Poincaré conjecture and via-topological spaces

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Abstract: The notion of a via-topological space is introduced on the basis of directed paths connecting its points and different various examples of such spaces are presented. Further the corresponding to their structure definitions of continuity, homeomorphism, and homogeneity are given, which are interpreted in the language of category theory. In terms of these new notions a theorem about the existence on the homogeneous via-topological spaces a structure of a smooth manifold diffeomorphic to the three-dimensional sphere is proved.

Key words: Poincaré conjecture, via-topological space, homogeneous space.

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1. Introduction

In 1904, Henri Poincaré in his work [1] posed a question that has come to be known as the Poincaré conjecture. Its standard modern formulation states that a simply connected, compact, three-dimensional manifold without boundary is homeomorphic to the 3-sphere $\mathbb{S}^3$. In 1961, Smale [2] provided a proof of an analogous theorem concerning smooth manifolds of dimension five and higher. Subsequently, in 1982, Freedman [3] presented a rather complex proof for 4-manifolds.

We shall not reproduce here the history of the problem's resolution — a subject covered in detail in many other sources — as our objective is different. First, let us consider the following figures: the total volume of the three well-known works in Arxiv by Perelman (2002–2003), which provides the final proof of the conjecture for 3-manifolds, spans 68 pages, while the subsequent text by Kleiner and Lott [4], which includes various additions and clarifications, runs to 269 pages. Furthermore, while these works confirm the validity of the conjecture from a technical standpoint, they offer more insight into the behavior of solutions to a certain nonlinear partial differential equation than they do into the nature of three-dimensional topology itself. In fairness, it should be noted that the results of Freedman and Perelman, beyond answering the original question, have made a significant contribution to the general classification of manifolds. Nevertheless, a natural question arises: if proving an intuitively clear theorem required techniques so complex, then is the very concept of a topological structure truly appropriate in every respect?

The modern standard definition of the topology of a space, which is expressed in terms of open subsets, was apparently first introduced by Hausdorff only in 1914 [5], a full decade after the Poincaré conjecture. In essence this concept can already be discerned in Cauchy's definition of the continuity of a function, in the «epsilon-delta» language. In what sense did Riemann and Poincaré originally understand topology? Riemann studied curves and surfaces endowed with a topology induced by the Euclidean structure of real spaces; he used the term «analysis situs» to refer to a theory of continuous quantities that abstracts away from measurement, focusing instead solely on relationships of mutual position and inclusion. This is discussed, for instance, in his work "Theorie der Abelschen Funktionen" (Crelle, Vol. 54, 1857); see also the entry "Topological Spaces" in Bourbaki's book [6].

Poincaré understands topology in the very same sense, defining manifolds by means of equations and inequalities, and setting within this branch of geometry the problem of describ-

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ing the relative positions of points, lines, and surfaces without regard for their magnitude[2] [7]. Moreover, the primary tools employed in Poincaré 's proofs are precisely curves, cycles, and polyhedra [8].

In a simple everyday sense «topology» of landscapes, cities, buildings, and communication networks is defined by the paths — whether physically possible or permitted by specific rules of movement — that connect points within these structures. In the following section, utilizing subsets referred to as paths, we provide a precise definition for an alternative concept of topology. In the context of the 3-sphere, this concept is naturally suggested by the well-known Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \to \mathbb{S}^2$. Furthermore, in the broader study of three-dimensional manifolds, this approach is justified by the effectiveness of knot and link theory [9].

To a certain extent, the structures that we introduce are analogous to topological graphs [10]. Next, we define the continuity of mappings and the concept of homeomorphism of spaces, interpreting these new concepts in the language of category theory. In section 3, we introduce homogeneous spaces and the notion of their dimension. In section 4, we prove a remote analogue of the Poincaré conjecture.

2. Basic concepts

To distinguish the new concept to be introduced from the standard concept of topology, we add the prefix «via» meaning a path in Latin.

Definition 1. *A via-topological space — or a via-space for short — is a pair (X, V), where X is a non-empty set whose elements are called points, and V — referred to as a via-topology — is a collection of non-empty ordered*[3] *subsets $v(a, b)$ of X. These subsets are defined for every ordered pair of points (a, b) so that $a \preceq_v b$ and $\{a, b\} \subset v(a, b)$; they are called paths directed from point a to point b. This system must satisfy the additivity condition: if $v(a, b) \in V$ and $v(b, c) \in V$, then $v(a, b) \cup v(b, c) \in V$. Furthermore, $V = \bigcup_{a,b \in X} V(a, b)$, where $V(a, b) = \{v(a, b)\}$ is the set of all paths from point a to point b. Paths of the form $v(a, a)$ are called loops, and it is always assumed that $\{a\} \in V(a, a)$ for any point $a \in X$. The dimension of all points is put to be zero, while the dimension of paths that are not points is put to be one.*

Lemma 1. *Any via-space (X, V) is a category whose objects are the points of the set X and whose morphisms are the paths of the via-topology V.*

Proof. The statement of the lemma is obtained immediately from Definition 1 by elementary checking the axioms of a category [11, Ch. 1, §1].

Definition 2. *Let (X, V) be a via-space. A pair (X', V') is called a subspace of (X, V) if $X' \subseteq X$ and the collection $V' \subseteq V$ consists of all paths that lie entirely within X'. In this case we write $(X', V') \hookrightarrow (X, V)$.*

Example 1. Consider (C, V^+), where C is a circle in the plane and

$$V^+ = \left\{ \{v(p, p) = \{p\}\}_{p \in C}, \{A^+(a, b)\}_{a,b \in C} \right\},$$

where $A^+(a, b)$ are arcs of the circle oriented only in one direction.

[2] The original text states: "...une branche de la Géométrie que l'on a appelée Analysis situs, et qui décrit la situation relative des points des lignes et des surfaces, sans aucune considération de leur grandeur."

[3] The ordering relation $\preceq_v$ generally depends on a path. If $a = b$, then when ordering the set $v(a, a)$, the point a is counted only once as the starting point.

Example 2. Following the same principle of anisotropy, one can consider a 4-point space (T_4, Σ_4), where $T_4 = \{s, h, l, r\}$,

$$\Sigma_4 = \{\{s\}, \{h\}, \{l\}, \{r\}, v(s,h), v(h,r), v(h,l), v(r,s), v(l,s)\} \cup \Sigma^*,$$

and the set Σ^* consists of paths formed by concatenating the ordered two-point paths already listed.

Example 3. On the phase plane $\{(p,q) \in \mathbb{R}^2\}$ for systems of differential equations of the form $\dot{p} = F(t)$, $\dot{q} = p$, one can define a family of via-topologies parametrized by $\varepsilon > 0$, by taking as paths those trajectories that satisfy the constraint $|F(t)| < \varepsilon$.

Example 4. Consider the projective plane $\mathbb{R}\mathrm{P}^2$, realized as the set of lines passing through the origin in $\mathbb{R}^3$, and endow it with a via-topology by defining its paths to be composed of connected segments of orbits under the action of one-dimensional subgroups of the group $\mathrm{SO}(3)$.

Example 5. We may endow the space $\mathbb{R}^n$ with a via-topology in which the paths are piecewise smooth curves of rational length (note that this property is additive). Although a metric was used here, it can be disregarded once the via-topology has been defined. This example admits variations if the set of rational numbers is replaced by other semigroups of $\mathbb{R}^+$.

Example 6. Let us consider the three-dimensional sphere $\mathbb{S}^3$ equipped with a contact structure [12, Ch. 3, Sect. 3.1]. We may define a via-topology on this space by taking its paths to be composed of connected segments of one-dimensional Legendre submanifolds.

Example 7. On the seven-dimensional «algebraic» sphere realized as the quaternionic Hopf fibration $\mathrm{Sp}(1) \hookrightarrow \mathbb{S}^7 \to \mathbb{H}\mathrm{P}^1$, one can define a via-topology by constructing paths from connected segments of orbits of one-dimensional subgroups of $\mathrm{Spin}(4)$, which acts within the fibers $\mathrm{Sp}(1) \approx \mathbb{S}^3$, and from connected segments of orbits of one-dimensional subgroups of $\mathrm{Sp}(2)$, which acts on the base space — the quaternionic projective line $\mathbb{H}\mathrm{P}^1 \approx \mathbb{S}^4$.

Example 8. Let us equip the function space $C^1[0,1]$ with a via-topology by defining the paths $v(f_1, f_2)$ to be smooth transformations such that the number of critical points of any function $f \in v(f_1, f_2)$ does not exceed $\max\{N_1, N_2\} + 2$, where N_1 and N_2 are the numbers of critical points of the initial and final functions, f_1 and f_2, respectively.

Definition 3. *Let (X, V) and (X^*, V^*) be two via-spaces. A mapping $F : X \to X^*$ is called continuous if $F(v) \in V^*$ for every path $v \in V$.*

Lemma 2. *For any two via-spaces (X, V) and (X^*, V^*) a continuous mapping $F : X \to X^*$ induces a covariant functor from the category (X, V) to the category (X^*, V^*).*

Proof. The statement of the lemma is obtained from Definition 1 and Lemma 1 by elementary checking the axioms of a covariant functor [11, Ch. 1, §3].

Remark 1. Since, according to definition 1, a via-topology is composed of paths, the standard definition of continuity for mappings in terms of preimages is entirely unsuitable. Indeed, consider the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x,y) = x^2 + y^2,$$

which is naturally considered to be continuous. However, the preimage of the path $[0,1]$ is the disk $\{(x,y) : x^2 + y^2 \leqslant 1\}$, but not a path.

Definition 4. *Let (X, V) and (X^*, V^*) be via-spaces. If there exists a bijective mapping $F : X \to X^*$ that induces a bijective mapping $\Phi : V \to V^*$ such that for any points $a \in X$ and $b \in X$ and for any path $v(a, b) \in V$, the following condition holds*:

$$F(v(a, b)) = \Phi(v(a, b))(F(a), F(b)),$$

then these via-spaces are called homeomorphic, which is denoted by the formula

$$(X, V) \overset{\text{via}}{\approx} (X^*, V^*).$$

Remark 2. To avoid introducing additional terminology, within the specific context of via-topological spaces, continuity and homeomorphism are understood in the sense of definitions 3 and 4, respectively.

Lemmas 1 and 2 suggest the following simple statement.

Theorem 1. *For any two homeomorphic via-spaces (X, V) and (X^*, V^*), the corresponding categories are equivalent.*

Proof. According to Definition 4 and Lemma 2, by virtue of the homeomorphism

$$(X, V) \overset{\text{via}}{\approx} (X^*, V^*)$$

we have two bijective mappings $F : X \to X^*$, $\Phi : V \to V^*$, and two covariant functors

$$(F, \Phi) : (X, V) \to (X^*, V^*), \qquad (F^{-1}, \Phi^{-1}) : (X^*, V^*) \to (X, V)$$

such that for any $a \in X$, $b \in X$, $v(a, b) \in V$, $a^* \in X^*$, $b^* \in X^*$, $v^*(a^*, b^*) \in V^*$ the following conditions hold:

$$F(v(a, b)) = \Phi(v(a, b))(F(a), F(b)), \qquad F^{-1}(v^*(a^*, b^*)) = \Phi^{-1}(v^*(a^*, b^*))(a, b).$$

Then we obtain the trivial isomorphisms of functors

$$(F^{-1}, \Phi^{-1})(F, \Phi) \cong \mathrm{id}_{(X,V)}, \qquad (F, \Phi)(F^{-1}, \Phi^{-1}) \cong \mathrm{id}_{(X^*,V^*)},$$

which imply the equivalence of categories (X, V) and (X^*, V^*) [11, Ch. 1, §6].

3. Homogeneous via-spaces

To formulate of an analogue of the Poincaré conjecture for via-spaces we must first introduce the concept of their dimension.

Definition 5. *Let (X, V) be a via-space, let G be a group acting transitively on the set X, and let $\eta : V \to G$ be a surjective mapping such that $\eta(v(a, b)) = g \Rightarrow ga = b$. Furthermore, suppose that the correspondences $\eta(v_1(a, b)) = g_1$ and $\eta(v_2(b, c)) = g_2$ imply the correspondence $\eta(v_1(a, b) \cup v_2(b, c)) = g_2 g_1$. Then (X, V) is called a homogeneous via-space with transformation group G denoted by (X, V, G, η).*

Remark 3. The implication «⇒» in definition 5 is not in general an equivalence, since one can move from one point to another by different paths and by applying different elements of the group G. It should be added that each element of the transformation group induces a functor from the category (X, V) to itself.

Definition 6. *Let (X, V, G, η) be a homogeneous via-space. Suppose that there exists a subsystem of paths $T \subseteq V$ and a group L such that (X, T, L, θ) is also a homogeneous space, where $\theta : T \to L$ is surjective, and the transformation group L is isomorphic to a Lie group. If such a subsystem is not unique, we choose the one for which the Lie group has the smallest topological dimension, and we define the dimension of the space (X, V, G, η) to be equal to the dimension of this Lie group.*

Remark 4. A Lie group describes generally far fewer paths than could theoretically be defined on a given set; this is the reason for definition 6. It should be added that the decision to employ homogeneous spaces in this work was due to difficulties encountered in defining the dimensions of via-spaces in the general case.

Theorem 2. *The transformation groups of any two homeomorphic homogeneous via-spaces, (X, V, G, η) and (X^*, V^*, G^*, η^*), are isomorphic.*

Proof. Let a homeomorphism between homogeneous via-spaces

$$(X, V, G, \eta) \overset{\text{via}}{\approx} (X^*, V^*, G^*, \eta^*)$$

be defined by bijective mappings $F : X \to X^*$ and $\Phi : V \to V^*$, and let these spaces have transformation groups G and G^* with corresponding mappings $\eta : V \to G$ and $\eta^* : V^* \to G^*$. Define a mapping $\varphi : G \to G^*$ according to the following rule: if $v^* = \Phi(v)$, then $\varphi : \eta(v) \mapsto \eta^*(v^*)$. Now, consider three arbitrary points $a_1, a_2, a_3 \in X$ and two paths $v_1(a_1, a_2)$ and $v_2(a_2, a_3)$. Let $a_j^* = F(a_j)$ for $j \in \{1, 2, 3\}$; let $v_i^* = \Phi(v_i)$, $g_i = \eta(v_i)$, and $g_i^* = \eta^*(v_i^*)$ for $i \in \{1, 2\}$. Then from definition 5, it follows that

$$\eta : v_1(a_1, a_2) \cup v_2(a_2, a_3) \mapsto g_2 g_1, \qquad \eta^* : v_1^*(a_1^*, a_2^*) \cup v_2^*(a_2^*, a_3^*) \mapsto g_2^* g_1^*,$$

which leads to the following identity relations:

$$\varphi(g_2 g_1) = g_2^* g_1^* = \eta^*(v_2^*)\eta^*(v_1^*) = \varphi(\eta(v_2))\varphi(\eta(v_1)) = \varphi(g_2)\varphi(g_1),$$

that is, φ is a homomorphism. Using the bijectivity of the mappings F and Φ, we obtain a mapping φ^{-1} with the property $\varphi^{-1}(g_2^* g_1^*) = g_2 g_1$, which implies that the groups G and G^* are isomorphic.

Theorem 2 immediately implies the following conclusion.

Theorem 3. *Homogeneous via-spaces of different dimensions are not homeomorphic.*

Definition 7. *A via-space (X, V) is called isotropic if the via-topology V, which is also called isotropic, along with every path contains the reverse path, that is a path consisting of the same points but with the opposite ordering.*

Example 9. Consider a 3-point isotropic via-space (P_3, V_{21}) that is representable in the form of a one-dimensional loop, where $P_3 = \{a, b, c\}$ and V_{21} consists of three single-point loops, six 2-point paths, six 3-point paths with distinct endpoints, and six 3-point loops. Now consider an anisotropic via-space (P_3, V_{12}), where V_{12} consists of three single-point loops, three 2-point unidirectional paths, three 3-point paths with distinct endpoints, and three 3-point loops. Both via-spaces, (P_3, V_{12}) and (P_3, V_{21}), are homogeneous with transformation groups isomorphic to $\mathbb{Z}_3$. Thus, *non-homeomorphic spaces may have isomorphic transformation groups.*

4. Poincaré conjecture in the context of via-topology

Lemma 3. *Let (X, V, G, η) be a homogeneous via-space, where the transformation group G is a Lie group with the trivial stationary subgroup, that is a trivial isotropy group* $\mathrm{Stab}(G)$. *Then the set X can be endowed with the structure of a C^∞-smooth manifold that is diffeomorphic to the transformation group G.*

Proof. Since the action of the transformation group $A : G \times X \to X$ is transitive, the restriction $A : G \times \{a\} \to X$ is surjective for any point $a \in X$. It follows that one can construct a bijective mapping $\beta : X \to G$ by utilizing the surjection $\eta : V \to G$. This allows one to introduce a smooth manifold structure on the set X that is diffeomorphic to the quotient group $G/\mathrm{Stab}(G)$. Since the isotropy group $\mathrm{Stab}(G)$ is trivial, we obtain the statement of the lemma.

Theorem 4. *Let (X, V, G, η) be a homogeneous via-space with a simply connected, compact, three-dimensional transformation group G, and a trivial isotropy group. Then this via-space admits a structure of a C^∞-smooth manifold diffeomorphic to a three-dimensional sphere that is consistent with the via-topology V.*

Proof. Using Lemma 3, we construct a bijection between X and a manifold $\widetilde{X}$ that is diffeomorphic to the transformation group G. Thus, $\widetilde{X}$ is a simply connected, compact, three-dimensional manifold. It is evident that $\widetilde{X}$ is a homogeneous space in the standard sense [13, Ch. X, §1]. From the complete classification of homogeneous three-dimensional spaces [14, §3], it follows that there exists a unique (up to a diffeomorphism) simply connected, compact, homogeneous three-dimensional space: namely, the sphere $\mathbb{S}^3$ with the transformation group $\mathrm{SU}(2)$. Consequently, the structure of a manifold diffeomorphic to $\mathbb{S}^3$ consistent with the path structure V is determined on the set X.

Remark 5. The condition that the manifold has no boundary — a requirement in the statement of the Poincaré conjecture — is implicitly included into the assumption that the via-space is homogeneous. Other topological conditions had to be transferred into a transformation group due to inherent difficulties in defining them directly in terms of via-topology. It is possible that in the future one might be able to define independently the notions of simple connectivity, compactness, and dimension. It may be admittedly done by employing category theory, which is suggested by Lemmas 1–2, Theorem 1, and the possibility of defining homological dimension for certain categories; see, for example, [11, Ch. 7, §8].

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