DERIVATION LIE ALGEBRAS OF NEW k-TH LOCAL ALGEBRAS OF ISOLATED HYPERSURFACE SINGULARITIES

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Abstract. Let $(V, 0) = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : f(z_1, \cdots, z_n) = 0\}$ be an isolated hypersurface singularity with $mult(f) = m$. Let $J_k(f)$ be the ideal generated by all k-th order partial derivative of f. For $1 \leq k \leq m-1$, the new object $\mathcal{L}_k(V)$ is defined to be the Lie algebra of derivations of the new k-th local algebra $M_k(V)$, where $M_k(V) := \mathcal{O}_n/(f + J_1(f) + \cdots + J_k(f))$. Its dimension is denoted as $\delta_k(V)$. This number $\delta_k(V)$ is a new numerical analytic invariant. In this article we compute $\mathcal{L}_3(V)$ for fewnomial isolated singularities (binomial, trinomial) and obtain the formulas of $\delta_3(V)$. We also formulate a sharp upper estimate conjecture for the $\delta_k(V)$ of weighted homogeneous isolated hypersurface singularities and verify this conjecture for large class of singularities. Furthermore, we formulate another inequality conjecture: $\delta_{(k+1)}(V) < \delta_k(V), k \geq 1$ and verify it for low-dimensional fewnomial singularities.

Keywords. isolated hypersurface singularity, Lie algebra, local algebra.
MSC(2010). 14B05, 32S05.

1. Introduction

Finite dimensional Lie algebras are semi-direct product of the semi-simple Lie algebras and solvable Lie algebras. Simple Lie algebras have been well understood, but not the solvable (nilpotent) Lie algebras. Brieskorn [Br] gave a beautiful connection between simple Lie algebras and simple singularities([EK]). Thus it is extremely important to establish connection between singularities and solvable (nilpotent) Lie algebras.

The algebra of germs of holomorphic functions at the origin of $\mathbb{C}^n$ is denoted as $\mathcal{O}_n$. Clearly, $\mathcal{O}_n$ can be naturally identified with the algebra of convergent power series in n indeterminates with complex coefficients. As a ring $\mathcal{O}_n$ has a unique maximal ideal $\mathfrak{m}$, the set of germs of holomorphic functions which vanish at the origin. For any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^n, 0)$ where $V = \{f = 0\}$ Yau considers the Lie algebra of derivations of moduli algebra $A(V) := \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$, i.e., $L(V) = \mathrm{Der}(A(V), A(V))$. It is known that $L(V)$ is a finite dimensional solvable Lie algebra ([Ya2], [Ya3]). $L(V)$ is called the Yau algebra of V in [Yu] and [Khi] in order to distinguish from Lie algebras of other types appearing in singularity theory ([AVZ], [AM]). The Yau algebra plays an important role in singularities [SY]. Yau and his collabrators have been systematically studying various derivation Lie algebras of isolated hypersurface singularities begin from eighties (see, e.g., [Ya1], [Ch1, Ch2, CXY] [BY], [XY], [YZ1, YZ2], [CYZ], [CCYZ], [HYZ1]-[HYZ8], [CHYZ], [MYZ1, MYZ2], [HYZ]).

Both Yau and Zuo are supported by NSFC Grants 11961141005. Zuo is supported by NSFC Grant 11771231. Yau is supported by Tsinghua University start-up fund and Tsinghua University Education Foundation fund (042202008). Naveed is supported by innovation team project of Humanities and Social Sciences in Colleges and universities of Guangdong Province (No.: 2020wcxtd008).

In the theory of isolated singularities, one always wants to find invariants associated to the isolated singularities. Hopefully with enough invariants found, one can distinguish between isolated singularities. However, not many invariants are known. Recently, in [CHYZ, HYZ2, HYZ3, HYZ8, MYZ2], Yau, Zuo, Hussain, and their collaborators gave many new natural connections between the set of complex analytic isolated hypersurface singularities and the set of finite dimensional solvable (nilpotent) Lie algebras. They introduced three different ways to associate Lie algebras to isolated hypersurface singularities. These constructions are helpful to understand the solvable (nilpotent) Lie algebras from the geometric point of view ([CHYZ]).

Firstly, a new series of derivation Lie algebras $L_k(V), 0 \leq k \leq n$ associated to the isolated hypersurface singularity $(V, 0)$ defined by the holomorphic function $f(x_1, \cdots, x_n)$ are introduced in [HYZ8]. Let $Hess(f)$ be the Hessian matrix (f_{ij}) of the second order partial derivatives of f and $h(f)$, the Hessian of f, be the determinant of the matrix $Hess(f)$. More generally, for each k satisfying $0 \leq k \leq n$ we denote by $h_k(f)$ the ideal in $\mathcal{O}_n$ generated by all $k \times k$-minors in the matrix $Hess(f)$. In particular, $h_0(f) = 0$, the ideal $h_n(f) = (h(f))$ is a principal ideal. For each k as above, the graded k-th Hessian algebra of the polynomial f is defined by

$$H_k(f) = \mathcal{O}_n/(f + J(f) + h_k(f)).$$

It is known that the isomorphism class of the local k-th Hessian algebra $H_k(f)$ is contact invariant of f, i.e. depends only on the isomorphism class of the germ $(V, 0)$ ([DS], Lemma 2.1). In [HYZ8], we investigated the new Lie algebra $L_k(V)$ which is the Lie algebra of derivations of k-th Hessian algebra $H_k(f)$. The dimension of $L_k(V)$, denoted by $\lambda_k(V)$, is a new numerical analytic invariant of an isolated hypersurface singularity.

In particular, when $k = 0$, those are exactly the previous Yau algebra and Yau number, i.e., $L_0(V) = L(V), \lambda_0(V) = \lambda(V)$. Thus, the $L_k(V)$ is a generalization of Yau algebra $L(V)$. Moreover, $L_n(V)$ has been investigated intensively and many interesting results were obtained. In [CHYZ], it was shown that $L_n(V)$ completely distinguish ADE singularities. Furthermore, the authors have proven Torelli-type theorems for some simple elliptic singularities. Therefore, this new Lie algebra $L_n(V)$ is a subtle invariant of isolated hypersurface singularities. It is a natural question whether we can distinguish singularities by only using part of information of $L_n(V)$. In [HYZ4], we studied generalized Cartan matrices of the new Lie algebra $L_n(V)$ for simple hypersurface singularities and simple elliptic singularities. We introduced many other numerical invariants, namely, dimension of the maximal nilpotent subalgebras (i.e., nilradical of nilpotent Lie algebra) $g(V)$ of $L_n(V)$; dimension of maximal torus of $g(V)$, etc. We have proven that the generalized Cartan matrix of $L_n(V)$ can be used to characterize the ADE singularities except the pair of A_6 and D_5 singularities [HYZ4].

Secondly, recall that the Mather-Yau theorem was slightly generalized in ([GLS], Theorem 2.26):

Theorem 1.1. *Let $f, g \in \mathfrak{m} \subset \mathcal{O}_n$. The following are equivalent:*
1) $(V(f), 0) \cong (V(g), 0)$;
2) For all $k \geq 0$, $\mathcal{O}_n/(f, \mathfrak{m}^k J(f)) \cong \mathcal{O}_n/(g, \mathfrak{m}^k J(g))$ as $\mathbb{C}$-algebra;
3) There is some $k \geq 0$ such that $\mathcal{O}_n/(f, \mathfrak{m}^k J(f)) \cong \mathcal{O}_n/(g, \mathfrak{m}^k J(g))$ as $\mathbb{C}$-algebra, where $J(f) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$.

In particular, if $k = 0$ and $k = 1$ above, then the claim of the equivalence of 1) and 3) is exactly the Mather-Yau theorem [MY].

Motivated from Theorem 1.1, in [HYZ2, HYZ3], we introduced the new series of k-th Yau algebras $L^k(V)$ (or $L^k((V,0))$) which are defined to be the Lie algebra of derivations of the moduli algebra $A^k(V) = \mathcal{O}_n/(f, \mathfrak{m}^k J(f)), k \geq 0$, where $\mathfrak{m}$ is the maximal ideal, i.e., $L^k(V) := \mathrm{Der}(A^k(V), A^k(V))$. Its dimension is denoted as $\lambda^k(V)$ (or $\lambda^k((V,0))$). This series of integers $\lambda^k(V)$ are new numerical analytic invariants of singularities. It is natural to call it k-th Yau number. In particular, when $k = 0$, those are exactly the previous Yau algebra and Yau number, i.e., $L^0(V) = L(V), \lambda^0(V) = \lambda(V)$. In [Ya1], Yau observed that the Yau algebra for the one-parameter family of simple elliptic singularities $\tilde{E}_6$ is constant. It turns out that the 1-st Yau algebra $L^1(V)$ is also constant for the family of simple elliptic singularities $\tilde{E}_6$. However, Torelli-type theorem for $L^k(V)$ for all $k > 1$ do hold on $\tilde{E}_6$ ([HYZ]). In general, the invariant $L^k(V), k \geq 1$ are more subtle than the Yau algebra $L(V)$.

Finally, in [MYZ2], the authors introduce a new series of invariants to singularities. Let $(V,0)$ be an isolated hypersurface singularity defined by a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The multiplicity $mult(f)$ of the singularity $(V,0)$ is defined to be the order of the lowest nonvanishing term in the power series expansion of f at 0.

Definition 1.1. *Let $(V,0) = \{(x_1, \cdots, x_n) \in \mathbb{C}^n : f(x_1, \cdots, x_n) = 0\}$ be an isolated hypersurface singularity with $mult(f) = m$. Let $J_k(f)$ be the ideal generated by all the k-th order partial derivative of f, i.e., $J_k(f) =< \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} \mid 1 \leq i_1, \cdots, i_k \leq n >$. For $1 \leq k \leq m$, we define the new k-th local algebra, $M_k(V) := \mathcal{O}_n/(f + J_1(f) + \cdots + J_k(f))$. In particular, $M_m(V) = 0$, $M_1(V) = A(V)$, and $M_2(V) = H_1(V)$.*

Remark 1.1. *If f defines a weighted homogeneous isolated singularity at the origin, then $f \in J_1(f) \subset J_2(f) \subset \cdots \subset J_k(f)$, thus $M_k(V) = \mathcal{O}_n/(f + J_1(f) + \cdots + J_k(f)) = \mathcal{O}_n/(J_k(f))$.*

The isomorphism class of the k-th local algebra $M_k(V)$ is a contact invariant of $(V,0)$, i.e. depends only on the isomorphism class of the germ $(V,0)$. The dimension of $M_k(V)$ is denoted by $d_k(V)$ which is new numerical analytic invariant of an isolated hypersurface singularity.

Theorem 1.2. [MYZ2] *Suppose $(V,0) = \{(x_1, \cdots, x_n) \in \mathbb{C}^n : f(x_1, \cdots, x_n) = 0\}$ and $(W,0) = \{(x_1, \cdots, x_n) \in \mathbb{C}^n : g(x_1, \cdots, x_n) = 0\}$ are isolated hypersurface singularities. If $(V,0)$ is biholomorphically equivalent to $(W,0)$, then $M_k(V)$ is isomorphic to $M_k(W)$ as a $\mathbb{C}$-algebra for all $1 \leq k \leq m$, where $m = mult(f) = mult(g)$.*

Based on Theorem 1.2, it is natural to introduce the new series of k-th derivation Lie algebras $\mathcal{L}_k(V)$ which are defined to be the Lie algebra of derivations of the k-th local algebra $M_k(V)$, i.e., $\mathcal{L}_k(V) = Der(M_k(V), M_k(V))$. Its dimension is denoted as $\delta_k(V)$. This number $\delta_k(V)$ is also a new numerical analytic invariant. In particular, $\mathcal{L}_1(V) = L_0(V) = L(V), \mathcal{L}_2(V) = L_1(V)$. In [MYZ2], the authors have proven that the $\mathcal{L}_k(V)$ are non-negatively graded for weighted homogeneous isolated hypersurface singularities in low dimension.

We have seen that these $L^k(V), L_k(V), \mathcal{L}_k(V)$ are generalization of the Yau algebra $L(V)$. These are subtle invariants of singularities. We have reasons to believe that these three new series of derivation Lie algebras will also play an important role in the study of singularities.

A natural interesting question is: can we bound sharply the analytic invariant $\delta_k(V)$ by only using the topological invariant ([Sa]) weight types of the weighted homogeneous isolated hypersurface singularities? We propose the following sharp upper estimate conjecture.

Conjecture 1.1. *For each $0 \leq k \leq n$, assume that $\delta_k(\{x_1^{a_1} + \cdots + x_n^{a_n} = 0\}) = h_k(a_1, \cdots, a_n)$. Let $(V, 0) = \{(x_1, x_2, \cdots, x_n) \in \mathbb{C}^n : f(x_1, x_2, \cdots, x_n) = 0\}, (n \geq 2)$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \cdots, x_n)$ of weight type $(w_1, w_2, \cdots, w_n; 1)$. Then $\delta_k(V) \leq h_k(1/w_1, \cdots, 1/w_n)$.*

Moreover, we also propose the following inequality conjecture.

Conjecture 1.2. *With the above notations, let $(V, 0)$ be an isolated hypersurface singularity defined by $f \in \mathcal{O}_n$, $n \geq 2$. Then*

$$\delta_{(k+1)}(V) < \delta_k(V), k \geq 1.$$

Similar conjectures are investigated for $\lambda_k(V)$ and $\lambda^k(V)$ (cf. [HYZ1], [HYZ5], [YZ2], [HYZ8]). Note that $\mathcal{L}_1(V) = L_0(V) = L(V), \mathcal{L}_2(V) = L_1(V)$, thus $\delta_1 = \lambda_0 = \lambda^0, \delta_2 = \lambda_1$. The Conjecture 1.1 is true for the following cases:

1) binomial singularities (see Definition 2.4) when $k = 1$ [YZ2],
2) trinomial singularities (see Definition 2.4) when $k = 1$ [HYZ1],
3) binomial and trinomial singularities when $k = 2$ [HYZ8].

The Conjecture 1.2 is true for binomial and trinomial singularities when $k = 1$ [HYZ8].

The main purpose of this paper is to verify the Conjecture 1.1 and Conjecture 1.2 for binomial and trinomial singularities when k is small. We obtain the following main results.

Theorem A. *Let $(V, 0) = \{(x_1, x_2, \cdots, x_n) \in \mathbb{C}^n : x_1^{a_1} + \cdots + x_n^{a_n} = 0\}, (n \geq 2;\ a_i \geq 5,\ 1 \leq i \leq n)$. Then*

$$\delta_3(V) = h_3(a_1, \cdots, a_n) = \sum_{j=1}^{n} \frac{a_j - 4}{a_j - 3} \prod_{i=1}^{n} (a_i - 3).$$

Theorem B. *Let $(V, 0)$ be a binomial singularity defined by the weighted homogeneous polynomial $f(x_1, x_2)$ (see corollary 2.1) with weight type $(w_1, w_2; 1)$ and $mult(f) \geq 5$. Then*

$$\delta_3(V) \leq h_3(\frac{1}{w_1}, \frac{1}{w_2}) = \sum_{j=1}^{2} \frac{\frac{1}{w_j} - 4}{\frac{1}{w_j} - 3} \prod_{i=1}^{2} (\frac{1}{w_i} - 3).$$

Theorem C. *Let $(V, 0)$ be a fewnomial singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, x_3)$ (see Proposition 2.2) with weight type $(w_1, w_2, w_3; 1)$ and $mult(f) \geq 5$. Then*

$$\delta_3(V) \leq h_3(\frac{1}{w_1}, \frac{1}{w_2}, \frac{1}{w_3}) = \sum_{j=1}^{3} \frac{\frac{1}{w_j} - 4}{\frac{1}{w_j} - 3} \prod_{i=1}^{3} (\frac{1}{w_i} - 3).$$

Theorem D. *Let $(V,0)$ be a binomial singularity defined by the weighted homogeneous polynomial $f(x_1,x_2)$ (see corollary 2.1) with weight type $(w_1,w_2;1)$ and $mult(f)\geq 5$. Then*

$$\delta_{(k+1)}(V)<\delta_k(V), k=1,2.$$

Theorem E. *Let $(V,0)$ be a trinomial singularity defined by the weighted homogeneous polynomial $f(x_1,x_2,x_3)$ (see Proposition 2.2) with weight type $(w_1,w_2,w_3;1)$ and $mult(f)\geq 5$. Then*

$$\delta_{(k+1)}(V)<\delta_k(V), k=1,2.$$

2. Generalities on Derivation Lie algebras of isolated Singularities

In this section we shall briefly defined the basic definitions and important results which are helpful to solve the problem. The following basic concepts and results will be used to compute the derivation Lie algebras of isolated hypersurface singularities.

Let A,B be associative algebras over $\mathbb{C}$. The subalgebra of endomorphisms of A generated by the identity element and left and right multiplications by elements of A is called multiplication algebra $M(A)$ of A. The centroid $C(A)$ is defined as the set of endomorphisms of A which commute with all elements of $M(A)$. Obviously, $C(A)$ is a unital subalgebra of $\mathrm{End}(A)$. The following statement is a particular case of a general result from Proposition 1.2 of [Bl]. Let $S=A\otimes B$ be a tensor product of finite dimensional associative algebras with units. Then

$$\mathrm{Der}S\cong(\mathrm{Der}A)\otimes C(B)+C(A)\otimes(\mathrm{Der}B).$$

We will only use this result for commutative associative algebras with unit, in which case the centroid coincides with the algebra itself and one has following result for commutative associative algebras A,B:

Theorem 2.1. ([Bl]) *For commutative associative algebras A,B,*

$$DerS\cong(DerA)\otimes B+A\otimes(DerB). \tag{2.1}$$

We shall use this formula in the sequel.

Definition 2.1. *Let J be an ideal in an analytic algebra S. Then $Der_JS\subseteq Der_{\mathbb{C}}S$ is Lie subalgebra of all $\sigma\in Der_{\mathbb{C}}S$ for which $\sigma(J)\subset J$.*

We shall use the following well-known result to compute the derivations.

Theorem 2.2. ([YZ2]) *Let J be an ideal in $R=\mathbb{C}\{x_1,\cdots,x_n\}$. Then there is a natural isomorphism of Lie algebras*

$$(Der_JR)/(J\cdot Der_{\mathbb{C}}R)\cong Der_{\mathbb{C}}(R/J).$$

Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab)=D(a)b+aD(b)$. Thus for such an algebra A one can consider the Lie algebra of its derivations $Der(A,A)$ with the bracket defined by the commutator of linear endomorphisms.

Definition 2.2. *Let $(V,0)$ be an isolated hypersurface singularity. The new series of k-th derivation Lie algebras $\mathcal{L}_k(V)$ (or $\mathcal{L}_k((V,0))$) which are defined to be the Lie algebra of derivations of the k-th local algebra $M_k(V)$, i.e.,$\mathcal{L}_k(V)=Der(M_k(V),M_k(V))$. Its* *dimension is denoted as $\delta_k(V)$ (or $\delta_k((V,0))$). This number $\delta_k(V)$ is also a new numerical analytic invariant*

Definition 2.3. *A polynomial $f \in \mathbb{C}[x_1, x_2, \cdots, x_n]$ is called quasi-homogeneous (or weighted homogeneous) if there exist positive rational numbers $w_1, \ldots, w_n$ (called weights of indeterminates x_j) and d such that, for each monomial $\prod x_j^{k_j}$ appearing in f with non-zero coefficient, one has $\sum w_j k_j = d$. The number d is called the quasi-homogeneous degree (w-degree) of f with respect to weights w_j and is denoted $\deg f$. The collection $(w; d) = (w_1, \cdots, w_n; d)$ is called the quasi-homogeneity type (qh-type) of f.*

Definition 2.4. [Kh] *An isolated hypersurface singularity in $\mathbb{C}^n$ is fewnomial if it can be defined by a n-nomial in n variables and it is a weighted homogeneous fewnomial isolated singularity if it can be defined by a weighted homogeneous fewnomial. The 2-nomial (resp. 3-nomial) isolated hypersurface singularity is also called binomial (resp. trinomial) singularity.*

Proposition 2.1. *Let f be a weighted homogeneous fewnomial isolated singularity with mult$(f) \geq 3$. Then f analytically equivalent to a linear combination of the following three series:*

Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$, $n \geq 1$,
Type B. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n}$, $n \geq 2$,
Type C. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1$, $n \geq 2$.

Proposition 2.1 has an immediate corollary.

Corollary 2.1. *Each binomial isolated singularity is analytically equivalent to one from the three series: A) $x_1^{a_1} + x_2^{a_2}$, B) $x_1^{a_1} x_2 + x_2^{a_2}$, C) $x_1^{a_1} x_2 + x_2^{a_2} x_1$.*

Wolfgang and Atsushi [ET] give the following classification of weighted homogeneous fewnomial singularities in case of three variables.

Proposition 2.2. ([ET]) *Let $f(x_1, x_2, x_3)$ be a weighted homogeneous fewnomial isolated singularity with mult$(f) \geq 3$. Then f is analytically equivalent to following five types:*

Type 1. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$,
Type 2. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3}$,
Type 3. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3} x_1$,
Type 4. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3} x_1$,
Type 5. $x_1^{a_1} x_2 + x_2^{a_2} x_1 + x_3^{a_3}$.

3. Proof of theorems

In order to prove the main theorems, we need to prove following propositions.

Proposition 3.1. *Let $(V, 0)$ be a weighted homogeneous fewnomial isolated singularity which is defined by $f = x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$ ($a_i \geq 5, i = 1, 2, \cdots, n$) with weight type $(\frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_n}; 1)$. Then*

$$\delta_3(V) = \sum_{j=1}^{n} \frac{a_j - 4}{a_j - 3} \prod_{i=1}^{n} (a_i - 3).$$

Proof. The generalized moduli algebra $M_3(V)$ has dimension $\prod_{i=1}^{n}(a_i-3)$ and has a monomial basis of the form

$$\{x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}, 0\le i_1\le a_1-4, 0\le i_2\le a_2-4,\cdots,0\le i_n\le a_n-4\},$$

with following relations:

$$x_1^{a_1-3}=0, x_2^{a_2-3}=0, x_3^{a_3-3}=0,\cdots,x_n^{a_n-3}=0. \tag{3.1}$$

In order to compute a derivation D of $M_3(V)$ it suffices to indicate its values on the generators $x_1,x_2,\cdots,x_n$ which can be written in terms of the monomial basis. Without loss of generality, we write

$$Dx_j=\sum_{i_1=0}^{a_1-4}\sum_{i_2=0}^{a_2-4}\cdots\sum_{i_n=0}^{a_n-4}c^j_{i_1,i_2,\cdots,i_n}x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n},\ j=1,2,\cdots,n.$$

Using the above relations (3.1) one easily finds the necessary and sufficient conditions defining a derivation of $M_3(V)$ as follows:

$$\begin{aligned}
c^1_{0,i_2,i_3,,\cdots,i_n}&=0; 0\le i_2\le a_2-4, 0\le i_3\le a_3-4,\cdots,0\le i_n\le a_n-4;\\
c^2_{i_1,0,i_3,,\cdots,i_n}&=0; 0\le i_1\le a_1-4, 0\le i_3\le a_3-4,\cdots,0\le i_n\le a_n-4;\\
c^3_{i_1,i_2,0,\cdots,i_n}&=0; 0\le i_1\le a_1-4, 0\le i_2\le a_2-4,\cdots,0\le i_n\le a_n-4;\\
&\vdots\\
c^n_{i_1,i_2,i_3,\cdots,i_{n-1},0}&=0; 0\le i_1\le a_1-4, 0\le i_2\le a_2-4,\cdots,0\le i_{n-1}\le a_{n-1}-4.
\end{aligned}$$

Therefore we obtain the following description of Lie algebras in question:

$$\begin{gathered}
x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\partial_1,\ 1\le i_1\le a_1-4, 0\le i_2\le a_2-4, 0\le i_3\le a_3-4,\cdots,0\le i_n\le a_n-4;\\
x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\partial_2,\ 0\le i_1\le a_1-4, 1\le i_2\le a_2-4, 0\le i_3\le a_3-4,\cdots,0\le i_n\le a_n-4;\\
x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\partial_3,\ 0\le i_1\le a_1-4, 0\le i_2\le a_2-4, 1\le i_3\le a_3-4, 0\le i_4\le a_4-4,\\
0\le i_5\le a_5-4, 0\le i_6\le a_6-4,\cdots,0\le i_n\le a_n-4;\\
\vdots\\
x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\partial_n,\ 0\le i_1\le a_1-4, 0\le i_2\le a_2-4, 0\le i_3\le a_3-4,\cdots,1\le i_n\le a_n-4.
\end{gathered}$$

Therefore we have the following formula

$$\delta_3(V)=\sum_{j=1}^{n}\frac{a_j-4}{a_j-3}\prod_{i=1}^{n}(a_i-3).$$

Q.E.D.

Remark 3.1. *Let $(V,0)$ be a weighted homogeneous fewnomial isolated singularity of type A which is defined by $f=x_1^{a_1}+x_2^{a_2}$ $(a_1\ge 5,\ a_2\ge 5)$ with weight type $(\frac{1}{a_1},\frac{1}{a_2};1)$. Then it follows from Proposition 3.1 that*

$$\delta_3(V)=2a_1a_2-7(a_1+a_2)+24.$$

Proposition 3.2. *Let $(V,0)$ be a binomial isolated singularity of type B which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(a_1 \geq 4,\ a_2 \geq 5)$ with weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. Then*

$$\delta_3(V) = 2a_1a_2 - 7(a_1+a_2) + 27.$$

Furthermore, assuming that $mult(f) \geq 5$, we have

$$2a_1a_2 - 7(a_1+a_2) + 27 \leq \frac{2a_1a_2^2}{a_2-1} - 7(\frac{a_1a_2}{a_2-1} + a_2) + 24.$$

Proof. It follows that the generalized moduli algebra

$$M_3(V) = \mathbb{C}\{x_1,x_2\}/(f_{x_1x_1x_1}, f_{x_2x_2x_2}, f_{x_1x_2x_2}, f_{x_1x_1x_2})$$

has dimension $a_1a_2 - 3(a_1+a_2) + 10$ and has a monomial basis of the form

$$\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1-4; 0 \leq i_2 \leq a_2-4; x_1^{a_1-3}\}. \tag{3.2}$$

Similar as the computation in Proposition 3.1, we obtain the following derivations which form a basis of Der$M_3(V)$:

$$x_1^{i_1}x_2^{i_2}\partial_1, 1 \leq i_1 \leq a_1-4, 0 \leq i_2 \leq a_2-4; x_1^{i_1}x_2^{i_2}\partial_2, 0 \leq i_1 \leq a_1-4, 1 \leq i_2 \leq a_2-4;$$
$$x_2^{a_2-4}\partial_1; x_1^{a_3-1}\partial_1; x_1^{a_1-3}\partial_2.$$

Therefore we have the following formula

$$\delta_3(V) = 2a_1a_2 - 7(a_1+a_2) + 27.$$

It follows from Proposition 3.1 that we have

$$h_3(a_1,a_2) = 2a_1a_2 - 7(a_1+a_2) + 24.$$

After putting the weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$ of binomial isolated singularity of type B we have

$$h_3(\frac{1}{w_1}, \frac{1}{w_2}) = \frac{2a_1a_2^2}{a_2-1} - 7(\frac{a_1a_2}{a_2-1} + a_2) + 24.$$

Finally we need to show that

$$2a_1a_2 - 7(a_1+a_2) + 27 \leq \frac{2a_1a_2^2}{a_2-1} - 7(\frac{a_1a_2}{a_2-1} + a_2) + 24. \tag{3.3}$$

After solving 3.3 we have $a_1(a_2-7) + a_2(a_1-3) + 3 \geq 0$. Q.E.D.

Proposition 3.3. *Let $(V,0)$ be a binomial isolated singularity of type C which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ $(a_1 \geq 4, a_2 \geq 4)$ with weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$.*

$$\delta_3(V) = \begin{cases} 2a_1a_2 - 7(a_1+a_2) + 30; & a_1 \geq 5, a_2 \geq 5 \\ a_2; & a_1 = 4, a_2 \geq 4. \end{cases}$$

Furthermore, assuming that $mult(f) \geq 6$, we have

$$2a_1a_2 - 7(a_1+a_2) + 30 \leq \frac{2(a_1a_2-1)^2}{(a_1-1)(a_2-1)} - 7(a_1a_2-1)(\frac{a_1+a_2-2}{(a_1-1)(a_2-1)}) + 24.$$

Proof. The generalized moduli algebra $M_3(V)$ has dimension $a_1a_2 - 3(a_1+a_2)+11$ and has a monomial basis of the form

$$\{x_1^{i_1}x_2^{i_2}, 0\le i_1\le a_1-4; 0\le i_2\le a_2-4; x_1^{a_1-3}; x_2^{a_2-3}\}. \tag{3.4}$$

Similarly, we obtain the following derivations which form a basis of $\mathrm{Der}M_3(V)$:

$$x_1^{i_1}x_2^{i_2}\partial_1, 1\le i_1\le a_1-4, 0\le i_2\le a_2-4; x_1^{i_1}x_2^{i_2}\partial_2, 0\le i_1\le a_1-4, 1\le i_2\le a_2-4;$$
$$x_2^{a_2-4}\partial_1; x_2^{a_2-3}\partial_1; x_1^{a_1-3}\partial_1; x_2^{a_2-3}\partial_2; x_1^{a_1-4}\partial_2; x_1^{a_1-3}\partial_2.$$

Therefore we have the following formula

$$\delta_3(V) = 2a_1a_2 - 7(a_1+a_2)+30.$$

In case of $a_1 = 4, a_2\ge 4$, we have following bases of Lie algebra:

$$x_2^{i_2}\partial_2, 1\le i_2\le a_2-3; x_2^{a_2-3}\partial_1; x_1\partial_1; x_1\partial_2.$$

It follows from Proposition 3.1 and binomial isolated singularity of type C, we have

$$h_3(\frac{1}{w_1},\frac{1}{w_2}) = \frac{2(a_1a_2-1)^2}{(a_1-1)(a_2-1)} - 7(\frac{a_1a_2-1}{a_2-1}+\frac{a_1a_2-1}{a_1-1})+24.$$

Finally we need to show that

$$2a_1a_2 - 7(a_1+a_2)+30 \le \frac{2(a_1a_2-1)^2}{(a_1-1)(a_2-1)} - 7(a_1a_2-1)(\frac{a_1+a_2-2}{(a_1-1)(a_2-1)})+24. \tag{3.5}$$

After solving (3.5), we have
$a_1a_2^2[(a_2-2)(a_1-2)-a_1(a_2-5)]+a_2^3+4a_1^2a_2+10a_2^2(a_1-3)+6a_1a_2(a_1-3)$
$+3a_1^2(a_2-3)+a_1a_2(a_1-3)+15a_1+2(a_2-3)\ge 0.$
In case of $a_1 = 4, a_2\ge 4$, we need to show that

$$a_2 \le \frac{2(4a_2-1)^2}{3(a_2-1)} - 7(4a_2-1)(\frac{a_2+2}{3(a_2-1)})+24.$$

After simplification we get

$$a_2(a_2+10)-56.$$

Q.E.D.

Remark 3.2. *Let $(V,0)$ be a fewnomial surface isolated singularity of type 1 (see Proposition 2.2) which is defined by $f = x_1^{a_1}+x_2^{a_2}+x_3^{a_3}$ $(a_1\ge 5, a_2\ge 5, a_3\ge 5)$ with weight type $(\frac{1}{a_1},\frac{1}{a_2},\frac{1}{a_3};1)$. Then it follows from Proposition 3.1 that*

$$\delta_3(V) = 3a_1a_2a_3 + 33(a_1+a_2+a_3) - 10(a_1a_2+a_1a_3+a_2a_3) - 108.$$

Proposition 3.4. *Let $(V,0)$ be a fewnomial surface isolated singularity of type 2 which is defined by $f = x_1^{a_1}x_2+x_2^{a_2}x_3+x_3^{a_3}$ $(a_1\ge 4, a_2\ge 4, a_3\ge 5)$ with weight type $(\frac{1-a_3+a_2a_3}{a_1a_2a_3},\frac{a_3-1}{a_2a_3},\frac{1}{a_3};1)$. Then*

$$\delta_3(V) = \begin{cases} 3a_1a_2a_3 - 10(a_1a_2+a_1a_3+a_2a_3)+37(a_1+a_3) & \\ +33a_2-135; & a_1\ge 4, a_2\ge 5, a_3\ge 5 \\ 2a_1a_3-3a_1-5a_2+5; & a_1\ge 4, a_2=4, a_3\ge 5. \end{cases}$$

Furthermore, assuming that $a_1 \geq 4, a_2 \geq 5, a_3 \geq 5$, we have

$$3a_1a_2a_3 - 10(a_1a_2 + a_1a_3 + a_2a_3) + 37(a_1 + a_3) + 33a_2 - 135 \leq \frac{3a_1a_2^2a_3^3}{(1 - a_3 + a_2a_3)(a_3 - 1)}$$
$$- 10(\frac{a_1a_2^2a_3^2}{(1 - a_3 + a_2a_3)(a_3 - 1)} + \frac{a_1a_2a_3^2}{1 - a_3 + a_2a_3} + \frac{a_2a_3^2}{a_3 - 1}) + 33(\frac{a_1a_2a_3}{1 - a_3 + a_2a_3}$$
$$+ \frac{a_2a_3}{a_3 - 1} + a_3) - 108.$$

Proof. The moduli algebra $M_3(V)$ has dimension $(a_1a_2a_3 - 3(a_1a_2 + a_1a_3 + a_2a_3) + 10(a_1 + a_3) + 9a_2 - 33)$ and has a monomial basis of the form:

$$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 0 \leq i_1 \leq a_1 - 4; 0 \leq i_2 \leq a_2 - 4; 0 \leq i_3 \leq a_3 - 4; x_1^{a_1-3}x_3^{i_3}, 0 \leq i_3 \leq a_3 - 4;$$
$$x_1^{i_1}x_3^{a_3-3}, 0 \leq i_1 \leq a_1 - 4\}.$$

The following derivations form a basis in $\mathrm{Der}M_3(V)$:

$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 1 \leq i_1 \leq a_1 - 4, 0 \leq i_2 \leq a_2 - 4, 0 \leq i_3 \leq a_3 - 4; x_1^{a_1-3}x_3^{i_3}\partial_1,\ 0 \leq i_3 \leq a_3 - 4,$$
$$x_2^{a_2-4}x_3^{i_3}\partial_1,\ 1 \leq i_3 \leq a_3 - 4; x_1^{i_1}x_2^{a_2-3}\partial_1,\ 0 \leq i_1 \leq a_1 - 4,$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\ 0 \leq i_1 \leq a_1 - 4, 1 \leq i_2 \leq a_2 - 4, 0 \leq i_3 \leq a_3 - 4; x_1^{a_1-3}x_3^{i_3}\partial_2,\ 0 \leq i_3 \leq a_3 - 4,$$
$$x_1^{i_1}x_2^{a_2-3}\partial_2,\ 0 \leq i_1 \leq a_1 - 4; x_1^{i_1}x_3^{a_3-4}\partial_2,\ 1 \leq i_1 \leq a_1 - 4,$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3,\ 0 \leq i_1 \leq a_1 - 4, 0 \leq i_2 \leq a_2 - 4, 1 \leq i_3 \leq a_3 - 4, x_1^{i_1}x_2^{a_2-3}\partial_3,\ 0 \leq i_1 \leq a_1 - 4,$$
$$x_1^{a_1-3}x_3^{i_3}\partial_3,\ 1 \leq i_3 \leq a_3 - 4.$$

Therefore we have

$$\delta_3(V) = 3a_1a_2a_3 - 10(a_1a_2 + a_1a_3 + a_2a_3) + 37(a_1 + a_3) + 33a_2 - 135.$$

In case of $a_1 \geq 4, a_2 = 4, a_3 \geq 5$, we obtain the following basis:

$$x_1^{i_1}x_3^{i_3}\partial_1,\ 1 \leq i_1 \leq a_1 - 3, 0 \leq i_3 \leq a_3 - 4; x_1^{i_1}x_2\partial_1,\ 0 \leq i_1 \leq a_1 - 4,$$
$$x_1^{i_1}x_2\partial_2,\ 0 \leq i_1 \leq a_1 - 4; x_1^{i_1}x_3^{a_3-4}\partial_2,\ 1 \leq i_1 \leq a_1 - 3,$$
$$x_1^{i_1}x_3^{i_3}\partial_3,\ 0 \leq i_1 \leq a_1 - 3, 1 \leq i_3 \leq a_3 - 4; x_1^{i_1}x_2\partial_3,\ 0 \leq i_1 \leq a_1 - 4.$$

We have

$$\delta_3(V) = 2a_1a_3 - 3a_1 - 5a_3 + 5.$$

Next, we need to show that when $a_1 \geq 4, a_2 \geq 5, a_3 \geq 5$, then

$$3a_1a_2a_3 - 10(a_1a_2 + a_1a_3 + a_2a_3) + 37(a_1 + a_3) + 33a_2 - 135 \leq \frac{3a_1a_2^2a_3^3}{(1 - a_3 + a_2a_3)(a_3 - 1)}$$
$$- 10(\frac{a_1a_2^2a_3^2}{(1 - a_3 + a_2a_3)(a_3 - 1)} + \frac{a_1a_2a_3^2}{1 - a_3 + a_2a_3} + \frac{a_2a_3^2}{a_3 - 1}) + 33(\frac{a_1a_2a_3}{1 - a_3 + a_2a_3}$$
$$+ \frac{a_2a_3}{a_3 - 1} + a_3) - 108.$$

After simplification we get
$(a_1 - 2)^3(a_2 - 4)a_3 + (a_2 - 3)a_1a_3((a_3 - 2)(a_1 - 4) + (a_2 - 2)(a_3 - 2)) + a_2(3a_3 - 3)(a_1 - 2) + a_2(a_1 - 1) + 6 \geq 0.$

We also need to show that when $a_1 \geq 4, a_3 \geq 5$, then

$$2a_1a_3 - a_1 - 3a_3 - 1 \leq \frac{48a_1a_3^3}{(1+3a_3)(a_3-1)} + 33(\frac{4a_1a_3}{1+3a_3} + \frac{4a_3}{a_3-1} + a_3) - 10(\frac{16a_1a_3^2}{(1+3a_3)(a_3-1)} + \frac{4a_1a_3^2}{1+3a_3} + \frac{4a_3^2}{a_3-1}) - 108.$$

After simplification we get

$$\frac{4a_1a_3^3}{(1+3a_3)(a_3-3)} + \frac{a_3^2(a_1a_3-4)+4a_3^3(a_1-3)}{(1+3a_3)(a_3-2)} + \frac{15a_1a_3^2(a_3-3)}{(1+3a_3)(a_3-2)} + \frac{4a_1a_3^2}{(1+3a_3)(a_3-2)} + \frac{35a_1a_3^2}{(1+a_3)(a_3-3)} + 16(a_3-4) + \frac{48a_3}{(a_3-3)} + 8 \geq 0.$$

Q.E.D.

Proposition 3.5. *Let $(V,0)$ be a fewnomial surface isolated singularity of type 3 which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$ $(a_1 \geq 4, a_2 \geq 4, a_3 \geq 4)$ with weight type*

$$(\frac{1-a_3+a_2a_3}{1+a_1a_2a_3}, \frac{1-a_1+a_1a_3}{1+a_1a_2a_3}, \frac{1-a_2+a_1a_2}{1+a_1a_2a_3}; 1).$$

Then

$$\delta_3(V) = \begin{cases} 3a_1a_2a_3 + 37(a_1+a_2+a_3) - 10(a_1a_2+a_1a_3+a_2a_3) & \\ -147; & a_1 \geq 5, a_2 \geq 5, a_3 \geq 5 \\ 2a_2a_3 - 5a_2 - 3a_3 + 9; & a_1 = 4, a_2 \geq 5, a_3 \geq 4 \\ 2a_1a_3 - 3a_1 - 5a_3 + 9; & a_1 \geq 4, a_2 = 4, a_3 \geq 4 \\ 2a_1a_2 - 5a_1 - 3a_2 + 9; & a_1 \geq 5, a_2 \geq 5, a_3 = 4. \end{cases}$$

Furthermore, assuming that $a_1 \geq 5, a_2 \geq 5, a_3 \geq 5$, we have

$$3a_1a_2a_3 + 37(a_1+a_2+a_3) - 10(a_1a_2+a_1a_3+a_2a_3) - 147 \leq \frac{3(1+a_1a_2a_3)^3}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)(1-a_2+a_1a_2)} + 33(\frac{1+a_1a_2a_3}{1-a_3+a_2a_3} + \frac{1+a_1a_2a_3}{1-a_1+a_1a_3} + \frac{1+a_1a_2a_3}{1-a_2+a_1a_2}) - 10(\frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)} + \frac{(1+a_1a_2a_3)^2}{(1-a_1+a_1a_3)(1-a_2+a_1a_2)} + \frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_2+a_1a_2)}) - 108.$$

Proof. The moduli algebra $M_3(V)$ has dimension $(a_1a_2a_3 - 3(a_1a_2+a_1a_3+a_2a_3) + 10(a_1 + a_2 + a_3) - 36)$ and has a monomial basis of the form

$$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 0 \leq i_1 \leq a_1-4; 0 \leq i_2 \leq a_2-4; 0 \leq i_3 \leq a_3-4; x_1^{a_1-3}x_3^{i_3}, 0 \leq i_3 \leq a_3-4; x_2^{i_2}x_3^{a_3-3}, 0 \leq i_2 \leq a_2-4; x_1^{i_1}x_2^{a_2-3}, 0 \leq i_1 \leq a_1-4\}.$$

We obtain the following description of Lie algebras in question:

$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 1 \leq i_1 \leq a_1-4, 0 \leq i_2 \leq a_2-4, 0 \leq i_3 \leq a_3-4; x_2^{i_2}x_3^{a_3-3}\partial_1,\ 0 \leq i_2 \leq a_2-5,$$
$$x_2^{a_2-4}x_3^{i_3}\partial_1,\ 1 \leq i_3 \leq a_3-3; x_1^{i_1}x_2^{a_2-3}\partial_1,\ 0 \leq i_1 \leq a_1-4; x_1^{a_1-3}x_3^{i_3}\partial_1,\ 0 \leq i_3 \leq a_3-4,$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\ 0 \leq i_1 \leq a_1-4, 1 \leq i_2 \leq a_2-4, 0 \leq i_3 \leq a_3-4; x_1^{a_1-3}x_3^{i_3}\partial_2,\ 0 \leq i_3 \leq a_3-4,$$
$$x_1^{i_1}x_2^{a_2-3}\partial_2,\ 0 \leq i_1 \leq a_1-4; x_1^{i_1}x_3^{a_3-4}\partial_2,\ 1 \leq i_1 \leq a_1-4; x_2^{i_2}x_3^{a_3-3}\partial_2,\ 0 \leq i_2 \leq a_2-4,$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3,\ 0 \leq i_1 \leq a_1-4, 0 \leq i_2 \leq a_2-4, 1 \leq i_3 \leq a_3-4; x_1^{i_1}x_2^{a_2-3}\partial_3,\ 0 \leq i_1 \leq a_1-4,$$
$$x_1^{a_1-4}x_2^{i_2}\partial_3,\ 1 \leq i_2 \leq a_2-4; x_2^{i_2}x_3^{a_3-3}\partial_3,\ 0 \leq i_2 \leq a_2-4; x_1^{a_1-3}x_3^{i_3}\partial_3,\ 0 \leq i_3 \leq a_3-4.$$

Therefore we have

$$\delta_3(V) = 3a_1a_2a_3 + 37(a_1 + a_2 + a_3) - 10(a_1a_2 + a_1a_3 + a_2a_3) - 147.$$

In case of $a_1 = 4, a_2 \geq 5, a_3 \geq 4$, we obtain the following basis:

$$x_2^{a_2-4}x_3^{i_3}\partial_1,\ 1 \leq i_3 \leq a_3 - 3; x_1x_3^{i_3}\partial_1,\ 0 \leq i_3 \leq a_3 - 4; x_2^{a_2-3}\partial_1; x_3^{a_3-3}\partial_2,$$
$$x_1x_3^{i_3}\partial_2,\ 0 \leq i_3 \leq a_3 - 4; x_2^{a_2-3}\partial_2; x_2^{i_2}x_3^{i_3}\partial_2,\ 1 \leq i_2 \leq a_2 - 4, 0 \leq i_3 \leq a_3 - 3,$$
$$x_2^{i_2}x_3^{i_3}\partial_3,\ 0 \leq i_2 \leq a_2 - 4, 1 \leq i_3 \leq a_3 - 3, x_1x_3^{i_3}\partial_3,\ 0 \leq i_3 \leq a_3 - 4; x_2^{a_2-3}\partial_3.$$

Therefore we have

$$\delta_3(V) = 2a_2a_3 - 5a_2 - 3a_3 + 9.$$

Similarly, we obtain the basis of Lie algebra for $a_1 \geq 4, a_2 = 4, a_3 \geq 4$ and $a_1 \geq 5, a_2 \geq 5, a_3 = 4$.
Furthermore, we need to show that when $a_1 \geq 5, a_2 \geq 5, a_3 \geq 5$, then the inequality in Proposition 3.5 holds. After simplification we get
$5(a_1a_2 + a_2a_3 + a_1a_3) + a_1(a_2 - 4) + a_2(a_3 - 4) + a_3(a_1 - 4) + 4a_1^2[a_2(a_3 - 4) + a_3(a_2 - 4)] + 4a_2^2[a_1(a_3 - 4) + a_3(a_1 - 4)] + 6a_3^2[a_1(a_2 - 4) + a_2(a_1 - 4)] + 3(a_1^2 + a_2^2 + a_3^2) + 4(a_1^3a_2 + a_2^3a_3 + a_3^3a_1) + 2a_1^2a_2^2a_3^2 + 6(a_1a_2^2a_3 + a_1a_2a_3^2) + 2a_1^2a_2a_3 + a_1a_2a_3[2a_1 - 8] + a_1^3a_2a_3^2(a_3 - 4)(a_2 - 4) + a_1^2a_3^2(a_3 - 4)(a_1a_2 - 4) + a_1^2a_2a_3^2(a_3 + a_2 - 5) + 4a_1a_2a_3^3(a_1 - 4) + a_1^2a_2^3a_3(a_3 - 4)(a_1 - 4) + a_1^2a_2^2(a_1 - 4)(a_2a_3 - 3) + a_1^3a_2a_3(a_2 - 4) + a_1^2a_2^2a_3(a_1 - 4 + (a_3 - 4)) + a_1a_2^2a_3^3(a_2 - 4)(a_1 - 4) + a_2^2a_3^2(a_2 - 4)(a_1a_3 - 4) + 8 \geq 0.$
Similarly we can prove Conjecture 1.1 for $a_1 \geq 4, a_2 = 4, a_3 \geq 4; a_1 \geq 5, a_2 \geq 5, a_3 = 4$ and $a_1 = 4, a_2 \geq 5, a_3 \geq 4$. Q.E.D.

Proposition 3.6. *Let $(V, 0)$ be a fewnomial surface isolated singularity of type 4 which is defined by $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}x_2$ $(a_1 \geq 5, a_2 \geq 5, a_3 \geq 4)$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}, \frac{a_2-1}{a_2a_3}; 1)$. Then*

$$\delta_3(V) = 3a_1a_2a_3 + 37a_1 + 33(a_2 + a_3) - 10(a_1a_2 + a_1a_3 + a_2a_3) - 121.$$

Furthermore, assuming that mult$(f) \geq 5$, we have
$3a_1a_2a_3 + 37a_1 + 33(a_2 + a_3) - 10(a_1a_2 + a_1a_3 + a_2a_3) - 121 \leq \frac{3a_2^2a_1a_3}{a_2-1} + 33(a_1 + a_2 + \frac{a_2a_3}{a_2-1}) - 10(a_1a_2 + \frac{a_1a_2a_3}{a_2-1} + \frac{a_2^2a_3}{a_2-1}) - 108.$

Proof. The moduli algebra $M_3(V)$ has dimension $(a_1a_2a_3 - 3(a_1a_2 + a_1a_3 + a_2a_3) + 9(a_2 + a_3) + 10a_1 - 30)$ and has a monomial basis of the form

$$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 0 \leq i_1 \leq a_1 - 4; 0 \leq i_2 \leq a_2 - 4; 0 \leq i_3 \leq a_3 - 4; x_1^{i_1}x_3^{a_3-3}, 0 \leq i_2 \leq a_2 - 4\}.$$

We obtain the following description of the Lie algebra in question:

$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 1 \leq i_1 \leq a_1 - 4, 0 \leq i_2 \leq a_2 - 4, 0 \leq i_3 \leq a_3 - 4; x_1^{i_1}x_3^{a_3-3}\partial_1,\ 1 \leq i_1 \leq a_1 - 4,$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\ 1 \leq i_1 \leq a_1 - 4, 1 \leq i_2 \leq a_2 - 4, 0 \leq i_3 \leq a_3 - 4; x_1^{i_1}x_3^{a_3-3}\partial_2,\ 0 \leq i_1 \leq a_1 - 4,$$
$$x_2^{i_2}x_3^{i_3}\partial_2,\ 1 \leq i_2 \leq a_2 - 4, 0 \leq i_3 \leq a_3 - 4, x_1^{i_1}x_2^{a_2-4}\partial_3,\ 0 \leq i_1 \leq a_1 - 4$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3,\ 0 \leq i_1 \leq a_1 - 4, 0 \leq i_2 \leq a_2 - 4, 1 \leq i_3 \leq a_3 - 4, x_1^{i_1}x_3^{a_3-3}\partial_3,\ 0 \leq i_1 \leq a_1 - 4.$$

Therefore we have

$$\delta_3(V) = 3a_1a_2a_3 + 37a_1 + 33(a_2 + a_3) - 10(a_1a_2 + a_1a_3 + a_2a_3) - 121.$$

Furthermore, we need to show that when $a_1 \geq 5, a_2 \geq 5, a_3 \geq 4$, then the inequality in Proposition 3.6 holds. After simplifying the inequality, we get

$$\frac{a_1a_3(2a_2-8)}{a_2-4} + 2a_2a_3 + a_3(a_2-2) + \frac{8a_3}{a_2-3} + \frac{a_1[a_2(a_3-3)+5]}{a_2-3} \geq 0.$$

Q.E.D.

Proposition 3.7. *Let $(V,0)$ be a fewnomial surface isolated singularity of type 5 which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$ ($a_1 \geq 4, a_2 \geq 4, a_3 \geq 5$) with weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}, \frac{1}{a_3}; 1)$. Then*

$$\delta_3(V) = \begin{cases} 3a_1a_2a_3 + 33(a_1+a_2) + 41a_3 - 10(a_1a_2 + a_1a_3 + a_2a_3) & \\ -134; & a_1 \geq 5, a_2 \geq 5, a_3 \geq 5 \\ 2a_2a_3 - 7a_2 - a_3 + 4; & a_1 = 4, a_2 \geq 4, a_3 \geq 5 \end{cases}$$

Furthermore, assuming that $a_1 \geq 5, a_2 \geq 5, a_3 \geq 5$, we have $3a_1a_2a_3 + 33(a_1+a_2) + 41a_3 - 10(a_1a_2+a_1a_3+a_2a_3) - 134 \leq \frac{3a_3(a_1a_2-1)^2}{(a_2-1)(a_1-1)} + 33(\frac{a_1a_2-1}{a_2-1} + \frac{a_1a_2-1}{a_1-1} + a_3) - 10(\frac{(a_1a_2-1)^2}{(a_2-1)(a_1-1)} + \frac{a_3(a_1a_2-1)}{a_1-1} + \frac{a_3(a_1a_2-1)}{a_2-1}) - 108.$

Proof. It is easy to see that the moduli algebra $M_3(V)$ has dimension $a_1a_2a_3 - 3(a_1a_2 + a_1a_3 + a_2a_3) + 9(a_1+a_2) + 11a_3 - 33$ and has a monomial basis of the form

$$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 0 \leq i_1 \leq a_1-4; 0 \leq i_2 \leq a_2-4; 0 \leq i_3 \leq a_3-4; x_1^{a_1-3}x_3^{i_3}, 0 \leq i_3 \leq a_3-4; x_2^{a_2-3}x_3^{i_3}, 0 \leq i_3 \leq a_3-4\}.$$

We obtain the following description of the Lie algebra in question:

$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 1 \leq i_1 \leq a_1-4, 0 \leq i_2 \leq a_2-4, 0 \leq i_3 \leq a_3-4; x_1^{a_1-3}x_3^{i_3}\partial_1,\ 0 \leq i_3 \leq a_3-4,$$
$$x_2^{a_2-3}x_3^{i_3}\partial_1,\ 0 \leq i_3 \leq a_3-4; x_2^{a_2-4}x_3^{i_3}\partial_1,\ 0 \leq i_3 \leq a_3-4,$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\ 0 \leq i_1 \leq a_1-4, 1 \leq i_2 \leq a_2-4, 0 \leq i_3 \leq a_3-4; x_1^{a_1-3}x_3^{i_3}\partial_2,\ 0 \leq i_3 \leq a_3-4,$$
$$x_2^{a_2-3}x_3^{i_3}\partial_2,\ 0 \leq i_3 \leq a_3-4; x_1^{a_1-4}x_3^{i_3}\partial_2,\ 0 \leq i_3 \leq a_3-4,$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3,\ 0 \leq i_1 \leq a_1-4, 0 \leq i_2 \leq a_2-4, 1 \leq i_3 \leq a_3-4; x_1^{a_1-3}x_3^{i_3}\partial_3,\ 1 \leq i_3 \leq a_3-4,$$
$$x_2^{a_2-3}x_3^{i_3}\partial_3,\ 1 \leq i_3 \leq a_3-4.$$

Therefore we have

$$\delta_3(V) = 3a_1a_2a_3 + 33(a_1+a_2) + 41a_3 - 10(a_1a_2 + a_1a_3 + a_2a_3) - 134.$$

In case of $a_1 = 4, a_2 \geq 4, a_3 \geq 5$, we obtain the following basis:

$$x_2^{i_2}x_3^{i_3}\partial_2,\ 1 \leq i_2 \leq a_2-4, 0 \leq i_3 \leq a_3-4; x_2^{a_2-3}x_3^{i_3}\partial_1,\ 0 \leq i_3 \leq a_3-4,$$
$$x_1x_3^{i_3}\partial_1,\ 0 \leq i_3 \leq a_3-4; x_2^{a_2-3}x_3^{i_3}\partial_2,\ 0 \leq i_3 \leq a_3-4,$$
$$x_2^{i_2}x_3^{i_3}\partial_3,\ 0 \leq i_2 \leq a_2-4, 1 \leq i_3 \leq a_3-4; x_1x_3^{i_3}\partial_2,\ 0 \leq i_3 \leq a_3-4,$$
$$x_1x_3^{i_3}\partial_3,\ 1 \leq i_3 \leq a_3-4.$$

We have

$$\delta_3(V) = 2a_2a_3 - 7a_2 - a_3 + 4.$$

Next, we need to show that when $a_1 \geq 5, a_2 \geq 5, a_3 \geq 5$, then the inequality in Proposition 3.7 holds. After simplification, we get

$$\begin{aligned}
&a_1(a_1-5)(a_2-3)(a_3+(a_1-2)a_2(a_2-4)a_3)+a_1^2(a_3-3)(a_2-4)+a_2^2a_1+6a_1(a_2-5)\\
&+6a_2(a_1-5)+6a_3(a_1-4)+16a_1a_2+15a_1a_3+4a_2a_3+30+35a_2+a_1a_2(a_1-5)\\
&+(a_1-2)a_2(a_2-5)(a_3-4)+(a_1-3)(a_3-4)\geq 0.
\end{aligned}$$

similarly, we can prove that the Conjecture 1.1 also true for $a_1 = 4, a_2 \geq 4, a_3 \geq 5$, Q.E.D.

Proof of Theorem A.

Proof. It follows from Proposition 3.1 that Theorem A is true. Q.E.D.

Proof of Theorem B.

Proof. Let $f \in \mathbb{C}\{x_1, x_2\}$ be a weighted homogeneous fewnomial isolated singularity. Then f can be divided into the following three types:

Type A. $x_1^{a_1} + x_2^{a_2}$,

Type B. $x_1^{a_1}x_2 + x_2^{a_2}$,

Type C. $x_1^{a_1}x_2 + x_2^{a_2}x_1$.

The Theorem B is an immediate corollary of Remark 3.1, Proposition 3.2, and Proposition 3.3. Q.E.D.

Proof of Theorem C.

Proof. Let $f \in \mathbb{C}\{x_1, x_2, x_3\}$ be a weighted homogeneous fewnomial isolated surface singularity. Then f can be divided into the following five types:

Type 1. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$,

Type 2. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$,

Type 3. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$,

Type 4. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}x_1$,

Type 5. $x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$.

The Theorem C is an immediate corollary of Remark 3.2, Propositions 3.4, 3.5, 3.6, and 3.7. Q.E.D.

Proof of Theorem D.

Proof. It is easy to see, from Remark 3.1, Propositions 3.2-3.3, Propositions 4.1-4.3 in [YZ2] and Remark 3.1, Propositions 3.2-3.3 in [HYZ8], the inequality $\delta_{(k+1)}(V) < \delta_k(V), k = 1, 2$ holds true. Q.E.D.

Proof of Theorem E.

Proof. It is easy to see, from Remark 3.2, Propositions 3.4-3.7, Proposition 4.1 in [YZ2], Propositions 3.1, 3.2 in [HYZ1], Propositions 3.4, 3.5 in [HYZ3] and Remark 3.4, Propositions 3.4-3.7 in [HYZ8], the inequality $\delta_{(k+1)}(V) < \delta_k(V), k = 1, 2$ holds true. Q.E.D.

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