NESTED HILBERT SCHEMES ON SURFACES: VIRTUAL FUNDAMENTAL CLASS

AMIN GHOLAMPOUR AND ARTAN SHESHMANI AND SHING-TUNG YAU

Abstract. We construct natural virtual fundamental classes for nested Hilbert schemes on a nonsingular projective surface S. This allows us to define new invariants of S that recover some of the known important cases such as Poincaré invariants of Dürr-Kabanov-Okonek and the stable pair invariants of Kool-Thomas. In the case of the nested Hilbert scheme of points, we can express these invariants in terms of integrals over the products of Hilbert scheme of points on S, and relate them to the vertex operator formulas found by Carlsson-Okounkov. The virtual fundamental classes of the nested Hilbert schemes play a crucial role in the Donaldson-Thomas theory of local-surface-threefolds that we study in [GSY17b].

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1. Introduction

Hilbert scheme of points on a nonsingular surface S have been vastly studied. They are nonsingular varieties with rich geometric structures some of which have applications in physics (see [N99] for a survey). We are mainly interested in the enumerative geometry of Hilbert schemes of points [G90, L99, CO12, GS16]. This has applications in curve counting problems on S [LT14]. The first two authors of this paper have studied the relation of some of these enumerative problems to the Donaldson-Thomas theory of 2-dimensional sheaves in threefolds and to S-duality conjectures [GS13]. In contrast, Hilbert scheme of curves on S can be badly behaved and singular. They were studied in details by Dürr-Kabanov-Okonek [DKO07] in the context of Poincaré invariants (algebraic Seiberg-Witten invariants). More recently, the stable pair invariants of surfaces have been employed in the context of curve counting problems [PT10, MPT10, KT14, KST11]. We define new invariants for the nested Hilbert scheme of curves and points on S. Our main application of these invariants is in the study of Donaldson-Thomas theory of local surfaces, that is carried out in [GSY17b].

1.1. **Nested Hilbert schemes on surfaces.** Suppose that $\boldsymbol{n} := n_1, n_2, \dots, n_r$ is a sequence of $r \geq 1$ nonnegative integers, and $\boldsymbol{\beta} := \beta_1, \dots, \beta_{r-1}$ is a sequence of classes in $H^2(S, \mathbb{Z})$ such that $\beta_i \geq 0$. We denote the corresponding nested Hilbert scheme by $S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}$. A closed point of $S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}$ corresponds to

$$(Z_1, Z_2, \dots, Z_r), \quad (C_1, \dots, C_{r-1})$$

where $Z_i \subset S$ is a 0-dimensional subscheme of length n_i, and $C_i \subset S$ is a divisor with $[C_i] = \beta_i$, and Z_{i+1} is a subscheme of $Z_i \cup C_i$ for any $i < r$, or equivalently

$$I_{Z_i}(-C_i) \subset I_{Z_{i+1}}. \tag{1}$$

To be able to define invariants for the nested Hilbert schemes (see Definitions 2.12, 2.13), we construct a virtual fundamental class $[S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}]^{\text{vir}}$ and then we integrate against it. More precisely, we construct a natural perfect obstruction theory over $S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}$. This is done by studying the deformation/obstruction theory of the maps of coherent sheaves given by the natural inclusions (1) following Illusie. As we will see, this in particular provides a uniform way of studying all known obstruction theories of the Hilbert schemes of points and curves, as well as the stable pair moduli spaces on S. The first main result of the paper is (Propositions 2.4, 2.7 and Corollary 2.8):

Theorem 1. *Let S be a nonsingular projective surface over $\mathbb{C}$ and ω_S be its canonical bundle.The nested Hilbert scheme $S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}$ with $r \geq 2$ carries a natural virtual fundamental class*

$$[S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}]^{\text{vir}} \in A_d(S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}), \qquad d = n_1 + n_r + \frac{1}{2}\sum_{i=1}^{r-1} \beta_i \cdot (\beta_i - c_1(\omega_S)).$$

1.2. **Special cases.** In the simplest special case, i.e. when $r=1$, we have $S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]} = S^{[n_1]}$ is the Hilbert scheme of n_1 points on S which is nonsingular of dimension $2n_1$, and hence it has a well-defined fundamental class $[S^{[n_1]}] \in A_{2n_1}(S^{[n_1]})$. For $r>1$ and $\beta_i = 0$, $S^{[\boldsymbol{n}]} := S^{[\boldsymbol{n}]}_{(0,\dots,0)}$ is the nested Hilbert scheme of points on S parameterizing flags of 0-dimensional subschemes $Z_r \subset \cdots \subset Z_2 \subset Z_1 \subset S$. $S^{[\boldsymbol{n}]}$ is in general singular of actual dimension $2n_1$.

We are specifically interested in the case $r=2$ in this paper: $S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]} = S_{\beta}^{[n_1,n_2]}$ for some $\beta \in H^2(S,\mathbb{Z})$. Interestingly, the invariants of nested Hilbert schemes recover the Poincaré and the stable pair invariants of S that were previously studied in the context of algebraic Seiberg-Witten invariants and curve counting problems. The following theorem is proven in Section 3.

Theorem 2. *The virtual fundamental class of Theorem 1 recovers the following known cases:*

1. *If $\beta = 0$ and $n_1 = n_2 = n$ then $S^{[n,n]}_{\beta=0} \cong S^{[n]}$ and $[S^{[n,n]}_{\beta=0}]^{\mathrm{vir}} = [S^{[n]}]$ is the fundamental class of the Hilbert scheme of n points.*
2. *If $\beta = 0$ and $n_2 = 0$, then $S^{[n,0]}_{\beta=0} \cong S^{[n]}$ and*
$$[S^{[n,0]}_{\beta=0}]^{\mathrm{vir}} = (-1)^n [S^{[n]}] \cap c_n(\omega_S^{[n]}),$$
where $\omega_S^{[n]}$ is the rank n tautological vector bundle over $S^{[n]}$ associated to the canonical bundle ω_S of S.[1]
3. *If $\beta = 0$ and $n = n_2 = n_1 - 1$, then it is known that $S^{[n+1,n]}_{\beta=0} \cong \mathbb{P}(\mathcal{I}^{[n]})$ is nonsingular, where $\mathcal{I}^{[n]}$ is the universal ideal sheaf over $S \times S^{[n]}$* [L99, Section 1.2]. *Then,*
$$[S^{[n+1,n]}_{\beta=0}]^{\mathrm{vir}} = -[S^{[n+1,n]}_{\beta=0}] \cap c_1(\mathcal{O}_{\mathbb{P}}(1) \boxtimes \omega_S).$$
4. *If $n_1 = n_2 = 0$ and $\beta \neq 0$, then $S^{[0,0]}_{\beta}$ is the Hilbert scheme of divisors in class β, and $[S^{[0,0]}_{\beta}]^{\mathrm{vir}}$ coincides with virtual cycle used to define Poincaré invariants in* [DKO07].
5. *If $n_1 = 0$ and $\beta \neq 0$, then $S^{[0,n_2]}_{\beta}$ is the relative Hilbert scheme of points on the universal divisor over $S^{[0,0]}_{\beta}$, which as shown in* [PT10], *is a moduli space of stable pairs and $[S^{[0,n_2]}_{\beta}]^{\mathrm{vir}}$ is the same as the virtual fundamental class constructed in* [KT14] *in the context of stable pair theory. If $p_g(S) = 0$ this class was used in* [KT14] *to define stable pair invariants.*

In certain cases, we construct a reduced virtual fundamental class for $S^{[n_1,n_2]}_{\beta}$ by reducing the perfect obstruction theory leading to Theorem 1 (Propositions 2.9, 2.11):

[1] We where notified about this identity by Richard Thomas.

Theorem 3. *Let S be a nonsingular projective surface with $p_g(S) > 0$, and the class β be such that the natural map*

$$H^1(T_S) \xrightarrow{*\cup\beta} H^2(\mathcal{O}_S) \quad \text{is surjective,}$$

then, $[S_\beta^{[n_1,n_2]}]^{\mathrm{vir}} = 0$. In this case the nested Hilbert scheme $S_\beta^{[n_1,n_2]}$ carries a reduced virtual fundamental class

$$[S_\beta^{[n_1,n_2]}]^{\mathrm{vir}}_{\mathrm{red}} \in A_d(S_\beta^{[n_1,n_2]}), \qquad d = n_1 + n_2 + \frac{1}{2}\beta\cdot(\beta - K_S) + p_g.$$

The reduced virtual fundamental classes $[S_\beta^{[0,0]}]^{\mathrm{vir}}_{\mathrm{red}}$ and $[S_\beta^{[0,n_2]}]^{\mathrm{vir}}_{\mathrm{red}}$ match with the reduced virtual cycles constructed in [DKO07, KT14] *in cases 3 and 4 of Theorem 2. $[S_\beta^{[0,n_2]}]^{\mathrm{vir}}_{\mathrm{red}}$ was used in* [KT14] *to define the stable pair invariants of S in this case.*

1.3. **Nested Hilbert scheme of points.** We study the nested Hilbert schemes of points

$$S^{[n_1\geq n_2]} := S^{[n_1,n_2]}_{\beta=0}$$

in much more details. Let $\iota : S^{[n_1\geq n_2]} \hookrightarrow S^{[n_1]}\times S^{[n_2]}$ be the natural inclusion. If S is toric with the torus $\mathbf{T}$ and the fixed set $S^{\mathbf{T}}$, in Section 4.1 we provide a purely combinatorial formula for computing $[S^{[n_1\geq n_2]}]^{\mathrm{vir}}$ by torus localization along the lines of [MNOP06]:

Theorem 4. *For a toric nonsingular surface S the $\mathbf{T}$-fixed set of $S^{[n_1,n_2]}$ is isolated and given by tuple of nested partitions of n_2, n_1:*

$$\left\{(\mu_{2,P}\subseteq \mu_{1,P})_P \mid P\in S^{\mathbf{T}}, \quad \mu_{i,P}\vdash n_i\right\}.$$

Moreover, the $\mathbf{T}$-character of the virtual tangent bundle $\mathcal{T}$ of $S^{[n_1\geq n_2]}$ at the fixed point $Q = (\mu_{2,P}\subseteq\mu_{1,P})_P$ is given by

$$\mathrm{tr}_{\mathcal{T}_Q^{\mathrm{vir}}}(t_1,t_2) = \sum_{P\in S^{\mathbf{T}}} \mathsf{V}_P,$$

where t_1,t_2 are the torus characters and V_P is a Laurent polynomial in t_1,t_2 that is completely determined by the partitions $\mu_{2,P}$ and $\mu_{1,P}$ and is given by the right hand side of formula (25)*.*

When S is toric, by torus localization, we can express $[S^{[n_1\geq n_2]}]^{\mathrm{vir}}$ in terms of the fundamental class of the product of Hilbert schemes $S^{[n_1]}\times S^{[n_2]}$ (Proposition 4.5):

Theorem 5. *If S is a nonsingular projective toric surface, then,*

$$\iota_*[S^{[n_1\geq n_2]}]^{\mathrm{vir}} = [S^{[n_1]}\times S^{[n_2]}]\cap c_{n_1+n_2}(\mathsf{E}^{n_1,n_2}),$$

where E^{n_1,n_2} is the first relative extension sheaf of the universal ideal sheaves $\mathcal{I}^{[n_1]}$ and $\mathcal{I}^{[n_2]}$ (Definition 4.3).

Theorem 5 holds in particular for $S = \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, which are the generators of the cobordism ring of nonsingular projective surfaces. We use a refinement of this fact together with a degeneration formula developed for $[S^{[n_1 \geq n_2]}]^{\mathrm{vir}}$ (Proposition 5.1) to prove (Corollary 5.9, Remark 5.10, Proposition 5.11):

Theorem 6. *If S is a nonsingular projective surface, M is a line bundle on S, and $\alpha_M^{n_1,n_2}$ is a cohomology class in $H^{n_1+n_2}(S^{[n_1]} \times S^{[n_2]})$ with the following properties* [2]*:*

- *$\alpha_M^{n_1,n_2}$ is universally defined for any pair (S, M) and any n_1, n_2,*
- *For any $n_1 \geq n_2$, the restriction $\iota^* \alpha_M^{n_1,n_2}$ is well-behaved under good degenerations of S,*

then

$$\int_{[S^{[n_1 \geq n_2]}]^{\mathrm{vir}}} \iota^* \alpha_M^{n_1,n_2} = \int_{S^{[n_1]} \times S^{[n_2]}} \alpha_M^{n_1,n_2} \cup c_{n_1+n_2}(\mathsf{E}^{n_1,n_2}),$$

and E^{n_1,n_2} is the alternating sum (in the K-group) of all the relative extension sheaves of the universal ideal sheaves $\mathcal{I}^{[n_1]}$ and $\mathcal{I}^{[n_2]}$ (Definition 4.3).

The operators

$$\int_{S^{[n_1]} \times S^{[n_2]}} - \cup \, c_{n_1+n_2}(\mathsf{E}_M^{n_1,n_2})$$

were studied by Carlsson-Okounkov in [CO12]. Here $M \in \mathrm{Pic}(S)$, and $\mathsf{E}_M^{n_1,n_2}$ is the alternating sum of all the relative extension sheaves of $\mathcal{I}^{[n_1]}$ and $\mathcal{I}^{[n_2]} \boxtimes M$ (Definition 4.3). They were able to express these operators in terms of explicit vertex operators. As an application of Theorem 6 and a result of [CO12], we prove the following explicit formula (Proposition 6.2):

Theorem 7. *Let S be a nonsingular projective surface, ω_S be its canonical bundle, and $K_S = c_1(\omega_S)$. Then,*

$$\sum_{n_1 \geq n_2 \geq 0} (-1)^{n_1+n_2} \int_{[S^{[n_1 \geq n_2]}]^{\mathrm{vir}}} \iota^* c(\mathsf{E}_M^{n_1,n_2}) q_1^{n_1} q_2^{n_2} =$$
$$\prod_{n>0} \left(1 - q_2^{n-1} q_1^n\right)^{\langle K_S, K_S - M \rangle} \left(1 - q_1^n q_2^n\right)^{\langle K_S - M, M \rangle - e(S)},$$

where $\langle -, - \rangle$ is the Poincaré paring on S.

1.4. **DT theory of local surfaces.** In this section we give an overview of the results of [GSY17b], which is a motivation behind this paper and at the same time an important application of it. Let (S, h) be a nonsingular simply connected projective surface with $h = c_1(\mathcal{O}_S(1))$. Let ω_S be the canonical bundle of S with the

[2] See Remark 5.10 for more precise statements of these properties.

projection map q to S, and X be the total space of ω_S [3]. X is a noncompact Calabi-Yau threefold and we define the DT invariants of X by using $\mathbb{C}^*$-localization, where $\mathbb{C}^*$-acts on X by scaling the fibers of ω_S. More precisely, let

$$v = (r, \gamma, m) \in \oplus_{i=0}^{2} H^{2i}(S, \mathbb{Q})$$

be a Chern character vector with $r \geq 1$, and $\mathcal{M}_h^{\omega_S}(v)$ be the moduli space of compactly supported 2-dimensional stable sheaves $\mathcal{E}$ on X such that $\mathrm{ch}(\mathrm{q}_* \mathcal{E}) = v$. Here stability is defined by means of the slope of $\mathrm{q}_* \mathcal{E}$ with respect to the polarization h. We always assume semistability implies stability and provide $\mathcal{M}_h^{\omega_S}(v)$ with a perfect obstruction theory by reducing the natural perfect obstruction theory given by [T98]. The fixed locus $\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*}$ of the moduli space is compact and the reduced obstruction theory gives a virtual fundamental class over it, that we denote by $[\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*}]^{\mathrm{vir}}_{\mathrm{red}}$. We define two types of DT invariants:

$$\mathrm{DT}_h^{\omega_S}(v;\alpha) = \int_{[\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*}]^{\mathrm{vir}}_{\mathrm{red}}} \frac{\alpha}{\mathrm{Nor}^{\mathrm{vir}}} \in \mathbb{Q}[\mathbf{s},\mathbf{s}^{-1}], \qquad \alpha \in H^*_{\mathbb{C}^*}(\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*}, \mathbb{Q})_{\mathbf{s}}$$
$$\mathrm{DT}_h^{\omega_S}(v) = \chi^{\mathrm{vir}}(\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*}) \in \mathbb{Z},$$

where $\mathrm{Nor}^{\mathrm{vir}}$ is the virtual normal bundle of $\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*} \subset \mathcal{M}_h^{\omega_S}(v)$, $\chi^{\mathrm{vir}}(-)$ is the virtual Euler characteristic [FG10], and $\mathbf{s}$ is the equivariant parameter.

If $\alpha = 1$ then it can be shown that

$$\mathrm{DT}_h^{\omega_S}(v;1) = \mathbf{s}^{-p_g}\,\mathrm{VW}_h(v),$$

where $\mathrm{VW}_h(-)$ is the Vafa-Witten invariant defined by Tanaka and Thomas [TT] and is expected to have modular properties based on S-duality conjecture [VW94].

The $\mathbb{C}^*$-fixed locus $\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*}$ consists of sheaves supported on S (the zero section of ω_S) and its thickenings. We write $\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*}$ as a disjoint union of several types of components, where each type is indexed by a partition of r. Out of these component types, there are two types that are in particular important for us. One of them (we call it type I) is identified with $\mathcal{M}_h(v)$, the moduli space of rank r torsion free stable sheaves on S. The other type (we call it type II) can be identified with the nested Hilbert scheme $S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$ for a suitable choice of $\boldsymbol{n}, \boldsymbol{\beta}$ depending on v. The reason that types I and II are more interesting for us is the following result proven in [GSY17b]:

Theorem ([GSY17b]). *The restriction of $[\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*}]^{\mathrm{vir}}_{\mathrm{red}}$ to the type I component $\mathcal{M}_h(v)$ is identified with $[\mathcal{M}_h(v)]^{\mathrm{vir}}_0$ induced by the natural trace free perfect obstruction theory over $\mathcal{M}_h(v)$. The restriction of $[\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*}]^{\mathrm{vir}}_{\mathrm{red}}$ to a type II component $S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$ is identified with $[S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}]^{\mathrm{vir}}$.*

[3] In [GSY17b], we consider a more general case in which X is the total space of an arbitrary line bundle $\mathcal{L}$ with $H^0(\mathcal{L} \otimes \omega_S^{-1}) \neq 0$.

When $r = 2$, then types I and II are the only component types of $\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*}$. This leads us to the following result:

Theorem ([GSY17b]). *Suppose that $v = (2, \gamma, m)$. Then,*

$$\mathrm{DT}_h^{\omega_S}(v;\alpha) = \mathrm{DT}_h^{\omega_S}(v;\alpha)_{\mathrm{I}} + \sum_{n_1,n_2,\beta} \mathrm{DT}_h^{\omega_S}(v;\alpha)_{\mathrm{II},S_\beta^{[n_1,n_2]}},$$

$$\mathrm{DT}_h^{\omega_S}(v) = \chi^{\mathrm{vir}}(\mathcal{M}_h(v)) + \sum_{n_1,n_2,\beta} \chi^{\mathrm{vir}}(S_\beta^{[n_1,n_2]}]),$$

where the sum is over all n_1, n_2, β (depending on v) for which $S_\beta^{[n_1,n_2]}$ is a type II component of $\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*}$, and the indices I and II indicate the contributions of type I and II components to the invariant $\mathrm{DT}_h^{\omega_S}(v;\alpha)$.

The stability of sheaves imposes a strong condition on n_1, n_2, β appearing in the summation in the theorem above. For example, if S is a generic complete intersection in a projective space, then for any n_1, n_2, β for which $S_\beta^{[n_1,n_2]}$ is a type II component of $\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*}$, the condition in Theorem 3 (leading to the vanishing $[S_\beta^{[n_1,n_2]}]^{\mathrm{vir}} = 0$) is not satisfied.

The invariants $\chi^{\mathrm{vir}}(S_\beta^{[n_1,n_2]}])$ and $\mathrm{DT}_h^{\omega_S}(v;\alpha)_{\mathrm{II},S_\beta^{[n_1,n_2]}}$ (for a suitable choice of class α e.g. $\alpha = 1$) appearing in the theorem above are special types of the invariants

$$\mathsf{N}_S(n_1, n_2, \beta; -)$$

that we have defined in this paper by integrating against $[S_\beta^{[n_1,n_2]}]^{\mathrm{vir}}$ (Definition 2.12). One advantage of this viewpoint is that it enables us to apply some of the techniques that we developed in this paper (such as Theorems 4, 5, 6, 7) to evaluate these invariants in certain cases.

Mochizuki in [M02] expresses certain integrals against the virtual cycle of $\mathcal{M}_h(v)$ in terms of Seiberg-Witten invariants and integrals $\mathsf{A}(\gamma_1, \gamma_2, v; -)$ over the product of Hilbert scheme of points on S. Using this result we are able to prove the following:

Theorem ([GSY17b]). *Suppose that $p_g(S) > 0$, and $v = (2, \gamma, m)$ is such that $\gamma \cdot h > 2K_S \cdot h$ and $\chi(v) := \int_S v \cdot td_S \geq 1$. Then,*

$$\mathrm{DT}_h^{\omega_S}(v;1) = -\sum_{\substack{\gamma_1+\gamma_2=\gamma \\ \gamma_1 \cdot h < \gamma_2 \cdot h}} \mathrm{SW}(\gamma_1) \cdot 2^{2-\chi(v)} \cdot \mathsf{A}(\gamma_1, \gamma_2, v; \mathsf{P}_1) + \sum_{n_1,n_2,\beta} \mathsf{N}_S(n_1, n_2, \beta; \mathcal{P}_1).$$

$$\mathrm{DT}_h^{\omega_S}(v) = -\sum_{\substack{\gamma_1+\gamma_2=\gamma \\ \gamma_1 \cdot h < \gamma_2 \cdot h}} \mathrm{SW}(\gamma_1) \cdot 2^{2-\chi(v)} \cdot \mathsf{A}(\gamma_1, \gamma_2, v; \mathsf{P}_2) + \sum_{n_1,n_2,\beta} \mathsf{N}_S(n_1, n_2, \beta; \mathcal{P}_2).$$

Here $\mathrm{SW}(-)$ is the Seiberg-Witten invariant of S, P_i and $\mathcal{P}_i$ are certain universally defined (independent of S) explicit integrands, and the second sum in the formulas *is over all* n_1, n_2, β *(depending on* v*) for which* $S_\beta^{[n_1,n_2]}$ *is a type II component of* $\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*}$.

If S *is a* $K3$ *surface or* S *is isomorphic to one of the five types of generic complete intersections*

$$(5) \subset \mathbb{P}^3,\ (3,3) \subset \mathbb{P}^4,\ (4,2) \subset \mathbb{P}^4,\ (3,2,2) \subset \mathbb{P}^5,\ (2,2,2,2) \subset \mathbb{P}^6,$$

the DT invariants $\mathrm{DT}_h^{\omega_S}(v;1)$ *and* $\mathrm{DT}_h^{\omega_S}(v)$ *can be completely expressed as the sum of integrals over the product of the Hilbert schemes of points on* S.

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2. Nested Hilbert schemes

Let S be a nonsingular projective surface over $\mathbb{C}$. We denote the canonical line bundle on S by ω_S and $K_S := c_1(\omega_S)$. For any nonnegative integer m and effective curve class $\beta \in H^2(S,\mathbb{Z})$, we denote by $S_\beta^{[m]}$ the Hilbert scheme of 1-dimensional subschemes $Z \subset S$ such that

$$[Z] = \beta, \quad c_2(I_Z) = m.$$

If $\beta = 0$ we drop it from the notation and denote by $S^{[m]}$ the Hilbert scheme of m points on S. Similarly, in the case $m = 0$ but $\beta \neq 0$ we drop m from the notation and use S_β to denote the Hilbert scheme of curves in class β. There are natural morphisms

$$\mathrm{div} : S_\beta^{[m]} \to S_\beta, \quad \det : S_\beta^{[m]} \to \mathrm{Pic}(S), \quad \mathrm{pts} : S_\beta^{[m]} \to S^{[m]},$$

where div sends a 1-dimensional subscheme $Z \subset S$ to its underlying divisor on S, $\det(Z) := \mathcal{O}(\mathrm{div}(Z))$, and $\mathrm{pts}(Z)$ is the 0-dimensional subscheme of S defined by the ideal $I_Z(\mathrm{div}(Z))$ (see [KM77]). From this description it is easy to see that $S_\beta^{[m]} \cong S^{[m]} \times S_\beta$.

Notation. *We will denote the universal ideal sheaves of* $S_\beta^{[m]}$, $S^{[m]}$, *and* S_β *respectively by* $\mathcal{I}_{-\beta}^{[m]}$, $\mathcal{I}^{[m]}$, *and* $\mathcal{I}_{-\beta}$, *and the corresponding universal subschemes respectively by* $\mathcal{Z}_\beta^{[m]}$, $\mathcal{Z}^{[m]}$, *and* $\mathcal{Z}_\beta$. *We will use the same symbol for the pull backs of* $\mathcal{I}^{[m]}$ *and* $\mathcal{I}_{-\beta} = \mathcal{O}(-\mathcal{Z}_\beta)$ *via* $\mathrm{id} \times \mathrm{pts}$ *and* $\mathrm{id} \times \mathrm{div}$ *to* $S \times S_\beta^{[m]}$. *We will also* *write* $\mathcal{I}^{[m]}_{\beta}$ *for* $\mathcal{I}^{[m]} \otimes \mathcal{O}(\mathcal{Z}_\beta)$. *Using the universal property of the Hilbert scheme, it can be seen that* $\mathcal{I}^{[m]}_{-\beta} \cong \mathcal{I}^{[m]} \otimes \mathcal{O}(-\mathcal{Z}_\beta)$, *and hence it is consistent with the chosen notation. Let* $\pi : S \times S^{[m]}_{\beta} \to S^{[m]}_{\beta}$ *be the projection, we denote the derived functor* $\mathbf{R}\pi_* \mathbf{R}\mathcal{H}om$ *by* $\mathbf{R}\mathcal{H}om_\pi$ *and its i-th cohomology sheaf by* $\mathcal{E}xt^i_\pi$.

It is well known that $S^{[m]}$ is a nonsingular variety of dimension $2m$. The tangent bundle of $S^{[m]}$ is identified with

$$T_{S^{[m]}} \cong \mathcal{H}om_\pi\left(\mathcal{I}^{[m]}, \mathcal{O}_{\mathcal{Z}^{[m]}}\right) \cong \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[m]}, \mathcal{I}^{[m]}\right)_0[1] \cong \mathcal{E}xt^1_\pi\left(\mathcal{I}^{[m]}, \mathcal{I}^{[m]}\right)_0, \tag{2}$$

where the index 0 indicates the the trace-free part.

The main object of study in this paper is the following

Definition 2.1. *Suppose that* $\boldsymbol{n} := n_1, n_2, \ldots, n_r$ *is a sequence of* $r \geq 1$ *nonnegative integers, and* $\boldsymbol{\beta} := \beta_1, \ldots, \beta_{r-1}$ *is a sequence of classes in* $H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$ *such that* $\beta_i \geq 0$. *The nested Hilbert scheme is the closed subscheme*

$$\iota : S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}} \hookrightarrow S^{[n_1]}_{\beta_1} \times \cdots \times S^{[n_{r-1}]}_{\beta_{r-1}} \times S^{[n_r]} \tag{3}$$

naturally defined by the r*-tuples* $(Z'_1, \ldots, Z'_r)$ *of subschemes of* S *such that* $\operatorname{pts}(Z'_i) \subset Z'_{i-1}$ *is a subscheme for any* $1 < i \leq r$. *We drop* $\boldsymbol{\beta}$ *or* $\boldsymbol{n}$ *from the notation respectively when* $\beta_i = 0$ *for any* i *or* $n_i = 0, \beta_j \neq 0$ *for any* i, j.

Equivalently, $S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}$ is given by the tuples of subschemes

$$(Z_1, \ldots, Z_r) \in S^{[n_1]} \times \cdots \times S^{[n_r]}, \quad (C_1, \ldots, C_{r-1}) \in S_{\beta_1} \times \cdots \times S_{\beta_{r-1}}$$

together with the nonzero maps $\phi_i : I_{Z_i} \to I_{Z_{i+1}}(C_i)$ for any $1 \leq i < r$. Note that each ϕ_i is necessarily injective.

Remark 2.2. *Note that* $S^{[\boldsymbol{n}]}$ *is the nested Hilbert scheme of 0-dimensional subschemes*

$$Z_1 \supset Z_2 \supset \cdots \supset Z_r$$

of S. $S^{[\boldsymbol{n}]}$ *is a projective scheme but it is singular in general* [C98]. *It is proven in* [C98] *that* $S^{[\boldsymbol{n}]}$ *with* $n_r < \cdots < n_2 < n_1$ *is nonsingular only if* $r = 2$ *and* $n_1 - n_2 = 1$.
The Hilbert scheme of curves in class β, S_β, *is also known to be singular unless* $H^{i\geq 1}(L) = 0$ *for any effective line bundle with* $c_1(L) = \beta$. *This two special cases show that the general nested Hilbert scheme* $S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}$ *is expected to be highly singular.*

By the construction of nested Hilbert schemes, the maps ϕ_i above are induced from the universal maps

$$\Phi_i : \mathcal{I}^{[n_i]} \to \mathcal{I}^{[n_{i+1}]}_{\beta_i} \qquad 1 \leq i < r$$

defined over $S \times S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}$.

Notation. *Let* pr_i *be the closed immersion* (3) *followed by the projection to the* i*-th factor, and let* $\pi : S \times S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]} \to S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$ *be the projection. Then we have the fibered square*

$$\begin{array}{ccc} S \times S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]} & \xhookrightarrow{\iota'} & S \times S_{\beta_1}^{[n_1]} \times \cdots \times S_{\beta_{r-1}}^{[n_{r-1}]} \times S^{[n_r]} \\ \downarrow \pi & & \downarrow \pi' \\ S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]} & \xhookrightarrow{\iota} & S_{\beta_1}^{[n_1]} \times \cdots \times S_{\beta_{r-1}}^{[n_{r-1}]} \times S^{[n_r]} \end{array} \tag{4}$$

where π' *is projection and* $\iota' = \mathrm{id} \times \iota$.

Convention. *Throughout the paper we slightly abuse the notation and use the same symbol for the universal objects (which are flat) on Hilbert schemes or line bundles on* S *and their pullbacks to the products of the Hilbert schemes via projections and other natural morphisms defined above, possibly followed by the restriction to the nested Hilbert schemes embedded in the product. This convention makes the notation much simpler. For example, in the definition of* Φ_i *above,* $\mathcal{I}^{[n_i]}$ *is pulled back from* $S \times S^{[n_i]}$ *via the composition of*

$$\mathrm{id} \times \mathrm{pts} : S \times S_{\beta_i}^{[n_i]} \to S \times S^{[n_i]}, \quad \mathrm{id} \times \mathrm{pr}_i : S \times S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]} \to S \times S_{\beta_i}^{[n_i]}.$$

Remark 2.3. *As before we denote*

$$\mathbf{R}\mathcal{H}om_{\pi}(-,-) := \mathbf{R}\pi_* \mathbf{R}\mathcal{H}om(-,-), \quad \mathbf{R}\mathcal{H}om_{\pi'}(-,-) := \mathbf{R}\pi'_* \mathbf{R}\mathcal{H}om(-,-).$$

By the flatness of the universal families and the flatness of π' *in diagram* (4) *we have*[4]

$$\mathbf{R}\mathcal{H}om_{\pi}(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_i]}) \cong \mathbf{L}\iota^* \mathbf{R}\mathcal{H}om_{\pi'}(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_i]}),$$
$$\mathbf{R}\mathcal{H}om_{\pi}(\mathcal{I}^{[n_i]}, \mathcal{I}_{\beta_i}^{[n_{i+1}]}) \cong \mathbf{L}\iota^* \mathbf{R}\mathcal{H}om_{\pi'}(\mathcal{I}^{[n_i]}, \mathcal{I}_{\beta_i}^{[n_{i+1}]}),$$

where we use the convention above to write $\mathcal{I}^{[n_i]}$ *for* $\mathbf{L}\iota'^* \mathcal{I}^{[n_i]} = \iota'^* \mathcal{I}^{[n_i]}$ *and* $\mathcal{I}_{\beta_i}^{[n_{i+1}]}$ *for* $\mathbf{L}\iota'^* \mathcal{I}_{\beta_i}^{[n_{i+1}]} = \iota'^* \mathcal{I}_{\beta_i}^{[n_{i+1}]}$.

Applying the functors $\mathbf{R}\mathcal{H}om_{\pi}\left(-, \mathcal{I}_{\beta_i}^{[n_{i+1}]}\right)$ and $\mathbf{R}\mathcal{H}om_{\pi}\left(\mathcal{I}^{[n_i]}, -\right)$ to the universal map Φ_i, we get the following morphisms of the derived category

$$\mathbf{R}\mathcal{H}om_{\pi}\left(\mathcal{I}^{[n_{i+1}]}, \mathcal{I}^{[n_{i+1}]}\right) \xrightarrow{\Xi_i} \mathbf{R}\mathcal{H}om_{\pi}\left(\mathcal{I}^{[n_i]}, \mathcal{I}_{\beta_i}^{[n_{i+1}]}\right),$$

$$\mathbf{R}\mathcal{H}om_{\pi}\left(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_i]}\right) \xrightarrow{\Xi'_i} \mathbf{R}\mathcal{H}om_{\pi}\left(\mathcal{I}^{[n_i]}, \mathcal{I}_{\beta_i}^{[n_{i+1}]}\right).$$

By Remark 2.2 the projective scheme $S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$ is highly singular in general. The following proposition that implies Theorem 1 is proven in Section 2.3:

[4]See Lemma 18.3 in http://stacks.math.columbia.edu/download/perfect.pdf.

Proposition 2.4. $S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$ *is equipped with the perfect absolute obstruction theory* $F^\bullet$ *with the derived dual*

$$F^{\bullet\vee} \cong \operatorname{Cone}\left(\left[\bigoplus_{i=1}^{r} \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_i]}\right)\right]_0 \xrightarrow{[(\Xi_i', \Xi_i)]_i} \bigoplus_{i=1}^{r-1} \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_i]}, \mathcal{I}_{\beta_i}^{[n_{i+1}]}\right)\right),$$

where $[-]_0$ *means the trace-free part.*

2.1. **2-step nested Hilbert schemes.** In this section we study $S_\beta^{[n_1,n_2]} := S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$ in the case $r = 2$. Recall from Definition 2.1 that for a pair of nonnegative integers n_1, n_2 and an effective curve class $\beta \in H^2(S, \mathbb{Z})$, we defined the projective scheme

$$S_\beta^{[n_1,n_2]} = \left\{(Z_1, C, Z_2, \phi) \mid Z_i \in S^{[n_i]},\ C \in S_\beta,\ \ 0 \neq \phi : I_{Z_1} \to I_{Z_2}(C)\right\} \subset S_\beta^{[n_1]} \times S^{[n_2]}.$$

There are universal objects defined over $S \times S_\beta^{[n_1,n_2]}$ as before:

$$\Phi : \mathcal{I}^{[n_1]} \to \mathcal{I}_\beta^{[n_2]}.$$

Let π be the projection $S \times S_\beta^{[n_1,n_2]} \to S_\beta^{[n_1,n_2]}$. π is a smooth morphism of relative dimension 2 and hence by Grothendieck-Verdier duality $\pi^!(-) := \pi^*(-) \otimes \omega_\pi[2]$ is a right adjoint of $\mathbf{R}\pi_*$. This fact will be exploited soon.

Applying the functors $\mathbf{R}\mathcal{H}om_\pi\left(-, \mathcal{I}_\beta^{[n_2]}\right)$ and $\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, -\right)$ to the universal map Φ, we get the following morphisms of the derived category

$$\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}, \mathcal{I}^{[n_2]}\right) \xrightarrow{\Xi} \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}_\beta^{[n_2]}\right) \tag{5}$$

$$\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}, \mathcal{I}^{[n_2]}\right) \xrightarrow{\Xi'} \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}\right).$$

Let T be any scheme over the $\mathbb{C}$-scheme U, and let

$$\begin{array}{ccc} T & \hookrightarrow & \overline{T} \\ {\scriptstyle a}\downarrow & \swarrow_{\bar a} & \\ U & & \end{array}$$

be a square zero extension over U with the ideal J. As $J^2 = 0$, J can be considered as an $\mathcal{O}_T$-module. Suppose we have the Cartesian diagram

$$\begin{array}{ccc} T & \xrightarrow{g} & S_\beta^{[n_1,n_2]} \\ \Big\downarrow{\scriptstyle a} & & \Big\downarrow{\scriptstyle p:=\mathrm{pts}\circ \mathrm{pr}_1} \\ U & \longrightarrow & S^{[n_1]}. \end{array} \tag{6}$$

The bottom row of the (6) corresponds to the U-point $Z_{1,U}$ of $S^{[n_1]}$, and the top row corresponds to the T-point

$$(Z_{1,T}, C_T, Z_{2,T}, \phi_T) \tag{7}$$

of $S_\beta^{[n_1,n_2]}$ in which $Z_{1,T}=(\mathrm{id},a)^{-1}(Z_{1,U})$. Let π_T be the projections from $S\times T\to T$. By [Ill, Prop. IV.3.2.12] and [JS12, Thm 12.8], there exists an element

$$\mathrm{ob}:=\mathrm{ob}(\phi_T,J)\in \mathrm{Ext}^2_{S\times T}\left(\mathrm{coker}(\phi_T),\pi_T^*J\otimes I_{Z_{2,T}}(C_T)\right),$$

whose vanishing is necessary and sufficient to extend the T-point (7) to a $\overline{T}$-point

$$(Z_{1,\overline{T}},C_{\overline{T}},Z_{2,\overline{T}},\phi_{\overline{T}})$$

of $S_\beta^{[n_1,n_2]}$ such that $Z_{1,\overline{T}}=(\mathrm{id},\overline{a})^{-1}(Z_{1,U})$. In fact by [Ill, Prop. IV.3.2.12], ob is the obstruction for deforming the morphism ϕ_T while the deformation $I_{Z_{1,\overline{T}}}$ of $I_{Z_{1,T}}$ is given. Suppose that $\phi_{\overline{T}}:I_{Z_{1,\overline{T}}}\to\mathcal{F}$ is such a deformation, where $\mathcal{F}$ is a flat family of rank 1 torsion free sheaves with $\mathcal{F}|_{S\times T}=I_{Z_{2,T}}(C_T)$. Then by [K90, Lemma 6.13] the double dual $\mathcal{F}^{**}$ is a line bundle. Now $\phi_{\overline{T}}^{**}:\mathcal{O}_{S\times\overline{T}}\to\mathcal{F}^{**}$ is fiberwise injective, and hence by [HL10, Lemma 2.1.4], $\mathrm{coker}(\phi_{\overline{T}}^{**})$ is also flat over $\overline{T}$. Thus, there exists a $\overline{T}$-flat subscheme $C_{\overline{T}}\subset S\times\overline{T}$ that restricts to C_T and $\mathcal{F}^{**}\cong\mathcal{O}(C_{\overline{T}})$. We conclude that $\mathcal{F}\cong I_{Z_{2,\overline{T}}}(C_{\overline{T}})$ for some $\overline{T}$-flat subscheme $Z_{2,\overline{T}}\subset S\times\overline{T}$ restricting to $Z_{2,T}$.

If ob $=0$ then [Ill, Prop. IV.3.2.12] again, the set of isomorphism classes of deformations forms a torsor under

$$\mathrm{Ext}^1_{S\times T}\left(\mathrm{coker}(\phi_T),\pi_T^*J\otimes I_{Z_{2,T}}(C_T)\right).$$

Furthermore, by [JS12, Thm 12.9]), the element ob is the cup product of the Atiyah class

$$\mathrm{At}(\phi_T)\in\mathrm{Ext}^1_{S\times T}\left(\mathrm{coker}(\phi_T),\pi_T^*\mathbb{L}_a^\bullet\otimes I_{Z_{2,T}}(C_T)\right), \tag{8}$$

that only depends on the data $(C_T,Z_{2,T},\phi_T)$[5], and the pullback of

$$e(\overline{T})\in\mathrm{Ext}^1_T(\mathbb{L}_a^\bullet,J) \tag{9}$$

associated to the square zero extension $T\hookrightarrow\overline{T}$, twisted by $I_{Z_{2,T}}(C_T)$.

[5]The only slight change in the proof of [JS12, Thm 12.9], is that in the diagram (12.17) of [ibid], the first vertical arrow must be replaced with

$$\mathrm{At}_{\mathcal{O}_{S\times T}/\mathcal{O}_{S\times U}}(\phi_T):\mathrm{coker}(\phi_T)\to k^1\left(\mathbb{L}^{\bullet,\mathrm{gr}}_{(\mathcal{O}_{S\times T}\oplus I_{Z_{1,T}})/\mathcal{O}_{S\times U}}\otimes(\mathcal{O}_{S\times T}\oplus I_{Z_{2,T}}(C_T))\right)[1].$$

By flatness of $I_{Z_{1,U}}$ over $\mathcal{O}_U$, we see that $\mathcal{O}_{S\times U}\oplus I_{Z_{1,U}}$ is flat over $\mathcal{O}_U$ and hence

$$(\mathcal{O}_{S\times U}\oplus I_{Z_{1,U}})\overset{\mathbf{L}}{\otimes}_{\mathcal{O}_{S\times U}}\mathcal{O}_{S\times T}\cong(\mathcal{O}_{S\times U}\oplus I_{Z_{1,U}})\otimes_{\mathcal{O}_{S\times U}}\mathcal{O}_{S\times T}\cong\mathcal{O}_{S\times T}\oplus I_{Z_{1,T}},$$

so we can write (see [Ill, II.2.2])

$$\mathbb{L}^{\bullet,\mathrm{gr}}_{(\mathcal{O}_{S\times T}\oplus I_{Z_{1,T}})/\mathcal{O}_{S\times U}}\cong\left(\mathbb{L}^\bullet_{\mathcal{O}_{S\times T}/\mathcal{O}_{S\times U}}\otimes(\mathcal{O}_{S\times T}\oplus I_{Z_{1,T}})\right)\oplus\left(\mathbb{L}^{\bullet,\mathrm{gr}}_{(\mathcal{O}_{S\times U}\oplus I_{Z_{1,U}})/\mathcal{O}_{S\times U}}\otimes(\mathcal{O}_{S\times T}\oplus I_{Z_{1,T}})\right),$$

and hence as in the proof of [JS12, Thm 12.9], composing $\mathrm{At}_{\mathcal{O}_{S\times T}/\mathcal{O}_{S\times U}}(\phi_T)$ with the projection

$$\mathbb{L}^{\bullet,\mathrm{gr}}_{(\mathcal{O}_{S\times T}\oplus I_{Z_{1,T}})/\mathcal{O}_{S\times U}}\to\mathbb{L}^\bullet_{\mathcal{O}_{S\times T}/\mathcal{O}_{S\times U}}\cong\pi_T^*\mathbb{L}_a^\bullet,$$

we arrive at the definition of $\mathrm{At}(\phi_T)$ in (8).

Notation. *For any line bundle L on S we define $L^D := L^{-1} \otimes \omega_S$. Similarly, for any class $\beta \in H^2(S, \mathbb{Z})$ we define $\beta^D := K_S - \beta$.*

Proposition 2.5. *The complex*

$$F^\bullet_{\mathrm{rel}} := \mathrm{Cone}(\Xi)^\vee \cong \mathbf{R}\mathcal{H}om_\pi \left(\mathrm{coker}(\Phi), \mathcal{I}^{[n_2]}_\beta\right)^\vee [-1]$$

defines a relative perfect obstruction theory for the morphism $\mathrm{p} : S^{[n_1,n_2]}_\beta \to S^{[n_1]}$. In other words, $F^\bullet_{\mathrm{rel}}$ is perfect with amplitude $[-1, 0]$, and there exists a morphism of the derived category $\alpha : F^\bullet_{\mathrm{rel}} \to \mathbb{L}^\bullet_{\mathrm{p}}$, such that $h^0(\alpha)$ and $h^{-1}(\alpha)$ are respectively isomorphism and epimorphism. The rank of $F^\bullet_{\mathrm{rel}}$ is equal to

$$\mathrm{Rank}\, [F^\bullet_{\mathrm{rel}}] = n_2 - n_1 - \frac{\beta \cdot \beta^D}{2}.$$

Proof. **Step 1:** *(perfectness)* We show that the complex $F^{\bullet\vee}_{\mathrm{rel}}$ is perfect with amplitude $[0, 1]$. By the base change and the same argument as in the proof of [HT10, Lemma 4.2], it suffices to show that $h^i(\mathbf{L}t^* F^{\bullet\vee}_{\mathrm{rel}}) = 0$ for $i \neq 0, 1$, where $t : P \hookrightarrow S^{[n_1,n_2]}_\beta$ is the inclusion of an arbitrary closed point $P = (Z_1, C, Z_2, \phi) \in S^{[n_1,n_2]}_\beta$. Note that by the definition of the universal families, $\mathbf{L}t^*\mathcal{I}^{[n_1]} = I_{Z_1}$ and $\mathbf{L}t^*\mathcal{I}^{[n_2]}_\beta = I_{Z_2}(C)$. Therefore, by the definition of $F^{\bullet\vee}_{\mathrm{rel}}$ we get the exact sequence

$$\cdots \to \mathrm{Ext}^i_S(I_{Z_1}, I_{Z_2}(C)) \to h^i(\mathbf{L}t^* F^{\bullet\vee}_{\mathrm{rel}}) \to \mathrm{Ext}^{i+1}_S(I_{Z_2}, I_{Z_2}) \to \ldots.$$

All the Ext^i_S for $i \neq 0, 1, 2$ vanish, so we deduce easily that $h^i(\mathbf{L}t^* F^{\bullet\vee}_{\mathrm{rel}}) = 0$ for $i \neq -1, 0, 1, 2$. From the sequence above we see that

$$h^{-1}(\mathbf{L}t^* F^{\bullet\vee}_{\mathrm{rel}}) = \ker\left(\mathrm{Hom}_S(I_{Z_2}, I_{Z_2}) \to \mathrm{Hom}_S(I_{Z_1}, I_{Z_2}(C))\right).$$

But by definition this morphism is induced by applying $\mathrm{Hom}_S(-, I_{Z_2}(C))$ to the map $\phi : I_{Z_1} \to I_{Z_2}(C)$. Since $\mathrm{coker}(\phi)$ is at most 1-dimensional we deduce that

$$\mathrm{Hom}_S(I_{Z_2}, I_{Z_2}) \to \mathrm{Hom}_S(I_{Z_1}, I_{Z_2}(C))$$

is injective, and hence $h^{-1}(\mathbf{L}t^* F^{\bullet\vee}_{\mathrm{rel}}) = 0$.

To prove $h^2(\mathbf{L}t^* F^{\bullet\vee}_{\mathrm{rel}}) = 0$, we show that the map

$$\mathrm{Ext}^2_S(I_{Z_2}, I_{Z_2}) \to \mathrm{Ext}^2_S(I_{Z_1}, I_{Z_2}(C))$$

in the exact sequence above is surjective, or equivalently by Serre duality, the dual map

$$\mathrm{Hom}_S(I_{Z_2}, I_{Z_1} \otimes \omega_S(-C)) \to \mathrm{Hom}_S(I_{Z_2}, I_{Z_2} \otimes \omega_S)$$

is injective. But this follows after applying the left exact functor $\mathrm{Hom}_S(I_{Z_2}, -)$ to the injection $I_{Z_1} \otimes \omega_S(-C) \to I_{Z_2} \otimes \omega_S$ that is induced by tensoring the map ϕ above by $\omega_S(-C)$.

Step 2: *(map to the cotangent complex)* We construct a morphism of derived category $\alpha : F^\bullet_{\mathrm{rel}} \to \mathbb{L}^\bullet_{\mathrm{p}}$. Consider the Atiyah class (8) in the case $T = S^{[n_1,n_2]}_\beta$ and $U = S^{[n_1]}$. It defines an element in

$$\begin{aligned}
&\operatorname{Ext}^1_{S\times S^{[n]}_\beta}\left(\operatorname{coker}(\Phi), \pi^*\mathbb{L}^\bullet_{\mathrm{p}}\otimes \mathcal{I}^{[n_2]}_\beta\right)\cong\\
&\operatorname{Ext}^1_{S\times S^{[n]}_\beta}\left(\mathbf{R}\mathcal{H}om\left(\mathcal{I}^{[n_2]}_\beta, \operatorname{coker}(\Phi)\right), \pi^*\mathbb{L}^\bullet_{\mathrm{p}}\right)\cong \quad \text{(by the definition of } \pi^!)\\
&\operatorname{Ext}^1_{S\times S^{[n]}_\beta}\left(\mathbf{R}\mathcal{H}om\left(\mathcal{I}^{[n_2]}_\beta, \operatorname{coker}(\Phi)\otimes\omega_\pi[2]\right), \pi^!\mathbb{L}^\bullet_{\mathrm{p}}\right)\cong \quad \text{(by Grothendieck-Verdier duality)}\\
&\operatorname{Ext}^1_{S^{[n]}_\beta}\left(\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}_\beta, \operatorname{coker}(\Phi)\otimes\omega_\pi[2]\right), \mathbb{L}^\bullet_{\mathrm{p}}\right)\cong\\
&\operatorname{Hom}_{S^{[n]}_\beta}\left(\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}_\beta, \operatorname{coker}(\Phi)\otimes\omega_\pi[1]\right), \mathbb{L}^\bullet_{\mathrm{p}}\right).
\end{aligned}$$

So under the identification above, the Atiyah class defines a morphism of the derived category

$$\alpha : \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}_\beta, \operatorname{coker}(\Phi)\otimes\omega_\pi[1]\right)\to \mathbb{L}^\bullet_{\mathrm{p}}.$$

But by Grothendieck-Verdier duality again,

$$\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}_\beta, \operatorname{coker}(\Phi)\otimes\omega_\pi[1]\right)\cong \mathbf{R}\mathcal{H}om_\pi\left(\operatorname{coker}(\Phi), \mathcal{I}^{[n_2]}_\beta\right)^\vee[-1]\cong F^\bullet_{\mathrm{rel}},$$

and hence we are done.

Step 3: *(obstruction theory)* We show $h^0(\alpha)$ and $h^{-1}(\alpha)$ are respectively isomorphism and epimorphism. Suppose we are in the situation of the diagram (6). Define

$$f := (\mathrm{id}, g) : S\times T\to S\times S^{[n_1,n_2]}_\beta.$$

Composing $e(\overline{T})$ (given in (9)) and the natural morphism of cotangent complexes $\mathbf{L}g^*\mathbb{L}^\bullet_{\mathrm{p}}\to\mathbb{L}^\bullet_a$ gives the element $\varpi(g)\in\operatorname{Ext}^1_T(\mathbf{L}g^*\mathbb{L}^\bullet_{\mathrm{p}}, J)$ whose image under α is denoted by

$$\alpha^*\varpi(g)\in\operatorname{Ext}^1_T\left(\mathbf{L}g^*F^\bullet_{\mathrm{rel}}, J\right).$$

For $i=0,1$, we will use the following identifications:

$$\begin{aligned}
\operatorname{Ext}^i_T\left(\mathbf{L}g^*F^\bullet_{\mathrm{rel}}, J\right) &\cong \operatorname{Ext}^i_T\left(\mathbf{L}g^*\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}_\beta, \operatorname{coker}(\Phi)\otimes\omega_\pi[1]\right), J\right)\\
&\cong \operatorname{Ext}^i_{S^{[n]}_\beta}\left(\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}_\beta, \operatorname{coker}(\Phi)\otimes\omega_\pi[1]\right), \mathbf{R}g_*J\right)\\
&\cong \operatorname{Ext}^i_{S\times S^{[n]}_\beta}\left(\mathbf{R}\mathcal{H}om\left(\mathcal{I}^{[n_2]}_\beta, \operatorname{coker}(\Phi)\otimes\omega_\pi[1]\right), \pi^!\mathbf{R}g_*J\right)\\
&\cong \operatorname{Ext}^{i+1}_{S\times S^{[n]}_\beta}\left(\mathbf{R}\mathcal{H}om\left(\mathcal{I}^{[n_2]}_\beta, \operatorname{coker}(\Phi)\right), \pi^*\mathbf{R}g_*J\right).
\end{aligned}$$

Here we have used the fact that $\mathbf{L}g^*\dashv\mathbf{R}g_*$ i.e. $\mathbf{L}g^*$ is the left adjoint of $\mathbf{R}g_*$, and Grothendieck-Verdier duality. Now using $\mathbf{R}f_*\pi_T^* = \pi^*\mathbf{R}g_*$ in the last Ext above, we get

$$\begin{aligned}
&\operatorname{Ext}^{i+1}_{S\times S^{[n]}_\beta}\left(\mathbf{R}\mathcal{H}om\left(\mathcal{I}^{[n_2]}_\beta,\operatorname{coker}(\Phi)\right),\mathbf{R}f_*\pi_T^*J\right)\cong \quad (\text{by } \mathbf{L}f^*\dashv \mathbf{R}f_*)\\
&\operatorname{Ext}^{i+1}_{S\times T}\left(\mathbf{L}f^*\mathbf{R}\mathcal{H}om\left(\mathcal{I}^{[n_2]}_\beta,\operatorname{coker}(\Phi)\right),\pi_T^*J\right)\cong\\
&\operatorname{Ext}^{i+1}_{S\times T}\left(\mathbf{R}\mathcal{H}om\left(\mathbf{L}f^*\mathcal{I}^{[n_2]}_\beta,\mathbf{L}f^*\operatorname{coker}(\Phi)\right),\pi_T^*J\right)\cong\\
&\operatorname{Ext}^{i+1}_{S\times T}\left(\mathbf{L}f^*\operatorname{coker}(\Phi),\pi_T^*J\otimes\mathbf{L}f^*\mathcal{I}^{[n_2]}_\beta\right)\cong \quad (\text{by flatness of } \operatorname{coker}(\Phi) \text{ and } \mathcal{I}^{[n_2]}_\beta)\\
&\operatorname{Ext}^{i+1}_{S\times T}\left(\operatorname{coker}(\phi_T),\pi_T^*J\otimes I_{Z_{2,T}}(C_T)\right).
\end{aligned}$$

Similar to the Step 2 it can be seen that the composition

$$\mathbf{L}g^*F^\bullet_{\mathrm{rel}}\xrightarrow{g^*\alpha}\mathbf{L}g^*\mathbb{L}^\bullet_{\mathrm{p}}\to\mathbb{L}^\bullet_a$$

has a lift to the Atiyah class $\operatorname{At}(\phi_T)$ over $S\times T$ (see (8)). Therefore, $\operatorname{ob}(\phi_T,J)$, which is the cup product of $\operatorname{At}(\phi_T)$ and the pullback of $e(\overline{T})$ twisted by $I_{Z_{2,T}}(C_T)$, is isomorphic to the element $\alpha^*\varpi(g)$ via the identifications above for $i=1$. Now the claim follows from the identifications above for $i=0$, and [BF97, Theorem 4.5].

Step 4: *(rank of $F^\bullet_{\mathrm{rel}}$)* The claim about the rank follows from

$$\begin{aligned}
\operatorname{Rank}[F^\bullet_{\mathrm{rel}}]&=\operatorname{Rank}[\operatorname{Cone}(\Xi)]\\
&=\operatorname{Rank}\left[\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]},\mathcal{I}^{[n_2]}_\beta\right)\right]-\operatorname{Rank}\left[\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]},\mathcal{I}^{[n_2]}\right)\right]\\
&=\chi(I_{Z_1},I_{Z_2}(C))-\chi(I_{Z_2},I_{Z_2})\\
&=-n_1-n_2+\chi(\mathcal{O}_S(C))+2n_2-\chi(\mathcal{O}_S)\\
&=n_2-n_1-\beta\cdot K_C/2+\beta^2/2,
\end{aligned}$$

where (Z_1,C,Z_2,ϕ) is a closed point of $S^{[n_1,n_2]}_\beta$.

□

Remark 2.6. *From the proof above, the reader can see that the relative obstruction theory $F^\bullet_{\mathrm{rel}}$ is obtained by from the deformation/obstruction theory of the universal map $\Phi:\mathcal{I}^{[n_1]}\to\mathcal{I}^{[n_2]}_\beta$ while the data $\mathcal{I}^{[n_1]}$ is kept fixed.*

Proposition 2.7. *$S^{[n_1,n_2]}_\beta$ is equipped with the perfect absolute obstruction theory $F^\bullet$ obtained from the perfect relative obstruction theory of Proposition 2.5. Its dual is given by*

$$F^{\bullet\vee}\cong\operatorname{Cone}\Bigg(\left[\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]},\mathcal{I}^{[n_1]}\right)\oplus\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]},\mathcal{I}^{[n_2]}\right)\right]_0 \xrightarrow{[(\Xi',\Xi)]}\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]},\mathcal{I}^{[n_2]}_\beta\right)\Bigg),$$

where $[-]_0$ means the trace-free part.

Proof. Since $S^{[n_1]}$ is nonsingular, by the standard techniques (see [MPT10, page 954])

$$F^\bullet := \text{Cone}\left(F^\bullet_{\text{rel}} \xrightarrow{\theta} \mathrm{p}^*\,\Omega_{S^{[n_1]}}[1]\right)[-1] \tag{10}$$

gives a perfect absolute obstruction theory over $S^{[n_1,n_2]}_\beta$, where θ is the composition of $\alpha : F^\bullet_{\text{rel}} \to \mathbb{L}^\bullet_{\mathrm{p}}$ and the Kodaira-Spencer map $c : \mathbb{L}^\bullet_{\mathrm{p}} \to \mathrm{p}^*\,\Omega_{S^{[n_1]}}[1]$. We claim that θ is given by

$$\begin{aligned} F^\bullet_{\text{rel}} \cong & \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}_\beta, \text{coker}(\Phi)\otimes\omega_\pi\right)[1] \to \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \text{coker}(\Phi)\otimes\omega_\pi\right)[1] \\ & \to \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_1]}\otimes\omega_\pi\right)_0[2] \cong \mathrm{p}^*\,\Omega_{S^{[n_1]}}[1], \end{aligned}$$

where the first and second maps are respectively induced by $\Phi : \mathcal{I}^{[n_1]} \to \mathcal{I}^{[n_2]}_\beta$ and the natural map $c' : \text{coker}(\Phi) \to \mathcal{I}^{[n_1]}[1]$. To see the claim consider the natural commutative diagram of graded algebras in each vertex of which the first summand has degree 0 and the second summand (if any) has degree 1:

$$\begin{array}{ccccc} \mathcal{O}_{S\times S^{[n_1]}} & \longrightarrow & \mathcal{O}_{S\times S^{[n_1,n_2]}_\beta} \oplus \mathcal{I}^{[n_1]} & \longrightarrow & \mathcal{O}_{S\times S^{[n_1,n_2]}_\beta} \oplus \mathcal{I}^{[n_2]}_\beta \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{O}_S & \longrightarrow & \mathcal{O}_{S\times S^{[n_1]}} & \longrightarrow & \mathcal{O}_{S\times S^{[n_1]}} \oplus \mathcal{I}^{[n_1]}, \end{array}$$

The compatibility of the exact triangles of transitivity for each each row in the diagram (see Step 2 in the proof of Lemma 2.15 for more details of a similar argument) implies the commutative diagram

$$\begin{array}{ccc} \text{coker}(\Phi) & \xrightarrow{\text{At}(\Phi)} & \mathbb{L}^\bullet_{\mathrm{p}} \otimes \mathcal{I}^{[n_2]}_\beta[1] \\ \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle c\otimes\text{id}} \\ \mathcal{I}^{[n_1]}[1] & \xrightarrow{(\text{id}\otimes\Phi)\circ\text{At}(\mathcal{I}^{[n_1]})} & \mathrm{p}^*\,\Omega_{S^{[n_1]}} \otimes \mathcal{I}^{[n_2]}_\beta[2] \end{array}$$

where $\text{At}(\mathcal{I}^{[n_1]}) : \mathcal{I}^{[n_1]} \to \pi^*\Omega_{S^{[n_1]}} \otimes \mathcal{I}^{[n_1]}$ is the Atiyah class. The claim now follows from this and the construction of the map α using the Atiyah class $\text{At}(\Phi)$ in Step 2 of proof of Proposition 2.5. For simplicity define

$$A^\bullet := \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_1]}\right), B^\bullet := \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}, \mathcal{I}^{[n_2]}\right), C^\bullet := \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}_\beta\right),$$

and denote by $E^\bullet$ the right hand side of the expression in the proposition. By Proposition 2.5, $F^{\bullet\vee}_{\text{rel}} = \text{Cone}\,(B^\bullet \to C^\bullet)$, so by (2) and the claim above, (10) can be rewritten as

$$F^{\bullet\vee} = \text{Cone}\left(A^\bullet_0 \xrightarrow{\theta^\vee} \text{Cone}\,(B^\bullet \to C^\bullet)\right).$$

Consider the commutative diagram

$$\begin{array}{ccccc} B^\bullet & \longrightarrow & A^\bullet \oplus B^\bullet & \longrightarrow & A^\bullet \\ & & \uparrow \scriptstyle{\mathrm{id}} & & \uparrow \scriptstyle{\mathrm{id}} \\ & & \mathbf{R}\pi_*\mathcal{O}_{S\times S_\beta^{[n_1,n_2]}} & = & \mathbf{R}\pi_*\mathcal{O}_{S\times S_\beta^{[n_1,n_2]}}. \end{array}$$

in which the top row is the natural exact triangle. Taking the cone of the diagram one gets the exact triangle

$$B^\bullet \to [A^\bullet \oplus B^\bullet]_0 \to A^\bullet_0$$

that fits into the following commutative diagram in which all the rows and columns are exact triangles:

$$\begin{array}{ccccc} B^\bullet & \longrightarrow & [A^\bullet \oplus B^\bullet]_0 & \longrightarrow & A^\bullet_0 \\ \| & & \downarrow \scriptstyle{[(\Xi',\Xi)]} & & \downarrow \scriptstyle{\theta^\vee} \\ B^\bullet & \xrightarrow{\Xi} & C^\bullet & \longrightarrow & \mathrm{Cone}\,(B^\bullet \to C^\bullet) \\ & & \downarrow & & \downarrow \\ & & E^\bullet & \longrightarrow & F^{\bullet\vee}. \end{array}$$

In fact the columns and the top and middle rows are exact triangles with commutative top squares, and the bottom row is induced from the rest of the diagram by taking the cone. Therefore, the bottom row must also be an exact triangle which means that $E^\bullet \cong F^{\bullet\vee}$ as desired. □

This finishes the proof of Proposition 2.4 in the case $r = 2$. Propositions 2.5 and 2.7 imply

Corollary 2.8. *The perfect obstruction theory $F^{\bullet\vee}$ defines a virtual fundamental class over $S_\beta^{[n_1,n_2]}$ denoted by*

$$[S_\beta^{[n_1,n_2]}]^{\mathrm{vir}} \in A_d(S_\beta^{[n_1,n_2]}), \qquad d = n_1 + n_2 - \frac{\beta\cdot\beta^D}{2}.$$

□

2.2. **Reduced obstruction theory and proof of Theorem 3.** In this section we assume that for any effective line bundle L on S with $c_1(L) = \beta$, we have

$$|L^D| = \emptyset \quad \text{or equivalently} \quad H^2(L) = 0. \tag{11}$$

Recall from Proposition 2.5 that the derived dual of

$$F^{\bullet\vee}_{\mathrm{rel}} = \mathrm{Cone}\left(\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}_\beta^{[n_2]}, \mathcal{I}_\beta^{[n_2]}\right) \to \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}_\beta^{[n_2]}\right)\right)$$

defines a relative perfect obstruction theory over $S_\beta^{[n_1,n_2]}$. By definition, we get a natural map

$$\mu : F^{\bullet\vee}_{\mathrm{rel}} \to \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}_\beta^{[n_2]}, \mathcal{I}_\beta^{[n_2]}\right)[1],$$

that induces

$$h^1(\mu) : h^1(F^{\bullet\vee}_{\mathrm{rel}}) \cong \mathcal{E}xt^2_\pi\left(\mathcal{I}^{[n_2]}_\beta/\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}_\beta\right) \to \mathcal{E}xt^2_\pi\left(\mathcal{I}^{[n_2]}_\beta, \mathcal{I}^{[n_2]}_\beta\right) \cong \mathbf{R}^2\pi_*\mathcal{O}_{S\times S^{[n_1,n_2]}_\beta} \cong \mathcal{O}^{p_g}_{S^{[n_1,n_2]}_\beta}.$$

We claim that $h^1(\mu)$ is surjective. To see this, by the base change, it suffices to prove that $h^1(\mu)$ is fiberwise surjective. Let $t : P \hookrightarrow S^{[n_1,n_2]}_\beta$ be the inclusion of an arbitrary closed point $P = (Z_1, C, Z_2, \phi) \in S^{[n_1,n_2]}_\beta$. Then, by the base change we have the natural exact sequence[6]

$$\cdots \to h^1(\mathbf{L}t^*F^{\bullet\vee}_{\mathrm{rel}}) \xrightarrow{h^1(\mu)_P} \mathrm{Ext}^2_S(I_{Z_2}(C), I_{Z_2}(C)) \xrightarrow{u} \mathrm{Ext}^2_S(I_{Z_1}, I_{Z_2}(C)) \to 0.$$

The surjectivity of the map u was established in Step 1 of the proof of Proposition 2.5. We have

$$\mathrm{Ext}^2_S(I_{Z_2}(C), I_{Z_2}(C)) \cong \mathrm{Ext}^2_S(I_{Z_2}, I_{Z_2}) \cong H^2(\mathcal{O}_S),$$

$$\mathrm{Ext}^2_S(I_{Z_1}, I_{Z_2}(C))^* \cong \mathrm{Hom}_S(I_{Z_2}, I_{Z_1}(C^D)) \subseteq \mathrm{Hom}_S(I_{Z_2}, \mathcal{O}_S(C^D)) \cong H^0(\mathcal{O}_S(C^D)).$$

By assumption (11), $H^0(\mathcal{O}_S(C^D)) = 0$, and hence $h^1(\mu)_P$ is surjective and the claim follows. We now have the diagram

$$\begin{array}{ccccccc} \mathrm{Ext}^1_S(I_{Z_1}, I_{Z_1})_0 & \xrightarrow{h^1(\theta^\vee)_P} & \mathrm{Ext}^2_S(I_{Z_2}(C)/I_{Z_1}, I_{Z_2}(C)) & \longrightarrow & h^1(\mathbf{L}t^*F^{\bullet\vee}) & \longrightarrow & 0 \\ & & \big\downarrow {\scriptstyle h^1(\mu)_P} & & & & \\ & & \mathrm{Ext}^2_S(I_{Z_2}(C), I_{Z_2}(C)) & & & & \end{array}$$

where the first row is exact by the proof Proposition 2.7. But since

$$h^1(\mu)_P \circ h^1(\theta^\vee)_P = 0,$$

the surjection $h^1(\mu)_P$ factors through $h^1(\mathbf{L}t^*F^{\bullet\vee})$. Therefore, by the base change again there exists a surjection $h^1(F^{\bullet\vee}) \to \mathcal{O}^{p_g}_{S^{[n_1,n_2]}_\beta}$. We have proven

Proposition 2.9. *If the condition* (11) *is satisfied and* $p_g(S) > 0$*, then,*

$$[S^{[n_1,n_2]}_\beta]^{\mathrm{vir}} = 0.$$

Proof. Under the assumptions of the proposition, by the discussion above, the obstruction theory $F^\bullet$ carries a trivial factor, and hence the associated virtual cycle vanishes by [KL13, Theorem 1.1]. □

Definition 2.10. *The map* $h^1(\mu)$ *induces the morphism of the derived category*

$$F^{\bullet\vee} \to h^1(F^{\bullet\vee})[-1] \to h^1(F^{\bullet\vee}_{\mathrm{rel}})[-1] \xrightarrow{h^1(\mu)} \mathcal{O}^{p_g}_{S^{[\boldsymbol{n}]}_\beta}[-1].$$

Dualizing gives a map $\mathcal{O}^{p_g}_{S^{[\boldsymbol{n}]}_\beta}[1] \to F^\bullet$*. Define* $F^\bullet_{\mathrm{red}}$ *to be its cone.*

[6] Note that $\mathrm{Ext}^3_S(\mathrm{coker}(\phi), I_{Z_2}(C)) = \mathrm{Ext}^3_S(I_{Z_2}(C), I_{Z_2}(C)) = 0$.

We show that under a slightly stronger condition than (11), $F^\bullet_{\text{red}}$ gives a perfect obstruction theory over $S_\beta^{[n_1,n_2]}$. This condition is[7]

$$H^1(T_S) \xrightarrow{*\cup\beta} H^2(\mathcal{O}_S) \quad \text{is surjective.} \tag{12}$$

To show $F^\bullet_{\text{red}}$ is a perfect obstruction theory, we use the beautiful idea of [KT14]. We sketch their method here and make some necessary changes; the reader can find the missing details in [KT14]. S is embedded as the central fiber of an algebraic twistor family $\mathcal{S} \to B$, where B is a first order Artinian neighborhood of the origin in a certain p_g-dimensional family of the first order deformations of S. Let

$$V \subset H^1(T_S)$$

be a subspace over which $* \cup \beta$ in (12) restricts to an isomorphism. Then, B is constructed so that T_B is naturally identified with V. By the construction of [KT14], $\mathcal{S}$ is transversal to the Noether-Lefschetz locus of the $(1,1)$-class β, and as a result, β does not deform outside of the central fiber of the family. Using this fact, as in [KT14, Proposition 2.3], one can show that

$$S_\beta^{[n_1,n_2]} \cong (\mathcal{S}/B)_\beta^{[n_1,n_2]}, \tag{13}$$

where the right hand side is the relative nested Hilbert scheme of the family $\mathcal{S} \to B$. We use the same symbols

$$\Phi : \mathcal{I}^{[n_1]} \to \mathcal{I}_\beta^{[n_2]}$$

as before to denote the universal objects over $\mathcal{S} \times_B (\mathcal{S}/B)_\beta^{[n_1,n_2]}$, and we let π be the projection to the second factor of $\mathcal{S} \times_B (\mathcal{S}/B)_\beta^{[n_1,n_2]}$. The arguments of Section 2.1 can be adapted with no changes to prove that

$$\text{Cone}\left(\left[\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_1]}\right) \oplus \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}, \mathcal{I}^{[n_2]}\right)\right]_0 \to \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}_\beta^{[n_2]}\right)\right)$$

is the dual of a perfect B-relative obstruction theory over $(\mathcal{S}/B)_\beta^{[n_1,n_2]}$, denoted by $G^\bullet_{\text{rel}}$. By construction,

$$G^\bullet := \text{Cone}\left(G^\bullet_{\text{rel}} \to \Omega_B[1]\right)[-1]$$

is an absolute perfect obstruction theory over $(\mathcal{S}/B)_\beta^{[n_1,n_2]}$ (see [KT14]). By the definitions of $F^\bullet$ and $G^\bullet_{\text{rel}}$, and the isomorphism (13), we see that $F^\bullet \cong G^\bullet_{\text{rel}}$. Now we claim that the composition

$$G^\bullet \to G^\bullet_{\text{rel}} \cong F^\bullet \to F^\bullet_{\text{red}}$$

is an isomorphism. By the definitions of $G^\bullet_{\text{rel}}$ and $F^\bullet_{\text{red}}$, to prove the claim, it suffices to show that

$$\mathcal{O}^{p_g}_{S_\beta^{[n]}} \to F^\bullet[-1] \cong G^\bullet_{\text{rel}}[-1] \to \Omega_B \tag{14}$$

[7]This is the condition (3) in [KT14], and (11) is the condition (2) in [ibid].

is an isomorphism. By the Nakayama lemma we may check this at a closed point $P = (Z_1, C, Z_2, \phi) \in S_\beta^{[n_1,n_2]}$. After dualizing and using the identifications above the derived pull back of (14) to P becomes [8]

$$T_B = V \subset H^1(T_S) \xrightarrow{\mathrm{At}(\phi)} \mathrm{Ext}^1(\mathrm{coker}(\phi), I_{Z_2}(C)) \xrightarrow{h^1(\mu)_P} \mathrm{Ext}^2(I_{Z_2}(C), I_{Z_2}(C)) \xrightarrow{\mathrm{tr}} H^2(\mathcal{O}_S).$$

By the naturality of the Atiyah classes,

$$h^1(\mu)_P \circ \mathrm{At}(\phi) = \mathrm{At}(I_{Z_2}(C)).$$

But by [BFl03, Prop 4.2],

$$\mathrm{tr} \circ \mathrm{At}(I_{Z_2}(C)) = - * \cup \beta,$$

which by condition (12) is an isomorphism when restricted to $V \subset H^1(T_S)$, and hence the claim is proven. We have shown

Proposition 2.11. *If the condition* (12) *is satisfied, then,* $F^\bullet_{\mathrm{red}}$ *is a perfect obstruction theory on* $S_\beta^{[n]}$*, and hence defines a reduced virtual fundamental class*

$$[S_\beta^{[n_1,n_2]}]^{\mathrm{vir}}_{\mathrm{red}} \in A_{d'}(S_\beta^{[n_1,n_2]}), \quad d' = n_1 + n_2 - \frac{\beta \cdot \beta^D}{2} + p_g(S).$$

□

Definition 2.12. *Let* $M \in \mathrm{Pic}(S)$*. Define the following elements in* $K(S_\beta^{[n_1,n_2]})$ *of ranks (from left to right)* $n_1 + n_2$ *and* $-\beta \cdot \beta^D/2 + \beta \cdot c_1(M)$*, respectively:*

$$\mathsf{K}^{n_1,n_2}_{\beta;M} := [\mathbf{R}\pi_* M(\mathcal{Z}_\beta)] - \left[\mathbf{R}\mathcal{H}om_\pi(\mathcal{I}^{[n_1]}, \mathcal{I}_\beta^{[n_2]} \otimes M)\right], \quad \mathsf{G}_{\beta;M} := \left[\mathbf{R}\pi_* M(\mathcal{Z}_\beta)|_{\mathcal{Z}_\beta}\right].$$

If $\beta = 0$ *we will instead use the notation* $\mathsf{K}_M^{[n_1 \geq n_2]} := \mathsf{K}^{n_1,n_2}_{0;M}$ *(see Definition 5.3). We also define the rank* $2n_i$ *twisted tangent bundles*

$$\mathsf{T}^M_{S^{[n_i]}} := [\mathbf{R}\pi_* M] - \left[\mathbf{R}\mathcal{H}om_\pi(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_i]} \otimes M)\right] = \left[\mathcal{E}xt^1_\pi\left(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_i]} \otimes M\right)_0\right].$$

Note that if $M = \mathcal{O}_S$ *then* $\mathsf{T}^M_{S^{[n_i]}} = [T_{S^{[n_i]}}]$.

Let $\mathcal{P} := \mathcal{P}(M, \beta, n_1, n_2)$ *be a polynomial in the Chern classes of* $\mathsf{K}^{n_1,n_2}_{\beta;M}$*,* $\mathsf{G}_{\beta;M}$*,* $\mathsf{T}_{S^{[n_1]}}$*, and* $\mathsf{T}_{S^{[n_2]}}$*, then, we can define the invariant*

$$\mathsf{N}_S(n_1, n_2, \beta; \mathcal{P}) := \int_{[S_\beta^{[n_1,n_2]}]^{\mathrm{vir}}} \mathcal{P}.$$

[8]Here one needs to use a similar argument as in the proof of Proposition 2.7 or the proof of [MPT10, Proposition 13] to deduce that the composition of $G^\bullet_{\mathrm{rel}} \to \mathbb{L}^\bullet_{(\mathcal{S}/B)_\beta^{[n_1,n_2]}/B}$ and the Kodaira-Spencer map $\mathbb{L}^\bullet_{(\mathcal{S}/B)_\beta^{[n_1,n_2]}/B} \to \Omega_B[1]$ for $(\mathcal{S}/B)_\beta^{[n_1,n_2]}$ coincides with the cup product of the Atiyah class and the Kodaira-Spencer class for S.

If the condition (12) *is satisfied, we can define the reduced invariants*

$$\mathsf{N}_S^{\mathrm{red}}(n_1, n_2, \beta; \mathcal{P}) := \int_{[S_\beta^{[n_1,n_2]}]^{\mathrm{vir}}_{\mathrm{red}}} \mathcal{P}.$$

Definition 2.13 (Generalized Poincaré Invariants)**.** *Let* $u := c_1(\mathcal{O}(\mathcal{Z}_\beta))$*. Define*

$$P_S(n_1, n_2, \beta; M) := \det{}_* \left(\left(\pi^*[S_\beta^{[n_1,n_2]}]^{\mathrm{vir}} \cap c_{n_1+n_2}\left(\mathsf{K}_{\beta;M}^{n_1,n_2}\right)\right) \cap \sum_{i\geq 0} u^i \right) \in H^*(\mathrm{Pic}(S)).$$

Remark 2.14. *The invariants* $\mathsf{N}_S^{\mathrm{red}}(0, n_2, \beta; \mathcal{P})$ *recover some of the stable pair invariants of* [KT14] *(see Section 3). In* [GSY17b]*, we express the localized DT invariants of S in terms of the invariants* $\mathsf{N}_S(n_1, n_2, \beta; \mathcal{P})$*. The invariants* $P_S(0, 0, \beta; M)$ *are the Poincaré invariants of* [DKO07] *(see Section 3). As in [ibid], it is interesting to study the properties of these invariants* $P_S(n_1, n_2, \beta; M)$ *such as wall-crossing, blow-up formula etc. This will be pursued in a future work.*

2.3. **Proof of Theorem 1 (Proposition 2.4).** In Section 2.1 we proved Proposition 2.4 in the case $r = 2$. We now use induction on r to prove the theorem in general. For the simplicity of the notation, we show in detail how the result of Section 2.1 can be used to prove Proposition 2.4 in the case $r = 3$. Other induction steps are completely similar and omitted.

Suppose that $\boldsymbol{n} := n_1, n_2, n_3$ is a sequence of nonnegative integers, and $\boldsymbol{\beta} := \beta_1, \beta_2$ is a sequence of effective curve classes in $H^2(S, \mathbb{Z})$. Define $\boldsymbol{n}' := n_1, n_2$. Our goal is to prove the expression in Proposition 2.4 for $r = 3$ is a perfect obstruction theory.

Consider the chain of natural forgetful morphisms and the associated exact triangle of cotangent complexes

$$S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]} \xrightarrow{f_2} S_{\beta_1}^{[\boldsymbol{n}']} \xrightarrow{f_1} S^{[n_1]}, \qquad \mathbb{L}_{f_2}^\bullet[-1] \xrightarrow{j_2} \mathbf{L}f_2^*(\mathbb{L}_{f_1}^\bullet) \xrightarrow{j_1} \mathbb{L}_f^\bullet \xrightarrow{j_3} \mathbb{L}_{f_2}^\bullet \tag{15}$$

where $f := f_1 \circ f_2 = \mathrm{pts} \circ \mathrm{pr}_1$, using the notation at the beginning of Section 2.

Proposition 2.5, provides the relative perfect obstruction theory for the morphism f_1, that we denote by

$$F_{f_1}^\bullet \xrightarrow{\alpha_1} \mathbb{L}_{f_1}^\bullet. \tag{16}$$

Lemma 2.15. *There exists a relative perfect obstruction theory* $F_{f_2}^\bullet \xrightarrow{\alpha_2} \mathbb{L}_{f_2}^\bullet$*, where*

$$F_{f_2}^\bullet = \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}_{\beta_2}^{[n_3]}, \mathrm{coker}(\Phi_2) \otimes \omega_S\right)[1] \tag{17}$$

Proof. The proof is along the line of the proof of Proposition 2.5 (see Step 2 of that proof for the corresponding expression in RHS of (17)). This time the obstruction theory is obtained by the deformation/obstruction theory theory of the universal map $\Phi_2 : \mathcal{I}^{[n_2]} \to \mathcal{I}_{\beta_2}^{[n_3]}$ while the data $(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}, \mathcal{Z}_{\beta_1}, \Phi_1)$ is kept fixed (see Remark 2.6). □

Lemma 2.16. *The obstruction theories $F^\bullet_{f_1}$ and $F^\bullet_{f_2}$ fit into the following commutative diagram:*

$$\begin{array}{ccc} F^\bullet_{f_2}[-1] & \xrightarrow{r} & \mathbf{L}f_2^*(F^\bullet_{f_1}) \\ \downarrow{\scriptstyle \alpha_2[-1]} & & \downarrow{\scriptstyle f_2^*(\alpha_1)} \\ \mathbb{L}^\bullet_{f_2}[-1] & \xrightarrow{j_2} & \mathbf{L}f_2^*(\mathbb{L}^\bullet_{f_1}). \end{array} \tag{18}$$

Proof. **Step 1:** *(Define the map r)* All the maps in diagram (18) except r are already defined above (see (15), (16), and Lemma 2.15). By the universal properties of the Hilbert schemes and using our convention in suppressing the pullback symbols from the universal ideal sheave, we can write

$$\mathbf{L}f_2^*(F^\bullet_{f_1}) = \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}_{\beta_1}, \operatorname{coker}(\Phi_1)\otimes\omega_S[1]\right) \tag{19}$$

Twisting by $\mathcal{O}(\mathcal{Z}_{\beta_1})$, we get $\Phi_2(\mathcal{Z}_{\beta_1}) : \mathcal{I}^{[n_2]}_{\beta_1} \to \mathcal{I}^{[n_3]}_{\beta_2}(\mathcal{Z}_{\beta_1})$, and hence (17) can be written as

$$F^\bullet_{f_2} = \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_3]}_{\beta_2}(\mathcal{Z}_{\beta_1}), \operatorname{coker}(\Phi_2(\mathcal{Z}_{\beta_1}))\otimes\omega_S[1]\right). \tag{20}$$

The chain of maps $\mathcal{I}^{[n_1]} \xrightarrow{\Phi_1} \mathcal{I}^{[n_2]}_{\beta_1} \xrightarrow{\Phi_2(\mathcal{Z}_{\beta_1})} \mathcal{I}^{[n_3]}_{\beta_2}(\mathcal{Z}_{\beta_1})$ induces the natural exact triangle

$$\operatorname{coker}(\Phi_2(\mathcal{Z}_{\beta_1})\circ\Phi_1) \xrightarrow{i_3} \operatorname{coker}(\Phi_2(\mathcal{Z}_{\beta_1})) \xrightarrow{i_2[1]} \operatorname{coker}(\Phi_1)[1]. \tag{21}$$

The maps $i_2[1]$ and $\Phi_2(\mathcal{Z}_{\beta_1})$ induce

$$\begin{aligned} &F^\bullet_{f_2}[-1] = \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_3]}_{\beta_2}(\mathcal{Z}_{\beta_1}), \operatorname{coker}(\Phi_2(\mathcal{Z}_{\beta_1}))\otimes\omega_S\right) \to \\ &\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}_{\beta_1}, \operatorname{coker}(\Phi_2(\mathcal{Z}_{\beta_1}))\otimes\omega_S\right) \to \\ &\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}_{\beta_1}, \operatorname{coker}(\Phi_1)\otimes\omega_S[1]\right) = \mathbf{L}f_2^*(F^\bullet_{f_1}). \end{aligned} \tag{22}$$

The map r in diagram (18) is then defined by composition of two maps in (22).

Step 2: (*Commutativity of diagram (18)*) We start with the following diagram in which the columns are the exact triangles (21) and (15):

$$\begin{array}{ccc}
\operatorname{coker}(\Phi_1)[1] & \xrightarrow{(\mathrm{id}\otimes\Phi_2(\mathcal{Z}_{\beta_1}))\circ \mathrm{At}(\Phi_1)[1]} & \pi^*\mathbf{L}f_2^*\left(\mathbb{L}^\bullet_{f_1}\right)[1]\otimes \mathcal{I}^{[n_3]}_{\beta_2}(\mathcal{Z}_{\beta_1})[1] \\
\uparrow{\scriptstyle i_2[1]} & & \uparrow{\scriptstyle (\pi^* j_2[1]\otimes \mathrm{id})[1]} \\
\operatorname{coker}(\Phi_2(\mathcal{Z}_{\beta_1})) & \xrightarrow{\mathrm{At}(\Phi_2(\mathcal{Z}_{\beta_1}))} & \pi^*\mathbb{L}^\bullet_{f_2}\otimes \mathcal{I}^{[n_3]}_{\beta_2}(\mathcal{Z}_{\beta_1})[1] \\
\uparrow{\scriptstyle i_3} & & \uparrow{\scriptstyle (\pi^* j_3\otimes \mathrm{id})[1]} \\
\operatorname{coker}(\Phi_2(\mathcal{Z}_{\beta_1})\circ\Phi_1) & \xrightarrow{\mathrm{At}(\Phi_2(\mathcal{Z}_{\beta_1})\circ\Phi_1)} & \pi^*\mathbb{L}^\bullet_{f}\otimes \mathcal{I}^{[n_3]}_{\beta_2}(\mathcal{Z}_{\beta_1})[1] \\
\uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle (\pi^* j_1\otimes \mathrm{id})[1]} \\
\operatorname{coker}(\Phi_1) & \xrightarrow{(\mathrm{id}\otimes\Phi_2(\mathcal{Z}_{\beta_1}))\circ \mathrm{At}(\Phi_1)} & \pi^*\mathbf{L}f_2^*\left(\mathbb{L}^\bullet_{f_1}\right)\otimes \mathcal{I}^{[n_3]}_{\beta_2}(\mathcal{Z}_{\beta_1})[1]
\end{array} \tag{23}$$

We prove diagram (23) is commutative. For this, consider the following natural commutative diagrams of sheaf of graded algebras:

$$\begin{array}{ccccc}
\mathcal{O}_{S\times S^{[n_1]}} & \longrightarrow & \mathcal{O}_{S\times S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}}\oplus \mathcal{I}^{[n_1]} & \longrightarrow & \mathcal{O}_{S\times S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}}\oplus \mathcal{I}^{[n_3]}_{\beta_2}(\mathcal{Z}_{\beta_1}) \\
\| & & \uparrow & & \uparrow \\
\mathcal{O}_{S\times S^{[n_1]}} & \longrightarrow & \mathcal{O}_{S\times S^{[\boldsymbol{n}']}_{\beta_1}}\oplus \mathcal{I}^{[n_1]} & \longrightarrow & \mathcal{O}_{S\times S^{[\boldsymbol{n}']}_{\beta_1}}\oplus \mathcal{I}^{[n_2]}_{\beta_1},
\end{array}$$

and

$$\begin{array}{ccccc}
\mathcal{O}_{S\times S^{[\boldsymbol{n}']}_{\beta_1}} & \longrightarrow & \mathcal{O}_{S\times S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}}\oplus \mathcal{I}^{[n_2]}_{\beta_1} & \longrightarrow & \mathcal{O}_{S\times S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}}\oplus \mathcal{I}^{[n_3]}_{\beta_2}(\mathcal{Z}_{\beta_1}) \\
\uparrow & & \uparrow & & \| \\
\mathcal{O}_{S\times S^{[n_1]}} & \longrightarrow & \mathcal{O}_{S\times S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}}\oplus \mathcal{I}^{[n_1]} & \longrightarrow & \mathcal{O}_{S\times S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}}\oplus \mathcal{I}^{[n_3]}_{\beta_2}(\mathcal{Z}_{\beta_1}),
\end{array}$$

where at each vertex of the diagram, the first summand is in degree zero, and the second summand (if any) is in degree 1. Associated to each row of two diagrams above is the exact triangle of transitivity of the relative graded cotangent complexes (see [Ill, IV.2.3]). The vertical maps in the two diagrams above, induce the natural maps between the corresponding cotangent complexes in the exact triangles of transitivity, in such a way that all the resulting squares are commutative. After applying $k^1(-)$, which takes the degree 1 graded piece of a graded object, to the resulting diagrams, we get the commutativity of the following two squares:

$$\begin{array}{ccc}
\operatorname{coker}(\Phi_2(\mathcal{Z}_{\beta_1})\circ\Phi_1) & \longrightarrow & \left(\pi^*\mathbb{L}^\bullet_f\otimes\mathcal{I}^{[n_3]}_{\beta_2}(\mathcal{Z}_{\beta_1})[1]\right)\oplus\mathcal{I}^{[n_2]}_{\beta_1}[1] \\
\uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle (\pi^*j_1\otimes\mathrm{id}\,\oplus\Phi_1)[1]} \\
\operatorname{coker}(\Phi_1) & \longrightarrow & \left(\pi^*\mathbf{L}f_2^*\left(\mathbb{L}^\bullet_{f_1}\right)\otimes\mathcal{I}^{[n_3]}_{\beta_2}(\mathcal{Z}_{\beta_1})[1]\right)\oplus\mathcal{I}^{[n_1]}[1],
\end{array}$$

and

$$\begin{array}{ccc}
\operatorname{coker}(\Phi_2(\mathcal{Z}_{\beta_1})) & \longrightarrow & \left(\pi^*\mathbb{L}^\bullet_{f_2}\otimes\mathcal{I}^{[n_3]}_{\beta_2}(\mathcal{Z}_{\beta_1})[1]\right)\oplus\mathcal{I}^{[n_3]}_{\beta_2}(\mathcal{Z}_{\beta_1})[1] \\
\uparrow{\scriptstyle i_3} & & \uparrow{\scriptstyle (\pi^*j_3\otimes\mathrm{id}\,\oplus\Phi_2(\mathcal{Z}_{\beta_1}))[1]} \\
\operatorname{coker}(\Phi_2(\mathcal{Z}_{\beta_1})\circ\Phi_1) & \longrightarrow & \left(\pi^*\mathbb{L}^\bullet_f\otimes\mathcal{I}^{[n_3]}_{\beta_2}(\mathcal{Z}_{\beta_1})[1]\right)\oplus\mathcal{I}^{[n_2]}_{\beta_1}[1].
\end{array}$$

Here in both squares, the horizontal arrows are parts of the triangles of transitivity (after taking the degree 1 graded pieces)[9]. Now projecting to the first factors in the second columns of the last two diagrams, and using the definition of $\mathrm{At}(-)$ given in Section 2.1, we obtain the commutativity of the bottom and middle squares of diagram (23). Since in diagram (23) both columns are exact triangles, the commutativity of the top square follows, and hence we have proven that the whole diagram (23) commutes.

Recall from Step 2 in the proof of Proposition 2.5, that the maps $\alpha_i : F^\bullet_{f_i}\to\mathbb{L}^\bullet_{f_i}$ are naturally induced from the Atiyah classes $\mathrm{At}(\Phi_1)$ and $\mathrm{At}(\Phi_2(\mathcal{Z}_{\beta_1}))$. Therefore, by the definition of the map r in Step 1 of the proof, the commutativity of diagram (18) is equivalent to the commutativity of the top square in diagram (23) proven above, and hence the proof of lemma is complete.

□

The dashed arrow in the diagram below is now induced by the commutativity of diagram (18) and the property of exact triangles, and it makes the whole diagram of exact triangles commutative:

[9] Here we have used the base change property for the cotangent complexes, as already employed in the footnote of Section 2.1 (see [Ill, II.2.2]), as well as the following isomorphisms

$$k^0\left(\mathbb{L}^{\bullet,\mathrm{gr}}_{(B_0\oplus B_1)/(A_0\oplus A_1)}\right)\cong\mathbb{L}^\bullet_{B_0/A_0},\quad k^1\left(\mathbb{L}^{\bullet,\mathrm{gr}}_{(A_0\oplus C_1)/(A_0\oplus A_1)}\right)\cong\operatorname{coker}(s),$$

where $A_0\oplus A_1\to B_0\oplus B_1$ and $A_0\oplus A_1\xrightarrow{\mathrm{id}\,\oplus s}A_0\oplus C_1$ are the homomorphism of graded $\mathbb{C}$-algebras (summands with index i are in degree i), and furthermore s is injective. These identities follow from [Ill, IV (2.2.4), (2.2.5), (3.2.10)].

$$\begin{array}{ccccc} F^\bullet_{f_2}[-1] & \xrightarrow{r} & \mathbf{L}f_2^*(F^\bullet_{f_1}) & \longrightarrow & \mathrm{Cone}(r) =: F^\bullet_f \\ \downarrow{\scriptstyle \alpha_2[-1]} & & \downarrow{\scriptstyle f_2^*(\alpha_1)} & & \downarrow{\scriptstyle \alpha_3} \\ \mathbb{L}^\bullet_{f_2}[-1] & \xrightarrow{j_2} & \mathbf{L}f_2^*(\mathbb{L}^\bullet_{f_1}) & \xrightarrow{j_1} & \mathbb{L}^\bullet_f, \end{array}$$

where the bottom row is the exact triangle (15). By (19), (20), (21) it is easy to see that

$$F^\bullet_f = \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_3]}_{\beta_2}(\mathcal{Z}_{\beta_1}), \mathrm{coker}(\Phi_2(\mathcal{Z}_{\beta_2}) \circ \Phi_1) \otimes \omega_S[1]\right). \tag{24}$$

Proposition 2.17. $\alpha_3 : F^\bullet_f \to \mathbb{L}^\bullet_f$ *is a relative perfect obstruction theory.*

Proof. By the exact same argument as in Step 1 of the proof of Proposition 2.5, we can see that $F^\bullet_f$ is perfect with amplitude contained in $[-1, 0]$.

By the commutativity of diagram (23) and the naturality of our construction, we can see that α_3 is induced by the Atiyah class $\mathrm{At}(\Phi_2(\mathcal{Z}_{\beta_1}) \circ \Phi_1)$ following the identifications in Step 2 of the proof of Proposition 2.5. Therefore, repeating the arguments in Step 3 of the proof of Proposition 2.5, we can see that α_3 is an obstruction theory. □

Proof of Proposition 2.4 (for $r = 3$). First note that by construction, for $i = 1, 2$,

$$F^{\bullet\vee}_{f_i} = \mathrm{Cone}\left(\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_{i+1}]}, \mathcal{I}^{[n_{i+1}]}\right) \xrightarrow{\Xi_i} \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_{i+1}]}_{\beta_i}\right)\right).$$

Thus, by a similar argument as in the proof of Proposition 2.7 we can see that

$$F^{\bullet\vee}_f = \mathrm{Cone}\Bigg(\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}, \mathcal{I}^{[n_2]}\right) \oplus \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_3]}, \mathcal{I}^{[n_3]}\right)$$
$$\xrightarrow{\begin{pmatrix} \Xi_1 & \Xi_2 \\ \Xi_2' & 0 \end{pmatrix}} \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}_{\beta_1}\right) \oplus \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}, \mathcal{I}^{[n_3]}_{\beta_2}\right)\Bigg).$$

As in the proof of Proposition 2.7, the fact that $S^{[n_1]}$ is nonsingular can be used to show that

$$F^\bullet := \mathrm{Cone}\left(F^\bullet_f \to f^*\Omega_{S^{[n_1]}}[1]\right)[-1]$$

is an absolute perfect obstruction theory for $S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}$, and then (using the expression above for $F^{\bullet\vee}_f$) to prove that

$$F^{\bullet\vee}_f = \mathrm{Cone}\Bigg(\Big[\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_1]}\right) \oplus \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}, \mathcal{I}^{[n_2]}\right) \oplus \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_3]}, \mathcal{I}^{[n_3]}\right)\Big]_0$$
$$\to \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}_{\beta_1}\right) \oplus \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}, \mathcal{I}^{[n_3]}_{\beta_2}\right)\Bigg).$$

□

3. Special cases and proof of Theorem 2

In this section we show that the perfect obstruction theory $F^\bullet$ that we constructed in Proposition 2.7 specializes to several interesting and important cases such as the ones arising from the algebraic Seiberg-Witten theory and the stable pair theory of surfaces:

(1) (Nested Hilbert scheme of points) If $\beta = 0$, the nested Hilbert scheme of points $S^{[n_1\geq n_2]} := S^{[n_1,n_2]}_{\beta=0}$ carries a virtual fundamental class

$$[S^{[n_1\geq n_2]}]^{\mathrm{vir}} \in A_{n_1+n_2}(S^{[n_1\geq n_2]}).$$

Note that by [C98], $S^{[n_1\geq n_2]}$ is nonsingular only in the following two cases:

• $n_1 = n_2$. In this case $S^{[n_1\geq n_2]} \cong S^{[n_1]}$, and $[S^{[n_1]}]^{\mathrm{vir}} = [S^{[n_1]}]$. This is because by Proposition 2.5, $F^\bullet_{\mathrm{rel}} \cong 0$, and so by Proposition 2.7, $F^\bullet \cong \Omega_{S^{[n_1]}}$.

• $n_1 = n_2 + 1$. In this case, since $S^{[n_2+1,n_2]}$ is nonsingular, $h^1(F^{\bullet\vee}) \cong h^1(F^{\bullet\vee}_{\mathrm{rel}}) \cong \mathcal{E}xt^2_\pi\left(\mathcal{I}^{[n_2]}/\mathcal{I}^{[n_2+1]}, \mathcal{I}^{[n_2]}\right)$. Then, we can write

$$[S^{[n_2+1,n_2]}]^{\mathrm{vir}} = [S^{[n_2+1,n_2]}] \cap c_1\left(\mathcal{E}xt^2_\pi\left(\mathcal{I}^{[n_2]}/\mathcal{I}^{[n_2+1]}, \mathcal{I}^{[n_2]}\right)\right),$$

where we have used [BF97, Proposition 5.6] to write $[S^{[n_1\geq n_2]}]^{\mathrm{vir}}$ as the fundamental class capped with the Euler class of the obstruction bundle. Note that $h^1(F^{\bullet\vee})$ is a line bundle by the base change, because for any closed point $(I_{Z_1} \subseteq I_{Z_2}) \in S^{[n_2+1,n_2]}$, we know $I_{Z_2}/I_{Z_1} \cong \mathcal{O}_p$ for some $p \in S$, and by Serre duality

$$\mathrm{Ext}^2_S(I_{Z_2}/I_{Z_1}, I_{Z_2}) = \mathrm{Hom}_S(I_{Z_2}, I_{Z_2}/I_{Z_1}\otimes\omega_X)^* = \mathrm{Hom}_S(I_{Z_2}, \mathcal{O}_p)^* \cong \mathbb{C}.$$

In fact, using the isomorphism $S^{[n_2+1,n_2]} \cong \mathbb{P}(\mathcal{I}^{[n_2]})$, where $\mathbb{P}(\mathcal{I}^{[n_2]}) \xrightarrow{(\rho_1,\rho_2)} S\times S^{[n_2]}$ is the projectivization of the universal ideal sheaf (see [L99, Section 1.2]), we can identify this line bundle with $(\mathcal{O}_{\mathbb{P}}(1)\boxtimes\omega_S)^*$. To see this note that by [L04, Equation (25)] we have $\mathcal{I}^{[n_2]}/\mathcal{I}^{[n_2+1]} \cong \mathcal{O}_{\mathbb{P}}(1)|_Y$ where Y is the pullback of the diagonal under the morphism $(\rho_1, \mathrm{id}) : \mathbb{P}(\mathcal{I}^{[n_2]})\times S \to S\times S$. Using this isomorphism and Grothendieck-Verdier duality we can write the obstruction bundle as

$$\begin{aligned}\mathcal{E}xt^2_\pi\left(\mathcal{I}^{[n_2]}/\mathcal{I}^{[n_2+1]}, \mathcal{I}^{[n_2]}\right) &\cong \mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}, \mathcal{O}_{\mathbb{P}}(1)|_Y\boxtimes\omega_S\right)^* \\ &\cong \pi_*\left(\mathcal{H}om\left(\mathcal{I}^{[n_2]}, \mathcal{O}_Y\right)\boxtimes\mathcal{O}_{\mathbb{P}}(1)\boxtimes\omega_S\right)^*.\end{aligned}$$

Since $\mathcal{H}om\left(\mathcal{I}^{[n_2]}, \mathcal{O}_Y\right) \cong \mathcal{O}_Y$ the claim follows by the projection formula.

• (General case) In Proposition 4.5, in case S is a toric surface, e.g. $S = \mathbb{P}^2$ or $\mathbb{P}^1\times\mathbb{P}^1$, we will express $[S^{[n_1\geq n_2]}]^{\mathrm{vir}}$ as the top Chern class of a rank n_1+n_2 vector bundle over $S^{[n_1]}\times S^{[n_2]}$. In Section 5, we study this case in further details, and relate some specific integrals against $[S^{[n_1\geq n_2]}]^{\mathrm{vir}}$ to integrations over $S^{[n_1]}\times S^{[n_2]}$. In Section 6, we related some of these integrals to Carlsson-Okounkov vertex operators [CO12].

(2) If $n_2 = 0$ and $\beta = 0$, then we get a perfect obstruction theory over the nonsingular Hilbert scheme of points $S^{[n_1]}$ that is arising from the natural obstruction theory of the Hilbert scheme. In fact in this case

$$
\begin{aligned}
F^{\bullet\vee} &\cong \operatorname{Cone}\left(\left[\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]},\mathcal{I}^{[n_1]}\right)\oplus \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{O},\mathcal{O}\right)\right]_0 \to \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]},\mathcal{O}\right)\right)\\
&\cong \operatorname{Cone}\left(\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]},\mathcal{I}^{[n_1]}\right)\to \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]},\mathcal{O}\right)\right)\\
&\cong \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]},\mathcal{O}_{\mathcal{Z}^{[n_1]}}\right).
\end{aligned}
$$

Note that

$$
T_{S^{[n_1]}} = h^0(F^{\bullet\vee}) \cong \mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]},\mathcal{O}_{\mathcal{Z}^{[n_1]}}\right), \quad h^1(F^{\bullet\vee})\cong \mathcal{E}xt^1_\pi\left(\mathcal{I}^{[n_1]},\mathcal{O}_{\mathcal{Z}^{[n_1]}}\right).
$$

Since $S^{[n_1]}$ is nonsingular of dimension $2n_1$, we see that the obstruction sheaf $h^1(F^{\bullet\vee})$ is a vector bundle of rank n_1, and hence by [BF97, Proposition 5.6]

$$
[S^{[n_1]}]^{\mathrm{vir}} = [S^{[n_1]}]\cap c_{n_1}\left(\mathcal{E}xt^1_\pi\left(\mathcal{I}^{[n_1]},\mathcal{O}_{\mathcal{Z}_1}\right)\right).
$$

We were notified by Richard Thomas that the obstruction bundle $\mathcal{E}xt^1_\pi\left(\mathcal{I}^{[n_1]},\mathcal{O}_{\mathcal{Z}_1}\right)$ can be identified with the dual of the tautological bundle $\omega_S^{[n]} := \pi_*\left(\omega_S|_{\mathcal{O}_{\mathcal{Z}_1}}\right)$. This can be seen by applying $\mathcal{H}om\left(\mathcal{O}_{\mathcal{Z}_1},-\right)$ to the short exact sequence

$$
0\to \mathcal{I}^{[n_1]}\otimes\omega_S\to\omega_S\to\omega_S|_{\mathcal{O}_{\mathcal{Z}_1}}\to 0
$$

over $S\times S^{[n_1]}$ to get the isomorphism

$$
\omega_S|_{\mathcal{O}_{\mathcal{Z}_1}}\cong \mathcal{H}om\left(\mathcal{O}_{\mathcal{Z}_1},\omega_S|_{\mathcal{O}_{\mathcal{Z}_1}}\right)\cong \mathcal{E}xt^1\left(\mathcal{O}_{\mathcal{Z}_1},\mathcal{I}^{[n_1]}\otimes\omega_S\right).
$$

Now pushing forward, we prove the claim

$$
\omega_S^{[n]}\cong \pi_*\mathcal{E}xt^1\left(\mathcal{O}_{\mathcal{Z}_1},\mathcal{I}^{[n_1]}\otimes\omega_S\right)\cong \mathcal{E}xt^1_\pi\left(\mathcal{O}_{\mathcal{Z}_1},\mathcal{I}^{[n_1]}\otimes\omega_S\right)\cong \mathcal{E}xt^1_\pi\left(\mathcal{I}^{[n_1]},\mathcal{O}_{\mathcal{Z}_1}\right)^*,
$$

where the second isomorphism is because of local to global spectral sequence (as $\mathcal{Z}_1$ is fiberwise 0-dimensional) and the third one is by Grothendieck-Verdier duality. This fact is used in [TT] in some explicit calculation of the Vafa-Witten invariants.

(3) If $n_1 = n_2 = 0$, and $\beta\neq 0$, the perfect obstruction theory $F^{\bullet\vee}$ on $S_\beta = S_\beta^{[0,0]}$ specializes to

$$
\begin{aligned}
F^{\bullet\vee} &\cong \operatorname{Cone}\left(\left[\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{O},\mathcal{O}\right)\oplus \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{O},\mathcal{O}\right)\right]_0\to \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{O},\mathcal{O}(\mathcal{Z}_\beta)\right)\right)\\
&\cong \mathbf{R}\pi_*\mathcal{O}_{\mathcal{Z}_\beta}(\mathcal{Z}_\beta).
\end{aligned}
$$

studied by Dürr-Kabanov-Okonek [DKO07] in the course of algebraic Seiberg-Witten invariants (Poincaré invariants). Moreover, one can see by inspection that under condition (12), $F^\bullet_{\mathrm{red}}$ coincides with the reduced perfect obstruction theory on S_β constructed in [DKO07].

(4) If $n_2 = 0$, $n_1\neq 0$ and $\beta\neq 0$, then $F^{\bullet\vee}$ gives a perfect obstruction theory over $S_\beta^{[n_1]} = S_\beta^{[n_1,0]}$ generalizing items (2) and (3) above. In this case $F^{\bullet\vee}$ is given by

$$F^{\bullet\vee} \cong \operatorname{Cone}\left(\left[\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_1]}\right) \oplus \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{O}, \mathcal{O}\right)\right]_0 \to \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{O}(\mathcal{Z}_\beta)\right) \right)$$
$$\cong \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{O}_{\mathcal{Z}_\beta^{[n_1]}}(\mathcal{Z}_\beta)\right).$$

(5) If $n_1 = 0$, $n_2 \neq 0$ and $\beta \neq 0$, then by [PT10, Prop B.8],

$$S_\beta^{[0,n_2]} = \operatorname{Hilb}^{n_2}(\mathcal{Z}_\beta/S_\beta) \cong P_{n_2-\beta\cdot(\beta+K_S)/2}(S,\beta),$$

where $\operatorname{Hilb}^{n_2}(\mathcal{Z}_\beta/S_\beta)$ is the relative Hilbert scheme of points on the universal curve $\mathcal{Z}_\beta$, and $P_-(S,-)$ is the moduli space of stable pairs on S. Let $\mathcal{O}_{S\times P} \to \mathbb{F}$ be the universal stable pair over $S \times P_{n_2-\beta\cdot(\beta+K_S)/2}(S,\beta)$, and let $\mathbb{I}^\bullet$ be the associated complex. In this case $F^{\bullet\vee}$ is given by

$$F^{\bullet\vee} \cong \operatorname{Cone}\left(\left[\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{O}, \mathcal{O}\right) \oplus \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}, \mathcal{I}^{[n_2]}\right)\right]_0 \to \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{O}, \mathcal{I}_\beta^{[n_2]}\right) \right)$$
$$\cong \operatorname{Cone}\left(\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}, \mathcal{I}^{[n_2]}\right) \to \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{O}(-\mathcal{Z}_\beta), \mathcal{I}^{[n_2]}\right)\right)$$
$$\cong \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}_{\mathcal{Z}^{[n_2]}\subset\mathcal{Z}_\beta}, \mathcal{I}^{[n_2]}\right)[1] \cong \mathbf{R}\mathcal{H}om_\pi\left(\mathbb{I}^\bullet, \mathbb{F}\right).$$

Here by $\mathcal{I}_{\mathcal{Z}^{[n_2]}\subset\mathcal{Z}_\beta}$ we mean the push-forward of the ideal sheaf of $\mathcal{Z}^{[n_2]}$ as a subscheme of $\mathcal{Z}_\beta \subset S \times S_\beta^{[0,n_2]}$. The last isomorphism above follows from (91) in [KT14]. We have shown that in this case $F^{\bullet\vee}$ coincides with the perfect obstruction theory of the stable pair moduli space $P_{n_2-\beta\cdot(\beta+K_S)/2}(S,\beta)$ studied in [KT14]. Moreover, one can see by inspection that under condition (12), $F^\bullet_{\mathrm{red}}$ coincides with the reduced perfect obstruction theory on $P_{n_2-\beta\cdot(\beta+K_S)/2}(S,\beta)$ constructed in [KT14].

Remark 3.1. *In all the items (2)-(5) the moduli space can be realized as the zero locus of a section of a vector bundle over a smooth ambient space, and the perfect obstruction theory $F^{\bullet\vee}$ in all these cases coincides with the natural obstruction theory associated to the section of the vector bundle. The authors do not know if this is the case for $S_\beta^{[n_1,n_2]}$ in general or even for $S^{[n_1\geq n_2]}$ in item (1) (see Remark 4.6).*

4. Nested Hilbert scheme of points

We will discuss a few tools for evaluating the virtual fundamental class $[S^{[n_1\geq n_2]}]^{\mathrm{vir}}$ constructed in Corollary 2.8. We first develop a localization formula (25) in the case that S is toric along the lines of [MNOP06]. When S is toric we express $\iota_*[S^{[n_1\geq n_2]}]^{\mathrm{vir}}$ as the top Chern class of a vector bundle over the product of Hilbert schemes $S^{[n_1]} \times S^{[n_2]}$ (see Proposition 4.5 and also Remark 4.6). We have not been able to prove such a formula for general projective surfaces. Instead, we prove a weaker statement for general projective surfaces in which the integral of certain cohomology classes against $[S^{[n_1 \geq n_2]}]^{\text{vir}}$ is expressed in terms of integrals over $S^{[n_1]} \times S^{[n_2]}$. This is done in Section 5 using degeneration and the double point relations (see Corollary 5.9, Remark 5.10, Proposition 5.11). Such integrals arise in all the applications that we have in mind, particularly, they are related to the localized DT invariants of S discussed in [GSY17b]. Finally, in Section 6 we express some of these integrals against $[S^{[n_1 \geq n_2]}]^{\text{vir}}$ in terms of Carlsson-Okounkov's vertex operators and as a result obtain explicit product formulas for their generating series.

Recall that

$$S^{[n_1 \geq n_2]} = \{(Z_1, Z_2, \phi) \mid Z_i \in S^{[n_i]},\ \ Z_1 \supseteq Z_2\} \subset S^{[n_1]} \times S^{[n_2]}.$$

For simplicity in this section, we denote by I_i the ideal sheaf I_{Z_i} of Z_i. Hence for any closed point $(Z_1, Z_2) \in S^{[n_1 \geq n_2]}$ we have $I_1 \subseteq I_2$. Sometimes, we denote the closed point above by the pair (I_1, I_2), or by $I_1 \subseteq I_2$, when we want to emphasize the inclusion of subschemes. As before, we have the universal objects over $S \times S^{[n_1 \geq n_2]}$:

$$0 \neq \Phi : \mathcal{I}^{[n_1]} \to \mathcal{I}^{[n_2]}.$$

A direct corollary of Proposition 2.8 is the following:

Corollary 4.1. *For any nonsingular surface S there exists a perfect obstruction theory $F^\bullet$ of rank $n_1 + n_2$, whose derived dual is given by*

$$F^{\bullet\vee} \cong \operatorname{Cone}\left(\left[\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_1]}\right) \oplus \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}, \mathcal{I}^{[n_2]}\right)\right]_0 \to \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}\right)\right).$$

□

We will use the following lemma in Section 4.1:

Lemma 4.2. *1. If $(I_1 \subseteq I_2) \in S^{[n_1, n_2]}$ is a closed point, then*

$$\operatorname{Hom}_S(I_1, I_2) = \operatorname{Hom}_S(I_1, I_1) = \operatorname{Hom}_S(I_2, I_2) = H^0(\mathcal{O}_S) \cong \mathbb{C}.$$

2. If $(I_1, I_2) \in S^{[n_1]} \times S^{[n_2]} \setminus S^{[n_1, n_2]}$ is a closed point then $\operatorname{Hom}_S(I_1, I_2) = 0$.
3. If $p_g(S) = 0$ and if $(I_1, I_2) \in S^{[n_1]} \times S^{[n_2]}$ is a closed point then $\operatorname{Ext}^2_S(I_i, I_j) = 0$.

Proof. Applying the functor $\operatorname{Hom}(I_1, -)$ to the short exact sequence $0 \to I_2 \to \mathcal{O}_S \to \mathcal{O}_{Z_2} \to 0$, we get the exact sequence

$$0 \to \operatorname{Hom}(I_1, I_2) \subseteq \operatorname{Hom}(I_1, \mathcal{O}_S) \cong H^0(\mathcal{O}_S) = \mathbb{C} \xrightarrow{u} \operatorname{Hom}(I_1, \mathcal{O}_{Z_2}),$$

where u composes any map $I_1 \to \mathcal{O}_S$ with the natural map $\mathcal{O}_S \to \mathcal{O}_{Z_2}$. In part 1 the inclusion $I_1 \subseteq I_2$ gives a nonzero element of $\operatorname{Hom}(I_1, I_2)$ and hence the claim follows. In part 2, $u(I_1 \subset \mathcal{O}_S) \neq 0$ because $I_1 \not\subset I_2$, and so the claim is proven. For part 3, applying the functor $\operatorname{Hom}(I_j, -)$ to the short exact sequence $0 \to I_i \otimes \omega_S \to \omega_S \to \mathcal{O}_{Z_i} \to 0$, we get

$$\operatorname{Hom}(I_j, I_i \otimes \omega_S) \subseteq \operatorname{Hom}_S(I_j, \omega_S) = H^0(\omega_S) = 0,$$

and so the claim follows by Serre duality. □

As it will become clear shortly, the following K-group element plays an important role in the rest of the paper:

Definition 4.3. *For any line bundles M on S, let $\mathsf{E}_M^{n_1,n_2} \in K(S^{[n_1]} \times S^{[n_2]})$ be the element of rank n_1+n_2 defined by*

$$\mathsf{E}_M^{n_1,n_2} := [\mathbf{R}\pi'_* p'^* M] - \left[\mathbf{R}\mathcal{H}om_{\pi'}(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]} \otimes p'^* M)\right],$$

where p' and π' are respectively the projections from $S \times S^{[n_1]} \times S^{[n_2]}$ to the first and the product of last two factors (see diagram (4)). Let i be the inclusion of the closed point $(I_1, I_2) \in S^{[n_1]} \times S^{[n_2]}$, then, we define

$$\mathsf{E}_M^{n_1,n_2}|_{(I_1,I_2)} := [\mathbf{L}i^*\mathbf{R}\pi'_* p'^* M] - \left[\mathbf{L}i^*\mathbf{R}\mathcal{H}om_{\pi'}(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]} \otimes p'^* M)\right] \in K(\operatorname{Spec}(\mathbb{C})).$$

If $M = \mathcal{O}$, we sometimes drop it from the notation. We also define the following generating series

$$Z_{\mathrm{prod}}(S, M) := \sum_{n_1 \geq n_2 \geq 0} q_1^{n_1} q_2^{n_2} \int_{S^{[n_1]} \times S^{[n_2]}} c(\mathsf{E}^{n_1,n_2}) \cup c(\mathsf{E}_M^{n_1,n_2}).$$

4.1. **Toric surfaces.** Let $(\mathbb{C}^2)^{[n_1 \geq n_2]}$ be the nested Hilbert scheme of points on $\mathbb{C}^2 = \operatorname{Spec}(R)$, where $R = \mathbb{C}[x_1, x_2]$. The 2-dimensional torus $\mathbf{T}$ acts on $\mathbb{C}^2$. We denote by t_1, t_2 the torus characters, so that the tangent space at $0 \in \mathbb{C}^2$ has the representation $\mathbb{C}t_1^{-1} \oplus \mathbb{C}t_2^{-1}$. The $\mathbf{T}$-fixed set

$$(\mathbb{C}^2)^{[n_1 \geq n_2], \mathbf{T}} \subset (\mathbb{C}^2)^{[n_1], \mathbf{T}} \times (\mathbb{C}^2)^{[n_2], \mathbf{T}}$$

is isolated, and is given by the inclusion of the monomial ideals $I_1 \subseteq I_2$ or equivalently the corresponding nested Young diagrams $\mu_2 \subseteq \mu_1$. By Proposition 2.7 and Lemma 4.4, the virtual tangent space at the $\mathbf{T}$-fixed point $I_1 \subseteq I_2$ is given by[10]

$$\mathcal{T}^{\mathrm{vir}}_{I_1 \subseteq I_2} = -\chi(I_1, I_1) - \chi(I_2, I_2) + \chi(I_1, I_2) + \chi(R, R),$$

where $\chi(-,-) = \sum_{i=0}^{2} (-1)^i \operatorname{Ext}^i_R(-,-)$. By the exact method as in [MNOP06, Section 4.6] using Čech complexes and Taylor resolutions, the $\mathbf{T}$-representation of $\mathcal{T}^{\mathrm{vir}}_{I_1 \subseteq I_2}$ can be explicitly written down as a Laurent polynomial in t_1 and t_2. We skip the details of calculation here. For a $\mathbf{T}$-fixed 0-dimensional subscheme $Z \subset \mathbb{C}^2$, corresponding to the Young diagram μ, we denote by Z the $\mathbf{T}$-character of Z which is given by the polynomial

$$\mathsf{Z} = \sum_{(k_1,k_2) \in \mu} t_1^{k_1} t_2^{k_2}.$$

Also, we define $\overline{\mathsf{Z}} := \mathsf{Z}(t_1^{-1}, t_2^{-2})$. Then, we have

$$\operatorname{tr}_{\mathcal{T}^{\mathrm{vir}}_{I_1 \subseteq I_2}} = \mathsf{Z}_1 + \frac{\overline{\mathsf{Z}}_2}{t_1 t_2} + \left(\overline{\mathsf{Z}}_1 \cdot \mathsf{Z}_2 - \overline{\mathsf{Z}}_1 \cdot \mathsf{Z}_1 - \overline{\mathsf{Z}}_2 \cdot \mathsf{Z}_2\right) \frac{(1-t_1)(1-t_2)}{t_1 t_2}. \tag{25}$$

[10]This is obtained by taking the derived restriction of the complex $F^\bullet$ to the point $I_1 \subseteq I_2$, and then taking the K-group class of the resulting complex. Also note that by slightly modifying the proof of part 1 of Lemma 4.2, $\operatorname{Hom}(I_1, I_1) = \operatorname{Hom}(I_1, I_2) = \operatorname{Hom}(I_2, I_2) = R$.

Now if S is a toric surface, then the set of $\mathbf{T}$-fixed points of $S^{[n_1\geq n_2]}\subset S^{[n_1]}\times S^{[n_2]}$ is again isolated (Lemma 4.4), and the $\mathbf{T}$-character of the virtual tangent space at any fixed point is obtained by summing over the expression (25) for all the $\mathbf{T}$-invariant open subsets of S. This finishes the proof of Theorem 4.

Lemma 4.4. *Suppose that S is a nonsingular projective toric surface, and $Z_2\subseteq Z_1$ is a $\mathbf{T}$-fixed point of $S^{[n_1\geq n_2]}$, then* $\operatorname{Ext}^2_S(I_1,I_1)=\operatorname{Ext}^2_S(I_2,I_2)=\operatorname{Ext}^2_S(I_1,I_2)=0$, *the $\mathbf{T}$-representations*

$$\operatorname{Ext}^1_S(I_1,I_1),\quad \operatorname{Ext}^1_S(I_2,I_2),\quad \operatorname{Ext}^1_S(I_1,I_2)$$

contain no trivial sub-representations.

Proof. The vanishings in the lemma follow from the fact that $p_g(S)=0$ and part 3 of Lemma 4.2. For any fixed point $\alpha\in S$, let $U_\alpha\cong\mathbb{C}^2$ be the $\mathbf{T}$-invariant open neighborhood of α, and let $I_{i,\alpha}=I_i|_\alpha$, and $\mathcal{O}_{i,\alpha}=\mathcal{O}_{Z_i}|_\alpha$. By [ES87, Lemma 3.2], $\operatorname{Hom}_{U_\alpha}(I_{i,\alpha},I_{i,\alpha})$ contains no trivial subrepresentations. Therefore,

$$\operatorname{Ext}^1_S(I_i,I_i)\cong\operatorname{Hom}_S(I_i,\mathcal{O}_{Z_i})=\bigoplus_\alpha\operatorname{Hom}_{U_\alpha}(I_{i,\alpha},\mathcal{O}_{i,\alpha})$$

contains no trivial representations either.

Next, applying $\operatorname{Hom}_S(I_1,-)$ to the natural short exact sequence $0\to I_2\to\mathcal{O}_S\to\mathcal{O}_{Z_2}\to 0$, we obtain the exact sequence

$$\operatorname{Hom}_S(I_1,\mathcal{O}_{Z_2})\to\operatorname{Ext}^1_S(I_1,I_2)\to\operatorname{Ext}^1_S(I_1,\mathcal{O}_S).\tag{26}$$

To finish the proof it suffices to show that the 1st and the 3rd terms in (26) contain no trivial representations. The claim for the 1st term in (26) follows from the natural inclusion $\operatorname{Hom}_S(I_1,\mathcal{O}_{Z_2})\subset\operatorname{Hom}_S(I_2,\mathcal{O}_{Z_2})$ and the first part of the proof. The claim for the 3rd term in (26) also follows because, applying $\operatorname{Hom}_S(-,\mathcal{O}_S)$ to the natural short exact sequence $0\to I_1\to\mathcal{O}_S\to\mathcal{O}_{Z_1}\to 0$, and using equivariant Serre duality, we get

$$\operatorname{Ext}^1_S(I_1,\mathcal{O}_S)\cong\operatorname{Ext}^2_S(\mathcal{O}_{Z_1},\mathcal{O}_S)\cong H^0(\mathcal{O}_{Z_1}\otimes\omega_S)^*.$$

But since Z_1 is zero dimensional and $\mathbf{T}$-fixed

$$H^0(\mathcal{O}_{Z_1}\otimes\omega_S)=\bigoplus_\alpha H^0(U_\alpha,\mathcal{O}_{Z_1}\otimes\omega_S).$$

For each α, let $\mu_{1,\alpha}$ be the Young diagram corresponding to $Z_1|_{U_\alpha}$, and suppose that as $\mathbf{T}$-representation $T_\alpha S\cong\mathbb{C}t_1^{-1}\oplus\mathbb{C}t_2^{-1}$ for some $\mathbf{T}$-characters t_1 and t_2, then, the fiber of ω_S at α has the $\mathbf{T}$-character t_1t_2, and therefore,

$$H^0(U_\alpha,\mathcal{O}_{Z_1}\otimes\omega_S)=t_1t_2\sum_{(k_1,k_2)\in\mu_{1,\alpha}}t_1^{k_1}t_2^{k_2}$$

has no trivial representations. ∎

4.2. **Proof of Theorem 5.** Suppose that S is a toric surface, and $(I_1, I_2) \in S^{[n_1]} \times S^{[n_2]}$ is a closed point. By Lemma 4.4

$$\operatorname{Ext}^2_S(I_i, I_j) = 0. \tag{27}$$

Therefore by the base change, the sheaves

$$\mathcal{E}xt^1_\pi(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_1]}), \quad \mathcal{E}xt^1_\pi(\mathcal{I}^{[n_2]}, \mathcal{I}^{[n_2]}), \quad \mathcal{E}xt^1_\pi(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]})$$

are vector bundles over $S^{[n_1 \geq n_2]}$ of ranks $2n_1, 2n_2, n_1+n_2$, respectively. Moreover, the perfect obstruction theory of Proposition 2.7, simplifies to the 2-term complex

$$F^{\bullet\vee} = \left\{\mathcal{E}xt^1_\pi(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_1]}) \oplus \mathcal{E}xt^1_\pi(\mathcal{I}^{[n_2]}, \mathcal{I}^{[n_2]}) \to \mathcal{E}xt^1_\pi(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]})\right\}. \tag{28}$$

Recall that the rank of $F^{\bullet\vee}$ is equal to $n_1 + n_2$, and recall the K-group elements E^{n_1,n_2} and $\mathsf{E}^{n_1,n_2}|_{(I_1,I_2)}$ of ranks n_1+n_2 from Definition 4.3. We have

$$\mathsf{E}^{n_1,n_2}|_{(I_1,I_2)} = \begin{cases} \operatorname{Ext}^1_S(I_1, I_2) & (I_1, I_2) \in S^{[n_1 \geq n_2]}, \\ H^0(\mathcal{O}_S) \oplus \operatorname{Ext}^1_S(I_1, I_2) & (I_1, I_2) \notin S^{[n_1 \geq n_2]}. \end{cases} \tag{29}$$

This is true because of the base change, the vanishing (27), the vanishing $H^1(\mathcal{O}_S) = H^2(\mathcal{O}_S) = 0$, and that by Lemma 4.2,

$$\operatorname{Hom}_S(I_1, I_2) = \begin{cases} H^0(\mathcal{O}_S) & I_1 \subseteq I_2, \\ 0 & I_1 \nsubseteq I_2. \end{cases}$$

Note that this is consistent with the fact that the dimension of $\operatorname{Ext}^1(I_1, I_2)$ jumps by 1 on $S^{[n_1,n_2]} \subset S^{[n_1]} \times S^{[n_2]}$, and that the dimension of $\mathsf{E}^{n_1,n_2}|_{(I_1,I_2)}$ is constant over $S^{[n_1]} \times S^{[n_2]}$.

Now we are ready to express the main result of this section relating the push forward of $[S^{[n_1 \geq n_2]}]^{\text{vir}}$ to the products of the fundamental classes of Hilbert scheme of points. The following proposition proves Theorem 5.

Proposition 4.5. *Suppose that S is a nonsingular projective toric surface, then,*

$$\iota_*[S^{[n_1 \geq n_2]}]^{\text{vir}} = c_{n_1+n_2}(\mathsf{E}^{n_1,n_2}) \cap [S^{[n_1]} \times S^{[n_2]}],$$

where ι is the natural inclusion $S^{[n_1 \geq n_2]} \hookrightarrow S^{[n_1]} \times S^{[n_2]}$.

Proof. Let i and j be inclusion of the fixed point set in $S^{[n_1 \geq n_2]}$ and $S^{[n_1]} \times S^{[n_2]}$, respectively. By (28) and Lemma 4.4, the virtual localization formula (see [GP99]) gives

$$\begin{aligned} [S^{[n_1 \geq n_2]}]^{\text{vir}} &= \sum_{(I_1 \subseteq I_2) \in S^{[n_1 \geq n_2], \mathbf{T}}} \frac{i_*[(I_1 \subseteq I_2)]}{e(\mathcal{T}^{\text{vir}}_{I_1 \subseteq I_2})} \\ &= \sum_{(I_1 \subseteq I_2) \in S^{[n_1 \geq n_2], \mathbf{T}}} \frac{e(\operatorname{Ext}^1_S(I_1, I_2))}{e(\operatorname{Ext}^1_S(I_1, I_1)) e(\operatorname{Ext}^1_S(I_2, I_2))} i_*[(I_1 \subseteq I_2)], \end{aligned}$$

where the sum is over the isolated $\mathbf{T}$-fixed points, and $e(-)$ indicates the equivariant Euler class. By Lemma 4.4, the coefficient of $i_*[(I_1 \subseteq I_2)]$ in the last sum is the product of the pure nontrivial torus weights. On the other hand, by Lemma 4.4 and the Atiyah-Bott localization formula

$$c_{n_1+n_2}(\mathsf{E}^{n_1,n_2}) \cap [S^{[n_1]} \times S^{[n_2]}] = \sum_{(I_1,I_2)\in S^{[n_1],\mathbf{T}}\times S^{[n_2],\mathbf{T}}} \frac{e(\mathsf{E}^{n_1,n_2}|_{(I_1,I_2)})}{e(T_{(I_1,I_2)}(S^{[n_1]} \times S^{[n_2]}))} j_*[(I_1,I_2)]$$

$$= \sum_{(I_1,I_2)\in S^{[n_1],\mathbf{T}}\times S^{[n_2],\mathbf{T}}} \frac{e(\mathsf{E}^{n_1,n_2}|_{(I_1,I_2)})}{e(\mathrm{Ext}^1_S(I_1,I_1))e(\mathrm{Ext}^1_S(I_2,I_2))} j_*[(I_1,I_2)]$$

$$= \sum_{I_1\subseteq I_2\in S^{[n_1\geq n_2],\mathbf{T}}} \frac{e(\mathrm{Ext}^1_S(I_1,I_2))}{e(\mathrm{Ext}^1_S(I_1,I_1))e(\mathrm{Ext}^1_S(I_2,I_2))} \iota_* \circ i_*[I_1 \subseteq I_2],$$

where the last equality is because of (29), and the fact that since $H^0(\mathcal{O}_S) \cong \mathbb{C}$ is the trivial $\mathbf{T}$-representation, we have $e(H^0(\mathcal{O}_S)) = 0$. The proposition is proven by comparing the outcomes of both localization formulas above. $\square$

Remark 4.6. *Let $S = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, and let ℓ be either a line on $S = \mathbb{P}^2$ or a $(1,1)$-type curve on $S = \mathbb{P}^1 \times \mathbb{P}^1$. Since*

$$\mathrm{Hom}(I_1, I_2(-\ell)) \cong \mathrm{Ext}^2(I_1, I_2(-\ell)) = 0$$

by Serre duality and the stability of the ideal sheaves, by the base change

$$V := \mathcal{E}xt^1_{\pi'}(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}(-\ell))$$

is a vector bundle (independent of the choice of ℓ) of rank $n_1 + n_2$ on $S^{[n_1]} \times S^{[n_2]}$. Define $U_\ell \subset S^{[n_1]} \times S^{[n_2]}$ to be the open locus of $(Z_1, Z_2) \in S^{[n_1]} \times S^{[n_2]}$ where $\ell \cap (Z_1 \cup Z_2) = \emptyset$. Over U_ℓ, we have the natural exact sequence

$$\begin{aligned} 0 \to \mathcal{H}om_{\pi'}(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}) \to \mathcal{H}om_{\pi'}(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}|_{U_\ell\times\ell}) &\xrightarrow{s_\ell} V \\ \to \mathcal{E}xt^1_{\pi'}(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}) &\to \mathcal{E}xt^1_{\pi'}(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}|_{U_\ell\times\ell}) \to 0. \end{aligned}$$

Note that $\mathcal{H}om_{\pi'}(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}) = 0$ because of the vanishing of the fibers $\mathrm{Hom}(I_1, I_2) = 0$ over the open dense locus $(I_1, I_2) \in U_\ell \setminus S^{[n_1\geq n_2]}$ (Lemma 4.2). Also by the definition of U_ℓ, we have $\mathcal{I}^{[n_1]}|_{U_\ell\times\ell} = \mathcal{I}^{[n_2]}|_{U_\ell\times\ell} = \mathcal{O}_{U_\ell\times\ell}$, and since $\mathcal{O}_\ell$ has no higher cohomologies, we see that

$$\mathcal{H}om_{\pi'}(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}|_{U_\ell\times\ell}) \cong \mathcal{O}_{U_\ell}, \qquad \mathcal{E}xt^1_{\pi'}(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}|_{U_\ell\times\ell}) = 0.$$

Thus, the exact sequence above simplifies to

$$0 \to \mathcal{O}_{U_\ell} \xrightarrow{s_\ell} V|_{U_\ell} \to \mathcal{E}xt^1_{\pi'}(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]})|_{U_\ell} \to 0.$$

Since $\mathrm{Rank}(\mathcal{E}xt^1_{\pi'}(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}))$ jumps by one over $S^{[n_1\geq n_2]} \cap U_\ell$ the short exact sequence above suggests that

$$\mathrm{Zero}(s_\ell) = S^{[n_1\geq n_2]} \cap U_\ell, \qquad \mathcal{E}xt^1_{\pi'}(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]})|_{S^{[n_1\geq n_2]}\cap U_\ell} = V|_{S^{[n_1\geq n_2]}\cap U_\ell}.$$

If we could show that the sections s_ℓ glue to a global section of V, we would obtain a geometric proof of Proposition 4.5 when $S = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. Motivated by this observation, in a future work, the first two authors together with Richard Thomas will give a geometric proof of Proposition 4.5 when S is a surface with $p_g(S) = 0$ (not necessarily toric), using the degeneracy loci and Portous' formula.

5. Relative nested Hilbert schemes and double point relation

The goal of this section is to prove Theorem 6. We first develop a degeneration formula for the virtual cycle of the nested Hilbert scheme of points and then we will use double point relation of Lee-Levine-Pandharipande to prove the theorem.

5.1. **Relative nested Hilbert schemes.** Let (S, D) be a pair of nonsingular projective surface and a nonsingular effective divisor. Li and Wu [LW15] introduced the notion of a *stable relative* ideal sheaf. $I \in S^{[n]}$ is said to be relative to D if the natural map

$$I \otimes \mathcal{O}_D \to \mathcal{O}_S \otimes \mathcal{O}_D \tag{30}$$

is injective (see also [MNOPII]). Relativity is an open condition in $S^{[n]}$. Li and Wu constructed a relative Hilbert scheme, denoted by $(S/D)^{[n]}$, by considering the equivalence classes of the stable relative ideal sheaves on the k-step semistable models $S[k]$ for $0 \leq k \leq n$. Let $D_0, \dots, D_{k-1}$ be the singular locus of $S[k]$ and $D_k \subset S[k]$ be the proper transform of D. $S[k]$ consists of $k+1$ irreducible components $\Delta_0, \dots, \Delta_k$ with $\Delta_0 = S$ and $D_i = \Delta_i \cap \Delta_{i+1}$ for $i = 0, \dots, k-1$. A relative ideal sheaf I on $S[k]$ satisfies (30) for $D = D_0, \dots, D_k$. Two relative ideal sheaves I and I' on $S[k]$ are equivalent if the quotients $\mathcal{O}_{S[k]}/I$ and $\mathcal{O}_{S[k]}/I'$ differ by an automorphism of $S[k]$ covering the identity on $\Delta_0 = S$. The stability of a relative ideal sheaf means that it has finitely many auto-equivalences as described above. $(S/D)^{[n]}$ is a smooth proper Deligne-Mumford stack of dimension $2n$.

Since by the relativity condition for any relative ideal sheaf I, $I|_{D_j} \cong \mathcal{O}_{D_j}$, the generalization of Li-Wu Hilbert schemes to the set up of the nested Hilbert schemes is straightforward. In other words, we can construct a proper Deligne-Mumford stack $(S/D)^{[n_1 \geq n_2]}$ as the moduli space of relative ideal sheaves I_1 and I_2 with I_1 stable and $I_1 \subseteq I_2$[11].

Notation. *Following* [LW15]*, let $\mathfrak{A}_\diamond$ be the stack of expanded degenerations for the pair (S, D), and let $\mathcal{S} \to \mathfrak{A}_\diamond$ be the universal family of surfaces over it. Let $(S/D)^{[n_1 \geq n_2]} \to \mathfrak{A}_\diamond$ be the natural morphism; it factors through the substack $\mathfrak{A}_\diamond^{[\boldsymbol{n}]} \subset \mathfrak{A}_\diamond$ corresponding to $\boldsymbol{n}$. We use the same notation as in the absolute case to denote*

[11] Note that if $Z_i \subset S[k]$ is the 0-dimension subscheme corresponding to I_i, then the number of the auto-equivalences of $Z_2 \subseteq Z_1 \subset S[k]$ is less than or equal to that of $Z_1 \subset S[k]$, which is finite by the stability of I_1.

the inclusion of the universal objects over $\mathcal{S} \times_{\mathfrak{A}_\diamond^{[n]}} (S/D)^{[\boldsymbol{n}]}$:

$$0 \neq \Phi : \mathcal{I}^{[n_1]} \to \mathcal{I}^{[n_2]}.$$

Let π be the projection to the second factor of $\mathcal{S} \times_{\mathfrak{A}_\diamond^{[n]}} (S/D)^{[\boldsymbol{n}]}$, and p be the projection to its first factor followed by the natural map $\mathcal{S} \to S$.

By the method of [MPT10, Section 3.9] and [LW15], one can see, after modifying our argument for the usual nested Hilbert schemes (Proposition 2.7), that the following defines a perfect obstruction theory $\mathcal{F}^\bullet_{\mathrm{rel}}$ over $(S/D)^{[n_1 \geq n_2]}$ relative to $\mathfrak{A}_\diamond^{[\boldsymbol{n}]}$:

$$\mathcal{F}^{\bullet\vee}_{\mathrm{rel}} := \mathrm{Cone}\Bigg(\left[\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_1]}\right) \oplus \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}, \mathcal{I}^{[n_2]}\right)\right]_0 \to \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}\right)\Bigg), \tag{31}$$

where $[-]_0$ means the trace-free part. We denote the corresponding virtual fundamental class by

$$[(S/D)^{[n_1 \geq n_2]}]^{\mathrm{vir}} \in A_{n_1+n_2}((S/D)^{[n_1 \geq n_2]}).$$

Let $S \rightsquigarrow S_0 := S_1 \cup_D S_2$ be a good degeneration of the surface S along D over a pointed curve $(C, 0)$, and let

$$\mathfrak{S} \to \mathfrak{C} \to C \tag{32}$$

be the universal family of surfaces over the stack of expanded degenerations $\mathfrak{C}$ (see [L01, L02, LW15]). Following the construction of Li and Wu [LW15], one can construct the nested Hilbert scheme of points, denoted by $\mathfrak{S}^{[n_1 \geq n_2]}$ on the fibers of $\mathfrak{S}$. The Hilbert scheme $\mathfrak{S}^{[n_1 \geq n_2]}$ is a proper Deligne-Mumford stack over $\mathfrak{C}$ and its structure morphism factors through the substack $\mathfrak{C}^{[\boldsymbol{n}]} \subset \mathfrak{C}$ corresponding to $\boldsymbol{n}$. A non-special fiber of $\mathfrak{S}^{[n_1 \geq n_2]}$ is isomorphic to $S^{[n_1 \geq n_2]}$, whereas the special fiber of $\mathfrak{S}^{[n_1 \geq n_2]}$, denoted by $S_0^{[n_1 \geq n_2]}$, can be written as the (non-disjoint) union

$$S_0^{[n_1 \geq n_2]} = \bigcup_{\boldsymbol{n} = \boldsymbol{n}' + \boldsymbol{n}''} (S_1/D)^{[n_1' \geq n_2']} \times (S_2/D)^{[n_1'' \geq n_2'']}, \tag{33}$$

where $\boldsymbol{n}' = (n_1', n_2')$ and $\boldsymbol{n}'' = (n_1'', n_2'')$ with $n_1' \geq n_2'$ and $n_1'' \geq n_2''$. Each component

$$S^{[\boldsymbol{n}]}_{0,\boldsymbol{n}',\boldsymbol{n}''} := (S_1/D)^{[n_1' \geq n_2']} \times (S_2/D)^{[n_1'' \geq n_2'']}$$

is the pull-back of a divisor $\mathfrak{C}^{[\boldsymbol{n}]}_{\boldsymbol{n}',\boldsymbol{n}''} \subset \mathfrak{C}^{[\boldsymbol{n}]}$. Let $\mathfrak{L}^{[\boldsymbol{n}]}_{\boldsymbol{n}',\boldsymbol{n}''}$ be the corresponding line bundle. We then have

$$\bigotimes_{\boldsymbol{n}=\boldsymbol{n}'+\boldsymbol{n}''} \mathfrak{L}^{[\boldsymbol{n}]}_{\boldsymbol{n}',\boldsymbol{n}''} \cong \mathfrak{L}_0$$

where $\mathfrak{L}_0$ is the line bundle associated to the pull back of the divisor $\{0\} \subset C$.

We denote the universal objects over $\mathfrak{S} \times_{\mathfrak{C}^{[\boldsymbol{n}]}} \mathfrak{S}^{[n_1 \geq n_2]}$ by

$$0 \neq \Phi : \mathfrak{I}^{[n_1]} \to \mathfrak{I}^{[n_2]}.$$

The restriction of Φ to the component $\mathfrak{S} \times_{\mathfrak{C}^{[\boldsymbol{n}]}} \left((S_1/D)^{[n_1' \geq n_2']} \times (S_2/D)^{[n_1'' \geq n_2'']}\right)$ is identified with the pair of universal maps

$$(\mathcal{I}^{[n_1']} \hookrightarrow \mathcal{I}^{[n_2']}, \mathcal{I}^{[n_1'']} \hookrightarrow \mathcal{I}^{[n_2'']}).$$

We denote by π the projection to the second factor of $\mathfrak{S} \times_{\mathfrak{C}^{[\boldsymbol{n}]}} \mathfrak{S}^{[n_1 \geq n_2]}$, and by p the projection to its first factor followed by the natural morphism to the total space of the good degeneration of S over C. Again by the method of Section 2.1 and [MPT10, LW15], one can construct the following perfect obstruction theory $\mathfrak{F}^\bullet_{\text{rel}}$ over $\mathfrak{S}^{[n_1 \geq n_2]}$ relative to $\mathfrak{C}^{[\boldsymbol{n}]}$:

$$\mathfrak{F}^{\bullet\vee}_{\text{rel}} =$$
$$\operatorname{Cone}\left(\left[\mathbf{R}\mathcal{H}om_\pi\left(\mathfrak{I}^{[n_1]}, \mathfrak{I}^{[n_1]}\right) \oplus \mathbf{R}\mathcal{H}om_\pi\left(\mathfrak{I}^{[n_2]}, \mathfrak{I}^{[n_2]}\right)\right]_0 \to \mathbf{R}\mathcal{H}om_\pi\left(\mathfrak{I}^{[n_1]}, \mathfrak{I}^{[n_2]}\right) \right).$$

Let $\mathfrak{F}^\bullet$ be the corresponding absolute perfect obstruction theory (see [MPT10]). The restriction of $\mathfrak{F}^\bullet_{\text{rel}}$ to $S_0^{[\boldsymbol{n}]}$ and its components $S^{[\boldsymbol{n}]}_{0,\boldsymbol{n}',\boldsymbol{n}''}$ induces perfect obstruction theories denoted by $\mathfrak{F}^\bullet_0$ and $\mathfrak{F}^\bullet_{0,\boldsymbol{n}',\boldsymbol{n}''}$, respectively. As in [MPT10], they satisfy the following compatibilities:

$$\mathfrak{F}^\bullet|_{S_0^{[\boldsymbol{n}]}} \to \mathfrak{F}^\bullet_0 \to \mathfrak{L}_0^\vee[1], \qquad \mathfrak{F}^\bullet|_{S^{[\boldsymbol{n}]}_{0,\boldsymbol{n}',\boldsymbol{n}''}} \to \mathfrak{F}^\bullet_{0,\boldsymbol{n}',\boldsymbol{n}''} \to \mathfrak{L}^{[\boldsymbol{n}]\vee}_{\boldsymbol{n}',\boldsymbol{n}''}[1], \tag{34}$$

where each sequence is an exact triangle.

A decomposition $S_0[k_1,k_2] := S_1[k_1] \cup_D S_2[k_2]$ yields the natural exact sequence

$$0 \to \mathcal{O}_{S_0[k_1,k_2]} \to \mathcal{O}_{S_1[k_1]} \oplus \mathcal{O}_{S_2[k_2]} \to \mathcal{O}_D \to 0.$$

Suppose that $I_1 \subseteq I_2$ is a nested pairs of relative ideal sheaves on $S_0[k_1,k_2]$, and let $I_i' := I_i|_{S_1[k_1]}$ and $I_i'' := I_i|_{S_2[k_2]}$. Using this and the relativity condition of ideal sheaves, one can conclude

$$\begin{aligned}
&\operatorname{Cone}\Big([\mathbf{R}\operatorname{Hom}(I_1,I_1) \oplus \mathbf{R}\operatorname{Hom}(I_2,I_2)]_0 \to \mathbf{R}\operatorname{Hom}(I_1,I_2) \Big) \cong \\
&\operatorname{Cone}\Big([\mathbf{R}\operatorname{Hom}(I_1',I_1') \oplus \mathbf{R}\operatorname{Hom}(I_2',I_2')]_0 \to \mathbf{R}\operatorname{Hom}(I_1',I_2') \Big) \bigoplus \\
&\operatorname{Cone}\Big([\mathbf{R}\operatorname{Hom}(I_1'',I_1'') \oplus \mathbf{R}\operatorname{Hom}(I_2'',I_2'')]_0 \to \mathbf{R}\operatorname{Hom}(I_1'',I_2'') \Big).
\end{aligned}$$

One of the upshots is that following the construction of [MPT10, LW15], we are led by the identity above to the following degeneration formula for the virtual integration over $S^{[n_1 \geq n_2]}$ ([MPT10, Thm. 16], [LW15, Prop. 6.5, Thm. 6.6]). This is done by using the compatibilities (34) and relating the relative prefect obstruction theory $\mathfrak{F}^\bullet_{\text{rel}}$ to the absolute perfect obstruction theories $F^\bullet$ (given in Proposition 2.7) and $\mathcal{F}^\bullet_{\text{rel}}$ (given by (31)):

Proposition 5.1. *Let α be a cohomology class in the total space of $\mathfrak{S}^{[n_1\geq n_2]}$, then,*

$$\int_{[S^{[n_1\geq n_2]}]^{\mathrm{vir}}}\alpha=\sum_{\boldsymbol{n}=\boldsymbol{n}'+\boldsymbol{n}''}\left(\int_{[(S_1/D)^{[n_1'\geq n_2']}]^{\mathrm{vir}}}\alpha\right)\cdot\left(\int_{[(S_2/D)^{[n_1'\geq n_2']}]^{\mathrm{vir}}}\alpha\right).$$

□

Remark 5.2. *In Proposition 5.1, if $n=n_1=n_2$ then $\mathfrak{S}^{[n_1\geq n_2]}\cong\mathfrak{S}^{[n]}$ constructed by* [LW15]*, and by the same argument as in proof of Theorem 2 part 1, one can recover the usual degeneration formula for the Hilbert schemes of points used in* [T12, LT14, GS16]*:*

$$\int_{S^{[n]}}\alpha=\sum_{n=n'+n''}\left(\int_{(S_1/D)^{[n']}}\alpha\right)\cdot\left(\int_{[(S_2/D)^{[n'']}}\alpha\right). \tag{35}$$

5.2. **Double point relation and proof of Theorem 6.** Let (S,D) be a pair of nonsingular projective surface and a nonsingular effective divisor as in Section 5.1, and let M be line bundle on S. Also, recall the definitions of

$$\pi:\mathcal{S}\times_{\mathfrak{A}_\diamond^{[\boldsymbol{n}]}}(S/D)^{[\boldsymbol{n}]}\to(S/D)^{[\boldsymbol{n}]},\qquad p:\mathcal{S}\times_{\mathfrak{A}_\diamond^{[\boldsymbol{n}]}}(S/D)^{[\boldsymbol{n}]}\to S.$$

Let $\mathcal{D}\subset\mathcal{S}$ be the proper transform of $D\subset S$ via p. Note that we use π and p for the similar natural morphisms from $\mathfrak{S}\times_{\mathfrak{C}^{[\boldsymbol{n}]}}\mathfrak{S}^{[n_1\geq n_2]}$ as well.

Definition 5.3. *Define the following element in $K((S/D)^{[n_1\geq n_2]})$ of rank n_1+n_2:*

$$\mathsf{K}_M^{[n_1\geq n_2]}:=[\mathbf{R}\pi_*p^*M]-\left[\mathbf{R}\mathcal{H}om_\pi(\mathcal{I}^{[n_1]},\mathcal{I}^{[n_2]}\otimes p^*M)\right].$$

Define the following generating series:

$$Z_{\mathrm{nest}}(S/D,M):=\sum_{n_1\geq n_2\geq 0}q_1^{n_1}q_2^{n_2}\int_{[(S/D)^{[n_1\geq n_2]}]^{\mathrm{vir}}}c(\mathsf{K}_M^{[n_1\geq n_2]}).$$

If $D=0$ we drop it from the notation.

Lemma 5.4. *Given a good degeneration $S\rightsquigarrow S_0:=S_1\cup_D S_2$, and the degeneration of line bundles*

$$\mathrm{Pic}(S)\ni M\rightsquigarrow M_i\in\mathrm{Pic}(S_i)\quad i=1,2,$$

we get the degeneration of the class $c(\mathsf{K}_M^{[n_1\geq n_2]})$ whose restriction to the component

$$(S_1/D)^{[n_1'\geq n_2']}\times(S_2/D)^{[n_1''\geq n_2'']}$$

of the central fiber of $\mathfrak{S}^{[n_1\geq n_2]}$ is $c(\mathsf{K}_{M_1}^{[n_1'\geq n_2']})\boxtimes c(\mathsf{K}_{M_2}^{[n_1''\geq n_2'']})$.

Proof. Let $\mathcal{M}$ be the line bundle over the total space of the good degeneration of S that gives the degeneration of M as in the lemma. The derived pullbacks of the perfect complexes $\mathbf{R}\mathcal{H}om_\pi(\mathfrak{I}^{[n_1]}, \mathfrak{I}^{[n_2]} \otimes p^*\mathcal{M})$ and $\mathbf{R}\pi_* p^*\mathcal{M}$ to the component $(S_1/D)^{[n_1' \geq n_2']} \times (S_2/D)^{[n_1'' \geq n_2'']}$ fits in the exact triangles

$$\begin{aligned}
&\mathbf{R}\mathcal{H}om_\pi(\mathfrak{I}^{[n_1]}, \mathfrak{I}^{[n_2]} \otimes p^*\mathcal{M}) \\
&\to \mathbf{R}\mathcal{H}om_\pi(\mathcal{I}^{[n_1']}, \mathcal{I}^{[n_2']} \otimes p^* M_1) \oplus \mathbf{R}\mathcal{H}om_\pi(\mathcal{I}^{[n_1'']}, \mathcal{I}^{[n_2'']} \otimes p^* M_2) \\
&\to \mathbf{R}\mathcal{H}om_\pi(\mathcal{O}_\mathcal{D}, \mathcal{O}_\mathcal{D} \otimes p^* M|_D) \cong \mathbf{R}\pi_* p^* M|_D,
\end{aligned}$$

and $\mathbf{R}\pi_* p^*\mathcal{M} \to \oplus_{i=1}^2 \mathbf{R}\pi_* p^* M_i \to \mathbf{R}\pi_* p^* M|_D$. Now taking the difference of the K-group classes from the exact triangles above, and applying the total Chern class, we conclude that $c(\mathsf{K}_M^{[n_1 \geq n_2]})$ degenerates to a class whose restriction to the component $(S_1/D)^{[n_1' \geq n_2']} \times (S_2/D)^{[n_1'' \geq n_2'']}$ is $c(\mathsf{K}_{M_1}^{[n_1' \geq n_2']} \boxtimes \mathsf{K}_{M_2}^{[n_1'' \geq n_2'']})$.

□

A direct corollary of Proposition 5.1 and Lemma 5.4 is

Proposition 5.5. *Given a good degeneration $S \rightsquigarrow S_0 := S_1 \cup_D S_2$, and the degeneration of line bundles*

$$\mathrm{Pic}(S) \ni M \rightsquigarrow M_i \in \mathrm{Pic}(S_i) \quad i = 1, 2,$$

we have

$$Z_{\mathrm{nest}}(S, M) = Z_{\mathrm{nest}}(S_1/D, M_1) \cdot Z_{\mathrm{nest}}(S_2/D, M_2). \tag{36}$$

□

In the situation of Proposition 5.5, Let $\mathbb{P}$ be either of the projective bundles $\mathbb{P}(\mathcal{O}_D + N_{S_1/D}) \cong \mathbb{P}(\mathcal{O}_D + N_{S_2/D})$, and let $M_\mathbb{P}$ be the pullback of $M|_D$ to $\mathbb{P}$. Applying Proposition 5.5 to the degeneration to the normal cone of $D \subset S_i$ gives

$$Z_{\mathrm{nest}}(S_i, M_i) = Z_{\mathrm{nest}}(S_i/D, M_i) \cdot Z_{\mathrm{nest}}(\mathbb{P}/D, M_\mathbb{P}). \tag{37}$$

Similarly, the degeneration to the normal cone of $D \subset \mathbb{P}$ gives

$$Z_{\mathrm{nest}}(\mathbb{P}, M_\mathbb{P}) = Z_{\mathrm{nest}}(\mathbb{P}/D, M_\mathbb{P}) \cdot Z_{\mathrm{nest}}(\mathbb{P}/D, M_\mathbb{P}). \tag{38}$$

Let $\mathcal{M}_{2,1}(\mathbb{C})^+$ be the group completion of the set of isomorphism classes of the pairs (S, M), where S is a smooth projective surface over $\mathbb{C}$ and M is a line bundle on S (see [LP12, Definition 3]).

Corollary 5.6. *$Z_{\mathrm{nest}}(-,-)$ satisfies the relation*[12]

$$Z_{\mathrm{nest}}(S, M) \cdot Z_{\mathrm{nest}}(S_1, M_1)^{-1} \cdot Z_{\mathrm{nest}}(S_2, M_2)^{-1} \cdot Z_{\mathrm{nest}}(\mathbb{P}, M_\mathbb{P}) = 1 \tag{39}$$

and hence it respects the double point relations in $\mathcal{M}_{2,1}(\mathbb{C})^+$. In other words, $Z_{\mathrm{nest}}(-,-)$ descends to a homomorphism

$$Z_{\mathrm{nest}}(-,-) : \omega_{2,1}(\mathbb{C}) \otimes_\mathbb{Z} \mathbb{Q} \to \mathbb{Q}[[q_1, q_2]]^*,$$

where $\omega_{2,1}(\mathbb{C})$ is the double point cobordism theory for line bundles on surfaces obtained by taking the quotient of $\mathcal{M}_{2,1}(\mathbb{C})^+$ by all the double point relations.

[12]This relation is the analog of the relation (0.10) in [LP09].

Proof. Relation (39) follows immediately from relations (36)-(38). ☐

It is known that $\omega_{2,1}(\mathbb{C})$ is generated by the following classes (see [T12, LP12])

$$[\mathbb{P}^2, \mathcal{O}], \quad [\mathbb{P}^2, \mathcal{O}(1)], \quad [\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}], \quad [\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,0)]. \tag{40}$$

Let B_1, B_2, B_3, B_4 be $Z_{\text{nest}}(S, M)$ where (S, M) is one of the pairs above respectively from left to right. Define

$$A_1 := B_1^{-1} B_2 B_3^{3/2} B_4^{-3/2}, \; A_2 := B_3^{1/2} B_4^{-1/2}, \; A_3 := B_1^{-1/3} B_3^{-1/4}, \; A_4 := B_1^{-2/3} B_3^{3/4}.$$

Proposition 5.7. *Let S be a nonsingular projective surface and M be a line bundle on S. Let $A_1, A_2, A_3, A_4 \in \mathbb{Q}[[q_1, q_2]]^*$ be defined as above (independent of S) then*

$$Z_{\text{nest}}(S, M) = A_1^{M^2} A_2^{M \cdot K_S} A_3^{K_S^2} A_4^{c_2(S)}.$$

Proof. By Remark 2.3 and [CO12, Eq. (5)],

$$c(\mathsf{E}_M^{n_1,n_2})|_{S^{[n_1 \geq n_2]}} = c(\mathsf{K}_M^{[n_1 \geq n_2]}), \qquad c_i(\mathsf{E}_M^{n_1,n_2}) = 0 \text{ for } i > n_1 + n_2,$$

and so by Proposition 4.5, we have $Z_{\text{nest}}(S, M) = Z_{\text{prod}}(S, M)$ if (S, M) is one of the generators (40). For a general (S, M) as in the proposition we can express the class $[S, M] \in \omega_{2,1}(\mathbb{C})$ as a linear combination of the generators (40), with coefficients are as in the proof of [T12, Proposition 4.1]. The result then follows by applying the homomorphism $Z_{\text{nest}}(-,-)$ of Corollary 5.6 and then rearranging the factors as in the proof of [T12, Proposition 4.1]. ☐

Corollary 5.8. *Let S be a nonsingular projective surface and M be a line bundle on S. Then the invariant*

$$\int_{[S^{[n_1 \geq n_2]}]} c(\mathsf{K}_M^{[n_1 \geq n_2]})$$

can be written as a degree $n_1 + n_2$ universal polynomial in $M^2, M \cdot K_S, K_S^2, c_2(S)$.

Proof. The integral in the proposition is the coefficient of $q_1^{n_1} q_2^{n_2}$ in $Z_{\text{nest}}(S, M)$. The result follows after expanding the right hand side of the formula in Proposition 5.7 and extracting the coefficient of $q_1^{n_1} q_2^{n_2}$. ☐

Corollary 5.9. *For any nonsingular projective surface S, and $M \in \operatorname{Pic}(S)$*

$$\int_{[S^{[n_1 \geq n_2]}]^{\text{vir}}} c(\mathsf{K}_M^{[n_1 \geq n_2]}) = \int_{S^{[n_1]} \times S^{[n_2]}} c_{n_1+n_2}(\mathsf{E}^{n_1,n_2}) \cup c(\mathsf{E}_M^{n_1,n_2}).$$

Proof. By Corollary 5.9 we know that the LHS of the corollary is a universal polynomial P_1 in $M^2, M \cdot K_S, K_S^2, c_2(S)$. On the other hand, using Grothendieck-Riemann-Roch formula and the induction scheme of Ellingsrud, Göttsche, and Lehn (see [EGL99, Sections 3, 4] and [CO12, Section 3]), we can express the RHS of the corollary in terms of a universal polynomial P_2 in $M^2, M \cdot K_S, K_S^2, c_2(S)$. But since the equality in the corollary holds for any (S, M) in which S is toric, we conclude that $P_1 = P_2$, and the result follows.

☐

Remark 5.10. *Suppose that* $\alpha_M^{[n_1\geq n_2]}$ *is a cohomology class in* $H^*(S^{[n_1\geq n_2]})$ *that can be defined universally for any pair of* (S,M) *as above and any* $n_1\geq n_2$*, and which is well behaved under good degenerations i.e. the restriction of the degeneration of* $\alpha_M^{[n_1\geq n_2]}$ *to the component* $(S_1/D)^{[n_1'\geq n_2']}\times(S_2/D)^{[n_1''\geq n_2'']}$ *of the central fiber of* $\mathfrak{S}^{[n_1\geq n_2]}$ *is* $\alpha_{M_1}^{[n_1'\geq n_2']}\boxtimes\alpha_{M_2}^{[n_1''\geq n_2'']}$ *for any good degenerations*

$$S\rightsquigarrow S_0:=S_1\cup_D S_2,\quad \operatorname{Pic}(S)\ni M\rightsquigarrow M_i\in\operatorname{Pic}(S_i)\quad i=1,2.$$

Lemma 5.4 shows that $c(\mathsf{K}_M^{[n_1\geq n_2]})$ *is an example of such a class* $\alpha_M^{[n_1\geq n_2]}$*. Then, all the above arguments can be applied more generally to* $\alpha_M^{[n_1\geq n_2]}$ *to conclude as in Corollary 5.8,*

$$\int_{[S^{[n_1\geq n_2]}]^{\mathrm{vir}}}\alpha_M^{[n_1\geq n_2]}$$

is a universal polynomial in $M^2, M\cdot K_S, K_S^2, c_2(S)$*. Now suppose that* $\alpha_M^{n_1,n_2}$ *is a cohomology class in* $H^*(S^{[n_1]}\times S^{[n_2]})$ *that can be defined universally for any pair of* (S,M) *and any* n_1,n_2 *with the property that if* $n_1\geq n_2$ *then*

$$\alpha_M^{n_1,n_2}|_{S^{[n_1\geq n_2]}}=\alpha_M^{[n_1\geq n_2]}.$$

For example, $\alpha_M^{n_1,n_2}=c(\mathsf{E}_M^{n_1,n_2})$ *has this property with respect to* $\alpha_M^{[n_1\geq n_2]}=c(\mathsf{K}_M^{[n_1\geq n_2]})$*. Then, all the argument leading to Corollary 5.9 can be repeated with no changes with the classes* $\alpha_M^{[n_1\geq n_2]}$ *and* $\alpha_M^{n_1,n_2}$ *to conclude*

$$\int_{[S^{[n_1\geq n_2]}]^{\mathrm{vir}}}\alpha_M^{[n_1\geq n_2]}=\int_{S^{[n_1]}\times S^{[n_2]}}c_{n_1+n_2}(\mathsf{E}^{n_1,n_2})\cup\alpha_M^{n_1,n_2}.$$

We can extend Corollary 5.9 to the following more general statement:

Proposition 5.11. *Let* $M_1,\dots,M_s$ *and* $N_1,\dots,N_t$ *be some line bundles on the nonsingular projective surface* S*. Then,*

$$\int_{[S^{[n_1\geq n_2]}]^{\mathrm{vir}}}\frac{c(\mathsf{K}_{M_1}^{[n_1\geq n_2]})\cup\cdots\cup c(\mathsf{K}_{M_s}^{[n_1\geq n_2]})}{c(\mathsf{K}_{N_1}^{[n_1\geq n_2]})\cup\cdots\cup c(\mathsf{K}_{N_t}^{[n_1\geq n_2]})}=$$
$$\int_{S^{[n_1]}\times S^{[n_2]}}c_{n_1+n_2}(\mathsf{E}^{n_1,n_2})\cup\frac{c(\mathsf{E}_{M_1}^{n_1,n_2})\cup\cdots\cup c(\mathsf{E}_{M_s}^{n_1,n_2})}{c(\mathsf{E}_{N_1}^{n_1,n_2})\cup\cdots\cup c(\mathsf{E}_{N_t}^{n_1,n_2})}$$

Proof. This follows by noting that the integrands in both sides satisfy the requirements of Remark 5.10. □

5.A. **Appendix: Another proof of Corollary 5.9.** In this appendix we only sketch another proof of Corollary 5.9 that uses the degeneration of the products of the Hilbert schemes (which could be interesting on its own) instead of the induction scheme of [EGL99]. Consider the product of Li-Wu Hilbert scheme of points $\mathfrak{S}^{[n_1]}\times_C\mathfrak{S}^{[n_2]}\to\mathfrak{C}^{[n_1]}\times_C\mathfrak{C}^{[n_2]}\to C$. As before, a non-special fiber of

$\mathfrak{S}^{[n_1]} \times_C \mathfrak{S}^{[n_2]}$ is isomorphic to $S^{[n_1]} \times S^{[n_2]}$, whereas the special fiber can be written as

$$\bigcup_{\substack{n_1 = n_1' + n_1'' \\ n_2 = n_2' + n_2''}} \left((S_1/D)^{[n_1']} \times (S_2/D)^{[n_1'']}\right) \times \left((S_1/D)^{[n_2']} \times (S_2/D)^{[n_2'']}\right). \tag{41}$$

Each component is the pull-back of a divisor with the associated line bundle $\mathfrak{L}^{[n_1]}_{n_1',n_1''} \boxtimes \mathfrak{L}^{[n_2]}_{n_2',n_2''}$ such that

$$\bigotimes_{\substack{n_1 = n_1' + n_1'' \\ n_2 = n_2' + n_2''}} \mathfrak{L}^{[n_1]}_{n_1',n_1''} \boxtimes \mathfrak{L}^{[n_2]}_{n_2',n_2''} \cong \mathfrak{L}_0.$$

By the same discussions in Section 5.1 and using (35), it is straightforward to prove a degeneration formula for the product of the Hilbert scheme of points:

Proposition 5.12. *Let γ be a cohomology class in the total space of $\mathfrak{S}^{[n_1]} \times_C \mathfrak{S}^{[n_2]}$, then,*

$$\int_{S^{[n_1]} \times S^{[n_2]}} \gamma = \sum_{\substack{n_1 = n_1' + n_1'' \\ n_2 = n_2' + n_2''}} \left(\int_{[(S_1/D)^{[n_1']} \times (S_1/D)^{[n_2']}]} \gamma\right) \cdot \left(\int_{[(S_2/D)^{[n_1'']} \times (S_2/D)^{[n_2'']}]} \gamma\right).$$

□

We also need to give a relative version of Definition 4.3, but this is not straightforward as the underlaying surfaces for the relative ideal sheaves $I_1 \in (S/D)^{[n_1]}$ and $I_2 \in (S/D)^{[n_2]}$ could be different semistable models of S. We proceed to define an auxiliary common underlying surface for I_1 and I_2 in a systematic way as follows. Let $\mathfrak{H}^{[n_1,n_2]}$ be the moduli stack of pairs of relative ideal sheaves (I_1, I_2) on $S[k]$ of co-lengths (n_1, n_2) such that if $k_i \leq k$ is the largest integer with I_i stable over $S[k_i]$, then, $I_i|_{\Delta_{k'}} = \mathcal{O}_{\Delta_{k'}}$ for $k \geq k' > k_i$, and moreover, at least one of k_1 or k_2 is equal to k. The pairs (I_1, I_2) and (I_1', I_2') on $S[k]$ are equivalent if $\sigma_i^* \mathcal{O}_{S[k]}/I_i' \cong \mathcal{O}_{S[k]}/I_i$ for $i = 1, 2$, where σ_i is an automorphism of $S[k]$ induced by the canonical $\mathbb{C}^{*k}$-action covering identity on $\Delta_0 = S$. Let $(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]})$ be the universal ideal sheaves over $\mathcal{S} \times_{\mathfrak{A}_\diamond} \mathfrak{H}^{[n_1,n_2]}$.

There is a natural morphism $\mathfrak{H}^{[n_1,n_2]} \to \mathfrak{A}_\diamond$ as before, and there is a forgetful morphism

$$f : \mathfrak{H}^{[n_1,n_2]} \to (S/D)^{[n_1]} \times (S/D)^{[n_2]}$$

that sends the pair (I_1, I_2) on $S[k]$ to $(I_1|_{S[k_1]}, I_2|_{S[k_2]})$, where k_is are determined as in the last paragraph. The morphism f and the natural projections define the morphisms

$$\pi' : \mathcal{S} \times_{\mathfrak{A}_\diamond} \mathfrak{H}^{[n_1,n_2]} \to (S/D)^{[n_1]} \times (S/D)^{[n_2]}, \quad p' : \mathcal{S} \times_{\mathfrak{A}_\diamond} \mathfrak{H}^{[n_1,n_2]} \to S.$$

Definition 5.13. *Let (S, M) be a pair of a nonsingular surface and a line bundle and $D \subset S$ a nonsingular divisor as before. Define*

$$\mathsf{E}_M^{n_1,n_2} := [\mathbf{R}\pi'_* p'^* M] - \left[\mathbf{R}\mathcal{H}om_{\pi'}(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]} \otimes p'^* M)\right] \in K((S/D)^{[n_1]} \times (S/D)^{[n_2]}).$$

If $M = \mathcal{O}_S$ we drop it from the notation. Define the following generating series

$$Z_{\text{prod}}(S/D, M) := \sum_{n_1 \geq n_2 \geq 0} q_1^{n_1} q_2^{n_2} \int_{(S/D)^{[n_1]} \times (S/D)^{[n_2]}} c(\mathsf{E}^{n_1,n_2}) \cup c(\mathsf{E}_M^{n_1,n_2}),$$

If $D = 0$ we drop it from the notation, and recover the generating series in Definition 4.3.

We can prove an exact analog of Lemma 5.4 by replacing $c(\mathsf{K}_M^{[n_1 \geq n_2]})$ with $c(\mathsf{E}^{n_1,n_2}) \cup c(\mathsf{E}_M^{n_1,n_2})$. This means that if we have a good degeneration as in Lemma 5.4, then we get a degeneration of the class $c(\mathsf{E}^{n_1,n_2}) \cup c(\mathsf{E}_M^{n_1,n_2})$ whose restriction to the component

$$\left((S_1/D)^{[n_1']} \times (S_1/D)^{[n_2']}\right) \times \left((S_2/D)^{[n_1'']} \times (S_2/D)^{[n_2'']}\right)$$

of (41) is given by

$$c(\mathsf{E}^{n_1',n_2'}) \cup c(\mathsf{E}_{M_1}^{n_1',n_2'}) \boxtimes c(\mathsf{E}^{n_1'',n_2''}) \cup c(\mathsf{E}_{M_2}^{n_1'',n_2''}).$$

Proposition 5.12 then implies

$$Z_{\text{prod}}(S, M) = Z_{\text{prod}}(S_1/D, M_1) \cdot Z_{\text{prod}}(S_1/D, M_1).$$

By the same argument as Corollary 5.6, we can finally prove $Z_{\text{prod}}(-,-)$ descends to a homomorphism

$$Z_{\text{prod}}(-,-) : \omega_{2,1}(\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}[[q_1, q_2]]^*.$$

The following proposition gives another proof of Corollary 5.9:

Proposition 5.14. *For any nonsingular projective surface S, and a line bundle M, we have*

$$Z_{\text{nest}}(S, M) = Z_{\text{prod}}(S, M).$$

Proof. By Proposition 4.5, we have the equality in the proposition when (S, M) is one of the pairs in (40), by noting that by Remark 2.3 and [CO12, Eq. (5)],

$$c(\mathsf{E}_M^{n_1,n_2})|_{S^{[n_1 \geq n_2]}} = c(\mathsf{K}_M^{[n_1 \geq n_2]}), \qquad c_i(\mathsf{E}_M^{n_1,n_2}) = 0 \text{ for } i > n_1 + n_2.$$

For a general pair (S, M) as in the proposition there exists a positive integer r such that we can express the class $r[S, M] \in \omega_{2,1}(\mathbb{C})$ as a linear combination of the generators (40) with integer coefficients, and then apply the homomorphisms $Z_{\text{nest}}(-,-)$ and $Z_{\text{prod}}(-,-)$ to the both sides of the resulting relation. By what we just said we can see that $Z_{\text{nest}}(S, M)^r = Z_{\text{prod}}(S, M)^r$. Finally, since

$$Z_{\text{nest}}(S, M)|_{q_1 = q_2 = 0} = Z_{\text{prod}}(S, M)|_{q_1 = q_2 = 0} = 1,$$

the proposition follows.

□

6. Vertex operator formulas and proof of Theorem 7

Let (S, M) be a pair of a projective nonsingular surface S and a line bundles M on S. Let $\mathsf{F} = \oplus_n H^*(S^{[n]}, \mathbb{Q})$. Carlsson and Okounkov defined the operator $W(M_1)$ in $\mathrm{End}(\mathsf{F})[[z_1, z_1^{-1}]]$ by

$$\langle W(M_1)\eta_1, \eta_2\rangle := z_1^{n_2-n_1}\int_{S^{[n_1]}\times S^{[n_2]}} p_1^*\eta_1 \cup p_2^*\eta_2 \cup c_{n_1+n_2}(\mathsf{E}_{M_1}^{n_1,n_2}),$$

where $\langle -, -\rangle$ is the Poincaré pairing, $\eta_i \in H^*(S^{[n_i]}, \mathbb{Q})$, p_i is the projection to the i-th factor of $S^{[n_1]} \times S^{[n_2]}$, and $\mathsf{E}_{M_1}^{n_1,n_2} \in K(S^{[n_1]} \times S^{[n_2]})$ is as in Definition 4.3. In other words, using [F13, Definition 16.1.2], $W(M_1)$ is the operator associated to the family of correspondences

$$c_{n_1+n_2}(\mathsf{E}_{M_1}^{n_1,n_2}) : S^{[n_1]} \vdash S^{[n_2]} \quad n_1, n_2 \geq 0.$$

If M_2 is another line bundle on S, we define

$$W(M_1, M_2)(z_1, z_2) := W(M_2)(z_2) \circ W(M_1)(z_1).$$

By [F13, Proposition 16.1.2],

$$\langle W(M_1, M_2)\eta_1, \eta_3\rangle =$$
$$\sum_{n_2} z_1^{n_2-n_1} z_2^{n_3-n_2}\int_{S^{[n_1]}\times S^{[n_2]}\times S^{[n_3]}} p_1^*\eta_1 \cup p_3^*\eta_3 \cup c_{n_1+n_2}(\mathsf{E}_{M_1}^{n_1,n_2}) \cup c_{n_2+n_3}(\mathsf{E}_{M_2}^{n_2,n_3}),$$

where $\eta_i \in H^*(S^{[n_i]}, \mathbb{Q})$, and p_i is the projection to the i-th factor of $S^{[n_1]} \times S^{[n_2]} \times S^{[n_3]}$.

Carlsson and Okounkov found an explicit formula for $W(-)$ in terms of vertex operators. Let $\alpha_\pm(-)$ denote Nakajima's annihilation/creation operators.

Theorem 6.1. *(Carlsson-Okounkov* [CO12]*)*

$$W(M_1) = \Gamma_-(-M_1, -z_1) \circ \Gamma_+(-M_1^D, z_1),$$

where

$$\Gamma_\pm(M_1, z_1) := \exp\left(\sum_{n>0} \frac{z_1^{\mp n}}{n}\alpha_{\pm n}(M_1)\right).$$

□

Note that the operators $\Gamma_\pm$ satisfy the commutation relations $[\Gamma_\pm, \Gamma_\pm] = 0$, and moreover,

$$\Gamma_+(M_2, z_2) \circ \Gamma_-(M_1, z_1) = (1 + \frac{z_1}{z_2})^{\langle M_1, M_2\rangle}\Gamma_-(M_1, z_1) \circ \Gamma_+(M_2, z_2).$$

Let $\mathsf{c} := (1 - \frac{z_1}{z_2})^{\langle M_1, M_2^D\rangle}$. Using these properties, we can write

$$\begin{aligned} W(M_1,M_2) &= \Gamma_-(-M_2,-z_2)\circ\Gamma_+(-M_2^D,z_2)\circ\Gamma_-(-M_1,-z_1)\circ\Gamma_+(-M_1^D,z_1)\\ &= \mathsf{c}\,\Gamma_-(-M_2,-z_2)\circ\Gamma_-(-M_1,-z_1)\circ\Gamma_+(-M_2^D,z_2)\circ\Gamma_+(-M_1^D,z_1). \end{aligned}$$

Let N be the number-of-points operator: $\mathsf{N}|_{\mathsf{F}_n} = n\,\mathrm{id}$. It satisfies

$$q^{\mathsf{N}}\circ\Gamma_-(M_i,z_i) = \Gamma_-(M_i,qz_i)\circ q^{\mathsf{N}}.$$

Starting with $\mathrm{str}\left(q^{\mathsf{N}}\circ W(M_1,M_2)\right)$ and using the commutation relation of the super-trace $\mathrm{str}(A\circ B)=\mathrm{str}(B\circ A)$, we obtain

$$\begin{aligned} &\mathsf{c}\,\mathrm{str}\left(q^{\mathsf{N}}\circ\Gamma_-(-M_2,-z_2)\circ\Gamma_-(-M_1,-z_1)\circ\Gamma_+(-M_2^D,z_2)\circ\Gamma_+(-M_1^D,z_1)\right) =\\ &\mathsf{c}\,\mathrm{str}\left(\Gamma_-(-M_2,-z_2q)\circ\Gamma_-(-M_1,-z_1q)\circ q^{\mathsf{N}}\circ\Gamma_+(-M_2^D,z_2)\circ\Gamma_+(-M_1^D,z_1)\right) =\\ &\mathsf{c}\,\mathrm{str}\left(q^{\mathsf{N}}\circ\Gamma_+(-M_2^D,z_2)\circ\Gamma_+(-M_1^D,z_1)\circ\Gamma_-(-M_2,-z_2q)\circ\Gamma_-(-M_1,-z_1q)\right) =\\ &(1-\frac{z_1q}{z_2})^{\langle M_1,M_2^D\rangle}(1-\frac{z_2q}{z_1})^{\langle M_1^D,M_2\rangle}(1-q)^{\langle M_1^D,M_1\rangle+\langle M_2^D,M_2\rangle}\\ &\mathsf{c}\,\mathrm{str}\left(q^{\mathsf{N}}\circ\Gamma_-(-M_2,-z_2q)\circ\Gamma_-(-M_1,-z_1q)\circ\Gamma_+(-M_2^D,z_2)\circ\Gamma_+(-M_1^D,z_1)\right). \end{aligned}$$

Iterating this process, we get

$$\begin{aligned} &\mathrm{str}\left(q^{\mathsf{N}}\circ W(M_1,M_2)\right) =\\ &\prod_{n>0}(1-\frac{z_1q^n}{z_2})^{\langle M_1,M_2^D\rangle}(1-\frac{z_2q^n}{z_1})^{\langle M_1^D,M_2\rangle}(1-q^n)^{\langle M_1^D,M_1\rangle+\langle M_2^D,M_2\rangle}\\ &\mathsf{c}\,\mathrm{str}\left(q^{\mathsf{N}}\circ\Gamma_+(-M_2^D,z_2)\circ\Gamma_+(-M_1^D,z_1)\right) =\\ &\prod_{n\geq 0}(1-\frac{z_1q^n}{z_2})^{\langle M_1,M_2^D\rangle}(1-q^n)^{\langle M_1^D,M_1\rangle+\langle M_2^D,M_2\rangle-e(S)}\prod_{n>0}(1-\frac{z_2q^n}{z_1})^{\langle M_1^D,M_2\rangle}, \end{aligned}$$

where for the last equality, we have used the fact that Γ_+ is a lower triangular operator, and Göttsche's formula

$$\sum_{n\geq 0} e(S^{[n]})q^n = \prod_{n\geq 0}(1-q^n)^{-e(S)}.$$

Define

$$q_1 := qz_1^{-2}, \quad q_2 := z_1^2,$$

then we have shown

(42)

$$\begin{aligned} &\mathrm{str}\left(q^{\mathsf{N}}\circ W(M_1,M_2)(z_1,1/z_1)\right) =\\ &\prod_{n\geq 0}(1-q_2^{n+1}q_1^n)^{\langle M_1,M_2^D\rangle}(1-(q_1q_2)^n)^{\langle M_1^D,M_1\rangle+\langle M_2^D,M_2\rangle-e(S)}\prod_{n>0}(1-q_2^{n-1}q_1^n)^{\langle M_1^D,M_2\rangle}. \end{aligned}$$

On the other hand, by [F13, Example 16.1.3],

(43)

$$\operatorname{str}\left(q^{\mathsf{N}} \circ W(M_1,M_2)(z_1,1/z_1)\right) = q^{n_1} z_1^{2(n_2-n_1)} \int_{S^{[n_1]}\times S^{[n_2]}} c_{n_1+n_2}(\mathsf{E}_{M_1}^{n_1,n_2}) c_{n_1+n_2}(\mathsf{E}_{M_2}^{n_2,n_1})$$
$$= (-1)^{n_1+n_2} q_1^{n_1} q_2^{n_2} \int_{S^{[n_1]}\times S^{[n_2]}} c_{n_1+n_2}(\mathsf{E}_{M_1}^{n_1,n_2}) c_{n_1+n_2}(\mathsf{E}_{M_2^D}^{n_1,n_2}),$$

where the last equality is because of Grothendieck-Verdier duality.

Notation. *If* $Z=\sum_{r_1,r_2\geq 0} a_{r_1,r_2} q_1^{r_1} q_2^{r_2}$ *is a formal series, we define*

$$Z\ [q_1^{n_1}q_2^{n_2}] := a_{n_1,n_2}.$$

The main result of this section is

Proposition 6.2. *Let* (S,M) *be a pair of a projective nonsingular surface* S *and a line bundles* M *on* S*, then,*

$$\int_{[S^{[n_1\geq n_2]}]^{\mathrm{vir}}} c_{n_1+n_2}(\mathsf{K}_M^{[n_1\geq n_2]}) =$$
$$(-1)^{n_1+n_2} \prod_{n>0} (1-q_2^{n-1}q_1^{n})^{\langle K_S, M^D\rangle} (1-(q_1q_2)^n)^{\langle M^D, M\rangle - e(S)}\ [q_1^{n_1}q_2^{n_2}]\,.$$

Proof. By (43),

$$\int_{S^{[n_1]}\times S^{[n_2]}} c_{n_1+n_2}(\mathsf{E}^{n_1,n_2}) \cup c_{n_1+n_2}(\mathsf{E}_M^{n_1,n_2}) =$$
$$(-1)^{n_1+n_2} \operatorname{str}\left(q^{\mathsf{N}} \circ W(\mathcal{O}_S, M^D)(z_1,1/z_1)\right) [q_1^{n_1}q_2^{n_2}]\,.$$

The result now follows immediately from (42) and Corollary 5.9. ☐

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amingh@math.umd.edu, University of Maryland
College Park, MD 20742-4015, USA

artan@qgm.au.dk, Centre for Quantum Geometry of Moduli Spaces, Aarhus University, Department of Mathematics Ny Munkegade 118, building 1530, 319, 8000 Aarhus C, Denmark

artan@cmsa.fas.harvard.edu, Center for Mathematical Sciences and Applications, Harvard University, Department of Mathematics, 20 Garden Street, Room 207, Cambridge, MA, 02139

yau@math.harvard.edu Department of Mathematics, Harvard University, Cambridge, MA 02138, USA