

ACCELERATING LANGEVIN DYNAMICS WITH WEIGHTED LAPLACIAN FLOW: GLOBAL WELL-POSEDNESS AND EXPONENTIAL CONVERGENCE *

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Abstract. We study an accelerated Langevin equation in which the classical overdamped Langevin drift is supplemented by a nonlocal gradient velocity determined through a target-weighted Poisson equation. The target p is only assumed to be strictly positive and need not be normalized. Writing the dynamics in terms of $\omega = \log(p/q)$ yields a coupled parabolic–elliptic system whose parabolic component contains the usual Langevin diffusion, whereas the elliptic component generates a global transport. The system is invariant under multiplication of p by a positive constant. On a bounded smooth domain with homogeneous Neumann boundary conditions, we establish global existence and uniqueness of classical solutions for strictly positive $C^{2+\alpha}$ target weights and compatible $C^{2+\alpha}$ initial data, with full α -Hölder regularity up to the initial time. For long-time behavior, the oscillation of ω decays at the explicit rate e^{-t} . For normalized initial densities, the solution density converges exponentially to the normalized target. Moreover, an exact entropy-dissipation identity holds relative to the normalized target, in which the relative Fisher information and the Jeffreys divergence give the two dissipation mechanisms associated with the Langevin and weighted-Poisson components. In particular, KL divergence also converges to zero with rate e^{-t} without a convexity assumption on the target.

Key words. Langevin dynamics, weighted Laplacian flow, parabolic-elliptic systems, global well-posedness, entropy dissipation, exponential convergence

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1. Introduction. Sampling from an unnormalized target is a fundamental problem in Bayesian computation, molecular dynamics, statistical physics, and machine learning. A canonical example is the Gibbs weight

$$p_\kappa(x) = e^{-U(x)/\kappa}, \quad \pi_\kappa(x) = Z_\kappa^{-1} p_\kappa(x), \quad Z_\kappa = \int p_\kappa(x) dx,$$

where U is an energy landscape and $\kappa > 0$ denotes the temperature. The normalizing constant Z_κ is often unavailable in sampling problems. Langevin dynamics is nevertheless well defined because its drift depends only on the logarithmic gradient of the target, which is unchanged by multiplication by a positive constant. The resulting connection between stochastic sampling and Fokker–Planck equations makes available a broad collection of analytic tools, including functional inequalities, Wasserstein gradient-flow methods, coupling arguments, spectral theory, and hypocoercivity.

Let $p = e^{-U}$ denote an unnormalized target weight and set

$$Z_p := \int_{\Omega} p(x) dx, \quad \pi := \frac{p}{Z_p}.$$

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Since $\nabla \log p = \nabla \log \pi$, the overdamped Langevin diffusion can be written as

$$dX_t = -\nabla U(X_t) dt + \sqrt{2} dW_t = \nabla \log p(X_t) dt + \sqrt{2} dW_t.$$

If q_t denotes the density of X_t , then

$$\partial_t q_t = \Delta q_t - \nabla \cdot (q_t \nabla \log p) = \nabla \cdot \left(q_t \nabla \log \frac{q_t}{\pi} \right) = \nabla \cdot \left(q_t \nabla \log \frac{q_t}{p} \right).$$

On a bounded domain the corresponding reflected dynamics is represented at the density level by the same Fokker–Planck equation together with a no-flux boundary condition. The variational formulation of Jordan, Kinderlehrer, and Otto [20] identifies this evolution as the 2-Wasserstein gradient flow of the relative entropy with respect to the normalized target π ,

$$\text{KL}(q \parallel \pi) = \int q \log \frac{q}{\pi} dx.$$

At the continuous level, the entropy satisfies

$$\frac{d}{dt} \text{KL}(q_t \parallel \pi) = - \int q_t \left| \nabla \log \frac{q_t}{\pi} \right|^2 dx = - \int q_t \left| \nabla \log \frac{q_t}{p} \right|^2 dx.$$

Consequently, a logarithmic Sobolev inequality yields exponential relaxation in relative entropy. This mechanism is classical for strongly log-concave targets through the Bakry–Émery criterion [1], Poincaré and transportation inequalities provide related L^2 and Wasserstein estimates [32]. Quantitative convergence beyond global convexity can be obtained by Lyapunov, coupling, and spectral methods [15, 35].

The discretization of Langevin diffusions has also been studied extensively. Non-asymptotic estimates for the unadjusted Langevin algorithm and related schemes have been established in total variation, Wasserstein distance, relative entropy, and Rényi divergence under a variety of convexity and regularity assumptions [8, 12, 13, 36, 33, 4, 29, 3]. These results demonstrate the robustness of the Langevin framework, while also showing that its quantitative performance may deteriorate substantially when the target is strongly nonconvex.

A principal obstruction is metastability. For multimodal Gibbs measures with well-separated local minima, transitions between metastable regions may occur on exponentially long time scales. Eyring–Kramers and potential-theoretic analyses make this dependence precise [2, 28]. In the small-temperature regime $p_\kappa \propto e^{-U/\kappa}$, characteristic transition times may scale exponentially in $1/\kappa$. This phenomenon motivates the construction of modified dynamics in which the target distribution is retained while the transport between distant regions of state space is enhanced.

Several acceleration mechanisms have been proposed. Underdamped and kinetic Langevin methods introduce momentum and exploit hypocoercive interactions between transport and dissipation [5, 9, 26, 30]. Nonreversible perturbations retain the invariant measure while adding a circulation that is divergence-free with respect to the target measure. The effects on spectral gaps, convergence, and asymptotic variance have been studied in [19, 21, 34, 11, 23]. At the density level, birth-death dynamics provides a genuinely nonlocal acceleration mechanism [24, 25]. A related principle arises in enhanced dissipation: conservative transport can redistribute a scalar so that diffusion acts more effectively, and quantitative forms of this mechanism have been developed in [6, 16, 7, 17, 38].

The present work is motivated by Weighted Laplacian Flow (WLF) [18], which constructs a deterministic transport velocity through a target-weighted elliptic problem. Consider Langevin dynamics with an auxiliary velocity field

$$dX_t = (\nabla \log p(X_t) + v(t, X_t)) dt + \sqrt{2} dW_t.$$

The associated Fokker–Planck equation implies that

$$\partial_t \omega + v \cdot \nabla \omega + |\nabla \omega|^2 = L_p \omega + \frac{1}{p} \nabla \cdot (p v).$$

Based on the weighted Laplacian flow, we choose $v = \nabla \phi$, where ϕ solves

$$L_p \phi = -(\omega - \bar{\omega}).$$

Here $\bar{\omega}$ is the unique scalar for which the Neumann problem is compatible, namely,

$$\int_{\Omega} p(\omega - \bar{\omega}) dx = 0.$$

Thus $\bar{\omega}$ may be regarded as an additional scalar unknown determined simultaneously with ϕ . The resulting parabolic–elliptic system is

$$\begin{cases} L_p \phi = -(\omega - \bar{\omega}), \\ \partial_t \omega + \nabla \phi \cdot \nabla \omega + |\nabla \omega|^2 = L_p \omega - (\omega - \bar{\omega}). \end{cases}$$

Homogeneous Neumann conditions are imposed on both ϕ and ω , and the additive constant in ϕ is fixed by a zero-mean normalization. In this formulation, $L_p \omega$ represents the local Langevin relaxation, whereas the weighted-Poisson equation produces a nonlocal transport correction.

The model possesses two complementary coercive structures. The first is a decay for ω , which yields exponential contraction of its spatial oscillation. The second is an exact entropy-dissipation identity. For a normalized density q , define

$$I(q \parallel \pi) := \int_{\Omega} q \left| \nabla \log \frac{q}{\pi} \right|^2 dx = \int_{\Omega} q \left| \nabla \log \frac{q}{p} \right|^2 dx$$

and

$$J(q, \pi) := \text{KL}(q \parallel \pi) + \text{KL}(\pi \parallel q).$$

We prove in Section 6 that

$$(1.1) \quad \frac{d}{dt} \text{KL}(q_t \parallel \pi) = -I(q_t \parallel \pi) - J(q_t, \pi).$$

The two terms on the right-hand side correspond, respectively, to the Langevin diffusion and the weighted-Poisson transport. Since $J(q, \pi) \geq \text{KL}(q \parallel \pi)$, identity (1.1) yields exponential convergence in relative entropy with exactly the same rate as in the normalized formulation.

A related class of nonlocal methods is Fisher-Rao sampling. For a smooth, strictly positive probability density q_t , write $\ell_t := \log(q_t/\pi)$ and $\langle f \rangle_{\nu} := \int_{\Omega} f(x) \nu(x) dx$ for a probability density ν . The Fisher-Rao gradient flow of $\text{KL}(q \parallel \pi)$ is the mass-preserving reaction equation

$$\partial_t q_t = -q_t (\ell_t - \langle \ell_t \rangle_{q_t}).$$

Combining this reaction with the Langevin diffusion yields the mass-preserving Wasserstein-Fisher-Rao gradient flow underlying birth–death accelerated sampling [24, 25]. Fisher-Rao paths can also be realized by transport: kernel Fisher-Rao flow constructs a Poisson-based velocity along a geometric-mixture path and approximates it in a reproducing kernel Hilbert space [27]. To compare the operators explicitly, a transport representation of the Fisher-Rao flow above would use a current-density-weighted potential $\psi = \psi(t, \cdot)$ satisfying

$$\frac{1}{q_t} \nabla \cdot (q_t \nabla \psi) = \ell_t - \langle \ell_t \rangle_{q_t}, \quad \partial_n \psi = 0,$$

so that $-\nabla \cdot (q_t \nabla \psi)$ equals the reaction term. In contrast, our feedback solves the target-weighted problem

$$L_p \phi = \ell_t - \langle \ell_t \rangle_\pi, \quad \partial_t q_t = \nabla \cdot (q_t \nabla \ell_t) - \nabla \cdot (q_t \nabla \phi).$$

Thus both approaches provide a nonlocal correction to Langevin relaxation, but their elliptic weights and centering measures differ: the present transport is not, in general, the standard Fisher-Rao reaction or its Poisson representation. Correspondingly, the additional entropy dissipation here is $J(q_t, \pi)$ in (1.1), whereas the Fisher-Rao KL reaction dissipates $\int_\Omega q_t (\ell_t - \langle \ell_t \rangle_{q_t})^2 dx$.

The principal contributions of this paper are as follows.

- (i) We derive the accelerated Langevin equation from an overdamped Langevin diffusion with an auxiliary gradient velocity determined by a weighted Neumann problem.
- (ii) We construct local $W^{1,q}$ mild solutions, prove a maximum principle, and obtain global continuation through an exponential change of variable. Compatible smooth initial data yield global strong solutions with classical regularity.
- (iii) We prove exponential contraction of $\text{osc}_\Omega \omega(t)$ and establish the exact Jeffreys–Fisher dissipation law.

The remainder of the paper is organized as follows. Section 2 formulates the model and states the main theorem. Section 3 constructs local $W^{1,q}$ mild solutions and gives a continuation criterion. Section 4 proves the maximum principle and oscillation estimate. Section 5 establishes global existence, and Section 5.3 obtains global strong solutions with classical regularity. Section 6 proves convergence and the entropy-dissipation law. The appendices contain the semigroup estimate, the singular Grönwall inequality, and the proof of the regularity lemma used both for the maximum principle and for the final regularity upgrade.

2. Model formulation and main result. This section formulates the accelerated Langevin equation considered throughout the paper. The derivation starts from an overdamped Langevin diffusion with an auxiliary velocity field and subsequently specifies this field through a target-weighted Poisson equation.

2.1. Notation and standing assumptions. Let $\Omega \subset \mathbb{R}^d$ be a bounded, connected $C^{2+\alpha}$ domain for some $0 < \alpha < 1$. The target is represented by a strictly positive weight p , not necessarily normalized, satisfying

$$(2.1) \quad p \in C^{2+\alpha}(\overline{\Omega}), \quad 0 < p_* \leq p(x) \leq p^*.$$

For a positive density $q = q(t, x)$, define

$$(2.2) \quad \omega(t, x) := \log \frac{p(x)}{q(t, x)}, \quad q(t, x) = p(x) e^{-\omega(t, x)}.$$

We write

$$(2.3) \quad L_p h := \frac{1}{p} \nabla \cdot (p \nabla h) = \Delta h + \nabla \log p \cdot \nabla h$$

for the weighted Laplacian associated with p .

For a function w , we denote by $\bar{w}$ the unique scalar determined by the compatibility condition

$$(2.4) \quad \int_{\Omega} p(x) (w(x) - \bar{w}) dx = 0.$$

Equivalently, $\bar{w}$ is the constant that makes the weighted Neumann problem

$$L_p \psi = -(w - \bar{w}), \quad \partial_n \psi = 0,$$

solvable. Explicitly,

$$\bar{w} = \frac{\int_{\Omega} p(x) w(x) dx}{\int_{\Omega} p(x) dx}.$$

2.2. Augmented Langevin dynamics. Consider an overdamped Langevin diffusion supplemented by a time-dependent velocity field $v = v(t, x)$. The drift depends on the target only through $\nabla \log p$, which is invariant under multiplication of p by a positive constant:

$$(2.5) \quad dX_t = (\nabla \log p(X_t) + v(t, X_t)) dt + \sqrt{2} dW_t.$$

On the bounded domain Ω , the reflecting boundary condition will be imposed at the level of the associated Fokker–Planck equation. Its interior evolution is

$$(2.6) \quad \partial_t q + \nabla \cdot (q(\nabla \log p + v)) = \Delta q.$$

Consequently, evolution of ω satisfies

$$(2.7) \quad \partial_t \omega + v \cdot \nabla \omega + |\nabla \omega|^2 = L_p \omega + \frac{1}{p} \nabla \cdot (p v).$$

Thus the auxiliary velocity contributes a transport term together with its p -weighted divergence.

We restrict the auxiliary velocity to gradient form,

$$(2.8) \quad v = \nabla \phi,$$

and determine the potential ϕ through

$$(2.9) \quad L_p \phi = -(\omega - \bar{\omega}).$$

The scalar $\bar{\omega}$ is determined together with ϕ by the compatibility condition (2.4). Then (2.7) reduces to

$$(2.10) \quad \partial_t \omega + \nabla \phi \cdot \nabla \omega + |\nabla \omega|^2 = L_p \omega - (\omega - \bar{\omega}).$$

In addition to the diffusion term $L_p \omega$, the auxiliary velocity introduces an explicit dissipation term which will imply exponential convergence.

We impose homogeneous Neumann boundary conditions

$$(2.11) \quad \partial_n \phi = \partial_n \omega = 0 \quad \text{on } \partial\Omega.$$

The PDE model we proposed is therefore

$$(2.12) \quad \left\{ \begin{array}{ll} L_p \phi = -(\omega - \bar{\omega}), & \text{in } \Omega, \\ \partial_t \omega + \nabla \phi \cdot \nabla \omega + |\nabla \omega|^2 = L_p \omega - (\omega - \bar{\omega}), & \text{in } \Omega, \\ \partial_n \phi = \partial_n \omega = 0, & \text{on } \partial\Omega, \\ \int_{\Omega} \phi \, dx = 0, \\ \omega(0, \cdot) = \omega_0. \end{array} \right.$$

The elliptic equation determines the auxiliary velocity instantaneously from the current logarithmic density-ratio discrepancy, while the parabolic equation governs the coupled local and nonlocal relaxation. Indeed, the PDE system (2.12) is not a separate sampling dynamics. Rather, it provides a way to construct, from the current logarithmic density ratio $\omega = \log(p/q)$, the nonlocal velocity field. With the nonlocal velocity field, the normally reflected accelerated Langevin dynamics is

$$dX_t = (\nabla \log p(X_t) + \nabla \phi(t, X_t)) \, dt + \sqrt{2} \, dW_t - n(X_t) \, d\Lambda_t,$$

where n is the outward unit normal and Λ_t is continuous and nondecreasing, starts at zero, and increases only when $X_t \in \partial\Omega$. Its Fokker–Planck equation is exactly the density equation obtained from (2.12) under the transformation

$$q_t = p e^{-\omega(t, \cdot)}.$$

For normalized q_0 , the marginal density of this reflected process is q_t , as justified after Theorem 5.2. Thus the evolution of q_t is recovered from ω , and we focus on the PDE system (2.12).

One of the main results is the global well-posedness.

THEOREM 2.1 (Global classical well-posedness). *Assume (2.1) and let*

$$\omega_0 \in C^{2+\alpha}(\bar{\Omega}), \quad \partial_n \omega_0 = 0 \quad \text{on } \partial\Omega.$$

Then the system (2.12) admits a unique global classical solution (ω, ϕ) . More precisely, for every $T > 0$,

$$\omega \in C^{1+\alpha/2, 2+\alpha}([0, T] \times \bar{\Omega}), \quad \phi \in C^{\alpha/2}([0, T]; C^{2+\alpha}(\bar{\Omega})).$$

The proof proceeds from local $W^{1,q}$ mild solutions to the maximum principle, global continuation, and finally strong and classical regularity. These steps are carried out in Sections 3–5.3.

3. Local well-posedness in $W^{1,q}$.

Remark 3.1 (Choice of the Sobolev exponent). For the well-posedness proof, fix once and for all a finite exponent

$$(3.1) \quad q > \frac{2(d+2)}{1-\alpha}, \quad r = q/2.$$

Then

$$\alpha < 1 - \frac{d+2}{r}, \quad \alpha < 1 - \frac{d}{q}.$$

The initial datum in Theorem 2.1 belongs to $W^{1,q}(\Omega)$ for every finite q , so this choice imposes no additional initial condition. The local contraction argument below requires only $q > \max\{d, 2\}$.

The local theory is developed after eliminating the elliptic variable through the solution operator associated with the weighted Neumann problem. The resulting scalar parabolic equation is treated by a contraction argument in $C([0, T]; W^{1,q}(\Omega))$.

3.1. The elliptic solution operator. For $w \in L^q(\Omega)$, let $\Phi[w]$ be the solution of

$$(3.2) \quad \begin{cases} L_p \Phi[w] = -(w - \bar{w}), \\ \partial_n \Phi[w] = 0, \\ \int_{\Omega} \Phi[w] dx = 0. \end{cases}$$

The compatibility condition is automatic.

LEMMA 3.2 (Elliptic estimates). *Let $1 < q < \infty$. There exists a constant $C = C(\Omega, p, q) > 0$ such that, for every $w, w_1, w_2 \in L^q(\Omega)$,*

$$(3.3) \quad \|\Phi[w]\|_{W^{2,q}} \leq C \|w\|_{L^q},$$

$$(3.4) \quad \|\Phi[w_1] - \Phi[w_2]\|_{W^{2,q}} \leq C \|w_1 - w_2\|_{L^q}.$$

If $q > d$, then, possibly by enlarging C ,

$$(3.5) \quad \|\nabla \Phi[w]\|_{L^\infty} \leq C \|w\|_{L^q},$$

$$(3.6) \quad \|\nabla(\Phi[w_1] - \Phi[w_2])\|_{L^\infty} \leq C \|w_1 - w_2\|_{L^q}.$$

Proof. The operator L_p has the divergence-form representation

$$L_p \Phi = \frac{1}{p} \nabla \cdot (p \nabla \Phi).$$

By (2.1), the associated Neumann problem is uniformly elliptic with $C^{1+\alpha}$ coefficients. The compatibility condition has already been verified, and the normalization $\int_{\Omega} \Phi dx = 0$ removes the kernel of constants. Standard $W^{2,q}$ -regularity for the Neumann problem therefore gives

$$\|\Phi[w]\|_{W^{2,q}} \leq C \|w - \bar{w}\|_{L^q}.$$

By the compatibility condition (2.4),

$$|\bar{w}| \int_{\Omega} p dx = \left| \int_{\Omega} p w dx \right| \leq \|p\|_{L^{q'}} \|w\|_{L^q}.$$

Since $p \geq p_* > 0$,

$$\int_{\Omega} p dx \geq p_* |\Omega|,$$

and therefore

$$|\bar{w}| \leq C \|w\|_{L^q}, \quad \|w - \bar{w}\|_{L^q} \leq C \|w\|_{L^q}.$$

This proves (3.3). Since Φ depends linearly on its datum, the same estimate applied to $\Phi[w_1] - \Phi[w_2] = \Phi[w_1 - w_2]$ yields (3.4). If $q > d$, the Sobolev embedding

$$W^{2,q}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$$

implies (3.5) and (3.6). $\square$

3.2. Local solvability by a contraction argument. Let $S(t) = e^{tL_p}$ denote the Neumann heat semigroup generated by L_p . Eliminating ϕ by $\phi = \Phi[\omega]$, the scalar equation becomes

$$(3.7) \quad \partial_t \omega = L_p \omega - \nabla \Phi[\omega] \cdot \nabla \omega - |\nabla \omega|^2 - (\omega - \bar{\omega}).$$

Hence a mild solution satisfies

$$(3.8) \quad \begin{aligned} \omega(t) = & S(t)\omega_0 - \int_0^t S(t-s)(\nabla \Phi[\omega(s)] \cdot \nabla \omega(s)) ds \\ & - \int_0^t S(t-s)|\nabla \omega(s)|^2 ds - \int_0^t S(t-s)(\omega(s) - \bar{\omega}(s)) ds. \end{aligned}$$

We use the following semigroup estimates, proved in detail in Appendix A; see in particular (A.4)–(A.5). For $0 < t \leq 1$,

$$(3.9) \quad \|S(t)g\|_{W^{1,q}} \leq Ct^{-1/2} \|g\|_{L^q},$$

$$(3.10) \quad \|S(t)g\|_{W^{1,q}} \leq Ct^{-\theta} \|g\|_{L^{q/2}}, \quad \theta := \frac{1}{2} + \frac{d}{2q} < 1.$$

In addition $S(t)$ is strongly continuous and locally bounded on $W^{1,q}$:

$$(3.11) \quad \sup_{0 \leq t \leq 1} \|S(t)g\|_{W^{1,q}} \leq C \|g\|_{W^{1,q}}.$$

Set

$$X_T := C([0, T]; W^{1,q}(\Omega)), \quad \|w\|_{X_T} := \sup_{0 \leq t \leq T} \|w(t)\|_{W^{1,q}},$$

and, for $R > 0$,

$$B_R := \{w \in X_T : \|w\|_{X_T} \leq R\}.$$

In the rest of the paper, we define $\Gamma : X_T \rightarrow X_T$ by the right-hand side of (3.8).

3.2.1. The self-map estimate. Let $w \in B_R$. By Lemma 3.2,

$$\|\nabla \Phi[w]\|_{L^\infty} \leq C \|w\|_{L^q} \leq CR.$$

Thus

$$(3.12) \quad \|\nabla \Phi[w] \cdot \nabla w\|_{L^q} \leq CR^2.$$

Moreover,

$$(3.13) \quad \| |\nabla w|^2 \|_{L^{q/2}} = \|\nabla w\|_{L^q}^2 \leq R^2,$$

and

$$(3.14) \quad \|w - \bar{w}\|_{L^q} \leq CR.$$

Using (3.11), (3.9), (3.10), and $T \leq 1$, we obtain

$$\begin{aligned}
 \|\Gamma(w)\|_{X_T} &\leq C \|\omega_0\|_{W^{1,q}} + CR^2 \int_0^T (T-s)^{-1/2} ds \\
 &\quad + CR^2 \int_0^T (T-s)^{-\theta} ds + CR \int_0^T (T-s)^{-1/2} ds \\
 (3.15) \quad &\leq C \|\omega_0\|_{W^{1,q}} + CT^{1/2}(R + R^2) + CT^\delta R^2.
 \end{aligned}$$

Here

$$(3.16) \quad \delta := 1 - \theta = \frac{1}{2} - \frac{d}{2q} = \frac{q-d}{2q} > 0.$$

The corresponding time-integrability calculation is recorded in Appendix A, cf. (A.6). Set

$$R = 2C \|\omega_0\|_{W^{1,q}} + 1.$$

Then $T > 0$ can be chosen sufficiently small, depending only on R and the fixed coefficients, so that $\Gamma(B_R) \subset B_R$.

3.2.2. The contraction estimate. Let $w_1, w_2 \in B_R$ and $z = w_1 - w_2$. First,

$$(3.17) \quad |\nabla w_1|^2 - |\nabla w_2|^2 = (\nabla w_1 + \nabla w_2) \cdot \nabla z,$$

hence

$$(3.18) \quad \||\nabla w_1|^2 - |\nabla w_2|^2\|_{L^{q/2}} \leq 2R \|z\|_{W^{1,q}}.$$

Next,

$$\begin{aligned}
 (3.19) \quad &\nabla \Phi[w_1] \cdot \nabla w_1 - \nabla \Phi[w_2] \cdot \nabla w_2 \\
 &= \nabla(\Phi[w_1] - \Phi[w_2]) \cdot \nabla w_1 + \nabla \Phi[w_2] \cdot \nabla z.
 \end{aligned}$$

Using Lemma 3.2,

$$(3.20) \quad \|\nabla \Phi[w_1] \cdot \nabla w_1 - \nabla \Phi[w_2] \cdot \nabla w_2\|_{L^q} \leq CR \|z\|_{W^{1,q}}.$$

Finally,

$$(3.21) \quad \|(w_1 - \bar{w}_1) - (w_2 - \bar{w}_2)\|_{L^q} \leq C \|z\|_{W^{1,q}}.$$

Therefore

$$(3.22) \quad \|\Gamma(w_1) - \Gamma(w_2)\|_{X_T} \leq C \left[(1+R)T^{1/2} + RT^\delta \right] \|w_1 - w_2\|_{X_T}.$$

Shrinking T if necessary, we may assume

$$C \left[(1+R)T^{1/2} + RT^\delta \right] < 1.$$

Hence Γ is a strict contraction on B_R .

PROPOSITION 3.3 (Local well-posedness). *Let $q > \max\{d, 2\}$ and $\omega_0 \in W^{1,q}(\Omega)$. Then there exists*

$$T = T(\|\omega_0\|_{W^{1,q}}; \Omega, p, q) > 0$$

such that (2.12) admits a unique mild solution satisfying

$$\omega \in C([0, T]; W^{1,q}(\Omega)), \quad \phi = \Phi[\omega] \in C([0, T]; W^{2,q}(\Omega)).$$

Moreover, on every common local existence interval, the solution depends continuously on the initial datum in the $W^{1,q}$ -topology.

Proof. The estimates (3.15) and (3.22) show that, after choosing R as above and taking $T > 0$ sufficiently small, the map Γ maps B_R into itself and is a strict contraction. Banach's fixed-point theorem therefore yields a unique fixed point $\omega \in B_R$, which is the mild solution of (3.7).

The kernels $(t-s)^{-1/2}$ and $(t-s)^{-\theta}$ are integrable because $\theta < 1$. Hence the Duhamel terms are continuous at $t = 0$, and the strong continuity of $S(t)$ on $W^{1,q}$ gives $\omega \in C([0, T]; W^{1,q})$. Lemma 3.2 then implies

$$\phi = \Phi[\omega] \in C([0, T]; W^{2,q}).$$

For uniqueness in the full mild class, let two solutions be defined on a common compact time interval, and let R bound both $W^{1,q}$ -norms there. For $z = \omega_1 - \omega_2$, set

$$E_{a,h} := \sup_{a \leq t \leq a+h} \|z(t)\|_{W^{1,q}}.$$

The estimates leading to (3.22), applied to the mild formulations restarted at a , give

$$E_{a,h} \leq C\|z(a)\|_{W^{1,q}} + C((1+R)h^{1/2} + Rh^\delta)E_{a,h},$$

whenever $0 < h \leq 1$ and $[a, a+h]$ is contained in the common interval. Choose h so that the last coefficient is at most $1/2$; this choice is independent of a . Absorption and finite iteration prove uniqueness for equal initial data. The same estimate, together with the common local ball provided by the contraction construction for nearby initial data, gives continuous dependence in $W^{1,q}$. $\square$

3.3. Continuation criterion. The local existence time in Proposition 3.3 depends only on a bound for the current $W^{1,q}$ -norm. This yields the following standard continuation criterion.

LEMMA 3.4 (Continuation criterion). *Let*

$$\omega \in C([0, T_{\max}); W^{1,q}(\Omega))$$

be the maximal mild solution obtained from Proposition 3.3. If $T_{\max} < \infty$, then

$$\limsup_{t \uparrow T_{\max}} \|\omega(t)\|_{W^{1,q}} = \infty.$$

Equivalently, a uniform $W^{1,q}$ -bound on the maximal existence interval precludes finite-time breakdown.

Proof. Assume, to the contrary, that $T_{\max} < \infty$ and

$$\sup_{0 \leq t < T_{\max}} \|\omega(t)\|_{W^{1,q}} \leq R_* < \infty.$$

Choose $t_n \uparrow T_{\max}$. By time translation and Proposition 3.3, the problem with initial datum $\omega(t_n)$ admits a mild solution on an interval of length $\tau = \tau(R_*) > 0$, independently of n . Local uniqueness identifies this restarted solution with the original one on their common interval. For n sufficiently large, $t_n + \tau > T_{\max}$, so the restarted solution extends ω beyond $T_{\max}$, contradicting maximality. $\square$

3.4. Regularity bridge for mild solutions. Proposition 3.3 constructs a solution only in the mild $W^{1,q}$ -class. The maximum-principle argument in the next section, however, requires pointwise evaluation of the equation at positive times. The following lemma provides the required bridge. Its first part will be used in Section 4, while its second part will be used in Section 5.3 to recover regularity up to the initial time.

LEMMA 3.5 (Regularity on a finite time interval). *Assume (2.1), fix q as in (3.1), and suppose that*

$$\omega \in C([0, T]; W^{1,q}(\Omega)), \quad \phi = \Phi[\omega],$$

is a mild solution of (2.12).

(i) *For every $0 < \tau < T$,*

$$\omega \in C^{1+\alpha/2, 2+\alpha}([\tau, T] \times \overline{\Omega}), \quad \phi \in C^{\alpha/2}([\tau, T]; C^{2+\alpha}(\overline{\Omega})).$$

(ii) *If, in addition,*

$$\omega_0 \in C^{2+\alpha}(\overline{\Omega}), \quad \partial_n \omega_0 = 0 \quad \text{on } \partial\Omega,$$

then the same conclusions hold on the closed interval $[0, T]$. Moreover, with $r = q/2$,

$$(4.23) \quad \omega \in W^{1,r}(0, T; L^r(\Omega)) \cap L^r(0, T; W^{2,r}(\Omega)).$$

The proof is deferred to Appendix C. It uses maximal L^r -regularity, parabolic Sobolev embedding, elliptic Schauder estimates, and a final parabolic Schauder bootstrap.

4. Maximum principle. We first control the range of the local mild solution. The argument requires classical regularity only at positive times, supplied by Lemma 3.5(i), whose proof is given in Appendix C. That lemma depends only on local mild solvability and the fixed coefficients.

4.1. Spatial extrema. We use the following elementary form of the envelope theorem. For a real-valued function g , write

$$D^+g(t) := \limsup_{h \downarrow 0} \frac{g(t+h) - g(t)}{h}, \quad D_+g(t) := \liminf_{h \downarrow 0} \frac{g(t+h) - g(t)}{h}.$$

LEMMA 4.1 (Dini derivatives of extrema). *Let $v \in C^1([a, b] \times \overline{\Omega})$, and set*

$$M_v(t) := \max_{\overline{\Omega}} v(t, \cdot), \quad m_v(t) := \min_{\overline{\Omega}} v(t, \cdot).$$

Then, for every $a \leq t < b$,

$$(4.1) \quad D^+M_v(t) \leq \max_{x \in \text{Argmax } v(t, \cdot)} \partial_t v(t, x),$$

$$(4.2) \quad D_+m_v(t) \geq \min_{x \in \text{Argmin } v(t, \cdot)} \partial_t v(t, x).$$

Moreover, for Lipschitz functions f, g on $[a, b]$,

$$(4.3) \quad D^+(f - g)(t) \leq D^+f(t) - D_+g(t).$$

If $y \in C([a, b]; [0, \infty))$ satisfies $D^+y(t) \leq -\lambda y(t)$ for $a \leq t < b$, with $\lambda \geq 0$, then

$$(4.4) \quad y(t) \leq e^{-\lambda(t-a)}y(a), \quad a \leq t \leq b.$$

In particular, $D^+f \leq 0$ implies that f is nonincreasing, while $D_+g \geq 0$ implies that g is nondecreasing.

Proof. For (4.1), choose $h_n \downarrow 0$ realizing the upper right limsup and points x_n with $M_v(t + h_n) = v(t + h_n, x_n)$. Since $v(t, x_n) \leq M_v(t)$,

$$\frac{M_v(t + h_n) - M_v(t)}{h_n} \leq \frac{v(t + h_n, x_n) - v(t, x_n)}{h_n}.$$

After passing to a subsequence, compactness gives $x_n \rightarrow x_* \in \text{Argmax } v(t, \cdot)$; continuity of $\partial_t v$ yields (4.1). Applying this result to $-v$, and using $M_{-v} = -m_v$ and $D^+(-m_v) = -D_+m_v$, gives (4.2). Moreover,

$$|M_v(t) - M_v(s)| + |m_v(t) - m_v(s)| \leq 2\|\partial_t v\|_{C^0([a, b] \times \bar{\Omega})}|t - s|,$$

so both extremal functions are Lipschitz on $[a, b]$ and the difference inequality below applies to them. Formula (4.3) is the elementary inequality $\limsup a_h - \liminf b_h \leq \limsup a_h - \liminf b_h$. Finally, the Dini monotonicity criterion for continuous real-valued functions, applied to $e^{\lambda t}y(t)$, gives (4.4). Applied directly to f and $-g$, the same criterion gives the two monotonicity assertions. $\square$

4.2. Invariant bounds and oscillation decay. Let $T_{\max}$ denote the maximal existence time obtained by iteration of Proposition 3.3, with the exponent fixed as in Remark 3.1. We derive estimates on $[0, T_{\max})$ that depend only on the initial range of ω and the fixed coefficients.

For subsequent use, define

$$m_0 := \min_{\bar{\Omega}} \omega_0, \quad M_0 := \max_{\bar{\Omega}} \omega_0.$$

LEMMA 4.2 (Maximum principle and oscillation decay). *Let*

$$\omega \in C([0, T_{\max}); W^{1,q}(\Omega))$$

be the maximal mild solution obtained in Proposition 3.3, with q chosen as in Remark 3.1. Then

$$(4.5) \quad m_0 \leq \omega(t, x) \leq M_0, \quad 0 \leq t < T_{\max}, \quad x \in \bar{\Omega}.$$

Moreover, the spatial maximum is nonincreasing, the spatial minimum is nondecreasing, and

$$(4.6) \quad \text{osc}_{\Omega} \omega(t) \leq e^{-(t-s)} \text{osc}_{\Omega} \omega(s), \quad 0 \leq s \leq t < T_{\max}.$$

Proof. Fix $0 < \tau < T < T_{\max}$. By Lemma 3.5(i), ω is classical on $[\tau, T] \times \bar{\Omega}$. Define

$$M(t) := \max_{\bar{\Omega}} \omega(t, \cdot), \quad m(t) := \min_{\bar{\Omega}} \omega(t, \cdot).$$

At a spatial maximum x_M of $\omega(t, \cdot)$, the Neumann condition and the second-derivative test give

$$\nabla\omega(t, x_M) = 0, \quad L_p\omega(t, x_M) \leq 0.$$

For a boundary maximum, all tangential first derivatives vanish, and $\partial_n\omega = 0$ gives the full gradient condition. Along each boundary curve the second derivative is nonpositive; the curvature term vanishes because the full gradient is zero. A one-sided Taylor expansion along the inward normal gives the same sign for the normal second derivative. Summing over an orthonormal tangential–normal frame yields $\Delta\omega \leq 0$, and therefore $L_p\omega = \Delta\omega \leq 0$. Thus (3.7) implies

$$\partial_t\omega(t, x_M) \leq -(M(t) - \bar{\omega}(t)).$$

Similarly, at a spatial minimum x_m ,

$$\nabla\omega(t, x_m) = 0, \quad L_p\omega(t, x_m) \geq 0,$$

and hence

$$\partial_t\omega(t, x_m) \geq \bar{\omega}(t) - m(t).$$

Lemma 4.1 yields

$$(4.7) \quad D^+M(t) \leq -(M(t) - \bar{\omega}(t)),$$

$$(4.8) \quad D_+m(t) \geq \bar{\omega}(t) - m(t).$$

Since $p > 0$,

$$m(t) \leq \bar{\omega}(t) \leq M(t),$$

and therefore

$$D^+M(t) \leq 0, \quad D_+m(t) \geq 0.$$

The Dini monotonicity criterion shows that M is nonincreasing and m is nondecreasing on $[\tau, T]$. Consequently,

$$(4.9) \quad m(\tau) \leq \omega(t, x) \leq M(\tau), \quad \tau \leq t \leq T.$$

For $\mathcal{O} = M - m$, the same inequalities and (4.3) give

$$(4.10) \quad D^+\mathcal{O} \leq -(M - \bar{\omega}) - (\bar{\omega} - m) = -\mathcal{O}.$$

Thus (4.4) yields $\mathcal{O}(t) \leq e^{-(t-s)}\mathcal{O}(s)$ for $\tau \leq s \leq t \leq T$.

Because $q > d$, the embedding

$$W^{1,q}(\Omega) \hookrightarrow C(\bar{\Omega})$$

is continuous. Hence

$$\omega(t) \rightarrow \omega_0 \quad \text{uniformly on } \bar{\Omega} \quad \text{as } t \downarrow 0,$$

and thus

$$M(\tau) \rightarrow M_0, \quad m(\tau) \rightarrow m_0.$$

Letting $\tau \downarrow 0$ proves (4.5), the monotonicity of the extrema, and (4.6), including $s = 0$. Since $T < T_{\max}$ is arbitrary, all assertions hold throughout the maximal existence interval. $\square$

Set

$$K_0 := M_0 - m_0.$$

Then

$$(4.11) \quad \|\omega(t) - \bar{\omega}(t)\|_{L^\infty} \leq K_0.$$

Applying the elliptic estimate to

$$L_p \phi = -(\omega - \bar{\omega})$$

gives, for every fixed $s > d$,

$$\|\phi(t)\|_{W^{2,s}} \leq C_s \|\omega(t) - \bar{\omega}(t)\|_{L^s} \leq C_s K_0.$$

Hence Sobolev embedding yields the uniform bound

$$(4.12) \quad \sup_{0 \leq t < T_{\max}} \|\nabla \phi(t)\|_{L^\infty} \leq C_0,$$

where C_0 depends only on the initial oscillation and the fixed coefficients.

5. Global existence of solution. The maximum principle bounds ω and the elliptic drift independently of the current Sobolev norm. We now use these bounds to control $\|\omega(t)\|_{W^{1,q}}$ on each finite time interval.

5.1. Global $W^{1,q}$ estimate. A direct estimate of (3.8) in the variable ω leads to a quadratic Volterra inequality of the schematic form

$$Y(t) \lesssim C + \int_0^t (t-s)^{-\theta} (Y(s) + Y(s)^2) ds, \quad Y(t) = \|\omega(t)\|_{W^{1,q}},$$

which is sufficient for local theory but does not by itself rule out finite-time growth. It is therefore advantageous, at the level of a priori estimates, to introduce the exponential variable.

Define

$$u := e^{-\omega}, \quad u_0 := e^{-\omega_0}.$$

By (4.5),

$$(5.1) \quad e^{-M_0} \leq u(t, x) \leq e^{-m_0} \quad (0 \leq t < T_{\max}).$$

By Lemma 3.5(i), the following computation is valid classically for every $t > 0$. Using

$$\nabla u = -u \nabla \omega, \quad L_p u = -u L_p \omega + u |\nabla \omega|^2,$$

and the ω -equation, the quadratic gradient term cancels and we obtain

$$(5.2) \quad u_t = L_p u - \nabla \phi \cdot \nabla u + (\omega - \bar{\omega})u.$$

The corresponding variation-of-constants formula holds first on $[\tau, T]$, $\tau > 0$, and then extends to $t = 0$ by the continuity of ω in $W^{1,q}$ and of $u = e^{-\omega}$ in the same space. Indeed, the Sobolev chain rule gives $\nabla u = -u \nabla \omega$, while $q > d$ and (4.5) give continuity of the exponential composition in $W^{1,q}$. The source in (5.2) belongs to

$C([0, T]; L^q)$, and the integrable kernel in (3.9) justifies passage to the limit in the integral term. Thus

$$(5.3) \quad u(t) = S(t)u_0 + \int_0^t S(t-s) [-\nabla\phi(s) \cdot \nabla u(s) + (\omega(s) - \bar{\omega}(s))u(s)] ds.$$

By (4.11), (4.12), and (5.1),

$$(5.4) \quad \begin{aligned} & \|-\nabla\phi(s) \cdot \nabla u(s) + (\omega(s) - \bar{\omega}(s))u(s)\|_{L^q} \\ & \leq C_0 \|\nabla u(s)\|_{L^q} + K_0 \|u(s)\|_{L^q} \leq C_1 \|u(s)\|_{W^{1,q}}. \end{aligned}$$

Fix a finite time horizon $H > 0$. By the semigroup property and local boundedness, there are constants $C_{S,H}$ and $C_{\text{sm},H}$, depending only on H, Ω, p, q , such that

$$\begin{aligned} \|S(t)g\|_{W^{1,q}} & \leq C_{S,H} \|g\|_{W^{1,q}}, & 0 \leq t \leq H, \\ \|S(t)f\|_{W^{1,q}} & \leq C_{\text{sm},H} t^{-1/2} \|f\|_{L^q}, & 0 < t \leq H. \end{aligned}$$

For $1 \leq t \leq H$, the second estimate follows from $S(t) = S(1/2)S(t-1/2)$, (3.9), and finite-time L^q -boundedness, enlarging the constant using $1 \leq \sqrt{H} t^{-1/2}$. Boundedness on $W^{1,q}$ follows by iterating (3.11). Applying these estimates to (5.3) and using (5.4), we obtain

$$(5.5) \quad Y_u(t) \leq A_H + B_H \int_0^t (t-s)^{-1/2} Y_u(s) ds, \quad Y_u(t) := \|u(t)\|_{W^{1,q}},$$

for $0 \leq t < \min\{H, T_{\max}\}$, where

$$A_H := C_{S,H} \|u_0\|_{W^{1,q}}, \quad B_H := C_{\text{sm},H} C_1.$$

Here C_1 depends only on the initial range and the fixed parameters. The constant A_H also depends on the initial Sobolev norm through $\|u_0\|_{W^{1,q}}$. Neither constant depends on $T_{\max} - t$. Apply Lemma B.1 first on any closed interval $[0, T']$ with $T' < \min\{H, T_{\max}\}$. Its bound is at most $A_H(1 + 2B_H\sqrt{H})e^{\pi B_H^2 H}$, independently of T' . Taking the supremum over these intervals therefore gives

$$(5.6) \quad \sup_{0 \leq t < \min\{H, T_{\max}\}} \|u(t)\|_{W^{1,q}} \leq C_H < \infty.$$

Finally, by (5.1),

$$\nabla\omega = -\frac{\nabla u}{u},$$

so

$$\|\nabla\omega(t)\|_{L^q} \leq e^{M_0} \|\nabla u(t)\|_{L^q}.$$

Together with (4.5), this yields

$$(5.7) \quad \sup_{0 \leq t < \min\{H, T_{\max}\}} \|\omega(t)\|_{W^{1,q}} \leq C_H < \infty.$$

5.2. Global continuation.

PROPOSITION 5.1 (Global Sobolev mild well-posedness). *Assume the hypotheses of Theorem 2.1, and fix q as in Remark 3.1. Then (2.12) admits a unique global mild solution*

$$\omega \in C([0, \infty); W^{1,q}(\Omega)), \quad \phi \in C([0, \infty); W^{2,q}(\Omega)).$$

For every $T > 0$,

$$\sup_{0 \leq t \leq T} (\|\omega(t)\|_{W^{1,q}} + \|\phi(t)\|_{W^{2,q}}) < \infty,$$

and (4.5) holds for all $t \geq 0$. Moreover, the solution depends continuously on the initial datum on every finite time interval in the $W^{1,q}$ -topology.

Proof. If $T_{\max} < \infty$, choose $H > T_{\max}$ in (5.7). The resulting uniform bound contradicts Lemma 3.4; hence $T_{\max} = \infty$. Lemma 3.2 gives the asserted continuity and finite-time bounds for $\phi = \Phi[\omega]$. Uniqueness and continuous dependence follow by iterating Proposition 3.3 over each finite interval. The local existence times can be chosen uniformly along this iteration because the initial $W^{1,q}$ -norm controls the initial range, and the preceding estimates are uniform for initial data in a bounded $W^{1,q}$ set. $\square$

5.3. Global strong solutions. It remains to include the initial time in the regularity estimates. We call a solution strong on $[0, T]$ if $\phi = \Phi[\omega]$, its parabolic component belongs, with $r = q/2$, to

$$W^{1,r}(0, T; L^r(\Omega)) \cap L^r(0, T; W^{2,r}(\Omega)),$$

the equations hold almost everywhere, the Neumann conditions hold in the trace sense, and the initial trace equals ω_0 . Under the initial assumptions of Theorem 2.1, the global mild solution has the stronger classical regularity stated below.

THEOREM 5.2 (Global strong and classical well-posedness). *Under the assumptions of Theorem 2.1, the global mild solution from Proposition 5.1 is the unique global strong solution of (2.12). For every $T > 0$,*

$$\omega \in C^{1+\alpha/2, 2+\alpha}([0, T] \times \bar{\Omega}), \quad \phi \in C^{\alpha/2}([0, T]; C^{2+\alpha}(\bar{\Omega})).$$

In particular, it is a global classical solution.

Proof. Fix $T > 0$. The exponent chosen before the local construction in (3.1) satisfies $\alpha < 1 - 2(d+2)/q$. The initial datum belongs to $C^{2+\alpha}$ and satisfies $\partial_n \omega_0 = 0$. Thus Lemma 3.5(ii), applied to the global mild solution on $[0, T]$, gives both claimed regularity statements and the strong-solution property.

Since $r > d + 2$, parabolic Sobolev embedding places any other strong solution in $C([0, T]; C^1(\bar{\Omega}))$, hence in $C([0, T]; W^{1,q}(\Omega))$. It also satisfies the mild formulation (3.8). Uniqueness in Proposition 5.1 identifies its parabolic component with ω , and uniqueness of the normalized elliptic problem identifies its potential with ϕ . Since $T > 0$ is arbitrary, this proves the theorem and completes the proof of Theorem 2.1. $\square$

Remark 5.3 (Probability interpretation). Assume $q_0 = pe^{-\omega_0}$ is normalized. On each finite time interval, $b(t, x) := \nabla \log p(x) + \nabla \phi(t, x)$ is continuous in time and uniformly Lipschitz in space. The normally reflected SDE in Section 2 is therefore well posed by the reflected-diffusion theory of [22]. To identify its marginal law, fix $t > 0$ and $g \in C^{2+\alpha}(\bar{\Omega})$ with $\partial_n g = 0$. By time reversal and [37, Theorem 1.17], the backward Neumann problem has a classical solution h :

$$h_s + \Delta h + b \cdot \nabla h = 0, \quad \partial_n h = 0, \quad h(t, \cdot) = g.$$

The transformed density $q_s = pe^{-\omega(s)}$ satisfies $\partial_s q_s = \Delta q_s - \nabla \cdot (b q_s)$ and $(b q_s - \nabla q_s) \cdot n = 0$. Itô's formula and integration by parts, respectively, give

$$\mathbb{E}[g(X_t)] = \int_{\Omega} q_0 h(0, \cdot) dx = \int_{\Omega} q_t g dx.$$

The reflection term vanishes since $\partial_n h = 0$. These Neumann test functions are dense in $C(\bar{\Omega})$, for example by Neumann heat-semigroup approximation. Hence the law of X_t has density q_t .

6. Long-time behavior and exponential convergence. This section characterizes the equilibrium selected by the conserved mass and establishes exponential convergence of the accelerated Langevin dynamics whose feedback velocity is constructed by the parabolic–elliptic PDE system. We prove two complementary forms of relaxation: uniform decay of the logarithmic density ratio and dissipation of relative entropy. We keep the unnormalized target p in the evolution and use

$$Z_p = \int_{\Omega} p(x) dx, \quad \pi = \frac{p}{Z_p}, \quad d\mu(x) := \pi(x) dx, \quad \int_{\Omega} d\mu = 1$$

whenever a probability measure is required.

THEOREM 6.1 (Exponential convergence to equilibrium). *Let (ω, ϕ) be the global classical solution of Theorem 2.1, and let*

$$(6.1) \quad \mathcal{M}_0 := \int_{\Omega} p(x) e^{-\omega_0(x)} dx = \int_{\Omega} q_0(x) dx.$$

Define

$$(6.2) \quad \omega_{\infty} := -\log \left(\frac{\mathcal{M}_0}{Z_p} \right) = \log \frac{Z_p}{\mathcal{M}_0}.$$

Then, for every $t \geq 0$,

$$(6.3) \quad \|\omega(t) - \omega_{\infty}\|_{L^{\infty}} \leq e^{-t} \operatorname{osc}_{\Omega} \omega_0.$$

In particular,

$$\omega(t, \cdot) \longrightarrow \omega_{\infty} \quad \text{and} \quad q_t = p e^{-\omega(t, \cdot)} \longrightarrow \frac{\mathcal{M}_0}{Z_p} p = \mathcal{M}_0 \pi$$

uniformly as $t \rightarrow \infty$. If q_0 is normalized, then $\mathcal{M}_0 = 1$, $\omega_{\infty} = \log Z_p$, and $q_t \rightarrow \pi = p/Z_p$.

Proof. Set

$$M(t) := \max_{\Omega} \omega(t, \cdot), \quad m(t) := \min_{\Omega} \omega(t, \cdot), \quad \mathcal{O}(t) := M(t) - m(t).$$

Lemma 4.2 already gives $\mathcal{O}(t) \leq e^{-t} \mathcal{O}(0)$ by (4.6).

Writing $q_t = p e^{-\omega(t)}$, the transformed equation is

$$\partial_t q_t = -\nabla \cdot (q_t (\nabla \omega + \nabla \phi)).$$

The Neumann conditions therefore imply

$$(6.4) \quad \frac{d}{dt} \int_{\Omega} p(x) e^{-\omega(t, x)} dx = - \int_{\partial \Omega} q_t (\partial_n \omega + \partial_n \phi) dS = 0,$$

and hence

$$(6.5) \quad \int_{\Omega} p(x) e^{-\omega(t, x)} dx = \mathcal{M}_0 = Z_p e^{-\omega_{\infty}}.$$

Since

$$Z_p e^{-M(t)} \leq \int_{\Omega} p(x) e^{-\omega(t,x)} dx \leq Z_p e^{-m(t)},$$

we obtain

$$e^{-M(t)} \leq \frac{\mathcal{M}_0}{Z_p} = e^{-\omega_{\infty}} \leq e^{-m(t)}.$$

Therefore $\omega_{\infty} \in [m(t), M(t)]$, and

$$\|\omega(t) - \omega_{\infty}\|_{L^{\infty}} \leq M(t) - m(t) \leq e^{-t} \operatorname{osc}_{\Omega} \omega_0,$$

which proves (6.3). Since $\omega_{\infty}, \omega(t, x) \in [m_0, M_0]$, the mean value theorem gives

$$\|q_t - \mathcal{M}_0 \pi\|_{L^{\infty}} \leq p^* e^{-m_0} \|\omega(t) - \omega_{\infty}\|_{L^{\infty}},$$

proving the asserted density convergence with the same exponential rate. $\square$

For normalized initial data, Remark 5.3 identifies the density q_t in Theorem 6.1 with the marginal density of the accelerated Langevin dynamics introduced in Section 2. Therefore Theorem 6.1 is an exponential convergence result for the accelerated Langevin dynamics itself: for normalized q_0 , the law of X_t converges to the normalized target $\pi = p/Z_p$, while the logarithmic density ratio contracts at the explicit rate e^{-t} .

The normalization of p affects only the additive equilibrium level of ω . For a probability density q_t , the limiting value is $\log Z_p$, while the limiting density is the normalized target $\pi = p/Z_p$. The evolution itself is unchanged by the unknown constant Z_p .

6.1. Exact entropy dissipation and KL convergence. For the entropy statements, q is a probability density and the reference probability density is $\pi = p/Z_p$. We use the relative Fisher information

$$(6.6) \quad I(q\|\pi) := \int_{\Omega} q \left| \nabla \log \frac{q}{\pi} \right|^2 dx = \int_{\Omega} q \left| \nabla \log \frac{q}{p} \right|^2 dx$$

and the Jeffreys divergence

$$(6.7) \quad J(q, \pi) := \operatorname{KL}(q\|\pi) + \operatorname{KL}(\pi\|q).$$

THEOREM 6.2 (Exact Jeffreys–Fisher entropy dissipation). *Let (ω, ϕ) be the global classical solution of Theorem 2.1, set*

$$q_t := p e^{-\omega(t, \cdot)},$$

and assume that q_0 is normalized:

$$\int_{\Omega} q_0 dx = 1.$$

Then q_t remains normalized for all $t \geq 0$, and

$$(6.8) \quad \frac{d}{dt} \operatorname{KL}(q_t\|\pi) = -I(q_t\|\pi) - J(q_t, \pi).$$

Proof. With

$$r := \log \frac{q_t}{\pi} = -\omega + \log Z_p,$$

define

$$H(t) := \text{KL}(q_t \| \pi) = \int_{\Omega} q_t r \, dx.$$

Mass conservation gives $H'(t) = \int_{\Omega} r \, \partial_t q_t \, dx$. Using the transformed density equation and $\partial_n r = -\partial_n \omega = 0$, $\partial_n \phi = 0$ in the integrations by parts below, we obtain

$$(6.9) \quad H'(t) = - \int_{\Omega} r \, \nabla \cdot (q_t \nabla \phi) \, dx + \int_{\Omega} r \, \nabla \cdot (q_t \nabla r) \, dx.$$

For the Langevin contribution, integration by parts yields

$$(6.10) \quad \begin{aligned} \int_{\Omega} r \, \nabla \cdot (q_t \nabla r) \, dx &= - \int_{\Omega} q_t |\nabla r|^2 \, dx \\ &= -I(q_t \| \pi). \end{aligned}$$

For the weighted-Poisson transport contribution,

$$(6.11) \quad - \int_{\Omega} r \, \nabla \cdot (q_t \nabla \phi) \, dx = \int_{\Omega} q_t \nabla r \cdot \nabla \phi \, dx.$$

Since $q_t = \pi e^r$,

$$q_t \nabla r = \pi \nabla(e^r).$$

Set

$$\bar{r}_{\pi} := \int_{\Omega} \pi r \, dx = -\bar{\omega} + \log Z_p, \quad L_p \phi = r - \bar{r}_{\pi}.$$

Using $L_{\pi} = L_p$, integration by parts gives

$$(6.12) \quad \begin{aligned} \int_{\Omega} q_t \nabla r \cdot \nabla \phi \, dx &= \int_{\Omega} \pi \nabla(e^r) \cdot \nabla \phi \, dx \\ &= - \int_{\Omega} e^r \nabla \cdot (\pi \nabla \phi) \, dx \\ &= - \int_{\Omega} q_t L_p \phi \, dx \\ &= - \int_{\Omega} q_t (r - \bar{r}_{\pi}) \, dx. \end{aligned}$$

Because both q_t and π are normalized,

$$\int_{\Omega} q_t (r - \bar{r}_{\pi}) \, dx = \int_{\Omega} q_t r \, dx - \bar{r}_{\pi}.$$

Moreover,

$$\int_{\Omega} q_t r \, dx = \text{KL}(q_t \| \pi), \quad \bar{r}_{\pi} = \int_{\Omega} \pi \log \frac{q_t}{\pi} \, dx = -\text{KL}(\pi \| q_t).$$

Hence

$$(6.13) \quad - \int_{\Omega} q_t (r - \bar{r}_{\pi}) \, dx = -[\text{KL}(q_t \| \pi) + \text{KL}(\pi \| q_t)] = -J(q_t, \pi).$$

Combining (6.9), (6.10), and (6.13) proves (6.8). $\square$

COROLLARY 6.3 (Exponential convergence in relative entropy). *Under the assumptions of Theorem 6.2,*

$$(6.14) \quad \text{KL}(q_t \| \pi) \leq e^{-t} \text{KL}(q_0 \| \pi), \quad t \geq 0.$$

Proof. Since relative entropy is nonnegative,

$$J(q, \pi) = \text{KL}(q \| \pi) + \text{KL}(\pi \| q) \geq \text{KL}(q \| \pi).$$

Thus (6.8) implies

$$\frac{d}{dt} \text{KL}(q_t \| \pi) \leq -\text{KL}(q_t \| \pi),$$

and Grönwall's inequality gives (6.14). $\square$

7. Conclusion. We have analyzed an accelerated Langevin dynamics in which the classical local Langevin relaxation is coupled to a nonlocal gradient velocity reconstructed from a target-weighted Poisson equation. The target p may be unnormalized. On a bounded $C^{2+\alpha}$ domain with homogeneous Neumann boundary conditions, the logarithmic ratio $\omega = \log(p/q)$ leads to a parabolic–elliptic system for which we established local $W^{1,q}$ mild solutions, a maximum principle, global continuation, and global strong solutions with $C^{1+\alpha/2, 2+\alpha}$ regularity up to the initial time. The exponential variable $u = e^{-\omega}$ converts the gradient nonlinearity into terms controlled by a linear singular Grönwall estimate. A common regularity lemma supplies both the positive-time regularity needed for the maximum principle and the regularity at the initial time.

The evolution is invariant under multiplication of p by a positive constant. Writing

$$Z_p = \int_{\Omega} p \, dx, \quad \pi = \frac{p}{Z_p},$$

the conserved quantity $\int_{\Omega} p e^{-\omega} \, dx = \int_{\Omega} q_t \, dx$ selects the limiting additive constant of ω , while its spatial oscillation contracts at the explicit rate e^{-t} . For normalized solution densities, $\omega(t) \rightarrow \log Z_p$ and $q_t \rightarrow \pi$ uniformly. At the entropy level, the exact Jeffreys–Fisher dissipation identity is naturally written relative to the normalized target π : the Langevin diffusion dissipates $I(q_t \| \pi)$, whereas the weighted-Poisson transport dissipates $J(q_t, \pi)$. Consequently, $\text{KL}(q_t \| \pi)$ decays at least as e^{-t} without a convexity assumption on the target.

These results provide a continuous-time analytical foundation for combining local diffusive sampling with nonlocal weighted-Laplacian transport. A particle implementation may not solve the evolution equation for ω . Given particles approximating q_t , one may estimate a scalar log-density potential $f_t \approx \log q_t + C_t$, form $\hat{\omega}_t = \log p - f_t$, and reconstruct the nonlocal velocity from a weighted-Poisson problem. The additive constant C_t is immaterial because the elliptic source depends only on the centered quantity $\omega - \bar{\omega}$. This observation makes scalar density-estimation methods that identify $\log q_t$ only up to an additive constant, such as constant-target energy matching (CTEM) [39], naturally compatible with the present model. In high dimension, the variational structure of the weighted-Poisson equation also suggests neural variational solvers of Deep Ritz type [14]. Moreover, the decomposition (1.1) suggests an intermittent numerical strategy: cheap Langevin evolution may be used by default, while the more expensive nonlocal field is recomputed only when the remaining local Fisher dissipation is small relative to the global Jeffreys mismatch. A full particle algorithm, its error analysis, and numerical experiments are left to subsequent work.

Appendix A. Semigroup smoothing estimates.

Let $S(t) = e^{tL_p}$ be the Neumann semigroup generated by

$$L_p = \Delta + \nabla \log p \cdot \nabla.$$

We record only the estimates used in the fixed-point and continuation arguments.

PROPOSITION A.1 (Short-time smoothing). *Let $1 < r \leq q < \infty$. There exists $C = C(r, q, p, \Omega)$ such that, for $0 < t \leq 1$,*

$$(A.1) \quad \|S(t)g\|_{L^q} \leq Ct^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})} \|g\|_{L^r},$$

$$(A.2) \quad \|S(t)g\|_{W^{1,q}} \leq Ct^{-\frac{1}{2}-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})} \|g\|_{L^r}.$$

Moreover,

$$(A.3) \quad \sup_{0 \leq t \leq 1} \|S(t)g\|_{W^{1,q}} \leq C \|g\|_{W^{1,q}}.$$

If, in addition, $q > \max\{d, 2\}$, then

$$(A.4) \quad \|S(t)g\|_{W^{1,q}} \leq Ct^{-1/2} \|g\|_{L^q},$$

$$(A.5) \quad \|S(t)g\|_{W^{1,q}} \leq Ct^{-\theta} \|g\|_{L^{q/2}}, \quad \theta := \frac{1}{2} + \frac{d}{2q} < 1,$$

and

$$(A.6) \quad \int_0^t (t-s)^{-\theta} ds = \frac{t^{1-\theta}}{1-\theta} = C_{d,q} t^{\frac{q-d}{2q}}.$$

Proof. The standard Gaussian upper bound for the Neumann heat kernel of the uniformly elliptic operator L_p gives (A.1). It remains to gain one spatial derivative. Analyticity of the semigroup on L^q yields, for $0 < s \leq 1$,

$$\|L_p S(s)h\|_{L^q} \leq Cs^{-1} \|h\|_{L^q}, \quad \|S(s)h\|_{L^q} \leq C \|h\|_{L^q}.$$

The Neumann elliptic estimate therefore implies

$$\|S(s)h\|_{W^{2,q}} \leq C(\|L_p S(s)h\|_{L^q} + \|S(s)h\|_{L^q}) \leq Cs^{-1} \|h\|_{L^q}.$$

Interpolating between L^q and $W^{2,q}$ gives

$$\|S(s)h\|_{W^{1,q}} \leq Cs^{-1/2} \|h\|_{L^q}.$$

Applying this estimate to $S(t)g = S(t/2)S(t/2)g$, and using (A.1) on the inner factor, proves (A.2). The boundedness (A.3) is the standard strong continuity/boundedness of the Neumann analytic semigroup on $W^{1,q}$. Taking $r = q$ and $r = q/2$ gives (A.4) and (A.5); the final integral is elementary, and $\theta < 1$ is exactly the assumption $q > d$. $\square$

Appendix B. Singular Grönwall inequality with square-root kernel.

LEMMA B.1 (Square-root singular Grönwall inequality). *Let $Y \in C([0, T]; [0, \infty))$ satisfy*

$$(B.1) \quad Y(t) \leq A + B \int_0^t (t-s)^{-1/2} Y(s) ds, \quad 0 \leq t \leq T,$$

for $A, B \geq 0$. Then

$$(B.2) \quad Y(t) \leq A(1 + 2B\sqrt{T})e^{\pi B^2 t}, \quad 0 \leq t \leq T.$$

Proof. Set

$$(Kf)(t) := \int_0^t (t-s)^{-1/2} f(s) ds.$$

The beta-integral gives

$$K^2 f(t) = \pi \int_0^t f(s) ds, \quad K1(t) = 2\sqrt{t}.$$

From $Y \leq A + BKY$ we therefore obtain

$$KY(t) \leq 2A\sqrt{t} + \pi B \int_0^t Y(s) ds,$$

and hence, for $t \leq T$,

$$Y(t) \leq A(1 + 2B\sqrt{T}) + \pi B^2 \int_0^t Y(s) ds.$$

The classical Grönwall inequality gives (B.2). □

Appendix C. Proof of the Regularity Lemma.

This appendix proves Lemma 3.5.

Proof of Lemma 3.5. Set $r = q/2$. By (3.1), we can choose $\alpha < \eta < 1 - (d+2)/r$. The reduced equation is

$$\omega_t - L_p \omega = \mathcal{N}(\omega), \quad \mathcal{N}(\omega) = -\nabla \phi \cdot \nabla \omega - |\nabla \omega|^2 - (\omega - \bar{\omega}).$$

Lemma 3.2 and $\omega \in C([0, T]; W^{1,q})$ give $\nabla \phi \in C([0, T]; L^\infty)$, and hence $\mathcal{N}(\omega) \in C([0, T]; L^r)$.

Let A_r be the Neumann realization of L_p on $L^r(\Omega)$, with

$$D(A_r) = \{w \in W^{2,r}(\Omega) : \partial_n w = 0\}.$$

For (ii), set $v = \omega$ and $a = 0$. The initial assumptions give $\omega_0 \in D(A_r) \subset (L^r, D(A_r))_{1-1/r, r}$. For (i), choose a smooth time cutoff χ that vanishes on $[0, \tau/4]$ and equals one on $[\tau/2, T]$; set $v = \chi\omega$ and $a = \tau/2$. Then $v(0) = 0$ and $v_t - L_p v = \chi\mathcal{N}(\omega) + \chi'\omega$, whose right-hand side belongs to $L^r(0, T; L^r)$. In both cases, the scalar uniformly elliptic Neumann operator satisfies the boundary hypotheses for maximal regularity; see [10, Lemma 2.4 and Theorem 6.14]. A spectral shift is removed on $[0, T]$ by multiplication by $e^{-\mu t}$. Thus

$$v \in W^{1,r}(0, T; L^r) \cap L^r(0, T; W^{2,r}).$$

The forcing is treated as prescribed, so linear uniqueness identifies this solution with the given v . This proves (3.23) in case (ii). In either case, $v = \omega$ on $J = [a, T]$.

Parabolic Sobolev embedding on $Q_J = J \times \bar{\Omega}$ gives

$$\omega \in C^{(1+\eta)/2, 1+\eta}(Q_J), \quad \nabla \omega \in C^{\alpha/2, \alpha}(Q_J).$$

The time exponents for ω and $\nabla \omega$ are $(1+\eta)/2$ and $\eta/2$, both larger than $\alpha/2$. Together with $C^1(\bar{\Omega}) \hookrightarrow C^\alpha(\bar{\Omega})$, this yields

$$(C.1) \quad \|\omega(t) - \omega(s)\|_{C^\alpha} \leq C \|\omega(t) - \omega(s)\|_{C^1} \leq C_J |t - s|^{\alpha/2}, \quad s, t \in J.$$

We next record the elliptic estimate needed for both parts. For $w \in C^\alpha$ and $\psi = \Phi[w]$, Lemma 3.2 gives $\|\nabla \psi\|_{C^\alpha} \leq C\|w\|_{C^\alpha}$, since $\alpha < 1 - d/q$. Rewrite the elliptic equation as

$$\Delta \psi = f_w := -(w - \bar{w}) - \nabla \log p \cdot \nabla \psi, \quad \|f_w\|_{C^\alpha} \leq C\|w\|_{C^\alpha}.$$

The Neumann condition implies $\int_\Omega f_w dx = 0$. The normalized Neumann Schauder estimate for the Laplacian [31, Theorems 3.1 and 4.1], together with uniqueness of its zero-mean weak solution, gives $\|\Phi[w]\|_{C^{2+\alpha}} \leq C\|w\|_{C^\alpha}$. By linearity,

$$(C.2) \quad \|\Phi[w_1] - \Phi[w_2]\|_{C^{2+\alpha}} \leq C\|w_1 - w_2\|_{C^\alpha}.$$

Applying this estimate to $\omega(t)$ and 0, and then to $\omega(t)$ and $\omega(s)$, proves by (C.1) that $\phi \in C^{\alpha/2, \alpha}(J; C^{2+\alpha})$. Both gradients therefore belong to the algebra $C^{\alpha/2, \alpha}(Q_J)$. The weighted mean is a bounded linear functional on C^0 , so $\mathcal{N}(\omega) \in C^{\alpha/2, \alpha}(Q_J)$.

To finish both parts with the same Schauder argument, take $\rho = 1$ in case (ii). In case (i), choose a smooth ρ that vanishes near $a = \tau/2$ and equals one on $[\tau, T]$. The function $z = \rho\omega$ satisfies

$$z_t - L_p z = \rho \mathcal{N}(\omega) + \rho' \omega, \quad \partial_n z = 0$$

on J , with initial value ω_0 in case (ii) and zero in case (i). Its forcing belongs to $C^{\alpha/2, \alpha}(Q_J)$. The principal matrix is the identity, $\nabla \log p \in C^{1+\alpha}$, the boundary is $C^{2+\alpha}$, and the initial value is $C^{2+\alpha}$ and satisfies the Neumann compatibility condition. After translating the initial time a to zero, [37, Theorem 1.17, p. 16] applies with $a_{ij} = \delta_{ij}$, $b_i = -\partial_i \log p$, $c = 0$, and $b = g = 0$, in that theorem's notation. Its conditions (C) and (E) hold, and its compatibility condition reduces to $\partial_n z(a, \cdot) = 0$. It yields a solution in $C^{1+\alpha/2, 2+\alpha}(Q_J)$. Both this solution and z belong to $W^{1,r}(a, T; L^r) \cap L^r(a, T; W^{2,r})$ and solve the same linear Neumann problem with the same initial value, so linear uniqueness identifies them. Restricting to the interval where $\rho = 1$ proves the asserted regularity of ω ; the estimate for ϕ was already obtained on J . $\square$

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