ENTIRE SOLUTIONS WITH MERGING THREE FRONTS TO THE ALLEN-CAHN EQUATION

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ABSTRACT. In this paper, we study entire solutions of the Allen-Chan equation in one-dimensional Euclidean space. This equation is a scalar reaction-diffusion equation with a bistable nonlinearity. It is well-known that this equation admits three different types of traveling fronts connecting two of its three constant states. Under certain conditions on the wave speeds, the existence of entire solutions with merging these three traveling fronts is shown by constructing a suitable pair of super-sub-solutions.

1. INTRODUCTION

In this paper, we consider the following reaction-diffusion equation

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R},\ t \in \mathbb{R}, \tag{1.1}$$

where the function $f(u) \in C^2(\mathbb{R})$ satisfies

$$f(0) = f(1) = 0, \quad f'(0),\ f'(1) < 0, \tag{1.2}$$

$$f(a) = 0,\ f'(a) > 0,\ a \in (0,1), \quad f(u) \neq 0 \text{ for } u \in (0,a) \cup (a,1), \tag{1.3}$$

$$\int_0^1 f(s)\,ds > 0. \tag{1.4}$$

A typical example of $f(u)$ is $u(1-u)(u-a)$, where $a \in (0, 1/2)$. This equation is often called the Allen-Cahn equation or the Nagumo equation. It is easy to see that the constant states $u = 0$ and $u = 1$ are stable and the constant state $u = a$ is unstable for the kinetic equation (i.e., (1.1) without diffusion term), since $f'(0) < 0$, $f'(1) < 0$ and $f'(a) > 0$. Due to the rich dynamics of this prototype equation (1.1), there have been a lot of research on the dynamical behaviors of (1.1).

One of the main concerns on the dynamics of (1.1) is the existence of entire solutions. Here an entire solution means a classical solution defined for all $(x,t) \in \mathbb{R}^2$. One of typical examples of entire solutions is the traveling wave solution. A solution u of (1.1) is called a traveling wave solution, if $u(x,t) = \Phi(x + vt)$ for some constant v (the wave speed) and

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some function Φ (the wave profile). A traveling wave solution is called a traveling front, if it connects two different constant states.

In fact, (1.1) admits three different kinds of traveling fronts connecting states $\{0,1\}$, $\{0,a\}$, $\{a,1\}$, respectively. The first one is the bistable connection and the latter two cases are the monostable connections. In this paper, the wave profiles of traveling front connecting states $\{0,1\}$, $\{0,a\}$, $\{a,1\}$, are denoted by ϕ, ψ_1, ψ_2, and the speeds are denoted by c, c_1, c_2, respectively.

By [5], there exists a unique (up to translations) traveling front $u(x,t) = \phi(x+ct)$ of (1.1) connecting $\{0,1\}$ with the unique speed c. Note that, by setting $z = x + ct$, ϕ satisfies

$$\begin{aligned}
&\phi''(z) - c\,\phi'(z) + f(\phi(z)) = 0,\ \ \phi'(z) > 0, \quad z \in \mathbb{R},\\
&\phi(-\infty) = 0, \quad \phi(\infty) = 1.
\end{aligned}$$

and the speed c is given by

$$c = \frac{\int_0^1 f(\phi)\, d\phi}{\int_{-\infty}^{\infty} (\phi'(z))^2 dz} > 0.$$

By [1, 11], there exists a constant $c_{1,max} \le -2\sqrt{f'(a)}$ such that a traveling front $u(x,t) = \psi_1(x+c_1 t)$ of (1.1) connecting $\{0,a\}$ with speed c_1 exists for each $c_1 \le c_{1,max}$. Set $z = x+c_1 t$. Then $\psi_1(z)$ satisfies

$$\begin{aligned}
&\psi_1''(z) - c_1\,\psi_1'(z) + f(\psi_1(z)) = 0,\ \ \psi_1'(z) > 0, \quad z \in \mathbb{R},\\
&\psi_1(-\infty) = 0, \quad \psi_1(\infty) = a.
\end{aligned}$$

Similarly, there exists a constant $c_{2,min} \ge 2\sqrt{f'(a)}$ such that a traveling front $u(x,t) = \psi_2(x+c_2 t)$ of (1.1) connecting $\{a,1\}$ with speed c_2 exists for each $c_2 \ge c_{2,min}$. Set $z = x+c_2 t$. Then $\psi_2(z)$ satisfies

$$\begin{aligned}
&\psi_2''(z) - c_2\,\psi_2'(z) + f(\psi_2(z)) = 0,\ \ \psi_2'(z) > 0, \quad z \in \mathbb{R},\\
&\psi_2(-\infty) = a, \quad \psi_2(\infty) = 1.
\end{aligned}$$

Note that $c > 0$, $c_1 < 0 < c_2$, and $0 < \phi < 1$, $0 < \psi_1 < a$, $a < \psi_2 < 1$ in $\mathbb{R}$.

In 1999, Hamel and Nadirashvili [9] constructed a new type of entire solutions with merging two fronts for a scalar reaction-diffusion equation (see also [10]). Since then, there have been many works devoted to the construction of entire solutions with merging two fronts. For the works on the scalar reaction-diffusion equations, we refer the reader to, e.g., [19, 6, 8, 2, 3, 12].

In [19], Yagisita proved the existence of entire solutions which behave as two traveling fronts $\phi(x + ct)$ and $\phi(-x + ct)$ on the left x-axis and right x-axis as $t \to -\infty$, respectively. Then, Fukao, Morita and Ninomiya [6] provide a simple proof for the results shown in [19] for the Allen-Cahn equation.

Next, for the function $f(u)$ satisfying (1.2), (1.3), according to the results shown in [9, 8], for any $c_{11}, c_{12} \le c_{1,max}$, there exists an entire solution of (1.1) which converges to $\psi_1(x+c_{11}t)$ and $\psi_1(-x+c_{12}t)$ on the left x-axis and right x-axis, respectively, as $t \to -\infty$. Similarly, the existence of an entire solution of (1.1) which converges to $\psi_2(-x+c_{21}t)$ and $\psi_2(x+c_{22}t)$ on the left x-axis and right x-axis, respectively, as $t \to -\infty$ can be shown for any $c_{21}, c_{22} \ge c_{2,min}$.

Later, in [12], Morita and Ninomiya proposed a unified method to construct all types of entire solutions with merging two fronts mentioned above. Besides the works on the scalar equations, there are many works on the entire solutions with merging two fronts for systems of two reaction-diffusion equations. We refer the reader to, for examples, [13, 7, 17, 18, 15, 20]. However, one might suspect whether there are other types of entire solutions with merging multiple (≥ 3) fronts for scalar equations. The main purpose of this work is to construct entire solutions with merging three fronts for the equation (1.1).

The following theorem is the main result of this paper.

Theorem 1.1. *Let (c,ϕ), (c_1,ψ_1) and (c_2,ψ_2) be described as above. Suppose that*

$$c > -c_1 > 0. \tag{1.5}$$

Then there exists an entire solution of (1.1) *such that*

$$\lim_{t\to-\infty}\Big\{\sup_{x\leq\omega_1(t)}|u(x,t)-\phi(-x+ct-\theta)|+\sup_{\omega_1(t)\leq x\leq\omega_2(t)}|u(x,t)-\psi_1(x+c_1t+\theta)| \\ +\sup_{x\geq\omega_2(t)}|u(x,t)-\psi_2(x+c_2t+\theta)|\Big\}=0 \tag{1.6}$$

for some constant θ, where

$$\omega_1(t) := \frac{-(-c+c_1)t}{2}, \quad \omega_2(t) := \frac{-(c_1+c_2)t}{2}.$$

Moreover, it holds

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|u(x,t)-1|=0.$$

This theorem shows us a new type of entire solution with merging three fronts. Note that the condition (1.5) on the speeds can be realized when we take the constant a such that $c > 2\sqrt{f'(a)}$. For example, when $f(u) = u(1-u)(u-a)$, we have $c = \sqrt{2}(1/2-a) > 2\sqrt{a(1-a)} = 2\sqrt{f'(a)}$ if we consider $0 < a < (3-\sqrt{6})/6$.

Since the comparison principle is available for (1.1), it is well-known that an entire solution exists if we can find a suitable pair of super-sub-solutions (cf., e.g., [6, 8]). Therefore, the main task of finding entire solutions with merging multiple fronts is to construct super-sub-solutions with the desired properties.

One of the main ideas in [12] is to find an auxiliary rational function with certain properties in order to construct a suitable pair of super-sub-solutions. The form of this auxiliary function depends on the four equilibrium states which are connected by those two traveling fronts under consideration. This method is very powerful and it can be applied to the construction of entire solutions with merging multiple fronts, if this auxiliary function can be found. However, the construction of this useful function is by no means trivial.

Another difficulty of constructing entire solutions of merging multiple fronts is to find the suitable curves (i.e. functions of time) of connecting adjacent fronts. Fortunately, in this work we are able to overcome these two major difficulties in the construction of entire solutions of merging three fronts for the nonlinearity $f(u)$ satisfies (1.2)-(1.4).

The rest of this paper is organized as follows. First, in §2, we first give an auxiliary function linking three traveling fronts of (1.1). Then we provide some useful properties of this auxiliary function and derive the key estimates (see Lemma 2.4) for the later construction of super-sub-solutions. Finally, in §3, we use this auxiliary function to construct a pair of super-sub-solutions and give a proof of the main theorem on the existence of entire solutions with merging three fronts.

2. Some function linking three-front dynamics

Let $v_1 := -c$, $v_2 := c_1 \le c_{1,max}$ and $v_3 := c_2 \ge c_{2,min}$. Let $\phi_i = \phi_i(x + v_i t)$, $i = 1, 2, 3$, be traveling fronts of (1.1) that satisfy

$$\begin{cases} \phi_i''(s) - v_i\phi_i'(s) + f(\phi_i(s)) = 0, \quad s \in \mathbb{R}, \\ \phi_i(-\infty) = \alpha_i, \quad \phi_i(\infty) = \omega_i, \end{cases} \tag{2.1}$$

where $(\alpha_1, \omega_1, \alpha_2, \omega_2, \alpha_3, \omega_3) = (1, 0, 0, a, a, 1)$. Here the prime denotes the derivative with respect to s. Note that $\phi_1(z) = \phi(-z)$ and $\phi_i = \psi_{i-1}$, $i = 2, 3$. In the sequel, we always assume

$$\phi_1(0) = \frac{a}{2}, \quad \phi_2(0) = \frac{a}{2}, \quad \phi_3(0) = \frac{1+a}{2}. \tag{2.2}$$

By the nondegenerate condition on f, for $p \le 0$, there are positive constants β_i,γ_i, $i = 1, 2, 3$, and $K > 0$ such that

$$\begin{cases} |\phi_i'(x+p)| \le K\exp(\beta_i(x+p)), \ x \le -p, \\ |\phi_i'(x+p)| \le K\exp(-\gamma_i(x+p)), \ x \ge -p. \end{cases} \tag{2.3}$$

In addition, there is a constant $\tau > 0$ such that

$$\begin{cases} \dfrac{|\phi_1(x-p)-1|}{|\phi_1'(x-p)|} \le \tau, \ x \le p, & \dfrac{|\phi_1(x-p)-0|}{|\phi_1'(x-p)|} \le \tau, \ x \ge p, \\ \dfrac{|\phi_2(x+p)-0|}{|\phi_2'(x+p)|} \le \tau, \ x \le -p, & \dfrac{|\phi_2(x+p)-a|}{|\phi_2'(x+p)|} \le \tau, \ x \ge -p, \\ \dfrac{|\phi_3(x+p)-a|}{|\phi_3'(x+p)|} \le \tau, \ x \le -p, & \dfrac{|\phi_3(x+p)-1|}{|\phi_3'(x+p)|} \le \tau, \ x \ge -p. \end{cases} \tag{2.4}$$

The key auxiliary function we found for linking three fronts is as follows.

Lemma 2.1. *Set*

$$Q(y, z, w) = z + (1-z)\frac{(1-y)z(w-a) + y(a-z)(1-w)}{(1-y)z(1-a) + (a-z)(1-w)}. \tag{2.5}$$

Then the following three statements hold:

(i) *Q can be rewritten as*

$$Q(y, z, w) = \begin{cases} y + (1-y)z\dfrac{(1-a)(w-y)}{(1-y)z(1-a) + (a-z)(1-w)}, \\ w + (a-z)(1-w)\dfrac{y-w}{(1-y)z(1-a) + (a-z)(1-w)}. \end{cases} \tag{2.6}$$

(ii) *There exist functions* Q_i, $i=1,2,3$, *such that*

$$\begin{aligned} Q_y(y,z,w) &= (a-z)(1-w)Q_1(y,z,w),\\ Q_z(y,z,w) &= (1-y)(1-w)Q_2(y,z,w),\\ Q_w(y,z,w) &= (1-y)zQ_3(y,z,w). \end{aligned}$$

(iii) *There exist functions* R_j, $j=1,\cdots,16$, *such that*

$$\begin{aligned} Q_{yy}(y,z,w) &= zR_1(y,z,w)=(a-z)R_2(y,z,w)=(1-w)R_3(y,z,w),\\ Q_{zz}(y,z,w) &= (1-y)R_4(y,z,w)=(1-w)R_5(y,z,w)\\ &= yR_6(y,z,w)+(w-a)R_7(y,z,w),\\ Q_{ww}(y,z,w) &= (1-y)R_8(y,z,w)=zR_9(y,z,w)=(a-z)R_{10}(y,z,w),\\ Q_{yz}(y,z,w) &= (1-w)R_{11}(y,z,w), \quad Q_{zw}(y,z,w)=(1-y)R_{12}(y,z,w),\\ Q_{yw}(y,z,w) &= (1-y)R_{13}(y,z,w)=zR_{14}(y,z,w)\\ &= (a-z)R_{15}(y,z,w)=(1-w)R_{16}(y,z,w). \end{aligned}$$

Proof. Obviously, the function $Q(y,z,w)$ defined by (2.5) allows the expression as (2.6).

By a simple calculation, we can derive

$$\begin{aligned} Q_y(y,z,w) &= \frac{a(1-z)(a-z)(1-w)^2}{[(1-y)z(1-a)+(a-z)(1-w)]^2},\\ Q_z(y,z,w) &= \frac{(1-a)a(1-y)(1-w)(w-y)}{[(1-y)z(1-a)+(a-z)(1-w)]^2},\\ Q_w(y,z,w) &= \frac{a(1-a)(1-y)^2z(1-z)}{[(1-y)z(1-a)+(a-z)(1-w)]^2}. \end{aligned}$$

Hence, the conclusion (ii) holds.

For the statement (iii), we compute the second derivative of function Q and obtain that

$$\begin{aligned} Q_{yy}(y,z,w) &= \frac{2(1-a)az(a-z)(1-z)(1-w)^2}{[(1-y)z(1-a)+(a-z)(1-w)]^3},\\ Q_{zz}(y,z,w) &= -\frac{2(1-a)a(1-y)(1-w)(w-y)[w-a-y(1-a)]}{[(1-y)z(1-a)+(a-z)(1-w)]^3},\\ Q_{ww}(y,z,w) &= \frac{2(1-a)a(1-y)^2z(a-z)(1-z)}{[(1-y)z(1-a)+(a-z)(1-w)]^3},\\ Q_{yz}(y,z,w) &= -\frac{(1-a)a(1-w)^2[(y-w)z+a(1-2y-z+w+yz)]}{[(1-y)z(1-a)+(a-z)(1-w)]^3},\\ Q_{yw}(y,z,w) &= -\frac{2(1-a)a(1-y)z(a-z)(1-z)(1-w)}{[(1-y)z(1-a)+(a-z)(1-w)]^3},\\ Q_{zw}(y,z,w) &= -\frac{(1-a)a(1-y)^2[(w-y)z+a(-1+w+z-2wz+yz)]}{[(1-y)z(1-a)+(a-z)(1-w)]^3}. \end{aligned}$$

Thus, we get the conclusion (iii) and the lemma is proved. □

With this auxiliary function Q, we can construct a suitable pair of super-sub-solutions. For this, we put $u(x,t) = U(\xi,t)$ with $\xi := x + \bar{c}t$ and $\bar{c} = (v_1 + v_2)/2$. Then (1.1) becomes

$$U_t = U_{\xi\xi} - \bar{c}U_\xi + f(U), \quad \xi \in \mathbb{R}. \tag{2.7}$$

We can easily check that (2.7) has traveling wave solutions

$$U = \phi_1(\xi - s_1 t),\ \phi_2(\xi + s_1 t),\ \phi_3(\xi + s_2 t),$$

where $s_1 := (v_2 - v_1)/2 > 0$ and $s_2 := v_3 - \bar{c} = (2v_3 - v_1 - v_2)/2 > s_1$, by (1.5).

Now we consider

$$U(\xi,t) = Q(\phi_1,\phi_2,\phi_3),\ \phi_1 = \phi_1(\xi - q_1(t)),\ \phi_2 = \phi_2(\xi + q_2(t)),\ \phi_3 = \phi_3(\xi + q_3(t)),$$

where $q_i(t) < 0$, $i = 1,2,3$, and $-q_2(t) < -q_3(t)$. Set

$$\mathcal{T}[U] := U_t - U_{\xi\xi} + \bar{c}U_\xi - f(U).$$

Then

$$\begin{aligned} \mathcal{T}[Q(\phi_1,\phi_2,\phi_3)] &= -Q_y\phi_1'(q_1' - s_1) + Q_z\phi_2'(q_2' - s_1) + Q_w\phi_3'(q_3' - s_2) \\ &\quad -G(\phi_1,\phi_2,\phi_3) - H(\phi_1,\phi_2,\phi_3) \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} G(\phi_1,\phi_2,\phi_3) &:= Q_{yy}\{\phi_1'\}^2 + Q_{zz}\{\phi_2'\}^2 + Q_{ww}\{\phi_3'\}^2 + 2[Q_{yz}\phi_1'\phi_2' + Q_{yw}\phi_1'\phi_3' + Q_{zw}\phi_2'\phi_3'], \\ H(\phi_1,\phi_2,\phi_3) &:= f(Q) - Q_y f(\phi_1) - Q_z f(\phi_2) - Q_w f(\phi_3). \end{aligned}$$

From (2.6) and Lemma 2.1 we see that

$$\begin{aligned} H(1,z,w) &= f(Q(1,z,w)) - Q_y(1,z,w)f(1) - Q_z(1,z,w)f(z) - Q_w(1,z,w)f(w) = 0, \\ H(y,0,w) &= f(Q(y,0,w)) - Q_y(y,0,w)f(y) - Q_z(y,0,w)f(0) - Q_w(y,0,w)f(w) = 0, \\ H(y,a,w) &= f(Q(y,a,w)) - Q_y(y,a,w)f(y) - Q_z(y,a,w)f(a) - Q_w(y,a,w)f(w) = 0, \\ H(y,z,1) &= f(Q(y,z,1)) - Q_y(y,z,1)f(y) - Q_z(y,z,1)f(z) - Q_w(y,z,1)f(1) = 0, \end{aligned}$$

which implies that there is a smooth function H_1 satisfying

$$H(y,z,w) = (1-y)z(a-z)(1-w)H_1(y,z,w).$$

Since $Q(0,z,a) = z$ and $Q_z(0,z,a) = 1$, we have

$$H(0,z,a) = f(Q(0,z,a)) - Q_z(0,z,a)f(z) = 0,$$

which implies $H_1(0,z,a) = 0$. Applying the mean value theorem to H_1 yields

$$\begin{aligned} H_1(y,z,w) &= \int_0^1 H_{1y}(\theta y, z, \theta w + (1-\theta)a)d\theta \cdot y \\ &\quad + \int_0^1 H_{1w}(\theta y, z, \theta w + (1-\theta)a)d\theta \cdot (w-a). \end{aligned}$$

Thus we obtain

$$\begin{cases} H(y,z,w) = (1-y)z[yH_{11}(y,z,w) + (w-a)H_{12}(y,z,w)], \\ H(y,z,w) = (1-w)(a-z)[yH_{21}(y,z,w) + (w-a)H_{22}(y,z,w)]. \end{cases} \tag{2.9}$$

Lemma 2.2. *For $q_1, q_2, q_3 \le -\delta < 0$, there exist positive constants ϵ_1, ϵ_2 and ϵ_3 such that*

$$\begin{aligned} &Q_y(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) \ge \epsilon_1 \text{ for } \xi \le -q_2,\\ &Q_z(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) \ge \epsilon_2 \text{ for } q_1 \le \xi \le -q_3,\\ &Q_w(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) \ge \epsilon_3 \text{ for } \xi \ge -q_2. \end{aligned}$$

Proof. Recall (2.2). Then

$$\begin{aligned} &\frac{a}{2} \le \phi_1(\xi - q_1) \le 1,\ 0 \le \phi_2(\xi + q_2) \le \frac{a}{2},\ a \le \phi_3(\xi + q_3) \le \frac{1+a}{2} \text{ for } \xi \le q_1,\\ &0 \le \phi_1(\xi - q_1) \le \frac{a}{2},\ 0 \le \phi_2(\xi + q_2) \le \frac{a}{2},\ a \le \phi_3(\xi + q_3) \le \frac{1+a}{2} \text{ for } q_1 \le \xi \le -q_2,\\ &0 \le \phi_1(\xi - q_1) \le \frac{a}{2},\ \frac{a}{2} \le \phi_2(\xi + q_2) \le a,\ a \le \phi_3(\xi + q_3) \le \frac{1+a}{2} \text{ for } -q_2 \le \xi \le -q_3,\\ &0 \le \phi_1(\xi - q_1) \le \frac{a}{2},\ \frac{a}{2} \le \phi_2(\xi + q_2) \le a,\ \frac{1+a}{2} \le \phi_3(\xi + q_3) \le 1 \text{ for } \xi \ge -q_3, \end{aligned}$$

when $q_1, q_2, q_3 \le -\delta$. Then we have

$$\frac{a(1-a)}{4} \le (1-a)(1-\phi_1)\phi_2 + (a - \phi_2)(1 - \phi_3) \le \frac{3a(1-a)}{2} \tag{2.10}$$

for $\xi \in \mathbb{R}$, $q_1, q_2, q_3 \le -\delta$.

By Lemma 2.1, for $q_1, q_2, q_3 < -\delta$, we derive that

$$\begin{aligned} &Q_y(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3))\\ &= \frac{a(1-\phi_2)(a-\phi_2)(1-\phi_3)^2}{[(1-\phi_1)\phi_2(1-a) + (a-\phi_2)(1-\phi_3)]^2}\\ &\ge \frac{a(1-a/2)(a/2)(1-(a+1)/2))^2}{[3a(1-a)/2]^2} = \frac{2-a}{36} \end{aligned}$$

for $\xi \le -q_2$,

$$\begin{aligned} &Q_z(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3))\\ &= \frac{(1-a)a(1-\phi_1)(1-\phi_3)(\phi_3-\phi_1)}{[(1-\phi_1)\phi_2(1-a) + (a-\phi_2)(1-\phi_3)]^2}\\ &\ge \frac{[(1-a)a(1-a/2)(1-(a+1)/2)(a-a/2)}{[3a(1-a)/2]^2} = \frac{2-a}{18} \end{aligned}$$

for $q_1 \le \xi \le -q_3$, and

$$\begin{aligned} &Q_w(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3))\\ &= \frac{a(1-a)(1-\phi_1)^2\phi_2(1-\phi_2)}{[(1-\phi_1)\phi_2(1-a) + (a-\phi_2)(1-\phi_3)]^2}\\ &\ge \frac{a(1-a)(1-a/2)^2(a/2)(1-a)}{[3a(1-a)/2]^2} = \frac{(2-a)^2}{18} \end{aligned}$$

for $\xi \ge -q_2$. Therefore, the lemma follows. □

From Lemma 2.1, it is easy to check that there exists a positive constant C such that

$$|R_j(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3))| \le C,$$
$$|H_{mn}(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3))| \le C,$$

for $\xi \in \mathbb{R}$, $q_1, q_2, q_3 < -\delta$, $j = 1, \cdots, 16$, and $m, n = 1, 2$. Now, we define a function $F(\phi_1, \phi_2, \phi_3)$ as follows

$$\begin{aligned} & F(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) \\ :=\ & -Q_y(\phi_1, \phi_2, \phi_3)\phi_1'(\xi - q_1) + Q_z(\phi_1, \phi_2, \phi_3)\phi_2'(\xi + q_2) + Q_w(\phi_1, \phi_2, \phi_3)\phi_3'(\xi + q_3). \end{aligned}$$

Then the function F is bounded above for $\xi \in \mathbb{R}$, $q_1, q_2, q_3 < -\delta$, since Q_y, Q_z, Q_w, $-\phi_1'$, ϕ_2' and ϕ_3' are bounded above for $\xi \in \mathbb{R}$, $q_1, q_2, q_3 < -\delta$.

The next lemma shows that the function F has a positive lower bound for $\xi \in \mathbb{R}$ and $q_1, q_2, q_3 < -\delta$, if δ is sufficiently large.

Lemma 2.3. *There exists a sufficiently large constant δ such that*

$$F(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) > 0 \text{ for } \xi \in \mathbb{R},\ q_1, q_2, q_3 \le -\delta. \tag{2.11}$$

Moreover, $F(\phi_1, \phi_2, \phi_3) = F(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3))$ satisfies

$$F(\phi_1, \phi_2, \phi_3) \ge \frac{1}{2} Q_y |\phi_1'(\xi - q_1)| \quad \text{for } \xi \le q_1, \tag{2.12}$$
$$F(\phi_1, \phi_2, \phi_3) \ge \frac{1}{2}\Big[Q_y |\phi_1'(\xi - q_1)| + Q_z |\phi_2'(\xi + q_2)|\Big] \quad \text{for } q_1 \le \xi \le -q_2, \tag{2.13}$$
$$F(\phi_1, \phi_2, \phi_3) \ge \frac{1}{2}\Big[Q_z |\phi_2'(\xi + q_2)| + Q_w |\phi_3'(\xi + q_3)|\Big] \quad \text{for } -q_2 \le \xi \le -q_3, \tag{2.14}$$
$$F(\phi_1, \phi_2, \phi_3) \ge \frac{1}{2} Q_w |\phi_3'(\xi + q_3)| \quad \text{for } \xi \ge -q_3, \tag{2.15}$$

when $q_1, q_2, q_3 \le -\delta$.

Proof. Since $\phi_1(-\infty) = 1$, $\phi_3(-\infty) = a$, $\phi_1(\xi - q_1)$ is decreasing and $\phi_3(\xi + q_3)$ is increasing for $\xi \in \mathbb{R}$, there exists a $q_0 < q_1$ such that $\phi_1(q_0 - q_1) = \phi_3(q_0 + q_3)$, $\phi_1(\xi - q_1) > \phi_3(\xi + q_3)$ for $\xi < q_0$ and $\phi_1(\xi - q_1) < \phi_3(\xi + q_3)$ for $q_0 < \xi \le q_1$. For $q_0 \le \xi \le q_1$, we have $Q_z(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) \ge 0$ and

$$F(\phi_1, \phi_2, \phi_3) = -Q_y\phi_1' + Q_z\phi_2' + Q_w\phi_3' \ge -Q_y\phi_1' \ge \frac{1}{2} Q_y |\phi_1'|$$

by $Q_w, \phi_2', \phi_3' \geq 0$. If $\xi < q_0$, we know that $Q_z(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) < 0$. From (2.3)-(2.4), we have $|\phi_1'| \geq (1 - \phi_1)/\tau$ and $|\phi_2'| \leq K e^{\beta_2 q_2}$. Then we compute that

$$
\begin{aligned}
&F(\phi_1, \phi_2, \phi_3) - \frac{1}{2}Q_y|\phi_1'| = \frac{1}{2}Q_y|\phi_1'| + Q_z\phi_2' + Q_w\phi_3' \geq \frac{1}{2}Q_y|\phi_1'| + Q_z\phi_2' \\
\geq\;& \frac{a(1-\phi_2)(a-\phi_2)(1-\phi_3)^2(1-\phi_1)}{2\tau[(1-\phi_1)\phi_2(1-a) + (a-\phi_2)(1-\phi_3)]^2} \\
&\qquad + \frac{(1-a)a(1-\phi_3)(1-\phi_1)(\phi_3-\phi_1)}{[(1-\phi_1)\phi_2(1-a) + (a-\phi_2)(1-\phi_3)]^2} K e^{\beta_2 q_2} \\
\geq\;& \frac{a(1-\phi_1)(1-\phi_3)}{2\tau[3a(1-a)/2]^2}\left[\left(1 - \frac{a}{2}\right)\left(a - \frac{a}{2}\right)\left(1 - \frac{a+1}{2}\right) + 2\tau(1-a)(a-1)Ke^{-\beta_2\delta}\right] \\
\geq\;& 0
\end{aligned}
$$

for δ sufficiently large. Therefore, (2.12) holds for $\xi \leq q_1$ and $q_1, q_2, q_3 \leq -\delta$.

For $\xi \geq q_1$, since $Q_y, Q_z, Q_w \geq 0$, $\phi_1' < 0$ and $\phi_2', \phi_3' > 0$, we get the conclusion. □

With Lemma 2.3, we now state and prove the following key lemma on the estimates to be used later in verifying super-sub-solutions.

Lemma 2.4. *There is a positive constant M such that*

$$
\left|\frac{H(\phi_1, \phi_2, \phi_3) + G(\phi_1, \phi_2, \phi_3)}{F(\phi_1, \phi_2, \phi_3)}\right| \leq \begin{cases} M(|\phi_2'| + |\phi_3'|) & \text{for } \xi \leq 0, \\ M(|\phi_1'| + |\phi_3'|) & \text{for } 0 \leq \xi \leq -\dfrac{q_3 + q_2}{2}, \\ M(|\phi_1'| + |\phi_2'|) & \text{for } \xi \geq -\dfrac{q_3 + q_2}{2}, \end{cases} \tag{2.16}
$$

for $q_1, q_2, q_3 < -\delta$ with $\delta \gg 1$.

Proof. For the simplicity of notation, we denote the functions $H_{ij}(\phi_1, \phi_2, \phi_3)$ $(i, j = 1, 2)$ by H_{ij}. Similarly we also omit (ϕ_1, ϕ_2, ϕ_3) for $H(\phi_1, \phi_2, \phi_3)$, $G(\phi_1, \phi_2, \phi_3)$, $Q_y(\phi_1, \phi_2, \phi_3)$ and so on.

First, we estimate $|H/F|$. For $\xi \leq q_1$, by (2.9), Lemma 2.2, (2.4) and (2.12), we have

$$
\begin{aligned}
\left|\frac{H}{F}\right| &\leq \left|\frac{2(1-\phi_1)\phi_2[\phi_1 H_{11} + (\phi_3 - a)H_{12}]}{Q_y|\phi_1'|}\right| \\
&\leq \frac{2\tau}{\epsilon_1}[C|\phi_2| + C|\phi_3 - a|] \leq \frac{2C\tau^2}{\epsilon_1}(|\phi_2'| + |\phi_3'|).
\end{aligned} \tag{2.17}
$$

For $q_1 \leq \xi \leq 0$, by (2.9), Lemma 2.2, (2.4) and (2.13), we have

$$
\begin{aligned}
\left|\frac{H}{F}\right| &\leq \left|\frac{2(1-\phi_1)\phi_2[\phi_1 H_{11} + (\phi_3 - a)H_{12}]}{Q_y|\phi_1'| + Q_z|\phi_2'|}\right| \\
&\leq \frac{2|1-\phi_1||\phi_2||\phi_1||H_{11}|}{Q_y|\phi_1'|} + \frac{2|1-\phi_1||\phi_2||\phi_3 - a||H_{12}|}{Q_z|\phi_2'|} \\
&\leq \frac{2C\tau^2|\phi_2'|}{\epsilon_1} + \frac{2C\tau^2|\phi_3'|}{\epsilon_2}.
\end{aligned} \tag{2.18}
$$

From (2.17)-(2.18), we obtain that

$$\left|\frac{H}{F}\right| \le M_1(|\phi_2'(\xi+q_2)|+|\phi_3'(\xi+q_3)|) \quad \text{for} \quad \xi \le 0. \tag{2.19}$$

For $0 \le \xi \le -q_2$, by (2.9), Lemma 2.2, (2.4) and (2.13), we have

$$\begin{aligned}\left|\frac{H}{F}\right| &\le \left|\frac{2(1-\phi_1)\phi_2[\phi_1 H_{11}+(\phi_3-a)H_{12}]}{Q_z|\phi_2'|}\right| \\ &\le \frac{2|1-\phi_1||\phi_2||\phi_1||H_{11}|}{Q_z|\phi_2'|}+\frac{2|1-\phi_1||\phi_2||\phi_3-a||H_{12}|}{Q_z|\phi_2'|} \\ &\le \frac{2C\tau^2|\phi_1'|}{\epsilon_2}+\frac{2C\tau^2|\phi_3'|}{\epsilon_2}.\end{aligned} \tag{2.20}$$

For $-q_2 \le \xi \le (-q_3-q_2)/2$, by (2.9), Lemma 2.2, (2.4) and (2.14), we have

$$\begin{aligned}\left|\frac{H}{F}\right| &\le \left|\frac{2(1-\phi_3)(a-\phi_2)[\phi_1 H_{21}+(\phi_3-a)H_{22}]}{Q_z|\phi_2'|}\right| \\ &\le \frac{2|1-\phi_3||a-\phi_2||\phi_1||H_{21}|}{Q_z|\phi_2'|}+\frac{2|1-\phi_3||a-\phi_2||\phi_3-a||H_{22}|}{Q_z|\phi_2'|} \\ &\le \frac{2C\tau^2|\phi_1'|}{\epsilon_2}+\frac{2C\tau^2|\phi_3'|}{\epsilon_2}.\end{aligned} \tag{2.21}$$

From (2.20)-(2.21), we obtain that

$$\left|\frac{H}{F}\right| \le M_2(|\phi_1'(\xi-q_1)|+|\phi_3'(\xi+q_3)|) \quad \text{for} \quad 0 \le \xi \le -\frac{q_3+q_2}{2}. \tag{2.22}$$

For $(-q_3-q_2)/2 \le \xi \le -q_3$, by (2.9), Lemma 2.2, (2.4) and (2.14), we have

$$\begin{aligned}\left|\frac{H}{F}\right| &\le \left|\frac{2(1-\phi_3)(a-\phi_2)[\phi_1 H_{21}+(\phi_3-a)H_{22}]}{Q_z|\phi_2'|+Q_w|\phi_3'|}\right| \\ &\le \frac{2|1-\phi_3||a-\phi_2||\phi_1||H_{21}|}{Q_z|\phi_2'|}+\frac{2|1-\phi_3||a-\phi_2||\phi_3-a||H_{22}|}{Q_w|\phi_3'|} \\ &\le \frac{2C\tau^2|\phi_1'|}{\epsilon_2}+\frac{2C\tau^2|\phi_2'|}{\epsilon_3}.\end{aligned} \tag{2.23}$$

For $\xi \ge -q_3$, by (2.9), Lemma 2.2, (2.4) and (2.15), we have

$$\begin{aligned}\left|\frac{H}{F}\right| &\le \left|\frac{2(1-\phi_3)(a-\phi_2)[\phi_1 H_{21}+(\phi_3-a)H_{22}]}{Q_w|\phi_3'|}\right| \\ &\le \frac{2\tau}{\epsilon_3}[C|\phi_1|+C|a-\phi_2|] \le \frac{2C\tau^2}{\epsilon_3}(|\phi_1'|+|\phi_2'|).\end{aligned} \tag{2.24}$$

From (2.23)-(2.24), we obtain that

$$\left|\frac{H}{F}\right| \le M_3(|\phi_1'(\xi-q_1)|+|\phi_2'(\xi+q_2)|) \quad \text{for} \quad \xi \ge -\frac{q_3+q_2}{2}. \tag{2.25}$$

Next, we estimate $|G/F|$. For $\xi \le q_1$, by Lemma 2.1(ii), Lemma 2.2, (2.4) and (2.12), we have

$$\begin{aligned}
\left|\frac{G}{F}\right| &\le 2\frac{|Q_{yy}(\phi_1')^2+Q_{zz}(\phi_2')^2+Q_{ww}(\phi_3')^2+2Q_{yz}\phi_1'\phi_2'+2Q_{yw}\phi_1'\phi_3'+2Q_{zw}\phi_2'\phi_3'|}{Q_y|\phi_1'|}\\
&\le 2\left[\frac{|\phi_2||R_1||\phi_1'|}{\epsilon_1}+\frac{|1-\phi_1||R_4||\phi_2'|^2}{\epsilon_1|\phi_1'|}+\frac{|1-\phi_1||R_8||\phi_3'|^2}{\epsilon_1|\phi_1'|}\right]\\
&\quad+\left[4\left(\frac{|Q_{yz}||\phi_2'|}{\epsilon_1}+\frac{|Q_{yw}||\phi_3'|}{\epsilon_1}+\frac{|1-\phi_1||R_{12}||\phi_2'||\phi_3'|}{\epsilon_1|\phi_1'|}\right)\right]\\
&\le 2\left[\frac{C\tau K}{\epsilon_1}|\phi_2'|+\frac{C\tau K}{\epsilon_1}|\phi_2'|+\frac{C\tau K}{\epsilon_1}|\phi_3'|+2\left(\frac{C}{\epsilon_1}|\phi_2'|+\frac{C}{\epsilon_1}|\phi_3'|+\frac{C\tau}{\epsilon_1}|\phi_2'||\phi_3'|\right)\right]\\
&\le M_4(|\phi_2'|+|\phi_3'|).
\end{aligned}$$

For $q_1 \le \xi \le 0$, by Lemma 2.1(ii), Lemma 2.2, (2.4) and (2.13), we have

$$\begin{aligned}
\left|\frac{G}{F}\right| &\le 2\frac{|Q_{yy}(\phi_1')^2+Q_{zz}(\phi_2')^2+Q_{ww}(\phi_3')^2+2Q_{yz}\phi_1'\phi_2'+2Q_{yw}\phi_1'\phi_3'+2Q_{zw}\phi_2'\phi_3'|}{Q_y|\phi_1'|+Q_z|\phi_2'|}\\
&\le 2\left(\frac{|\phi_2||R_1||\phi_1'|}{\epsilon_1}+\frac{|Q_{zz}||\phi_2'|}{\epsilon_2}+\frac{|\phi_2||R_9||\phi_3'|^2}{\epsilon_2|\phi_2'|}\right)\\
&\quad+4\left(\frac{|Q_{yz}||\phi_2'|}{\epsilon_1}+\frac{|Q_{yw}||\phi_3'|}{\epsilon_1}+\frac{|Q_{zw}||\phi_3'|}{\epsilon_2}\right)\\
&\le 2\left[\frac{C\tau K}{\epsilon_1}|\phi_2'|+\frac{C}{\epsilon_2}|\phi_2'|+\frac{C\tau K}{\epsilon_2}|\phi_3'|+2\left(\frac{C}{\epsilon_1}|\phi_2'|+\frac{C}{\epsilon_1}|\phi_3'|+\frac{C}{\epsilon_2}|\phi_3'|\right)\right]\\
&\le M_5(|\phi_2'|+|\phi_3'|).
\end{aligned}$$

Then we obtain that

$$\left|\frac{G}{F}\right| \le M_6(|\phi_2'(\xi+q_2)|+|\phi_3'(\xi+q_3)|) \quad \text{for} \quad \xi \le 0. \tag{2.26}$$

For $0 \le \xi \le -q_2$, by (2.9), Lemma 2.2, (2.4) and (2.13), we have

$$\begin{aligned}
\left|\frac{G}{F}\right| &\le 2\frac{|Q_{yy}(\phi_1')^2+Q_{zz}(\phi_2')^2+Q_{ww}(\phi_3')^2+2Q_{yz}\phi_1'\phi_2'+2Q_{yw}\phi_1'\phi_3'+2Q_{zw}\phi_2'\phi_3'|}{Q_y|\phi_1'|+Q_z|\phi_2'|}\\
&\le 2\left(\frac{|Q_{yy}||\phi_1'|}{\epsilon_1}+\frac{(|\phi_1||R_6|+|\phi_3-a||R_7|)|\phi_2'|}{\epsilon_2}+\frac{|\phi_2||R_9||\phi_3'|^2}{\epsilon_2|\phi_2'|}\right)\\
&\quad 4\left(\frac{|Q_{yz}||\phi_1'|}{\epsilon_2}+\frac{|Q_{yw}||\phi_3'|}{\epsilon_1}+\frac{|Q_{zw}||\phi_3'|}{\epsilon_2}\right)\\
&\le 2\left[\frac{C}{\epsilon_1}|\phi_1'|+\frac{C\tau K}{\epsilon_2}(|\phi_1'|+|\phi_3'|)+\frac{C\tau K}{\epsilon_2}|\phi_3'|+2\left(\frac{C}{\epsilon_2}|\phi_1'|+\frac{C}{\epsilon_1}|\phi_3'|+\frac{C}{\epsilon_2}|\phi_3'|\right)\right]\\
&\le M_7(|\phi_1'|+|\phi_3'|).
\end{aligned}$$

For $-q_2 \le \xi \le (-q_3 - q_2)/2$, by Lemma 2.1(ii), Lemma 2.2, (2.4) and (2.14), we have

$$\begin{aligned}
\left|\frac{G}{F}\right| &\le 2\frac{|Q_{yy}(\phi_1')^2 + Q_{zz}(\phi_2')^2 + Q_{ww}(\phi_3')^2 + 2Q_{yz}\phi_1'\phi_2' + 2Q_{yw}\phi_1'\phi_3' + 2Q_{zw}\phi_2'\phi_3'|}{Q_z|\phi_2'| + Q_w|\phi_3'|} \\
&\le 2\left(\frac{|a-\phi_2||R_2||\phi_1'|^2}{\epsilon_2|\phi_2'|} + \frac{(|\phi_1||R_6| + |\phi_3 - a||R_7|)|\phi_2'|}{\epsilon_2} + \frac{|Q_{ww}||\phi_3'|}{\epsilon_3}\right) \\
&\quad +4\left(\frac{|Q_{yz}||\phi_1'|}{\epsilon_2} + \frac{|Q_{yw}||\phi_1'|}{\epsilon_3} + \frac{|Q_{zw}||\phi_3'|}{\epsilon_2}\right) \\
&\le 2\left[\frac{C\tau K}{\epsilon_2}|\phi_1'| + \frac{C\tau K}{\epsilon_2}(|\phi_1'| + |\phi_3'|) + \frac{C}{\epsilon_3}|\phi_3'| + 2\left(\frac{C}{\epsilon_2}|\phi_1'| + \frac{C}{\epsilon_3}|\phi_1'| + \frac{C}{\epsilon_2}|\phi_3'|\right)\right] \\
&\le M_8(|\phi_1'| + |\phi_3'|).
\end{aligned}$$

Then we obtain that

$$\left|\frac{G}{F}\right| \le M_9(|\phi_1'(\xi - q_1)| + |\phi_3'(\xi + q_3)|) \quad \text{for } 0 \le \xi \le -\frac{q_3 + q_2}{2}. \tag{2.27}$$

For $(-q_3 - q_2)/2 \le \xi \le -q_3$, by Lemma 2.1(ii), Lemma 2.2, (2.4) and (2.14), we have

$$\begin{aligned}
\left|\frac{G}{F}\right| &\le 2\frac{|Q_{yy}(\phi_1')^2 + Q_{zz}(\phi_2')^2 + Q_{ww}(\phi_3')^2 + 2Q_{yz}\phi_1'\phi_2' + 2Q_{yw}\phi_1'\phi_3' + 2Q_{zw}\phi_2'\phi_3'|}{Q_z|\phi_2'| + Q_w|\phi_3'|} \\
&\le 2\left(\frac{|a-\phi_2||R_2||\phi_1'|^2}{\epsilon_2|\phi_2'|} + \frac{|Q_{zz}||\phi_2'|}{\epsilon_2} + \frac{|a-\phi_2||R_{10}||\phi_3'|}{\epsilon_3}\right) \\
&\quad +4\left(\frac{|Q_{yz}||\phi_1'|}{\epsilon_2} + \frac{|Q_{yw}||\phi_1'|}{\epsilon_3} + \frac{|Q_{zw}||\phi_2'|}{\epsilon_3}\right) \\
&\le 2\left[\frac{C\tau K}{\epsilon_2}|\phi_1'| + \frac{C}{\epsilon_2}|\phi_2'| + \frac{C\tau K}{\epsilon_3}|\phi_2'| + 2\left(\frac{C}{\epsilon_2}|\phi_1'| + \frac{C}{\epsilon_3}|\phi_1'| + \frac{C}{\epsilon_3}|\phi_2'|\right)\right] \\
&\le M_{10}(|\phi_1'| + |\phi_2'|).
\end{aligned}$$

For $\xi \ge -q_3$, by Lemma 2.1(ii), Lemma 2.2, (2.4) and (2.15), we have

$$\begin{aligned}
\left|\frac{G}{F}\right| &\le 2\frac{|Q_{yy}(\phi_1')^2 + Q_{zz}(\phi_2')^2 + Q_{ww}(\phi_3')^2 + 2Q_{yz}\phi_1'\phi_2' + 2Q_{yw}\phi_1'\phi_3' + 2Q_{zw}\phi_2'\phi_3'|}{Q_w|\phi_3'|} \\
&\le 2\left[\frac{|1-\phi_3||R_3||\phi_1'|^2}{\epsilon_3|\phi_3'|} + \frac{|1-\phi_3||R_5||\phi_2'|^2}{\epsilon_3|\phi_3'|} + \frac{|a-\phi_2||R_{10}||\phi_3'|}{\epsilon_3}\right] \\
&\quad +4\left(\frac{|1-\phi_3||R_{11}||\phi_1'||\phi_2'|}{\epsilon_3|\phi_3'|} + \frac{|Q_{yw}||\phi_1'|}{\epsilon_3} + \frac{|Q_{zw}||\phi_2'|}{\epsilon_3}\right) \\
&\le 2\left[\frac{C\tau K}{\epsilon_3}|\phi_1'| + \frac{C\tau K}{\epsilon_3}|\phi_2'| + \frac{C\tau K}{\epsilon_3}|\phi_2'| + 2\left(\frac{C\tau}{\epsilon_3}|\phi_1'||\phi_2'| + \frac{C}{\epsilon_3}|\phi_1'| + \frac{C}{\epsilon_3}|\phi_2'|\right)\right] \\
&\le M_{11}(|\phi_1'| + |\phi_2'|).
\end{aligned}$$

Then we obtain that

$$\left|\frac{G}{F}\right| \le M_{12}(|\phi_1'(\xi - q_1)| + |\phi_2'(\xi + q_2)|) \quad \text{for } \xi \ge -\frac{q_3 + q_2}{2}. \tag{2.28}$$

The lemma is proved by combining (2.19), (2.22), (2.25), (2.26), (2.27) and (2.28). □

Therefore, from (2.3) and (2.16), we have

$$
\begin{aligned}
&|G(\phi_1,\phi_2,\phi_3)+H(\phi_1,\phi_2,\phi_3)| \\
\le\ & F(\phi_1,\phi_2,\phi_3)KM\{e^{\beta_2(\xi+q_2)}+e^{\beta_3(\xi+q_3)}\} \\
\le\ & F(\phi_1,\phi_2,\phi_3)KM\{e^{\beta_2 q_2}+e^{\beta_3 q_3}\}
\end{aligned} \tag{2.29}
$$

for $\xi \le 0$;

$$
\begin{aligned}
&|G(\phi_1,\phi_2,\phi_3)+H(\phi_1,\phi_2,\phi_3)| \\
\le\ & F(\phi_1,\phi_2,\phi_3)KM\{e^{-\gamma_1(\xi-q_1)}+e^{\beta_3(\xi+q_3)}\} \\
\le\ & F(\phi_1,\phi_2,\phi_3)KM\{e^{\gamma_1 q_1}+e^{\beta_3(q_3-q_2)/2}\}
\end{aligned} \tag{2.30}
$$

for $0 \le \xi \le (-q_3-q_2)/2$; and

$$
\begin{aligned}
&|G(\phi_1,\phi_2,\phi_3)+H(\phi_1,\phi_2,\phi_3)| \\
\le\ & F(\phi_1,\phi_2,\phi_3)KM\{e^{-\gamma_1(\xi-q_1)}+e^{-\gamma_2(\xi+q_2)}\} \\
\le\ & F(\phi_1,\phi_2,\phi_3)KM\{e^{\gamma_1 q_1}+e^{\gamma_2(q_3-q_2)/2}\}
\end{aligned} \tag{2.31}
$$

for $\xi \ge (-q_3-q_2)/2$.

3. Existence of Entire solutions

In this section, we always assume (1.5) holds. Then $v_1 < v_2 < 0 < v_3$ and $s_2 > s_1 > 0$.

To construct the functions q_i, $i = 1, 2, 3$, in §2, we consider the following initial value problems (cf. [8, 12]):

$$p_1{}' = s_1 + Le^{\kappa p_1}, \quad -\infty < t < 0, \quad p_1(0) = p_0; \tag{3.1}$$

$$p_2{}' = s_2 + Le^{\kappa p_1}, \quad -\infty < t < 0, \quad p_2(0) = p_0; \tag{3.2}$$

$$r_1{}' = s_1 - Le^{\kappa r_1}, \quad -\infty < t < 0, \quad r_1(0) = r_0; \tag{3.3}$$

$$r_2{}' = s_2 - Le^{\kappa r_1}, \quad -\infty < t < 0, \quad r_2(0) = r_0, \tag{3.4}$$

where $L > 2KM$ is a positive constant and

$$\kappa := \min\left\{\gamma_1, \gamma_2, \beta_2, \beta_3, \frac{(s_2-s_1)\gamma_2}{4s_1}, \frac{(s_2-s_1)\beta_3}{4s_1}\right\}.$$

In fact, the solutions can be written explicitly as

$$
\begin{aligned}
p_1(t) &= s_1 t - \frac{1}{\kappa}\log\left[e^{-\kappa p_0} + \frac{L(1-e^{\kappa s_1 t})}{s_1}\right], \\
p_2(t) &= s_2 t - \frac{1}{\kappa}\log\left[e^{-\kappa p_0} + \frac{L(1-e^{\kappa s_1 t})}{s_1}\right], \\
r_1(t) &= s_1 t - \frac{1}{\kappa}\log\left[e^{-\kappa r_0} - \frac{L(1-e^{\kappa s_1 t})}{s_1}\right], \\
r_2(t) &= s_2 t - \frac{1}{\kappa}\log\left[e^{-\kappa r_0} - \frac{L(1-e^{\kappa s_1 t})}{s_1}\right].
\end{aligned}
$$

Now, we take p_0 and r_0 satisfying

$$p_0 = -\frac{1}{\kappa}\log\left(e^{-\kappa r_0} - \frac{2L}{s_1}\right) < -\delta, \quad r_0 < -\frac{1}{\kappa}\log\left(\frac{2L}{s_1} + e^{\kappa\delta}\right),$$

where δ is defined as in Lemma 2.3. Then we have

$$\begin{aligned}
&\lim_{t\to-\infty}(p_1(t) - r_1(t)) = \lim_{t\to-\infty}(p_2(t) - r_2(t)) = 0,\\
&\lim_{t\to-\infty}(p_1(t) - s_1 t) = \lim_{t\to-\infty}(p_2(t) - s_2 t) = -\frac{1}{\kappa}\log\left(e^{-\kappa p_0} + \frac{L}{s_1}\right),\\
&\lim_{t\to-\infty}(r_1(t) - s_1 t) = \lim_{t\to-\infty}(r_2(t) - s_2 t) = -\frac{1}{\kappa}\log\left(e^{-\kappa r_0} - \frac{L}{s_1}\right).
\end{aligned}$$

Also, there exists a positive constant N such that

$$0 < p_1(t) - r_1(t) = p_2(t) - r_2(t) \le N e^{\kappa s_1 t} \quad \text{for all } t \le 0, \tag{3.5}$$

and $p_1(t), p_2(t), r_1(t), r_2(t) \le -\delta$ for all $t \le 0$.

The next lemma shows the existence of super-sub-solutions of (2.7).

Lemma 3.1. *Define the functions $\overline{U}(\xi,t)$ and $\underline{U}(\xi,t)$ by*

$$\begin{aligned}
\overline{U}(\xi,t) &:= Q(\phi_1(\xi - p_1(t)), \phi_2(\xi + p_1(t)), \phi_3(\xi + p_2(t))),\\
\underline{U}(\xi,t) &:= Q(\phi_1(\xi - r_1(t)), \phi_2(\xi + r_1(t)), \phi_3(\xi + r_2(t))).
\end{aligned}$$

Then $(\overline{U},\underline{U})(\xi,t)$ is a pair of super-sub-solutions of (2.7) for $t \le t_0$ with some $t_0 < 0$. Moreover,

$$\overline{U}(\xi,t) \ge \underline{U}(\xi,t) \quad \textit{for} \quad \xi\in\mathbb{R},\ t \le t_0, \tag{3.6}$$

$$\sup_{\xi\in\mathbb{R}}\{\overline{U}(\xi,t) - \underline{U}(\xi,t)\} \le \mu e^{\kappa s_1 t} \quad \textit{for} \quad t \le t_0, \tag{3.7}$$

for some constant $\mu > 0$.

Proof. First, we prove $\overline{U}(\xi,t)$ is a super-solution of (2.7) for $t \le t_0$ with some $t_0 < 0$. By (2.29)-(2.31), we have

$$\begin{aligned}
&|G(\phi_1,\phi_2,\phi_3) + H(\phi_1,\phi_2,\phi_3)|\\
&\le \begin{cases}
F(\phi_1,\phi_2,\phi_3)KM(e^{\beta_2 p_1} + e^{\beta_3 p_2}) & \text{for } \xi \le 0,\\
F(\phi_1,\phi_2,\phi_3)KM(e^{\gamma_1 p_1} + e^{\beta_3(p_2-p_1)/2}) & \text{for } 0 \le \xi \le -\dfrac{p_1+p_2}{2},\\
F(\phi_1,\phi_2,\phi_3)KM(e^{\gamma_1 p_1} + e^{\gamma_2(p_2-p_1)/2})) & \text{for } \xi \ge -\dfrac{p_1+p_2}{2}.
\end{cases}
\end{aligned}$$

Moreover, we have

$$p_2(t) - p_1(t) = r_2(t) - r_1(t) = (s_2 - s_1)t \to -\infty$$

as $t\to-\infty$. Hence, by the choice of κ, there exists a $t_0 < 0$ such that

$$\frac{\beta_3(p_2(t)-p_1(t))}{2} < \kappa p_1(t) < 0, \quad \frac{\gamma_2(p_2(t)-p_1(t))}{2} < \kappa p_1(t) < 0, \tag{3.8}$$

$$\frac{\beta_3(r_2(t)-r_1(t))}{2} < \kappa r_1(t) < 0, \quad \frac{\gamma_2(r_2(t)-r_1(t))}{2} < \kappa r_1(t) < 0 \tag{3.9}$$

for all $t \le t_0$. Thus, by (3.8), we get

$$|G(\phi_1,\phi_2,\phi_3)+H(\phi_1,\phi_2,\phi_3)| \le 2F(\phi_1,\phi_2,\phi_3)KMe^{\kappa p_1}. \tag{3.10}$$

Then we obtain

$$\begin{aligned}\mathcal{T}[\overline{U}] &= -Q_y\phi_1'(p_1'-s_1)+Q_z\phi_2'(p_1'-s_1)+Q_w\phi_3'(p_2'-s_2)-G(\phi_1,\phi_2,\phi_3)-H(\phi_1,\phi_2,\phi_3)\\ &\ge F(\phi_1,\phi_2,\phi_3)(L-2KM)e^{\kappa p_1}\ge 0\end{aligned}$$

by (3.1), (3.2), (3.10) and Lemma 2.3. Hence $\overline{U}(\xi,t)$ is a super-solution of (2.7) for $t \le t_0$.

Next, we prove $\underline{U}(\xi,t)$ is a sub-solution of (2.7) for $t \le t_0$. By (2.29)-(2.31) and (3.9), we have

$$|G(\phi_1,\phi_2,\phi_3)+H(\phi_1,\phi_2,\phi_3)| \le 2F(\phi_1,\phi_2,\phi_3)KMe^{\kappa r_1}. \tag{3.11}$$

Then we obtain

$$\begin{aligned}\mathcal{T}[\underline{U}] &= -Q_y\phi_1'(r_1'-s_1)+Q_z\phi_2'(r_1'-s_1)+Q_w\phi_3'(r_2'-s_2)-G(\phi_1,\phi_2,\phi_3)-H(\phi_1,\phi_2,\phi_3)\\ &\le -F(\phi_1,\phi_2,\phi_3)(L-2KM)e^{\kappa r_1}\le 0\end{aligned}$$

by (3.3), (3.4), (3.11) and Lemma 2.3. Hence $\underline{U}(\xi,t)$ is a sub-solution of (2.7) for $t \le t_0$.

Finally, by (3.5), Lemma 2.3, the function F is bounded above and

$$\begin{aligned}&\overline{U}(\xi,t)-\underline{U}(\xi,t)\\ =\ & Q(\phi_1(\xi-p_1(t)),\phi_2(\xi+p_1(t)),\phi_3(\xi+p_2(t)))\\ &\quad -Q(\phi_1(\xi-r_1(t)),\phi_2(\xi+r_1(t)),\phi_3(\xi+r_2(t)))\\ =\ & \int_0^1 F(\phi_1(\xi-\theta p_1-(1-\theta)r_1),\phi_2(\xi+\theta p_1+(1-\theta)r_1),\\ &\qquad \phi_3(\xi+\theta p_2+(1-\theta)r_2))d\theta\cdot(p_1-r_1),\end{aligned}$$

(3.6) and (3.7) hold. Hence the lemma is proved. ☐

Now, we have a pair of super-sub-solutions of (2.7) satisfying (3.6). By using the same method as in [6, 8], the existence and uniqueness of entire solutions of (1.1) can be shown as follows.

Theorem 3.2. *There exists a unique entire solution $u(x,t)$ of (1.1) such that*

$$\underline{U}(x+\bar{c}t,t)\le u(x,t)\le \overline{U}(x+\bar{c}t,t)$$

for all $x\in\mathbb{R}$ and $t\le t_0$ where the functions $\underline{U}$ and $\overline{U}$ are defined as in Lemma 3.1.

Finally, we consider the asymptotic behavior of the entire solution in Theorem 3.2 as $t\to\pm\infty$. Since $r_2(t)-(s_1+s_2)t/2\to-\infty$ and $(s_1+s_2)t/2-r_1(t)\to-\infty$ as $t\to-\infty$, there exists a constant $T<0$ such that

$$r_2(t)<\frac{s_1+s_2}{2}t<r_1(t)$$

for $t<T$. Define

$$\theta := -\frac{1}{\kappa}\log\left(e^{-\kappa r_0}-\frac{L}{s_1}\right). \tag{3.12}$$

By a simple computation, there exists a constant $\rho > 0$ such that

$$-\rho e^{\kappa s_1 t} < r_1(t) - s_1 t - \theta = r_2(t) - s_2(t) - \theta \leq 0. \tag{3.13}$$

The next theorem shows the asymptotic behavior, as $t \to -\infty$, of the entire solution obtained in Theorem 3.2.

Theorem 3.3. *Let $u(x,t)$ be an entire solution obtained in Theorem 3.2. Then* (1.6) *holds for the constant θ defined by* (3.12).

Proof. Recall that $\xi := x + \overline{c}t$. For $x \leq -(v_1+v_2)t/2$, we have $\xi \leq 0 \leq -r_1(t)$. By (3.7), (2.3), (2.4), (2.6), (2.10) and (3.13), we derive that

$$\begin{aligned}
&|u(x,t) - \phi_1(x + v_1 t - \theta)| = |U(\xi,t) - \phi_1(\xi - s_1 t - \theta)| \\
&\leq |U(\xi,t) - \underline{U}(\xi,t)| + |\underline{U}(\xi,t) - \phi_1(\xi - s_1 t - \theta)| \\
&\leq |\overline{U}(\xi,t) - \underline{U}(\xi,t)| + |\phi_1(\xi - r_1(t)) - \phi_1(\xi - s_1 t - \theta)| + |\phi_2(\xi + r_1(t))| \cdot \\
&\quad \left| \frac{(1-a)(1-\phi_1(\xi - r_1(t)))(\phi_3(\xi + r_2(t)) - \phi_1(\xi - r_1(t)))}{(1-\phi_1(\xi - r_1(t)))\phi_2(\xi + r_1(t))(1-a) + (a - \phi_2(\xi + r_1(t)))(1 - \phi_3(\xi + r_2(t)))} \right| \\
&\leq |\overline{U}(\xi,t) - \underline{U}(\xi,t)| + |\phi_1(\xi - r_1(t)) - \phi_1(\xi - s_1 t - \theta)| + \eta_1 |\phi_2(\xi + r_1(t))| \\
&\leq \mu e^{\kappa s_1 t} + \sup_{\zeta \in \mathbb{R}} |\phi_1'(\zeta)| |r_1(t) - s_1 t - \theta| + \eta_1 \tau K e^{\beta_2(\xi + r_1(t))} \\
&\leq \mu e^{\kappa s_1 t} + K\rho e^{\kappa s_1 t} + \eta_1 \tau K e^{\beta_2 r_1(t)}
\end{aligned}$$

for $t \leq \min\{t_0, T\}$, where $\eta_1 = 8/a$.

Now we consider $-(v_1+v_2)t/2 \leq x \leq -(v_2+v_3)t/2$. This implies that $0 \leq \xi \leq -(s_1 + s_2)t/2$. Recall that $-(s_1+s_2)t/2 \leq -r_2(t)$ for $t \leq T$. By (3.7), (2.3), (2.4), (2.5), (2.10) and (3.13), we have

$$\begin{aligned}
&|u(x,t) - \phi_2(x + v_2 t + \theta)| = |U(\xi,t) - \phi_2(\xi + s_1 t + \theta)| \\
&\leq |U(\xi,t) - \underline{U}(\xi,t)| + |\underline{U}(\xi,t) - \phi_2(\xi + s_1 t + \theta)| \\
&\leq |\overline{U}(\xi,t) - \underline{U}(\xi,t)| + |\phi_2(\xi + r_1(t)) - \phi_2(\xi + s_1 t + \theta)| + |\phi_3(\xi + r_2(t)) - a| \cdot \\
&\quad \left| \frac{(1-\phi_2(\xi + r_2(t)))(1-\phi_1(\xi - r_1(t)))\phi_2(\xi + r_2(t))}{(1-\phi_1(\xi - r_1(t)))\phi_2(\xi + r_1(t))(1-a) + (a - \phi_2(\xi + r_1(t)))(1 - \phi_3(\xi + r_2(t)))} \right| \\
&\quad + |\phi_1(\xi - r_1(t))| \cdot \\
&\quad \left| \frac{(1-\phi_2(\xi + r_1(t)))(a - \phi_2(\xi + r_1(t)))(1 - \phi_3(\xi + r_2(t)))}{(1-\phi_1(\xi - r_1(t)))\phi_2(\xi + r_1(t))(1-a) + (a - \phi_2(\xi + r_1(t)))(1 - \phi_3(\xi + r_2(t)))} \right| \\
&\leq |\overline{U}(\xi,t) - \underline{U}(\xi,t)| + |\phi_2(\xi + r_1(t)) - \phi_2(\xi + s_1 t + \theta)| \\
&\qquad + \eta_2(|\phi_3(\xi + r_2(t)) - a| + |\phi_1(\xi - r_1(t))|) \\
&\leq \mu e^{\kappa s_1 t} + \sup_{\zeta \in \mathbb{R}} |\phi_2'(\zeta)| |r_1(t) - s_1 t - \theta| + \eta_2(|\phi_3(\xi + r_2(t)) - a| + |\phi_1(\xi - r_1(t))|) \\
&\leq \mu e^{\kappa s_1 t} + K\rho e^{\kappa s_1 t} + \eta_2 \tau K (e^{\beta_3(\xi + r_2(t))} + e^{-\gamma_1(\xi - r_1(t))}) \\
&\leq \mu e^{\kappa s_1 t} + K\rho e^{\kappa s_1 t} + \eta_2 \tau K (e^{\beta_3(-(s_1+s_2)t/2 + r_2(t))} + e^{\gamma_1 r_1(t)})
\end{aligned}$$

for $t \leq \min\{t_0, T\}$, where $\eta_2 = 4/(1-a)$.

For the case $x \geq -(v_2+v_3)t/2$, we have $\xi \geq -(s_1+s_2)t/2$. Also, for $t \leq T$, we know that $-(s_1+s_2)t/2 \geq -r_1(t)$. From (3.7), (2.3), (2.4), (2.6), (2.10) and (3.13), we show that

$$\begin{aligned}
&|u(x,t)-\phi_3(x+v_3t+\theta)| = |U(\xi,t)-\phi_3(\xi+s_2t+\theta)| \\
&\quad\leq |U(\xi,t)-\underline{U}(\xi,t)| + |\underline{U}(\xi,t)-\phi_3(\xi+s_2t+\theta)| \\
&\quad\leq |\overline{U}(\xi,t)-\underline{U}(\xi,t)| + |\phi_3(\xi+r_2(t))-\phi_3(\xi+s_2t+\theta)| + |a-\phi_2(\xi+r_1(t))| \cdot \\
&\qquad \left|\frac{(1-\phi_3(\xi+r_2(t)))(\phi_1(\xi-r_1(t))-\phi_3(\xi+r_2(t)))}{(1-\phi_1(\xi-r_1(t)))\phi_2(\xi+r_1(t))(1-a)+(a-\phi_2(\xi+r_1(t)))(1-\phi_3(\xi+r_2(t)))}\right| \\
&\quad\leq |\overline{U}(\xi,t)-\underline{U}(\xi,t)| + |\phi_3(\xi+r_2(t))-\phi_3(\xi+s_2t+\theta)| + \eta_3|a-\phi_2(\xi+r_1(t))| \\
&\quad\leq \mu e^{\kappa s_1 t} + \sup_{\zeta\in\mathbb{R}}|\phi_3'(\zeta)||r_2(t)-s_2t-\theta| + \eta_3\tau K e^{-\gamma_2(\xi+r_1(t))} \\
&\quad\leq \mu e^{\kappa s_1 t} + K\rho e^{\kappa s_1 t} + \eta_3\tau K e^{\gamma_2((s_1+s_2)t/2-r_1(t))}
\end{aligned}$$

for $t \leq \min\{t_0, T\}$, where $\eta_3 = 8/a$.

Therefore, the theorem is proved. □

Finally, the asymptotic behavior, as $t \to \infty$, of the entire solution obtained in Theorem 3.2 follows directly by a result in [4]. This completes the proof of our main theorem, Theorem 1.1.

References

[1] D. G. Aronson and H. F. Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation*, In Goldstein J. A. (ed.), Partial Differential Equations and Related Topics, Lecture Notes in Math. **446** (1975), 5-49, Springer.

[2] X. Chen and J.-S. Guo, *Existence and uniqueness of entire solutions for a reaction-diffusion equation*, J. Diff. Eq. **212** (2005), 62-84.

[3] X. Chen, J.-S. Guo, and H. Ninomiya, *Entire solutions of reaction diffusion equations with balanced bistable nonlinearities*, Proc. Royal Soc. Edinburgh A **136** (2006), 1207-1237.

[4] P. C. Fife, *Long time behavior of solutions of bistable nonlinear diffusion equations*, Arch. Ration. Mech. Anal. **70** (1979), 31-46.

[5] P. C. Fife, and J. B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling front solutions*, Arch. Ration. Mech. Anal. **65** (1977), 335-361.

[6] Y. Fukao, Y. Morita, and H. Ninomiya, *Some entire solutions of the Allen-Cahn equation*, Taiwanese J. Math. **8** (2004), 15-32.

[7] J.-S. Guo and C.-H. Wu, *Entire solutions for a two-component competition system in a lattice*, Tohoku Math. J. **62** (2010), 17-28.

[8] J.-S. Guo and Y. Morita, *Entire solutions of reaction-diffusion equations and an application to discrete diffusive equations*, Discrete Contin. Dyn. Syst. **12** (2005), 193-212.

[9] F. Hamel and N. Nadirashvili, *Entire solutions of the KPP equation*, Comm. Pure Appl. Math. **52** (1999), 1255-1276.

[10] F. Hamel and N. Nadirashvili, *Travelling fronts and entire solutions of the Fisher-KPP equation in* $\mathbb{R}^N$, Arch. Rat. Mech. Anal. **157** (2001), 91-163.

[11] A.N. Kolmogorov, I.G. Petrovsky, and N.S. Piskunov, *Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un probléme biologique*, Bull. Univ. Moskov. Ser. Internat., Sect. **A 1** (1937), 1-25.

[12] Y. Morita and H. Ninomiya, *Entire solutions with merging fronts to reaction-diffusion equations*, J. Dynam. Diff. Eq. **18** (2006), 841-861.

[13] Y. Morita and K. Tachibana, *An entire solution to the Lotka-Volterra competition-diffusion equations*, SIAM J. Math. Anal. **40** (2009) 2217-2240.

[14] Y.-J. Sun, W.-T. Li, Z.-C. Wang, *Entire solutions in nonlocal dispersal equations with bistable nonlinearity*, J. Diff. Eq. **251** (2011), 551-581.

[15] Y. Wang and X. Li, *Some entire solutions to the competitive reaction diffusion system*, J. Math. Anal. Appl. **430** (2015), 993-1008.

[16] S.-L. Wu and C.-H. Hsu, *Entire solutions of nonlinear cellular neural networks with distributed time delays*, Nonlinearity **25** (2012), 1-17.

[17] S.-L. Wu, Z.-X. Shih and F.-Y. Yang, *Entire solutions in periodic lattice dynamical systems*, J. Diff. Eq. **255** (2013), 62-84.

[18] S.-L. Wu and H. Wang, *Front-like entire solutions formonostable reaction-diffusion systems*, J. Dyn. Differ. Equ. **25** (2013), 505-533.

[19] H. Yagisita, *Backward global solutions characterizing annihilation dynamics of traveling fronts*, Publ. Res. Inst. Math. Sci. **39** (2003), 117-164.

[20] Li Zhang, W.-T. Li and S.-L. Wu, *Multi-type entire solutions in a nonlocal dispersal epidemic model*, J. Dynam. Diff. Eq. **28** (2016), 189-224.

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