

On multiple polylogarithms in characteristic p : v -adic vanishing versus ∞ -adic Eulerian

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ABSTRACT. In this paper, we give a simultaneous vanishing principle for the v -adic Carlitz multiple polylogarithms (abbreviated as CMPLs) at algebraic points, where v is a finite place of the rational function field over a finite field. This principle establishes the fact that the v -adic vanishing of CMPLs at algebraic points is equivalent to its ∞ -adic counterpart being Eulerian. This reveals a nontrivial connection between the v -adic and ∞ -adic worlds in positive characteristic.

1. INTRODUCTION

1.1. Motivation. The study of this paper is motivated by the classical theory and conjecture for special zeta values. Let $n \geq 2$ be an integer. Then the celebrated formula of Euler for the special values of the Riemann zeta function at even positive integers implies that

$$\zeta(n)/(2\pi\sqrt{-1})^n \in \mathbb{Q} \Leftrightarrow n \text{ is even.}$$

For a prime number p , we consider the Kubota-Leopoldt p -adic zeta function ζ_p , that interpolates the rational values of the Riemann zeta function at non-positive integers. When n is even, we know that $\zeta_p(n) = 0$ (cf. [KL64, Col82, Fo4]). So this reveals the following interesting phenomena between the archimedean world and its p -adic counterpart: for an integer $n \geq 2$,

$$\zeta(n)/(2\pi\sqrt{-1})^n \in \mathbb{Q} \Leftrightarrow n \text{ is even} \Rightarrow \zeta_p(n) = 0.$$

Conjecturally, the reverse direction above is valid. It is known by [S81] that for $m \in \mathbb{N}$, $\zeta_p(2m+1)$ is nonzero when p is regular or $(p-1) \mid 2m$.

We note that for an integer $n \geq 2$, $\zeta(n) = \text{Li}_n(1)$, where Li_n is the n th polylogarithm given by

$$\text{Li}_n(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^n}.$$

For a fixed prime number p , we let \mathbb{C}_p be the p -adic completion of a fixed algebraic closure of the p -adic numbers \mathbb{Q}_p . The series $\text{Li}_n(z)$ converges p -adically on the open unit disc and we denote this function by $\text{Li}_{n,p}$ which is called the p -adic n th polylogarithm. For each branch parameter $a \in \mathbb{C}_p$ of the p -adic logarithm, we note that by [Fo4, Col82] $\text{Li}_{n,p}$ can be analytically continued to $\mathbb{C}_p \setminus \{1\}$, and we denote by $\text{Li}_{n,p}^a$ its analytically

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continued function. Furusho [Fo4] showed that $\zeta_p(n)$ is the limit of $\text{Li}_{n,p}^a$ when “ $z \rightarrow 1$ ” in the sense of [Fo4], and $\zeta_p(n)$ is independent of choices of branch parameters a .

We fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Let u be a nonzero algebraic number for which $\text{Li}_n(u)$ converges with respect to the archimedean absolute value. Inspired by the conjectural equivalence between the rationality of $\zeta(n)/(2\pi\sqrt{-1})^n$ and the vanishing of $\zeta_p(n)$, for a fixed branch parameter $a \in \mathbb{C}_p$ of the p -adic logarithm one naturally asks whether there is a criterion \spadesuit_a for the rationality of $\text{Li}_n(u)/(2\pi\sqrt{-1})^n$ in terms of the vanishing of $\text{Li}_{n,p}^a(u)$:

$$\text{Li}_n(u)/(2\pi\sqrt{-1})^n \in \mathbb{Q} \Leftrightarrow \spadesuit_a \Leftrightarrow \text{Li}_{n,p}^a(u) = 0.$$

For example, for $n \geq 2$ and “ $u \rightarrow 1$ ”, conjecturally $\spadesuit_a = “n \text{ is even}”$.

The main purpose of this paper is to give a positive answer of the analogous question above for the Carlitz multiple polylogarithms (abbreviated as CMPLs), which was introduced by the first author of the present paper in [C14] generalizing the notion initiated by Anderson and Thakur [AT90].

1.2. Depth one case in function fields. Let $A := \mathbb{F}_q[\theta]$ be the polynomial ring in the variable θ over the finite field \mathbb{F}_q of q elements, and k be its quotient field. Denote by ∞ the infinity place of k with an associated absolute value $|\cdot|_\infty$. Let $k_\infty := \mathbb{F}_q((1/\theta))$ be the ∞ -adic completion of k . We fix an algebraic closure $\overline{k_\infty}$ of k_∞ , and let \mathbb{C}_∞ be the ∞ -adic completion of $\overline{k_\infty}$. We further fix a fundamental period $\tilde{\pi} \in \overline{k_\infty}^\times$ of the Carlitz $\mathbb{F}_q[t]$ -module \mathbb{C} , where t is an independent variable. Note that \mathbb{C} and $\tilde{\pi}$ play the analogous roles of \mathbb{G}_m and $2\pi\sqrt{-1}$ respectively in the function field setting. A positive integer n is called A -even if $(q-1)|n$, as $q-1$ is the cardinality of the unit group A^\times .

1.2.1. Special zeta values. Let A_+ be the set of monic polynomials in A and consider the Carlitz zeta value at $n \in \mathbb{N}$,

$$\zeta_A(n) := \sum_{a \in A_+} \frac{1}{a^n} \in k_\infty^\times.$$

In [Ca35], Carlitz derived an analogue of Euler’s formula on Riemann zeta function at even positive integers. More precisely, we have the following consequence: for a positive integer n , we have the equivalence

$$\zeta_A(n)/\tilde{\pi}^n \in k \Leftrightarrow n \text{ is } A\text{-even}.$$

Let v be a monic prime of A , and let k_v be the completion of k with respect to the normalized v -adic absolute value $|\cdot|_v$ associated to the place v . For a positive integer n , in analogy with the special value at n of the Kubota-Leopoldt p -adic zeta function $\zeta_p(n)$ we consider the v -adic Goss zeta function at n denoted by $\zeta_{A,v}(n) \in k_v$ (see [Go79]). Goss [Go79] showed that $\zeta_{A,v}(n)$ is vanishing for A -even n , and Yu [Yu91] showed the transcendence of $\zeta_{A,v}(n)$ for A -odd n (ie., $(q-1) \nmid n$), and therefore we have the following complete story: for $n \in \mathbb{N}$,

$$(1.2.1) \quad \zeta_A(n)/\tilde{\pi}^n \in k \Leftrightarrow n \text{ is } A\text{-even} \Leftrightarrow \zeta_{A,v}(n) = 0.$$

1.2.2. *Carlitz polylogarithms.* Put $L_0 := 1$ and $L_i := (\theta - \theta^q) \cdots (\theta - \theta^{q^i})$ for $i \in \mathbb{N}$. Let $\log_{\mathbf{C}}$ be the logarithm of the Carlitz $\mathbb{F}_q[t]$ -module \mathbf{C} given by the power series

$$\log_{\mathbf{C}}(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{L_i} \text{ (see [Go96, To4]).}$$

In analogy with the classical polylogarithm, Anderson and Thakur [AT90] defined the n th Carlitz polylogarithm for $n \in \mathbb{N}$:

$$\text{Li}_n(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{L_i^n}.$$

Unlike the simple identity between $\zeta(n)$ and the n th polylogarithm at 1 in the classical case, Anderson and Thakur [AT90] proved that $\zeta_A(n)$ is a k -linear combination of Li_n at some explicit integral points in A .

We fix an algebraic closure \bar{k}_v of k_v , and let \mathbf{C}_v be the v -adic completion of \bar{k}_v . Let \bar{k} be an algebraic closure of k , and fix embeddings $\bar{k} \hookrightarrow \bar{k}_{\infty}$ and $\bar{k} \hookrightarrow \bar{k}_v$. For a positive integer n , we denote by $\mathbf{C}^{\otimes n}$ the n th tensor power of the Carlitz module \mathbf{C} introduced by Anderson-Thakur [AT90]. Let $u \in \bar{k}^{\times}$ and put $\mathbf{v} := (0, \dots, 0, u)^{\text{tr}} \in \mathbf{C}^{\otimes n}(\bar{k})$. We further put the condition:

$$\clubsuit_1 := “\mathbf{v} = (0, \dots, 0, u)^{\text{tr}} \text{ is an } \mathbb{F}_q[t]\text{-torsion in } \mathbf{C}^{\otimes n}(\bar{k})”.$$

We assume $|u|_{\infty} < |\theta|_{\infty}^{\frac{nq}{q-1}}$ and note that $\text{Li}_n(u)$ converges in \mathbf{C}_{∞} . Then a consequence of the Anderson-Thakur theory [AT90] is the following equivalence:

$$(1.2.2) \quad \text{Li}_n(u) / \tilde{\pi}^n \in k \Leftrightarrow \clubsuit_1 \text{ is valid.}$$

Unlike the analytic continuation of classical p -adic polylogarithms, we shall mention here that the convergence domain of Li_n with respect to the v -adic absolute value is the open unit disc $\{z \in \mathbf{C}_v; |z|_v < 1\}$. We denote by $\text{Li}_n(z)_v$ when the series $\text{Li}_n(z)$ converges v -adically. We assume that the given algebraic element $u \in \bar{k}$ has v -adic absolute value less than one, so $\text{Li}_n(u)_v$ is defined. Then by Yu’s theory [Yu91] one can derive the following equivalence

$$(1.2.3) \quad \text{Li}_n(u)_v = 0 \Leftrightarrow \clubsuit_1 \text{ is valid,}$$

and hence one establishes the principle: for $u \in \bar{k}^{\times}$ with $|u|_{\infty} < |\theta|_{\infty}^{\frac{nq}{q-1}}$ and $|u|_v < 1$ we have

$$\text{Li}_n(u) / \tilde{\pi}^n \in k \Leftrightarrow \clubsuit_1 \text{ is valid} \Leftrightarrow \text{Li}_n(u)_v = 0.$$

The main result in this paper is to generalize this principle to higher depths.

1.3. **Higher depths case.** In analogy with the classical multiple polylogarithms, the first author of the present paper introduced the Carlitz multiple polylogarithm (abbreviated as CMPL) for each $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ (see [C14]):

$$\text{Li}_{\mathfrak{s}}(z_1, \dots, z_r) := \sum_{i_1 > \dots > i_r \geq 0} \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}},$$

whose weight is defined to be $\text{wt}(\mathfrak{s}) := \sum_{i=1}^r s_i$ and whose depth is defined to be r . It is shown in [C14] that each multizeta value $\zeta_A(\mathfrak{s})$ initiated by Thakur [To4] is a k -linear

combination of Li_s at some integral points in A^r , generalizing the formula of Anderson-Thakur to the higher depth case.

Fixing any $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ and $\mathbf{u} = (u_1, \dots, u_r) \in (\bar{k}^\times)^r$, one has an associated t -module $G := G_{\mathfrak{s}, \mathbf{u}}$ defined over \bar{k} and an associated algebraic point $\mathbf{v} := \mathbf{v}_{\mathfrak{s}, \mathbf{u}} \in G(\bar{k})$ given in [CPY14] (for avoiding the heavy notation, we drop \mathfrak{s} and \mathbf{u} without confusion when it is clear from the contents). Suppose that $\text{Li}_{(s_\ell, \dots, s_r)}(u_\ell, \dots, u_r)$ converges ∞ -adically for each $\ell = 1, \dots, r$. It is shown in [CPY14] that

$$\clubsuit_r := \text{“}\mathbf{v} \text{ is an } \mathbb{F}_q[t]\text{-torsion point in } G(\bar{k})\text{”}$$

is valid if and only if

$$\text{Li}_{(s_1, \dots, s_r)}(u_1, \dots, u_r) / \tilde{\pi}^{s_1 + \dots + s_r}, \text{Li}_{(s_2, \dots, s_r)}(u_2, \dots, u_r) / \tilde{\pi}^{s_2 + \dots + s_r}, \dots, \text{Li}_{s_r}(u_r) / \tilde{\pi}^{s_r}$$

are simultaneously in k . Note that the value $\text{Li}_{\mathfrak{s}}(\mathbf{u})$ is called *Eulerian* if the ratio $\text{Li}_{\mathfrak{s}}(\mathbf{u}) / \tilde{\pi}^{\text{wt}(\mathfrak{s})}$ lies in k (cf. [To4, CPY14]).

Our main result, stated as Theorem 4.1.2, is to prove the v -adic counterpart fitting into the reflection principle. Suppose that $|u_i|_v < 1$ for all i . We show that \clubsuit_r is valid if and only if

$$\text{Li}_{(s_r, \dots, s_1)}(u_r, \dots, u_1)_v = \text{Li}_{(s_r, \dots, s_2)}(u_r, \dots, u_2)_v = \dots = \text{Li}_{s_r}(u_r)_v = 0,$$

where the subscript v above denotes by the v -adic convergence. Note that when we restrict $r = 1$, then we recover the result in (1.2.3).

We shall mention that for the \mathfrak{s} and \mathbf{u} given above, if $\text{Li}_{(s_1, \dots, s_r)}(u_1, \dots, u_r)$ is Eulerian, then the following values

$$\text{Li}_{(s_2, \dots, s_r)}(u_2, \dots, u_r), \dots, \text{Li}_{s_r}(u_r)$$

are automatically Eulerian. This fact is proven in [CPY14] using the ABP-criterion [ABP04], which is a strong tool in the transcendence theory of ∞ -adic case. However, we do not know whether its analog is true in the v -adic case, which needs additional work on developing some v -adic transcendence theory.

We mention that the similar criterion for Thakur’s multizeta values (abbreviated MZVs) to be *Eulerian* is given in [CPY14]. One then naturally asks whether one has the criterion for its v -adic counterpart as v -adic MZVs were introduced in [To4]. Such a criterion is related to how to express the given v -adic MZV as some coordinate logarithm of certain t -module, which is not clear to the authors at this moment how to solve this issue. In the depth one case, the far reaching theory of [AT90] do relate both $\zeta_A(n)$ and $\zeta_{A,v}(n)$ to the logarithm of $\mathbb{C}^{\otimes n}$, but for higher depth MZVs it is an open question although certain ∞ -adic MZVs of higher depths are done in [C15] by t -motivic methods.

The first step of proving the main result above is to write down the t -module G and the algebraic point \mathbf{v} explicitly. We mention that in [CPY14], G and \mathbf{v} are only theoretically constructed without being written down explicitly. We further use some techniques in [AT90] to compute the coefficient matrices of the logarithm of G explicitly, and then show that the Carlitz multiple star polylogarithms (abbreviated as CMSPLs) defined in (2.2.1) occur as the coordinate logarithms of G . Then we establish an identity of the CMSPLs (stated as Lemma 4.1.3) in terms of linear combination of products of CMPLs and CMSPLs. These properties together with Yu’s theory [Yu91] enable us to derive the desired results.

2. PRELIMINARIES

2.1. **Notation.** We adopt the following notation.

\mathbb{F}_q	= the finite field with q elements, for q a power of a prime number p .
θ, t	= independent variables.
A	= $\mathbb{F}_q[\theta]$, the polynomial ring in the variable θ over \mathbb{F}_q .
v	= a monic irreducible polynomial in A .
k	= $\mathbb{F}_q(\theta)$, the fraction field of A .
k_v	= the completion of k with respect to the place v .
\overline{k}_v	= a fixed algebraic closure of k_v .
\overline{k}	= the algebraic closure of k in \overline{k}_v .
\mathbb{C}_v	= the completion of \overline{k}_v with respect to the canonical extension of v .
$ \cdot _v$	= a fixed absolute value on \mathbb{C}_v so that $ v _v = 1/q^{\deg v}$.
\mathbb{G}_a	= the additive group scheme over A .

2.2. **CMPLs and CMSPLs.** We recall the Carlitz multiple polylogarithms [C14] that are generalization of the polylogarithms initiated in [AT90]. Put $L_0 := 1$ and $L_i := (\theta - \theta^q) \cdots (\theta - \theta^{q^i})$ for $i \in \mathbb{N}$. Given any $\mathfrak{s} := (s_1, \dots, s_r) \in \mathbb{N}^r$, the associated Carlitz multiple polylogarithm (abbreviated as CMPL) and the Carlitz multiple star polylogarithm (abbreviated as CMSPL, compared with the terminology in [FKMT15]) are defined by the series

$$\text{Li}_{\mathfrak{s}}(z_1, \dots, z_r) := \sum_{i_1 > \dots > i_r \geq 0} \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}}$$

and

$$(2.2.1) \quad \text{Li}_{\mathfrak{s}}^*(z_1, \dots, z_r) := \sum_{i_1 \geq \dots \geq i_r \geq 0} \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}}.$$

We denote by $\text{Li}_{\mathfrak{s}}(z_1, \dots, z_r)_v$ and $\text{Li}_{\mathfrak{s}}^*(z_1, \dots, z_r)_v$ while working on the v -adic convergence. Both series converge when $|z_1|_v < 1$ and $|z_2|_v, \dots, |z_r|_v \leq 1$. Indeed, since v^2 does not divide $\theta - \theta^{q^i}$, if z_i 's satisfy the above condition, then we have

$$\left| \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \right|_v \leq |v|_v^{-(s_1 i_1 + \dots + s_r i_r)} |z_1|_v^{q^{i_1}} \cdots |z_r|_v^{q^{i_r}} \leq |v|_v^{-\text{wt}(\mathfrak{s}) i_1} |z_1|_v^{q^{i_1}} \rightarrow 0 \quad (i_1 \rightarrow \infty).$$

2.3. **t -modules.** In this section, we review the theory of t -modules introduced by Anderson [A86]. For an A -algebra R , we denote by τ the Frobenius q th power operator $\tau := (x \mapsto x^q) : R \rightarrow R$. For convenience, we extend the action of τ on the matrices with entries in R by componentwise action. We denote by $\mathbb{C}_v[\tau]$ the non-commutative polynomial generated by τ over \mathbb{C}_v subject to the relation

$$\tau \alpha = \alpha^q \tau \text{ for } \alpha \in \mathbb{C}_v.$$

For each $\varphi \in \text{Mat}_d(\mathbb{C}_v[\tau])$, we write $\varphi = \sum_{i=0}^{\infty} \alpha_i \tau^i$ with each $\alpha_i \in \text{Mat}_d(\mathbb{C}_v)$ and $\alpha_i = 0$ for $i \gg 0$, and further define $\partial \varphi := \alpha_0$.

Let t be a new variable and d be a positive integer. By a d -dimensional t -module, we mean a pair $G = (\mathbb{G}_a^d, \rho)$, where ρ is an \mathbb{F}_q -linear ring homomorphism

$$\rho : \mathbb{F}_q[t] \rightarrow \text{Mat}_d(\mathbb{C}_v[\tau])$$

so that $\partial\rho_t - \theta I_d$ is a nilpotent matrix. Here we denote by ρ_a the image of $a \in \mathbb{F}_q[t]$ by ρ . For a subring $A \subseteq R \subseteq \mathbb{C}_v$, we say that the t -module is defined over R if all the coefficient matrices of ρ_t are in $\text{Mat}_d(R)$. In this situation, $\mathbb{G}_a^d(R) = R^d$ has an $\mathbb{F}_q[t]$ -module via the map ρ .

Let K be either \bar{k} or \mathbb{C}_v and let $G = (\mathbb{G}_a^d, \rho)$ be a d -dimensional t -module defined over K . Then we have the *exponential function* of G which is an \mathbb{F}_q -linear d -variable power series of the form $\exp_G = I_d + \sum_{i=1}^{\infty} \alpha_i \tau^i$ with $\alpha_i \in \text{Mat}_d(K)$, satisfying the following identity:

$$(2.3.1) \quad \exp_G \circ \partial\rho_a = \rho_a \circ \exp_G \quad \text{for all } a \in \mathbb{F}_q[t].$$

The logarithm of G denoted by \log_G , is defined to be the formal inverse of \exp_G that has the property:

$$(2.3.2) \quad \log_G \circ \rho_a = \partial\rho_a \circ \log_G \quad \text{for all } a \in \mathbb{F}_q[t].$$

The logarithm \log_G will be the primary interest for our study as it could provide rich source of transcendental values (see [Yu91, Yu97]).

3. COMPUTATION ON THE LOGARITHMS

In this section, our goal is to give an explicit construction of an appropriate t -module G over \bar{k} associated to an index $\mathfrak{s} \in \mathbb{N}^r$ and an algebraic point $\mathbf{u} \in (\bar{k}^\times)^r$, and show that the CMSPLs in question occur as coordinate logarithms of G at an explicit algebraic point of G .

3.1. Construction of the t -module G . In what follows, we fix $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ and $\mathbf{u} = (u_1, \dots, u_r) \in (\bar{k}^\times)^r$. For $1 \leq \ell \leq r$, we set $d_\ell := s_\ell + \dots + s_r$ and $d := d_1 + \dots + d_r$. Let B be a $d \times d$ -matrix of the form

$$\left(\begin{array}{c|ccc|c} B[11] & \cdots & & B[1r] \\ \hline \vdots & & & \vdots \\ \hline B[r1] & \cdots & & B[rr] \end{array} \right)$$

where $B[\ell m]$ is a $d_\ell \times d_m$ -matrix for each ℓ and m . In this paper, $B[\ell m]$ is called the (ℓ, m) -th block matrix of B .

For $1 \leq \ell \leq m \leq r$, we set

$$N_\ell := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix} \in \text{Mat}_{d_\ell}(\bar{k}),$$

$$N := \begin{pmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_r \end{pmatrix} \in \text{Mat}_d(\bar{k}),$$

$$E[\ell m] := \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \text{Mat}_{d_\ell \times d_m}(\bar{k}) \quad (\text{if } \ell = m),$$

$$E[\ell m] := \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & \ddots & \vdots \\ (-1)^{m-\ell} \prod_{e=\ell}^{m-1} u_e & 0 & \cdots & 0 \end{pmatrix} \in \text{Mat}_{d_\ell \times d_m}(\bar{k}) \quad (\text{if } \ell < m),$$

$$E := \begin{pmatrix} E[11] & E[12] & \cdots & E[1r] \\ & E[22] & \ddots & \vdots \\ & & \ddots & E[r-1, r] \\ & & & E[rr] \end{pmatrix} \in \text{Mat}_d(\bar{k}).$$

We also define

$$E_m := \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & E[mm] & 0 \\ \hline 0 & 0 & 0 \end{array} \right) \in \text{Mat}_d(\bar{k})$$

to be the $d \times d$ -matrix such that the (m, m) -th block matrix is $E[mm]$ and the others are zero matrices.

We define the t -module $G = G_{\mathfrak{s}, \mathbf{u}} := (\mathbb{C}_a^d, \rho)$ by

$$(3.1.1) \quad \rho_t = \theta I_d + N + E\tau \in \text{Mat}_d(\bar{k}[\tau]).$$

Note that G depends only on u_1, \dots, u_{r-1} .

Example 3.1.2. When $r = 1$, we have

$$\rho_t = C_t^{\otimes s_1} := \begin{pmatrix} \theta & 1 & & \\ & \theta & 1 & \\ & & \ddots & \ddots \\ \tau & & & \theta & 1 \\ & & & & \theta \end{pmatrix} \in \text{Mat}_d(\bar{k}[\tau])$$

which is called the s_1 th tensor power of the Carlitz module.

When $r = 2$, we have

$$\rho_t = \left(\begin{array}{c|cc} \mathbf{C}_t^{\otimes(s_1+s_2)} & 0 & \cdots & 0 \\ \hline & \vdots & & \vdots \\ & -u_1\tau & \cdots & 0 \\ \hline 0 & & & \mathbf{C}_t^{\otimes s_2} \end{array} \right) \in \text{Mat}_d(\bar{k}[\tau]).$$

When $r = 3$, we have

$$\rho_t = \left(\begin{array}{c|cc|cc} \mathbf{C}_t^{\otimes(s_1+s_2+s_3)} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline & \vdots & & \vdots & \vdots & & \vdots \\ & -u_1\tau & \cdots & 0 & u_1u_2\tau & \cdots & 0 \\ \hline 0 & & & \mathbf{C}_t^{\otimes(s_2+s_3)} & 0 & \cdots & 0 \\ \hline & & & & \vdots & & \vdots \\ & & & & -u_2\tau & \cdots & 0 \\ \hline 0 & & 0 & & & & \mathbf{C}_t^{\otimes s_3} \end{array} \right) \in \text{Mat}_d(\bar{k}[\tau]).$$

3.2. Logarithm of G . We denote by

$$\log_G = \sum_{i \geq 0} P_i \tau^i$$

the logarithm of the t -module G , where we put

$$P_0 := I_d, \quad P_i := \begin{pmatrix} P_i[11] & \cdots & P_i[1r] \\ \vdots & & \vdots \\ P_i[r1] & \cdots & P_i[rr] \end{pmatrix} \in \text{Mat}_d(\bar{k}), \quad P_i[\ell m] \in \text{Mat}_{d_\ell \times d_m}(\bar{k}).$$

Proposition 3.2.1. *We have $P_i[\ell m] = 0$ for $\ell > m$. For $\ell \leq m$, if we denote by $y_i[\ell m] := P_i[\ell m]_{d_\ell d_m}$, which is the most lower right corner of $P_i[\ell m]$, then we have*

$$(3.2.2) \quad y_i[\ell m] = \frac{1}{L_i^{d_m}} \quad (\text{if } \ell = m),$$

and

$$(3.2.3) \quad y_i[\ell m] = (-1)^{m-\ell} \sum_{0 \leq i_\ell \leq \cdots \leq i_{m-1} < i} \frac{u_\ell^{q^{i_\ell}} \cdots u_{m-1}^{q^{i_{m-1}}}}{L_{i_\ell}^{s_\ell} \cdots L_{i_{m-1}}^{s_{m-1}} L_i^{d_m}} \quad (\text{if } \ell < m).$$

Proof. For any matrix $M = (M_{ij})$ with $M_{ij} \in \mathbf{C}_v$, we set $M^{(s)} := (M_{ij}^{q^s})$ for $s \in \mathbf{Z}$. From the functional equation

$$\log_G \circ \rho_t = \partial \rho_t \circ \log_G,$$

by comparing the coefficient matrices we have the following identity

$$P_{i+1}(\theta^{q^{i+1}} I_d + N) + P_i E^{(i)} = (\theta I_d + N) P_{i+1} \quad \text{for } i \in \mathbf{Z}_{\geq 0}.$$

Given two square matrices of the same size X, Y , we denote by $\text{ad}(X)^0(Y) := Y$ and $\text{ad}(X)^{j+1}(Y) := X(\text{ad}(X)^j(Y)) - (\text{ad}(X)^j(Y))X$. From the identity above, we obtain

$$P_{i+1} - \frac{\text{ad}(N)^1(P_{i+1})}{\theta^{q^{i+1}} - \theta} = -\frac{P_i E^{(i)}}{\theta^{q^{i+1}} - \theta}$$

and hence for each $i \in \mathbb{N}_{\geq 0}$,

$$(3.2.4) \quad P_{i+1} = -\sum_{j=0}^{2d_1-2} \frac{\text{ad}(N)^j(P_i E^{(i)})}{(\theta^{q^{i+1}} - \theta)^{j+1}}$$

because $\text{ad}(N)^{2d_1-1}(P_{i+1}) = 0$ from the fact $N^{d_1} = 0$. Therefore we have

$$(3.2.5) \quad P_i[\ell m] = 0 \text{ for } \ell > m$$

by the induction on i .

Letting

$$Y = \left(\begin{array}{c|c|c} * & * & * \\ \hline * & * & * \\ & & y \\ \hline * & * & * \end{array} \right)$$

be a $d \times d$ -matrix such that the most lower right corner of the (ℓ, m) -th block matrix is y , we note that $E_\ell^{\text{tr}} Y E_m$ is of the form

$$E_\ell^{\text{tr}} Y E_m = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline & y & \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right)$$

(the $d \times d$ -matrix such that the most upper left corner of the (ℓ, m) -th block matrix is y and the others are zero matrices).

Since $E_\ell^{\text{tr}} N = 0$ and $\text{ad}(N)^j(P_i E^{(i)})$ can be expressed as $\text{ad}(N)^j(P_i E^{(i)}) = NB + (-1)^j P_i E^{(i)} N^j$ for some $B \in \text{Mat}_d(\bar{k})$, we have

$$(3.2.6) \quad \begin{aligned} E_\ell^{\text{tr}} P_{i+1} E_m &= -\sum_{j=0}^{2d_1-2} \frac{E_\ell^{\text{tr}} \text{ad}(N)^j(P_i E^{(i)}) E_m}{(\theta^{q^{i+1}} - \theta)^{j+1}} \\ &= \sum_{j=0}^{2d_1-2} \frac{E_\ell^{\text{tr}} P_i E^{(i)} N^j E_m}{(\theta - \theta^{q^{i+1}})^{j+1}} \\ &= \frac{E_\ell^{\text{tr}} P_i E^{(i)} N^{d_m-1} E_m}{(\theta - \theta^{q^{i+1}})^{d_m}}. \end{aligned}$$

Note that the last equality above comes from the facts that $E^{(i)} N^j E_m = 0$ if $j \neq d_m - 1$, and

$$E^{(i)}N^{d_m-1}E_m = \left(\begin{array}{ccc} \overbrace{\hspace{10em}}^{d_1 + \cdots + d_{m-1}} & \overbrace{\hspace{2em}}^{d_m} & \overbrace{\hspace{10em}}^{d_{m+1} + \cdots + d_r} \\ \mathbf{0} & \begin{matrix} E[1m]^{(i)} \\ \vdots \\ E[mm]^{(i)} \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right) \left. \vphantom{\begin{matrix} \overbrace{\hspace{10em}} \\ \overbrace{\hspace{2em}} \\ \overbrace{\hspace{10em}} \end{matrix}} \right\} d_1 + \cdots + d_m$$

if $j = d_m - 1$. Note that we also have

$$E_\ell^{\text{tr}}P_i = \left(\begin{array}{ccccc} & \begin{matrix} d_1 \text{th} \\ \downarrow \end{matrix} & & \begin{matrix} (d_1+d_2) \text{th} \\ \downarrow \end{matrix} & & \begin{matrix} (d) \text{th} \\ \downarrow \end{matrix} \\ \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \\ * y_i[\ell 1] & * y_i[\ell 2] & \cdots & * y_i[\ell r] & \\ \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \end{array} \right) \leftarrow (d_1 + \cdots + d_{\ell-1} + 1) \text{th}$$

By comparing with the most upper left corner of the (ℓ, m) -th block matrix (which is the $(d_1 + \cdots + d_{\ell-1} + 1, d_1 + \cdots + d_{m-1} + 1)$ -th entry) of the both sides of the equation (3.2.6), we have from (3.2.5) that

$$(3.2.7) \quad L_{i+1}^{d_m} y_{i+1}[\ell m] = L_i^{d_m} y_i[\ell m] \quad (\text{if } \ell = m)$$

and that

$$(3.2.8) \quad L_{i+1}^{d_m} y_{i+1}[\ell m] = L_i^{d_m} \sum_{n=\ell}^{m-1} y_i[\ell n] (-1)^{m-n} \prod_{e=n}^{m-1} u_e^{q^i} + L_i^{d_m} y_i[\ell m] \quad (\text{if } \ell < m)$$

(note that $L_{i+1} = (\theta - \theta^{q^{i+1}})L_i$). By definition, we have $y_0[\ell m] = 1$ if $\ell = m$ and $y_0[\ell m] = 0$ if $\ell < m$. We fix ℓ and show the equalities (3.2.2) and (3.2.3) hold by the induction on m ($\geq \ell$).

When $m = \ell$, we can show this easily by the recurrence relation (3.2.7). Let $m > \ell$ and assume that the equality (3.2.3) is true for $y_i[\ell n]$ with $\ell \leq n < m$ and $i \geq 0$. By the recurrence relation (3.2.8), we have

$$\begin{aligned}
L_{i+1}^{d_m} y_{i+1}[\ell m] &= L_i^{d_m} \sum_{n=\ell+1}^{m-1} (-1)^{n-\ell} \sum_{0 \leq i_\ell \leq \dots \leq i_{n-1} < i} \frac{u_\ell^{q^{i_\ell}} \cdots u_{n-1}^{q^{i_{n-1}}}}{L_i^{s_\ell} \cdots L_{i_{n-1}}^{s_{n-1}} L_i^{d_n}} (-1)^{m-n} \prod_{e=n}^{m-1} u_e^{q^i} \\
&\quad + L_i^{d_m} \frac{1}{L_i^{d_\ell}} (-1)^{m-\ell} \prod_{e=\ell}^{m-1} u_e^{q^i} + L_i^{d_m} y_i[\ell m] \\
&= (-1)^{m-\ell} \sum_{n=\ell+1}^{m-1} \sum_{0 \leq i_\ell \leq \dots \leq i_{n-1} < i} \frac{u_\ell^{q^{i_\ell}} \cdots u_{n-1}^{q^{i_{n-1}}} u_n^{q^i} \cdots u_{m-1}^{q^i}}{L_i^{s_\ell} \cdots L_{i_{n-1}}^{s_{n-1}} L_i^{s_n} \cdots L_i^{s_{m-1}}} \\
&\quad + (-1)^{m-\ell} \frac{u_\ell^{q^i} \cdots u_{m-1}^{q^i}}{L_i^{s_\ell} \cdots L_i^{s_{m-1}}} + L_i^{d_m} y_i[\ell m] \\
&= (-1)^{m-\ell} \sum_{\substack{0 \leq i_\ell \leq \dots \leq i_{m-1} \\ i_{m-1} = i}} \frac{u_\ell^{q^{i_\ell}} \cdots u_{m-1}^{q^{i_{m-1}}}}{L_i^{s_\ell} \cdots L_{i_{m-1}}^{s_{m-1}}} + L_i^{d_m} y_i[\ell m].
\end{aligned}$$

Since $y_0[\ell m] = 0$, we have

$$\begin{aligned}
L_i^{d_m} y_i[\ell m] &= (-1)^{m-\ell} \sum_{i'=0}^{i-1} \sum_{\substack{0 \leq i_\ell \leq \dots \leq i_{m-1} \\ i_{m-1} = i'}} \frac{u_\ell^{q^{i_\ell}} \cdots u_{m-1}^{q^{i_{m-1}}}}{L_i^{s_\ell} \cdots L_{i_{m-1}}^{s_{m-1}}} \\
&= (-1)^{m-\ell} \sum_{0 \leq i_\ell \leq \dots \leq i_{m-1} < i} \frac{u_\ell^{q^{i_\ell}} \cdots u_{m-1}^{q^{i_{m-1}}}}{L_i^{s_\ell} \cdots L_{i_{m-1}}^{s_{m-1}}}
\end{aligned}$$

and hence the equalities (3.2.2) and (3.2.3) are true for each $m \geq \ell$ by the induction. \square

3.3. v -adic convergence of CMPLs. Next, we consider when the sum $\log_G \mathbf{x}$ converges in $\text{Lie } G(\mathbb{C}_v) = \mathbb{C}_v^d$ for $\mathbf{x} \in G(\mathbb{C}_v) = \mathbb{C}_v^d$. For any matrix $M = (M_{ij})$ with $M_{ij} \in \mathbb{C}_v$, we denote $|M|_v := \max\{|M_{ij}|_v\}$. Assume that $|u_m|_v \leq 1$ for each $1 \leq m < r$. Then $\log_G \mathbf{x}$ converges for each $\mathbf{x} \in G(\mathbb{C}_v)$ with $|\mathbf{x}|_v < 1$. Indeed, it is clear that $|N|_v = 1$ and $|E^{(i)}|_v \leq 1$. Since v^2 does not divide $\theta^{q^{i+1}} - \theta$ for each $i \geq 0$, we have $1 \geq |\theta^{q^{i+1}} - \theta|_v \geq |v|_v$. Thus by the equation (3.2.4), we obtain

$$|P_{i+1}|_v \leq |P_i|_v |\theta^{q^{i+1}} - \theta|_v^{-(2d_1-2+1)} \leq |P_i|_v |v|_v^{-(2d_1-1)}.$$

Since $P_0 = I_d$, we have an upper bound

$$|P_i|_v \leq |v|_v^{-i(2d_1-1)}$$

for each $i \geq 0$. Therefore if $|\mathbf{x}|_v < 1$, we have

$$|P_i \mathbf{x}^{(i)}|_v \leq |P_i|_v |\mathbf{x}|_v^{q^i} \leq |v|_v^{-i(2d_1-1)} |\mathbf{x}|_v^{q^i} \rightarrow 0 \quad (i \rightarrow \infty),$$

and hence the sum converges.

Let

$$(3.3.1) \quad \mathbf{v} = \mathbf{v}_{s,\mathbf{u}} := \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ (-1)^{r-1} u_1 \cdots u_r \\ 0 \\ \vdots \\ 0 \\ (-1)^{r-2} u_2 \cdots u_r \\ \vdots \\ 0 \\ \vdots \\ 0 \\ u_r \end{array} \right) \in G(\bar{k}).$$

$\left. \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right\} d_1$
 $\left. \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right\} d_2$
 $\left. \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right\} d_r$

It is clear that $|\mathbf{v}|_v < 1$ when $|u_m|_v \leq 1$ for each $1 \leq m < r$ and $|u_r|_v < 1$. Thus in this case, the sum $\log_G \mathbf{v}$ converges v -adically.

Remark 3.3.2. The authors do not know the precise v -adic convergence domain of \log_G .

Theorem 3.3.3. *Let $u_1, \dots, u_r \in \bar{k}^\times$ with $|u_m|_v \leq 1$ for each $1 \leq m < r$ and $|u_r|_v < 1$. Let G and \mathbf{v} be as above. Then we have*

$$\log_G \mathbf{v} = \left(\begin{array}{c} * \\ \vdots \\ * \\ (-1)^{r-1} \text{Li}_{(s_r, \dots, s_1)}^*(u_r, \dots, u_1)_v \\ * \\ \vdots \\ * \\ (-1)^{r-2} \text{Li}_{(s_r, \dots, s_2)}^*(u_r, \dots, u_2)_v \\ \vdots \\ * \\ \vdots \\ * \\ \text{Li}_{s_r}^*(u_r)_v \end{array} \right) \in \text{Lie } G(\mathbb{C}_v).$$

$\left. \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right\} d_1$
 $\left. \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right\} d_2$
 $\left. \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right\} d_r$

Proof. Let $1 \leq \ell \leq r$. By Proposition 3.2.1 and the definition of \mathbf{v} , the last coordinate of the ℓ -th block (= the $(d_1 + \dots + d_\ell)$ -th component) of the vector $\log_G \mathbf{v}$ is

$$\begin{aligned}
 & \sum_{i \geq 0} \sum_{m=\ell}^r y_i[\ell m] (-1)^{r-m} u_m^{q^i} \cdots u_r^{q^i} \\
 &= \sum_{i \geq 0} \left(\frac{1}{L_i^{d_\ell}} (-1)^{r-\ell} u_\ell^{q^i} \cdots u_r^{q^i} \right. \\
 & \quad \left. + \sum_{m=\ell+1}^r (-1)^{m-\ell} \sum_{0 \leq i_\ell \leq \dots \leq i_{m-1} < i} \frac{u_\ell^{q^{i_\ell}} \cdots u_{m-1}^{q^{i_{m-1}}}}{L_{i_\ell}^{s_\ell} \cdots L_{i_{m-1}}^{s_{m-1}} L_i^{d_m}} (-1)^{r-m} u_m^{q^i} \cdots u_r^{q^i} \right) \\
 &= (-1)^{r-\ell} \sum_{i \geq 0} \left(\frac{u_\ell^{q^i} \cdots u_r^{q^i}}{L_i^{s_\ell} \cdots L_i^{s_r}} + \sum_{m=\ell+1}^r \sum_{0 \leq i_\ell \leq \dots \leq i_{m-1} < i} \frac{u_\ell^{q^{i_\ell}} \cdots u_{m-1}^{q^{i_{m-1}}} u_m^{q^i} \cdots u_r^{q^i}}{L_{i_\ell}^{s_\ell} \cdots L_{i_{m-1}}^{s_{m-1}} L_i^{s_m} \cdots L_i^{s_r}} \right) \\
 &= (-1)^{r-\ell} \sum_{0 \leq i_\ell \leq \dots \leq i_r} \frac{u_\ell^{q^{i_\ell}} \cdots u_r^{q^{i_r}}}{L_{i_\ell}^{s_\ell} \cdots L_{i_r}^{s_r}} \\
 &= (-1)^{r-\ell} \text{Li}_{(s_r, \dots, s_\ell)}^*(u_r, \dots, u_\ell).
 \end{aligned}$$

□

Remark 3.3.4. The t -module G and algebraic point \mathbf{v} defined above are exactly identified with $\text{Ext}_{\mathcal{F}}^1(\mathbf{1}, M')$ and M respectively in [CPY14, Thm. 4.3.2] through [CPY14, Thms. 5.2.1, 5.2.3].

Remark 3.3.5. If we replace u_1, \dots, u_r by r independent variables t_1, \dots, t_r , then the t -module G is defined over $A[t_1, \dots, t_r]$ and the formula of $\log_G \mathbf{v}$ in Theorem 3.3.3 is still valid as power series in the variables t_1, \dots, t_r (when ignoring convergence). Multiple several variable zeta functions are studied in [ADR16], but the authors do not know whether the formula mentioned above has connection with those in [ADR16] or has application to v -adic MZVs.

4. v -ADIC VANISHING PRINCIPLE FOR CMPLS

In what follows, we fix a monic prime v of A and an r -tuple $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$. We further fix $\mathbf{u} = (u_1, \dots, u_r) \in (\bar{k})^r$ so that $|u_1|_v, \dots, |u_{r-1}|_v \leq 1$ and $|u_r|_v < 1$. Therefore

$$\text{Li}_{(s_r, \dots, s_\ell)}^*(u_r, \dots, u_\ell)_v$$

converges v -adically for $\ell = 1, \dots, r$.

4.1. The vanishing principle. We continue the notation and hypothesis above. We let $G = (\mathbb{G}_a^d, \rho)$ be the t -module defined in (3.1.1). Define $\mathbf{v} = \mathbf{v}_{\mathfrak{s}, \mathbf{u}} \in G(\bar{k})$ to be the algebraic point given in (3.3.1). Although we have the functional identity for the logarithm of a t -module 2.3.2, we can not evaluate at arbitrary points as the logarithm in question is not an entire function. The following lemma provides the situation fitting into our need for later study on \log_G .

Lemma 4.1.1. *Let $H = (\mathbb{G}_a^n, \sigma)$ be a t -module defined over the ring of integers of \mathbb{C}_v and $\mathbf{x} \in H(\mathbb{C}_v)$ be a point such that $|\mathbf{x}|_v < 1$. Then for each $a \in \mathbb{F}_q[t]$, we have $\log_H(\sigma_a(\mathbf{x})) = \partial \sigma_a(\log_H(\mathbf{x}))$.*

Proof. Let X be a new variable and consider a map $\log_{H,X}: (XC_v[[X]])^n \rightarrow (XC_v[[X]])^n$ by

$$\log_{H,X} \mathbf{g} := \sum_{i=0}^{\infty} Q_i \mathbf{g}^{(i)},$$

where $Q_i \in \text{Mat}_n(\bar{k})$ are the coefficient matrices of \log_H and $\mathbf{g}^{(i)}$ is obtained by raising each component of \mathbf{g} to the i th power. Then we have $\log_{H,X}(\sigma_a(\mathbf{g})) = \partial\sigma_a(\log_{H,X}(\mathbf{g}))$ for each $a \in \mathbb{F}_q[t]$ and $\mathbf{g} \in (XC_v[[X]])^n$. Let $\mathbf{C}_v\{X\}$ be the ring consisting of all power series which converge on $|X|_v \leq 1$, and we call $\mathbf{C}_v\{X\}$ the Tate algebra over \mathbf{C}_v .

Let \mathfrak{m}_v be the maximal ideal of the ring of integers of \mathbf{C}_v . For each $\mathbf{x} \in \mathfrak{m}_v^n$ and $i \geq 1$, it is clear that $\log_{H,X}(X^i \mathbf{x}) \in \mathbf{C}_v\{X\}^n$ and its values at $X = 1$ is $\log_H \mathbf{x}$. Since $\log_{H,X}$, \log_H and the evaluation map $X \mapsto 1$ are additive, we can extend these operations from $\{X^i \mathbf{x}\}$ to $(X\mathfrak{m}_v[X])^n$, and hence we have the commutative diagram

$$\begin{array}{ccc} (X\mathfrak{m}_v[X])^n & \longrightarrow & \mathfrak{m}_v^n \\ \log_{H,X} \downarrow & & \downarrow \log_H \\ \mathbf{C}_v\{X\}^n & \longrightarrow & \mathbf{C}_v^n = \text{Lie } H \end{array}$$

where the horizontal maps are the evaluating map $\mathbf{g}(X) \mapsto \mathbf{g}(1)$.

Let $\mathbf{x} \in H(\mathbf{C}_v)$ be a point such that $|\mathbf{x}|_v < 1$. Note that $\log_H(\sigma_a(\mathbf{x}))$ converges since $|\sigma_a(\mathbf{x})|_v \leq |\mathbf{x}|_v < 1$. From the fact that $\sigma_a(X\mathbf{x}) \in (X\mathfrak{m}_v[X])^n$, we have

$$\begin{aligned} \log_H(\sigma_a(\mathbf{x})) &= \log_H(\sigma_a(X\mathbf{x})|_{X=1}) = \log_{H,X}(\sigma_a(X\mathbf{x}))|_{X=1} \\ &= \partial\sigma_a(\log_{H,X}(X\mathbf{x}))|_{X=1} = \partial\sigma_a(\log_H(\mathbf{x})). \end{aligned}$$

□

Theorem 4.1.2. *Let notation and hypotheses be given as above and put $v(t) := v|_{\theta=t} \in \mathbb{F}_q[t]$. Then the following are equivalent.*

(i) *The following v -adic values are simultaneously vanishing:*

$$\text{Li}_{(s_r, \dots, s_1)}^*(u_r, \dots, u_1)_v = \text{Li}_{(s_r, \dots, s_2)}^*(u_r, \dots, u_2)_v = \dots = \text{Li}_{s_r}^*(u_r)_v = 0.$$

(ii) *\mathbf{v} is a $v(t)$ -power torsion point in $G(\bar{k})$.*

(iii) *\mathbf{v} is an $\mathbb{F}_q[t]$ -torsion point in $G(\bar{k})$.*

Furthermore, we assume that $|u_i|_v < 1$ for all i . Thus $\text{Li}_{(s_\ell, \dots, s_r)}(u_\ell, \dots, u_r)_v$ converges v -adically for $\ell = 1, \dots, r$. In this situation, the conditions above are also equivalent to the following one.

(iv) *The following v -adic values are simultaneously vanishing:*

$$\text{Li}_{(s_1, \dots, s_r)}(u_1, \dots, u_r)_v = \text{Li}_{(s_2, \dots, s_r)}(u_2, \dots, u_r)_v = \dots = \text{Li}_{s_r}(u_r)_v = 0.$$

Proof. (i) \Rightarrow (ii) To simplify the notation, we denote by $[-]$ the $\mathbb{F}_q[t]$ -actions on any t -modules. For $1 \leq i \leq r$, let G_i be the t -module whose t -action is defined by the square matrix of size $d_1 + \dots + d_i$ cut from the upper left square of the matrix (3.1.1). We also set $G_0 := 0$. Thus we have the exact sequence

$$0 \longrightarrow G_{i-1} \longrightarrow G_i \xrightarrow{\pi_i} \mathbf{C}^{\otimes d_i} \longrightarrow 0$$

for each $1 \leq i \leq r$. Note that $G_r = G$.

Let $\mathbf{v}_r := \mathbf{v}$. Since the operator $\partial[v(t)] - vI_{d_r}$ on $\text{Lie } \mathbf{C}^{\otimes d_r}$ is nilpotent, we have $\partial[v(t)^{p^n}] = (\partial[v(t)])^{p^n} = v^{p^n} I_{d_r}$ for $p^n \geq d_r$. Thus

$$\partial[v(t)^{p^n}](\log_{\mathbf{C}^{\otimes d_r}}(\pi_r(\mathbf{v}_r)))$$

lies in the convergence domain of $\exp_{\mathbf{C}^{\otimes d_r}}$ for $n \gg 0$, which coincides with

$$\log_{\mathbf{C}^{\otimes d_r}}([v(t)^{p^n}](\pi_r(\mathbf{v}_r)))$$

by Lemma 4.1.1. Note that by Theorem 3.3.3 the last coordinate of this vector is $v^{p^n} \text{Li}_{s_r}^*(u_r)_v = 0$. It follows by [Yu91, Thm. 3.7] that $\log_{\mathbf{C}^{\otimes d_r}}([v(t)^{p^n}](\pi_r(\mathbf{v}_r)))$ has to be zero. Since $\log_{\mathbf{C}^{\otimes d_r}}$ is the formal inverse of $\exp_{\mathbf{C}^{\otimes d_r}}$, we have

$$\pi_r([v(t)^{p^n}](\mathbf{v}_r)) = [v(t)^{p^n}](\pi_r(\mathbf{v}_r)) = 0.$$

Thus we conclude that there exists $n_r \in \mathbb{N}$ such that

$$\mathbf{v}_{r-1} := [v(t)^{p^{n_r}}](\mathbf{v}_r) \in \text{Ker } \pi_r = G_{r-1}(\bar{k}).$$

Repeating this argument, there exist n_{r-1}, \dots, n_1 such that

$$\begin{aligned} \mathbf{v}_{r-2} &:= [v(t)^{p^{n_{r-1}}}] (\mathbf{v}_{r-1}) \in \text{Ker } \pi_{r-1} = G_{r-2}(\bar{k}) \\ &\vdots \\ \mathbf{v}_1 &:= [v(t)^{p^{n_2}}] (\mathbf{v}_2) \in \text{Ker } \pi_2 = G_1(\bar{k}) \\ \mathbf{v}_0 &:= [v(t)^{p^{n_1}}] (\mathbf{v}_1) \in \text{Ker } \pi_1 = G_0(\bar{k}). \end{aligned}$$

Hence \mathbf{v} is a $v(t)$ -power torsion point.

(ii) \Rightarrow (iii) It is obvious.

(iii) \Rightarrow (i) Suppose that there exists a nonzero polynomial $a \in \mathbb{F}_q[t]$ for which $\rho_a(\mathbf{v}) = 0$. By Lemma 4.1.1 and Theorem 3.3.3, we have

$$0 = \log_G(\rho_a(\mathbf{v})) = \partial \rho_a(\log_G \mathbf{v}) = \begin{pmatrix} * \\ \vdots \\ * \\ a(\theta)(-1)^{r-1} \text{Li}_{(s_r, \dots, s_1)}^*(u_r, \dots, u_1)_v \\ * \\ \vdots \\ * \\ a(\theta)(-1)^{r-2} \text{Li}_{(s_r, \dots, s_2)}^*(u_r, \dots, u_2)_v \\ \vdots \\ * \\ \vdots \\ * \\ a(\theta) \text{Li}_{s_r}(u_r)_v \end{pmatrix} \begin{matrix} \left. \vphantom{\begin{matrix} * \\ \vdots \\ * \end{matrix}} \right\} d_1 \\ \left. \vphantom{\begin{matrix} * \\ \vdots \\ * \end{matrix}} \right\} d_2 \\ \left. \vphantom{\begin{matrix} * \\ \vdots \\ * \end{matrix}} \right\} d_r \end{matrix}$$

whence the desired result as $a(\theta)$ is nonzero.

Assume that $|u_i|_v < 1$ for all i .

(i) \Rightarrow (iv) This follows from Lemma 4.1.3.

(iv) \Rightarrow (i) This can be done inductively by using Lemma 4.1.3:

$$\begin{aligned}
& \text{Li}_{s_r}^*(u_r)_v = \text{Li}_{s_r}(u_r)_v = 0, \\
& \text{Li}_{(s_r, s_{r-1})}^*(u_r, u_{r-1})_v = \text{Li}_{s_{r-1}}(u_{r-1})_v \text{Li}_{s_r}^*(u_r)_v - \text{Li}_{(s_{r-1}, s_r)}(u_{r-1}, u_r)_v = 0, \\
& \text{Li}_{(s_r, s_{r-1}, s_{r-2})}^*(u_r, u_{r-1}, u_{r-2})_v \\
& = \text{Li}_{s_{r-2}}(u_{r-2})_v \text{Li}_{(s_r, s_{r-1})}^*(u_r, u_{r-1})_v - \text{Li}_{(s_{r-2}, s_{r-1})}(u_{r-2}, u_{r-1})_v \text{Li}_{(s_r)}^*(u_r)_v \\
& + \text{Li}_{(s_{r-2}, s_{r-1}, s_r)}(u_{r-2}, u_{r-1}, u_r)_v = 0, \\
& \vdots \\
& \text{Li}_{(s_r, \dots, s_1)}^*(u_r, \dots, u_1)_v \\
& = \sum_{\ell=2}^r (-1)^\ell \text{Li}_{(s_1, \dots, s_{\ell-1})}(u_1, \dots, u_{\ell-1})_v \text{Li}_{(s_r, \dots, s_\ell)}^*(u_r, \dots, u_\ell)_v \\
& + (-1)^{r+1} \text{Li}_{(s_1, \dots, s_r)}(u_1, \dots, u_r)_v = 0.
\end{aligned}$$

□

Lemma 4.1.3. For any $s_1, \dots, s_r \in \mathbb{N}$, we have the following identity as power series.

$$\begin{aligned}
& \text{Li}_{(s_r, \dots, s_1)}^*(z_r, \dots, z_1) \\
& = \sum_{\ell=2}^r (-1)^\ell \text{Li}_{(s_1, \dots, s_{\ell-1})}(z_1, \dots, z_{\ell-1}) \text{Li}_{(s_r, \dots, s_\ell)}^*(z_r, \dots, z_\ell) + (-1)^{r+1} \text{Li}_{(s_1, \dots, s_r)}(z_1, \dots, z_r)
\end{aligned}$$

Proof.

$$\begin{aligned}
\text{LHS} & = \sum_{i_1} \frac{z_1^{q^{i_1}}}{L_{i_1}^{s_1}} \sum_{i_r \geq \dots \geq i_2} \frac{z_r^{q^{i_r}} \dots z_2^{q^{i_2}}}{L_{i_r}^{s_r} \dots L_{i_2}^{s_2}} - \sum_{\substack{i_r \geq \dots \geq i_2 \\ i_1 > i_2}} \frac{z_r^{q^{i_r}} \dots z_1^{q^{i_1}}}{L_{i_r}^{s_r} \dots L_{i_1}^{s_1}} \\
& = \sum_{i_1} \frac{z_1^{q^{i_1}}}{L_{i_1}^{s_1}} \sum_{i_r \geq \dots \geq i_2} \frac{z_r^{q^{i_r}} \dots z_2^{q^{i_2}}}{L_{i_r}^{s_r} \dots L_{i_2}^{s_2}} - \sum_{i_1 > i_2} \frac{z_1^{q^{i_1}} z_2^{q^{i_2}}}{L_{i_1}^{s_1} L_{i_2}^{s_2}} \sum_{i_r \geq \dots \geq i_3} \frac{z_r^{q^{i_r}} \dots z_3^{q^{i_3}}}{L_{i_r}^{s_r} \dots L_{i_3}^{s_3}} + \sum_{\substack{i_r \geq \dots \geq i_3 \\ i_1 > i_2 > i_3}} \frac{z_r^{q^{i_r}} \dots z_1^{q^{i_1}}}{L_{i_r}^{s_r} \dots L_{i_1}^{s_1}} \\
& = \sum_{i_1} \frac{z_1^{q^{i_1}}}{L_{i_1}^{s_1}} \sum_{i_r \geq \dots \geq i_2} \frac{z_r^{q^{i_r}} \dots z_2^{q^{i_2}}}{L_{i_r}^{s_r} \dots L_{i_2}^{s_2}} - \sum_{i_1 > i_2} \frac{z_1^{q^{i_1}} z_2^{q^{i_2}}}{L_{i_1}^{s_1} L_{i_2}^{s_2}} \sum_{i_r \geq \dots \geq i_3} \frac{z_r^{q^{i_r}} \dots z_3^{q^{i_3}}}{L_{i_r}^{s_r} \dots L_{i_3}^{s_3}} \\
& + \sum_{i_1 > i_2 > i_3} \frac{z_1^{q^{i_1}} z_2^{q^{i_2}} z_3^{q^{i_3}}}{L_{i_1}^{s_1} L_{i_2}^{s_2} L_{i_3}^{s_3}} \sum_{i_r \geq \dots \geq i_4} \frac{z_r^{q^{i_r}} \dots z_4^{q^{i_4}}}{L_{i_r}^{s_r} \dots L_{i_4}^{s_4}} - \sum_{\substack{i_r \geq \dots \geq i_4 \\ i_1 > i_2 > i_3 > i_4}} \frac{z_r^{q^{i_r}} \dots z_1^{q^{i_1}}}{L_{i_r}^{s_r} \dots L_{i_1}^{s_1}} \\
& \vdots \\
& = \sum_{i_1} \frac{z_1^{q^{i_1}}}{L_{i_1}^{s_1}} \sum_{i_r \geq \dots \geq i_2} \frac{z_r^{q^{i_r}} \dots z_2^{q^{i_2}}}{L_{i_r}^{s_r} \dots L_{i_2}^{s_2}} - \sum_{i_1 > i_2} \frac{z_1^{q^{i_1}} z_2^{q^{i_2}}}{L_{i_1}^{s_1} L_{i_2}^{s_2}} \sum_{i_r \geq \dots \geq i_3} \frac{z_r^{q^{i_r}} \dots z_3^{q^{i_3}}}{L_{i_r}^{s_r} \dots L_{i_3}^{s_3}} + \dots \\
& \dots + (-1)^{r-1} \sum_{i_1 > \dots > i_{r-2}} \frac{z_1^{q^{i_1}} \dots z_{r-2}^{q^{i_{r-2}}}}{L_{i_1}^{s_1} \dots L_{i_{r-2}}^{s_{r-2}}} \sum_{i_r \geq i_{r-1}} \frac{z_r^{q^{i_r}} z_{r-1}^{q^{i_{r-1}}}}{L_{i_r}^{s_r} L_{i_{r-1}}^{s_{r-1}}} + (-1)^r \sum_{\substack{i_r \geq i_{r-1} \\ i_1 > \dots > i_{r-1}}} \frac{z_r^{q^{i_r}} \dots z_1^{q^{i_1}}}{L_{i_r}^{s_r} \dots L_{i_1}^{s_1}}
\end{aligned}$$

$$\begin{aligned}
 &= \sum_{i_1} \frac{z_1^{q^{i_1}}}{L_{i_1}^{s_1}} \sum_{i_r \geq \dots \geq i_2} \frac{z_r^{q^{i_r}} \dots z_2^{q^{i_2}}}{L_{i_r}^{s_r} \dots L_{i_2}^{s_2}} - \sum_{i_1 > i_2} \frac{z_1^{q^{i_1}} z_2^{q^{i_2}}}{L_{i_1}^{s_1} L_{i_2}^{s_2}} \sum_{i_r \geq \dots \geq i_3} \frac{z_r^{q^{i_r}} \dots z_3^{q^{i_3}}}{L_{i_r}^{s_r} \dots L_{i_3}^{s_3}} + \dots \\
 &\dots + (-1)^{r-1} \sum_{i_1 > \dots > i_{r-2}} \frac{z_1^{q^{i_1}} \dots z_{r-2}^{q^{i_{r-2}}}}{L_{i_1}^{s_1} \dots L_{i_{r-2}}^{s_{r-2}}} \sum_{i_r \geq i_{r-1}} \frac{z_r^{q^{i_r}} z_{r-1}^{q^{i_{r-1}}}}{L_{i_r}^{s_r} L_{i_{r-1}}^{s_{r-1}}} \\
 &+ (-1)^r \sum_{i_1 > \dots > i_{r-1}} \frac{z_1^{q^{i_1}} \dots z_{r-1}^{q^{i_{r-1}}}}{L_{i_1}^{s_1} \dots L_{i_{r-1}}^{s_{r-1}}} \sum_{i_r} \frac{z_r^{q^{i_r}}}{L_{i_r}^{s_r}} + (-1)^{r+1} \sum_{i_1 > \dots > i_r} \frac{z_r^{q^{i_r}} \dots z_1^{q^{i_1}}}{L_{i_r}^{s_r} \dots L_{i_1}^{s_1}} \\
 &= \text{RHS}
 \end{aligned}$$

□

Remark 4.1.4. To see the analogue of the identity above, we refer the reader to [IKOO11, Prop. 6], [SS15, Thm. 2.13] and [Z05, Thm. 3].

Corollary 4.1.5. *Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ and $\mathbf{u} = (u_1, \dots, u_r) \in (\bar{k}^\times)^r$. Assume that $\text{Li}_{\mathfrak{s}}(\mathbf{u})$ converges ∞ -adically and $|u_i|_v < 1$ for all i . Then $\text{Li}_{\mathfrak{s}}(\mathbf{u}) / \tilde{\pi}^{s_1 + \dots + s_r} \in k$ if and only if*

$$\text{Li}_{(s_1, \dots, s_r)}(u_1, \dots, u_r)_v = \text{Li}_{(s_2, \dots, s_r)}(u_2, \dots, u_r)_v = \dots = \text{Li}_{s_r}(u_r)_v = 0.$$

Proof. By [CPY14, Thm. 4.3.2] and Remark 3.3.4, $\text{Li}_{\mathfrak{s}}(\mathbf{u}) / \tilde{\pi}^{s_1 + \dots + s_r} \in k$ if and only if $\mathbf{v}_{\mathfrak{s}, \mathbf{u}}$ is an $\mathbb{F}_q[t]$ -torsion point in $G(\bar{k})$. Hence by Theorem 4.1.2 we have the desired result. □

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