$$

are orthogonal. Hence

$$(40) \quad \|P\varphi - y\|_{Y^s}^2 = \|P\varphi - \Pi_s y\|_{Y^s}^2 + \|(I - \Pi_s)y\|_{Y^s}^2.$$

Because $P : X^s \rightarrow \mathcal{R}_P^s$ is a bijection, there is exactly one $\varphi \in X^s$ for which $P\varphi = \Pi_s y$. Equation (40) proves that this φ is the unique minimizer and gives (32), (33), and (38).

Step 2: the normal equation. For $z \in Y^s$,

$$z \perp \mathcal{R}_P^s \iff \langle z, P\psi \rangle_{Y^s} = 0 \text{ for every } \psi \in X^s \iff P_s^* z = 0.$$

Applying this equivalence to $z = y - P\varphi_{\text{ls}}(y)$ proves (34).

Set $A_s = P_s^* P : X^s \rightarrow X^s$. It is bounded and self-adjoint, and

$$(41) \quad \langle A_s \varphi, \varphi \rangle_{X^s} = \|P\varphi\|_{Y^s}^2 \geq m_{\Gamma,s}^2 \|\varphi\|_{X^s}^2.$$

The Lax–Milgram theorem implies that A_s is boundedly invertible. Solving the normal equation gives $\varphi_{\text{ls}}(y) = P_s^\dagger y$ and justifies the definition in (31).

Step 3: projection identities and the sharp norm. Substitution into (31) gives $P_s^\dagger P = I_{X^s}$, while (32) gives $PP_s^\dagger = \Pi_s$. For $y \in Y^s$, the lower bound (29) and the second identity imply

$$\|P_s^\dagger y\|_{X^s} \leq m_{\Gamma,s}^{-1} \|PP_s^\dagger y\|_{Y^s} = m_{\Gamma,s}^{-1} \|\Pi_s y\|_{Y^s} \leq m_{\Gamma,s}^{-1} \|y\|_{Y^s}.$$

Conversely, for $0 \neq \varphi \in X^s$,

$$\frac{\|P_s^\dagger(P\varphi)\|_{X^s}}{\|P\varphi\|_{Y^s}} = \frac{\|\varphi\|_{X^s}}{\|P\varphi\|_{Y^s}}.$$

Taking the supremum proves (36). Applying the first estimate to $y_1 - y_2$ proves (37). $\square$

Remark 3.6 (Compatibility and residual). A prescribed datum y is compatible precisely when $r_s(y) = 0$, or equivalently $PP_s^\dagger y = y$. For incompatible y , the residual $(I - \Pi_s)y$ is exactly the component that cannot be realized as the averaged boundary divergence and curl of a vector density.

The exact datum

$$y_* := \left(\frac{1}{2}, 0\right) = Pn_\Omega$$

has the unique solution n_Ω . A perturbed value $y^\delta = y_* + \varepsilon$ need not belong to $\mathcal{R}_P^s$, so an exact solution of $P\varphi = y^\delta$ may not exist. Its least-squares reconstruction $\varphi^\delta := P_s^\dagger y^\delta$ satisfies

$$(42) \quad \varphi^\delta - n_\Omega = P_s^\dagger \varepsilon, \quad y^\delta - P\varphi^\delta = (I - \Pi_s)\varepsilon.$$

Only the compatible component $\Pi_s \varepsilon$ changes the reconstruction, whereas the orthogonal component remains as residual. In particular,

$$\|\varphi^\delta - n_\Omega\|_{X^s} \leq m_{\Gamma,s}^{-1} \|\Pi_s \varepsilon\|_{Y^s}, \quad \|y^\delta - P\varphi^\delta\|_{Y^s} = \|(I - \Pi_s)\varepsilon\|_{Y^s}.$$

4. ILLUSTRATIVE NUMERICAL EXPERIMENTS

The experiments test whether a regularized free-vector discretization of the div–curl system recovers the correct direction as the point cloud is refined. They do not constitute a benchmark against specialized orientation algorithms or a proof of discrete convergence. We report orientation statistics for four analytic surfaces.

For samples $x_1, \dots, x_N$ from μ , we regularize the Newtonian kernel by

$$(43) \quad \Phi_\varepsilon(z) = \frac{1}{4\pi\sqrt{|z|^2 + \varepsilon^2}}, \quad K_\varepsilon(z) = -\frac{z}{4\pi(|z|^2 + \varepsilon^2)^{3/2}},$$

and define

$$(44) \quad (B_{N,\varepsilon}q)_i = \frac{1}{N} \sum_{j=1}^N K_\varepsilon(x_i - x_j) \cdot q_j,$$

$$(45) \quad (C_{N,\varepsilon}q)_i = \frac{1}{N} \sum_{j=1}^N K_\varepsilon(x_i - x_j) \times q_j.$$

The discrete field is obtained from

$$(46) \quad \widehat{q} \in \operatorname{argmin}_{q \in (\mathbb{R}^3)^N} \left\| B_{N,\varepsilon}q - \frac{1}{2} \mathbf{1} \right\|_2^2 + \|C_{N,\varepsilon}q\|_2^2 + \lambda_N R_N(q),$$

where R_N is a coordinate-only graph Dirichlet energy. The recovered orientation is the normalized direction

$$(47) \quad \widehat{n}_i = \frac{\widehat{q}_i}{\|\widehat{q}_i\|}, \quad \widehat{q}_i \neq 0.$$

Thus the experiments evaluate orientation rather than the auxiliary magnitude of $\widehat{q}$.

The four test surfaces are as follows.

1. The unit sphere with

$$\rho_S(x) = \frac{1 + 0.6x_3}{4\pi};$$

2. A torus parameterized by

$$x(\theta, \phi) = ((R + r \cos \theta) \cos \phi, (R + r \cos \theta) \sin \phi, r \sin \theta),$$

with major radius $R = 2$, minor radius $r = 0.65$, and

$$\rho_T(\theta, \phi) = \frac{1 + 0.45 \sin \theta + 0.25 \cos \phi}{4\pi^2 R r};$$

3. The disconnected union of a unit sphere centered at $(-1.8, 0, 0)$ and a torus with $(R, r) = (2, 0.65)$, axis parallel to the x -axis, and center $(1.8, 0, 0)$. The separation gap is exactly $3.6 - 1 - 0.65 = 1.95$.
4. The boundary of the radius-3 ball with a toroidal cavity. The removed torus has $(R, r) = (1.35, 0.45)$, axis parallel to the x -axis, and center $(a, 0, 0)$, where

$$a = \sqrt{(3 - 0.8918 - r)^2 - R^2}.$$

Its containment gap is therefore 0.8918.

The disconnected and cavity examples are sampled uniformly with respect to surface area on the full boundary; component probabilities are therefore proportional to component areas. In the cavity case, the domain-outward normal on the inner torus is the negative of the usual solid-torus outward normal. For the first two cases, rejection sampling is applied to the displayed densities, including the torus area factor $r(R + r \cos \theta)$.

Let E_N be the symmetrization of the coordinate-only k -nearest-neighbor graph. We use the graph regularizer

$$R_N(q) = \sum_{\{i,j\} \in E_N} \exp\left(-\frac{|x_i - x_j|^2}{s_N^2}\right) |q_i - q_j|^2$$

with s_N equal to the median edge length. The softening parameter is

$$\varepsilon_N = c L_N N^{-1/4}, \quad L_N = \left(\frac{1}{N} \sum_{i=1}^N |x_i - \bar{x}|^2 \right)^{1/2}.$$

To make the meaning of the graph parameter independent of N , let $H_{dc} = P_{N,\varepsilon}^* P_{N,\varepsilon}$ and let L_G be the weighted graph Laplacian replicated in the three coordinate directions. We set

$$(48) \quad \lambda_N = \beta \frac{\operatorname{tr} H_{dc}}{\operatorname{tr} L_G}, \quad \operatorname{tr} H_{dc} = \frac{3}{N^2} \sum_{i,j=1}^N |K_{\varepsilon_N}(x_i - x_j)|^2, \quad \operatorname{tr} L_G = 6 \sum_{\{i,j\} \in E_N} w_{ij}.$$

Pilot samples disjoint from the reported trials were used once to fix

	sphere	torus	disconnected	cavity
(c, β)	(0.5, 50)	(0.6, 2)	(0.4, 5)	(0.6, 2)

These values were then held fixed at every reported sample size. We use a 12-nearest-neighbor graph for the first three geometries and an 8-nearest-neighbor graph for the cavity. All reported solves use the full div–curl system and double-precision, matrix-free CGLS, stopped when the relative normal residual is at most 10^{-8} .

The reference implementation records the base seed, every trial seed, all derived parameters, the stopping residual, and the trial-level and aggregate statistics. The recorded data and the script that generates the table and figures accompany the manuscript.

We use four dyadic sample sizes: 256–2048 on the sphere, 512–4096 on the torus, and 1024–8192 in cases 3 and 4. Each row of Table 1 is the median over 20 independent random point clouds. The pointwise angular error is

$$(49) \quad \alpha_i = \arccos \left(\frac{\widehat{q}_i \cdot n_i}{|\widehat{q}_i|} \right),$$

We also record the fractions of recovered directions lying within 5° and 10° of the exact normal. The results are reported in Table 1.

TABLE 1. Orientation statistics for the full div–curl system. Entries are medians over 20 independent random point clouds, and angles are measured in degrees. Every sampled point in all reported trials has the correct outward sign.

Surface	N	median α	95th perc.	99th perc.	within 5°	within 10°
Case 1	256	6.58	11.97	13.62	31.1%	86.9%
	512	4.01	8.49	10.35	65.1%	98.7%
	1024	2.55	5.71	7.41	90.8%	100.0%
	2048	1.60	3.74	5.02	99.0%	100.0%
Case 2	512	3.95	10.06	13.44	65.4%	94.8%
	1024	2.72	6.44	8.28	85.9%	99.9%
	2048	2.16	5.28	6.75	93.5%	100.0%
	4096	1.75	4.38	5.80	97.5%	100.0%
Case 3	1024	3.02	6.55	8.56	84.7%	99.8%
	2048	1.94	4.41	5.80	97.4%	100.0%
	4096	1.42	3.37	4.46	99.6%	100.0%
	8192	1.09	2.79	3.68	99.9%	100.0%
Case 4	1024	3.73	9.89	13.98	67.4%	95.2%
	2048	2.10	6.39	10.20	90.3%	98.9%
	4096	1.27	5.01	8.64	94.9%	99.5%
	8192	0.83	4.28	7.32	96.5%	99.8%

Across all four geometries, the reported median, 95th-percentile, and 99th-percentile angular errors decrease as N increases. The convergence trends for the sphere and torus are shown in Figure 1.

To isolate the role of the curl equation, we repeat selected experiments after omitting $\|C_{N,\varepsilon}q\|_2^2$ from (46), while leaving the kernel, graph, regularization rule, and solver unchanged. Figure 2 shows that the full system substantially reduces the 95th-percentile angular error on both surfaces. This finite-sample comparison is consistent with the continuum distinction between the noninjective scalar block B and the injective joint operator (B, C) .

Case 3 shows that a single global solve can orient separated components without post hoc sign alignment. The cavity experiment (case 4) recovers the domain-outward sign on the inner boundary, even though that sign is opposite to the outward normal of the removed solid. These observations agree with the continuum orientation characterization.

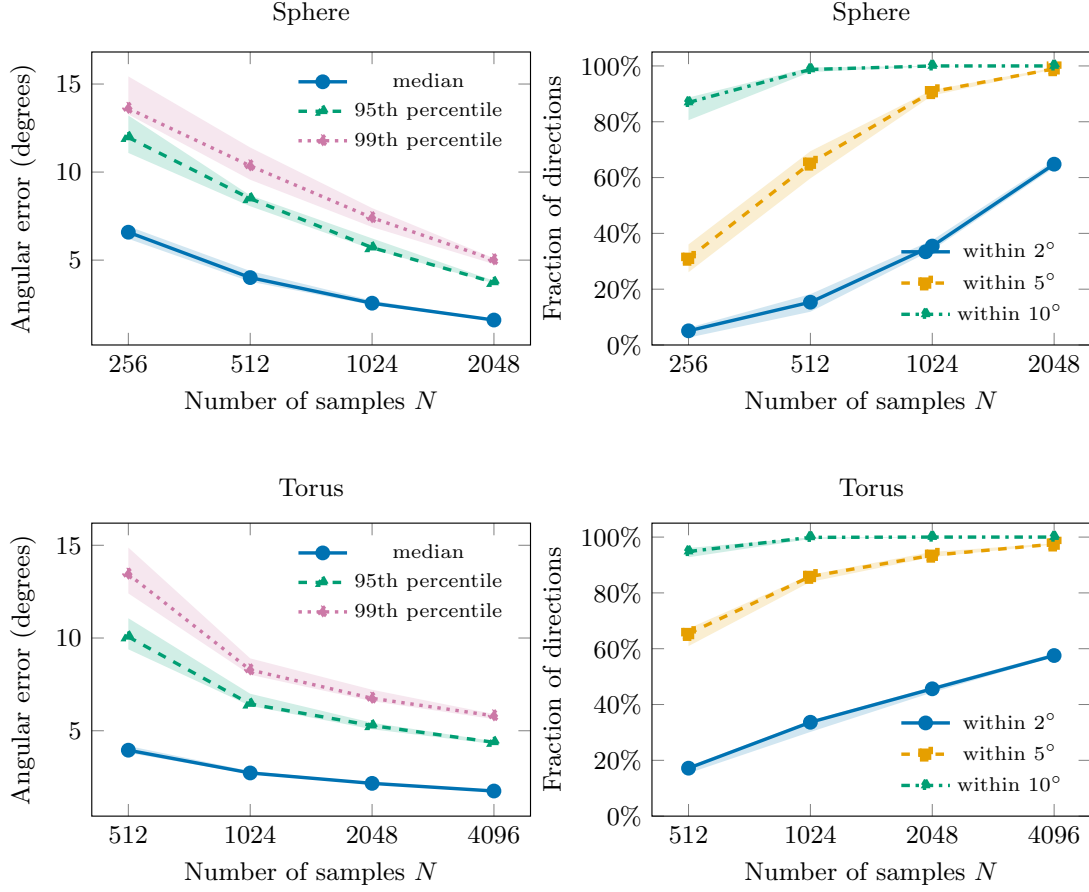


FIGURE 1. Orientation error under point-cloud refinement. The upper panels show the sphere, and the lower panels show the torus. Curves give medians over 20 trials; shaded bands span the 25th–75th percentiles.

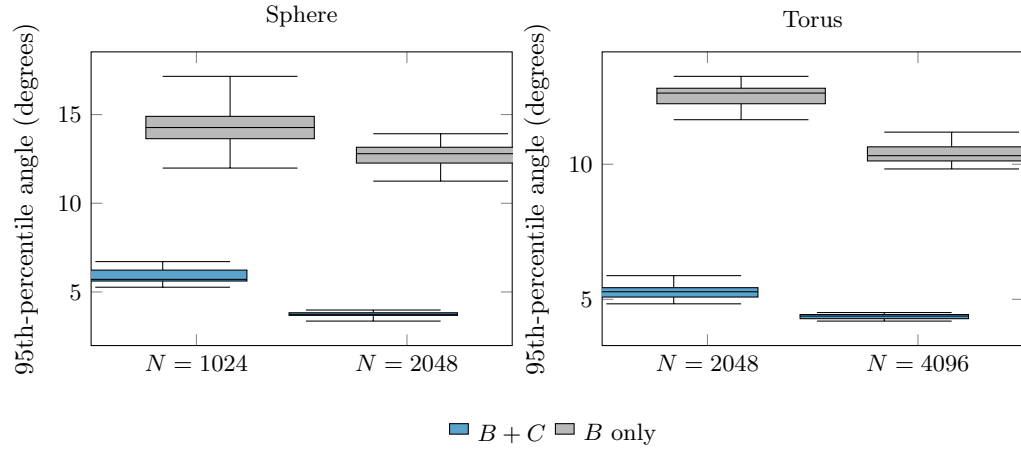


FIGURE 2. Ablation of the curl equation. Boxplots compare the 95th-percentile angular error of the full $B + C$ system with that of the scalar B block alone over 20 matched trials at each sample size. Boxes span the 25th–75th percentiles, the central line is the median, and whiskers show the observed range.

5. CONCLUSION

We have established a continuum div–curl characterization of outward orientation on smooth closed surfaces. The identity $T^2 = I/4$ and the underlying grade decomposition belong to the classical Clifford Cauchy framework. The new theorem is the same-order stability estimate for the particular joint boundary operator $P = (B, C)$. The problem-specific factorization $TM = JP$ converts the classical involution into this estimate and yields injectivity, a proper closed range with an explicit compatibility condition, and a stable least-squares inverse. In particular, the datum $(1/2, 0)$ uniquely recovers n_Ω ; the curl block is essential because the scalar Gauss block alone is noninjective on free vector densities. Under a positive sampling measure $d\mu = \rho \, dS$, the weighted system uniquely recovers $q = n_\Omega/\rho$, and normalization removes the unknown density.

The numerical tests are illustrative rather than a discrete convergence theorem. Angular errors decrease over the tested sample sizes, including for the disconnected and cavity geometries, and every reported direction has the correct domain-outward sign. A structure-preserving discretization theory that controls regularization and sampling error while retaining continuum stability remains the main open problem. Further directions include rough or incomplete boundaries and noisy or sparse data. The theorem gives a continuum reference point for the analysis of global orientation methods.

APPENDIX A. POTENTIAL-THEORETIC FACTS USED IN SECTION 2

We collect the analytic facts from Section 2 that are not specific to the div–curl recovery argument. They are standard consequences of Laplace layer-potential theory and the pseudodifferential calculus on smooth closed surfaces. We include the short reductions needed for the present sign conventions and cite the standard results used at each step.

A.1. Proof of Lemma 2.1.

Proof. Each component of $A\varphi$ is the Laplace single-layer potential with a smooth density. On a compact C^∞ surface there are $r_0, C > 0$ such that, uniformly for z in a fixed tubular neighborhood of Γ ,

$$\text{area}(\Gamma \cap B_r(z)) \leq Cr^2, \quad \int_{\Gamma \cap B_r(z)} \frac{dS_y}{|z - y|} \leq Cr \quad (0 < r < r_0).$$

The second estimate follows from the first by an annular decomposition and is the standard weak-singularity estimate for the single-layer kernel; see the layer-potential estimates in [25, Theorems 6.11–6.13] and [6].

Fix $x_0 \in \Gamma$. Split the integral into $\Gamma \cap B_{2r}(x_0)$ and its complement. The displayed estimate makes the contribution of the first set $O(r)$, uniformly for x sufficiently close to x_0 . On the complement the kernel $\Phi(x - y)$ is smooth in x and converges uniformly to $\Phi(x_0 - y)$. First choose r small and then let $x \rightarrow x_0$; this proves

$$A\varphi(x) \longrightarrow \int_{\Gamma} \Phi(x_0 - y)\varphi(y) \, dS_y.$$

The argument is independent of the side of approach, while away from Γ the potential is smooth. Hence $A\varphi$ extends continuously to all of $\mathbb{R}^3$ with the asserted boundary value. $\square$

A.2. Proof of Lemma 2.2.

Proof. Write $A\varphi = ((A\varphi)_1, (A\varphi)_2, (A\varphi)_3)$ and

$$K_{ij}\varphi(x) := \text{p. v.} \int_{\Gamma} \partial_{x_i} \Phi(x - y)\varphi_j(y) \, dS_y.$$

For the scalar Laplace single-layer potential, the tangential derivative is continuous across the boundary and the normal derivative has the classical jump by the density. Equivalently, with our outward-normal convention, the nontangential gradient traces satisfy

$$(50) \quad (\partial_{x_i}(A\varphi)_j)_{\text{int}} = K_{ij}\varphi + \frac{1}{2}(n_\Omega)_i\varphi_j,$$

and

$$(51) \quad (\partial_{x_i}(A\varphi)_j)_{\text{ext}} = K_{ij}\varphi - \frac{1}{2}(n_\Omega)_i\varphi_j.$$

These are the standard single-layer jump relations [25, Theorem 6.11, pp. 202–205]; see also [6]. The sign is consistent with $\Phi(z) = (4\pi|z|)^{-1}$: in the flat model with outward normal e_3 ,

$$\int_{\mathbb{R}^2} \partial_{x_3} \Phi((-u_1, -u_2, t)) du = -\frac{1}{2} \operatorname{sgn}(t),$$

so interior approach ($t < 0$) gives the plus sign in (50).

Taking the trace of the gradient matrix in (50)–(51) gives

$$(\operatorname{div} A\varphi)_{\text{int/ext}} = B\varphi \pm \frac{1}{2} n_\Omega \cdot \varphi.$$

Taking its antisymmetric contraction with the Levi–Civita tensor gives

$$(\operatorname{curl} A\varphi)_{\text{int/ext}} = C\varphi \pm \frac{1}{2} n_\Omega \times \varphi.$$

These are (14)–(15); averaging the two traces gives (16). $\square$

A.3. Proof of Lemma 2.3.

Proof. On a C^∞ closed surface, the boundary single-layer operator $S = A|_\Gamma$ is a classical pseudodifferential operator of order -1 , while the principal-value first-derivative operator $K = (K_{ij})_{i,j=1}^3$ is of order zero. These facts are part of the standard boundary pseudodifferential calculus for elliptic layer potentials; see [12, Theorem 11.8, pp. 314–315] and, for layer-potential mapping and jump properties, [6, 25]. The Sobolev mapping theorem for pseudodifferential operators [12, Theorem 8.5, pp. 206–207] therefore gives

$$S : H^m(\Gamma; \mathbb{R}^3) \rightarrow H^{m+1}(\Gamma; \mathbb{R}^3), \quad K : H^m(\Gamma; \mathbb{R}^3) \rightarrow H^m(\Gamma; \mathbb{R}^{3 \times 3})$$

for every $m \in \mathbb{R}$. Thus a smooth density has smooth boundary value and smooth one-sided gradient traces.

Equations (50) and (51) give

$$\gamma_{\text{int}}(\nabla A\varphi) - \gamma_{\text{ext}}(\nabla A\varphi) = n_\Omega \otimes \varphi,$$

which is (17). If $A\varphi$ were C^1 across Γ at a point x , the two gradient traces would agree there, hence $n_\Omega(x) \otimes \varphi(x) = 0$. Since $|n_\Omega(x)| = 1$, this forces $\varphi(x) = 0$. The final assertion follows. $\square$

A.4. Proof of Proposition 2.4.

Proof. The matrix operator

$$(K\varphi)_{ij}(x) = \text{p. v.} \int_\Gamma \partial_{x_i} \Phi(x - y) \varphi_j(y) dS_y$$

is a classical order-zero pseudodifferential operator on the smooth compact surface Γ ; equivalently, it is one of the order-zero boundary operators arising in the Calderón projector construction [12, Theorem 11.8, pp. 314–315]. Hence [12, Theorem 8.5, pp. 206–207]

$$\|K\varphi\|_{H^s(\Gamma; \mathbb{R}^{3 \times 3})} \leq C_{\Gamma, s} \|\varphi\|_{H^s(\Gamma; \mathbb{R}^3)} \quad (s \in \mathbb{R}).$$

The operators B and C are fixed algebraic contractions of K :

$$B\varphi = \sum_i (K\varphi)_{ii}, \quad (C\varphi)_i = \sum_{j,k} \epsilon_{ijk} (K\varphi)_{jk}.$$

Thus the same estimate gives (18). Density of $C^\infty(\Gamma; \mathbb{R}^3)$ in $H^s(\Gamma; \mathbb{R}^3)$ gives the unique extensions. $\square$

APPENDIX B. CLIFFORD FACTS USED IN THE DIV–CURL FACTORIZATION

This appendix records only the Clifford-algebra and Cauchy-operator facts needed in Theorem 3.1. Standard background on Clifford analysis may be found in [4, 10, 24]; the boundary Cauchy and Calderón theory used below is developed, in greater generality, in [27, 31].

B.1. Clifford conventions. Let $\text{Cl}_3 = \text{Cl}_{0,3}$, so the generators satisfy $e_i e_j + e_j e_i = -2\delta_{ij}$. We use the orthogonal grade decomposition

$$\text{Cl}_3 = \Lambda^0 \mathbb{R}^3 \oplus \Lambda^1 \mathbb{R}^3 \oplus \Lambda^2 \mathbb{R}^3 \oplus \Lambda^3 \mathbb{R}^3.$$

For vectors $v, w \in \mathbb{R}^3$,

$$(52) \quad vw = -v \cdot w + v \wedge w, \quad *(v \wedge w) = v \times w.$$

Thus

$$vw = -v \cdot w + *(v \times w).$$

All Clifford-valued Sobolev norms below are the componentwise norms associated with this finite-dimensional orthogonal decomposition.

B.2. The horizontal maps. For $s \in \mathbb{R}$, set

$$X^s = H^s(\Gamma; \mathbb{R}^3), \quad Y^s = H^s(\Gamma) \times H^s(\Gamma; \mathbb{R}^3), \quad Z^s = H^s(\Gamma; \text{Cl}_3),$$

and

$$Z_{\text{ev}}^s = H^s\left(\Gamma; \Lambda^0 \mathbb{R}^3 \oplus \Lambda^2 \mathbb{R}^3\right).$$

Proposition B.1 (The horizontal Clifford maps). *Define*

$$J(b, c) = -b + *c, \quad M\varphi = -n_\Omega \varphi.$$

For every $s \in \mathbb{R}$, $J : Y^s \rightarrow Z_{\text{ev}}^s$ is an isometry, $M : X^s \rightarrow Z_{\text{ev}}^s$ is bounded and injective, and

$$M^{-1}g = n_\Omega g \quad (g \in M(X^s)).$$

In particular,

$$\|\varphi\|_{X^s} \leq C_{\Gamma, s} \|M\varphi\|_{Z^s}.$$

For $s = 0$, M is an isometry onto its range, and for every $s \in \mathbb{R}$,

$$M(X^s) = \{a + *c \in Z_{\text{ev}}^s : c \cdot n_\Omega = 0\}.$$

Proof. The scalar and bivector grades are orthogonal and the Hodge star is a fixed orthogonal map, so

$$\|J(b, c)\|_{Z^s}^2 = \|b\|_{H^s(\Gamma)}^2 + \|c\|_{H^s(\Gamma; \mathbb{R}^3)}^2.$$

Hence J is an isometry.

Write $\varphi = an_\Omega + \tau$ with $a = n_\Omega \cdot \varphi$ and $\tau \cdot n_\Omega = 0$. By (52),

$$M\varphi = -n_\Omega \varphi = a - n_\Omega \wedge \tau = a + *c, \quad c = -n_\Omega \times \tau.$$

Thus $c \cdot n_\Omega = 0$, and pointwise $|M\varphi|^2 = |a|^2 + |\tau|^2 = |\varphi|^2$, proving the L^2 isometry. Moreover $n_\Omega^2 = -1$, so

$$n_\Omega(M\varphi) = \varphi.$$

Multiplication by a C^∞ coefficient on a compact smooth manifold is bounded on H^s for every $s \in \mathbb{R}$; this is the standard Sobolev multiplier property, obtainable for example from the order-zero pseudodifferential calculus [12, Chapter 8]. Since n_Ω is smooth, both M and left multiplication by n_Ω are therefore bounded on H^s . This proves boundedness, injectivity, and the displayed inverse estimate.

It remains only to identify the range. The preceding formula gives $M(X^s) \subset \{a + *c : c \cdot n_\Omega = 0\}$. Conversely, if $c \cdot n_\Omega = 0$, put

$$\tau = n_\Omega \times c, \quad \varphi = an_\Omega + \tau.$$

Then $-n_\Omega \times \tau = c$ by the vector triple-product identity, and the same decomposition yields $M\varphi = a + *c$. Hence the two spaces are equal. The range is closed because it is the kernel of the bounded map $a + *c \mapsto c \cdot n_\Omega$. $\square$

B.3. The Dirac operator and div–curl. Let

$$\mathcal{D} = \sum_{j=1}^3 e_j \partial_j.$$

The Clifford relations give, for a scalar field f_0 and a vector field v ,

$$(53) \quad \mathcal{D}(f_0 + v) = \nabla f_0 - \operatorname{div} v + *(\operatorname{curl} v), \quad \mathcal{D}^2 = -\Delta.$$

The first identity follows by separating the scalar and bivector grades of $\sum_{i,j} e_i e_j \partial_i v_j$; the second follows because the mixed derivatives cancel in pairs. These standard identities are the Moisil–Théodoresco form of the div–curl system; see [28, 4, 8]. In particular, for vector-valued v ,

$$\mathcal{D}v = 0 \iff \operatorname{div} v = 0 \text{ and } \operatorname{curl} v = 0.$$

B.4. The Cauchy kernel and its boundary operator. For $\Phi(z) = (4\pi|z|)^{-1}$, set

$$E(z) = \mathcal{D}\Phi(z) = \nabla\Phi(z) = -\frac{z}{4\pi|z|^3}.$$

By (53), $\mathcal{D}E = \delta_0$. For $g \in C^\infty(\Gamma; \mathbb{C}\ell_3)$ define

$$(Cg)(x) = \int_{\Gamma} E(x-y)n_{\Omega}(y)g(y) dS_y, \quad x \notin \Gamma,$$

and

$$(Tg)(x) = \text{p. v.} \int_{\Gamma} E(x-y)n_{\Omega}(y)g(y) dS_y, \quad x \in \Gamma.$$

Proposition B.2 (Boundary Cauchy operator). *The principal value defining Tg exists for $g \in C^\infty(\Gamma; \mathbb{C}\ell_3)$ and maps smooth fields to smooth fields. Moreover, for every $s \in \mathbb{R}$, T extends uniquely to a bounded operator*

$$T : H^s(\Gamma; \mathbb{C}\ell_3) \longrightarrow H^s(\Gamma; \mathbb{C}\ell_3).$$

Proof. The kernel $E(x-y)n_{\Omega}(y)$ is the classical boundary Cauchy kernel for the Euclidean Dirac operator. On a C^∞ boundary its principal homogeneous part is odd of degree -2 in the two tangential variables, so the principal value is well defined on smooth densities. This is the standard higher-dimensional Cauchy singular integral; see [24] and the general first-order elliptic-system treatment in [27, Sec. 2.3].

Equivalently, the Plemelj boundary operators are Calderón projectors. For smooth boundaries these are classical pseudodifferential operators of order zero; see [12, Theorem 11.8, pp. 314–315] and [27]. Since T differs from either Calderón projector by a multiple of the identity, $T \in \Psi^0(\Gamma; \mathbb{C}\ell_3)$. The Sobolev mapping theorem [12, Theorem 8.5, pp. 206–207] gives

$$T : H^s(\Gamma; \mathbb{C}\ell_3) \rightarrow H^s(\Gamma; \mathbb{C}\ell_3) \quad (s \in \mathbb{R}).$$

It also implies $T(C^\infty) \subset C^\infty$, and density of $C^\infty(\Gamma; \mathbb{C}\ell_3)$ gives uniqueness of the Sobolev extension. $\square$

REFERENCES

1. R. Abreu-Blaya, J. Bory-Reyes, and M. Shapiro, *The Cauchy transform for the Hodge/de Rham system and some of its properties*, Georgian Mathematical Journal **14** (2007), 1–20.
2. G. Barill, N. G. Dickson, R. Schmidt, D. I. W. Levin, and A. Jacobson, *Fast winding numbers for soups and clouds*, ACM Transactions on Graphics **37** (2018), Article 43, 1–12.
3. B. Booß-Bavnbek and K. P. Wojciechowski, *Elliptic boundary problems for Dirac operators*, Birkhäuser, Boston, 1993.
4. F. Brackx, R. Delanghe, and F. Sommen, *Clifford analysis*, Research Notes in Mathematics, vol. 76, Pitman, Boston, 1982.
5. A. P. Calderón, *Boundary value problems for elliptic equations*, Outlines of the Joint Soviet–American Symposium on Partial Differential Equations (Novosibirsk), 1963, November 1963, pp. 303–304.
6. M. Costabel, *Boundary integral operators on Lipschitz domains: elementary results*, SIAM Journal on Mathematical Analysis **19** (1988), 613–626.
7. B. B. Delgado and J. E. Macías-Díaz, *An exterior Neumann boundary-value problem for the div–curl system and applications*, Mathematics **9** (2021), 1609.
8. B. B. Delgado and R. M. Porter, *General solution of the inhomogeneous div–curl system and consequences*, Advances in Applied Clifford Algebras **27** (2017), 3015–3037.
9. R. Fu, J. Zheng, and L. Yu, *Consistent normal orientation for 3D point clouds via least squares on Delaunay graph*, Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition (CVPR), 2025, pp. 16932–16942.

10. J. E. Gilbert and M. A. M. Murray, *Clifford algebras and Dirac operators in harmonic analysis*, Cambridge Studies in Advanced Mathematics, vol. 26, Cambridge University Press, Cambridge, 1991.
11. C. Gotsman and K. Hormann, *A linear method to consistently orient normals of a 3D point cloud*, ACM SIGGRAPH 2024 Conference Papers, 2024, p. Article 55.
12. G. Grubb, *Distributions and operators*, Graduate Texts in Mathematics, vol. 252, Springer, New York, 2009.
13. S. Hofmann, E. Marmolejo-Olea, M. Mitrea, S. Pérez-Esteva, and M. Taylor, *Hardy spaces, singular integrals and the geometry of Euclidean domains of locally finite perimeter*, Geometric and Functional Analysis **19** (2009), 842–882.
14. H. Hoppe, T. DeRose, T. Duchamp, J. McDonald, and W. Stuetzle, *Surface reconstruction from unorganized points*, Proceedings of SIGGRAPH 1992, 1992, pp. 71–78.
15. F. Hou, C. Wang, W. Wang, H. Qin, C. Qian, and Y. He, *Iterative Poisson surface reconstruction (iPSR) for unoriented points*, ACM Transactions on Graphics **41** (2022), Article 128, 1–13.
16. G. Huang, Q. Fang, Z. Zhang, L. Liu, and X.-M. Fu, *Stochastic normal orientation for point clouds*, ACM Transactions on Graphics **43** (2024), Article 188, 1–12.
17. M. Kazhdan, M. Bolitho, and H. Hoppe, *Poisson surface reconstruction*, Symposium on Geometry Processing 2006, 2006, pp. 61–70.
18. Z. Li, F. Hou, W. Wang, X. Lu, and Y. He, *A divide-and-conquer approach for global orientation of non-watertight scene-level point clouds using 0-1 integer optimization*, ACM Transactions on Graphics **44** (2025), Article 48, 1–15.
19. S. Lin, Z. Shi, and Y. Liu, *Fast and globally consistent normal orientation based on the winding number normal consistency*, ACM Transactions on Graphics **43** (2024), Article 189, 1–19.
20. S. Lin, D. Xiao, Z. Shi, and B. Wang, *Surface reconstruction from point clouds without normals by parametrizing the Gauss formula*, ACM Transactions on Graphics **42** (2023), Article 14, 1–19.
21. Y. Ma, Y. Meng, D. Xiao, Z. Shi, and B. Wang, *Wavelet-based global orientation and surface reconstruction for sparse point clouds*, Computer Aided Geometric Design **128** (2026), 102560.
22. Y. Ma, J. Shen, Y. Meng, D. Xiao, Z. Shi, and B. Wang, *Anisotropic Gauss reconstruction and global orientation with octree-based acceleration*, Computer Graphics Forum **44** (2025), e70199.
23. Y. Ma, D. Xiao, Z. Shi, and B. Wang, *Convection augmented Gauss reconstruction for unoriented point clouds*, ACM Transactions on Graphics **44** (2025), Article 173, 1–18.
24. A. McIntosh, *Clifford algebras and the higher dimensional Cauchy integral*, Banach Center Publications **22** (1989), 253–267.
25. W. McLean, *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, Cambridge, 2000.
26. G. Metzger, R. Hanocka, D. Zorin, R. Giryes, D. Panozzo, and D. Cohen-Or, *Orienting point clouds with dipole propagation*, ACM Transactions on Graphics **40** (2021), Article 165, 1–14.
27. I. Mitrea, M. Mitrea, and M. Taylor, *Cauchy integrals, Calderón projectors, and Toeplitz operators on uniformly rectifiable domains*, Advances in Mathematics **268** (2015), 666–757.
28. G. C. Moisil and N. Théodoresco, *Fonctions holomorphes dans l'espace*, Mathematica (Cluj) **5** (1931), 142–159.
29. S. A. Sauter and C. Schwab, *Boundary element methods*, Springer Series in Computational Mathematics, vol. 39, Springer, Berlin, 2011.
30. N. Schertler, B. Savchynskyy, and S. Gumhold, *Towards globally optimal normal orientations for large point clouds*, Computer Graphics Forum **36** (2017), 197–208.
31. E. Schulz and R. Hiptmair, *First-kind boundary integral equations for the Dirac operator in 3-dimensional Lipschitz domains*, SIAM Journal on Mathematical Analysis **54** (2022), 616–648.
32. G. Schwarz, *Hodge decomposition: A method for solving boundary value problems*, Lecture Notes in Mathematics, vol. 1607, Springer, Berlin, 1995.
33. D. Xiao, Z. Shi, S. Li, B. Deng, and B. Wang, *Point normal orientation and surface reconstruction by incorporating isovalue constraints to Poisson equation*, Computer Aided Geometric Design **103** (2023), 102195.
34. R. Xu, Z. Dou, N. Wang, S. Xin, S. Chen, M. Jiang, X. Guo, W. Wang, and C. Tu, *Globally consistent normal orientation for point clouds by regularizing the winding-number field*, ACM Transactions on Graphics **42** (2023), Article 111, 1–15.
35. Z. Zhang, X.-M. Fu, and Q. Fang, *GPU-accelerated stochastic normal orientation*, Computer Aided Geometric Design **127** (2026), 102554.

YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING, CHINA 100084

YANQI LAKE BEIJING INSTITUTE OF MATHEMATICAL SCIENCES AND APPLICATIONS, BEIJING, CHINA 101408

Email address: zqshi@tsinghua.edu.cn

YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING, CHINA 100084

Email address: zhuh@tsinghua.edu.cn