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A sharp polynomial estimate of positive integral points in a 4-dimensional tetrahedron and a sharp estimate of the Dickman-de Bruijn function

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Dedicated to Professor Micheal Artin on the occasion of his 80th birthday

The estimate of integral points in right-angled simplices has many applications in number theory, complex geometry, toric variety and tropical geometry. In [24], [25], [27], the second author and other coworkers gave a sharp upper estimate that counts the number of positive integral points in n dimensional ($n \geq 3$) real right-angled simplices with vertices whose distance to the origin are at least $n-1$. A natural problem is how to form a new sharp estimate without the minimal distance assumption. In this paper, we formulate the Number Theoretic Conjecture which is a direct correspondence of the Yau Geometry conjecture. We have proved this conjecture for $n = 4$. This paper gives hope to prove the new conjecture in general. As an application, we give a sharp estimate of the Dickman-de Bruijn function $\psi(x, y)$ for $y < 11$.

1 Introduction

Let $\Delta(a_1, a_2, \ldots, a_n)$ be an n-dimensional simplex described by

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1, \quad x_1, x_2, \ldots, x_n \geq 0, \tag{1.1}$$

where $a_1 \geq a_2 \geq \cdots \geq a_n \geq 1$ are positive real numbers. Let $P_n = P(a_1, a_2, \ldots, a_n)$ and $Q_n = Q(a_1, a_2, \ldots, a_n)$ be defined as the number of positive and nonnegative integral solutions of (1.1) respectively. They are related by the following formula

$$Q(a_1, a_2, \ldots, a_n) = P(a_1(1+a), a_2(1+a), \ldots, a_n(1+a)), \tag{1.2}$$

where $a = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$. The estimate of integral points has many applications in number theory, complex geometry, toric variety and tropical geometry.

One of the central topics in computational number theory is the estimate of $\psi(x, y)$, the Dickman-de Bruijn function (see [4], [5], [6], [10]). Let $S(x, y)$ be the set of positive integers $\leq x$, composed only of prime factors $\leq y$. The Dickman-de Bruijn function $\psi(x, y)$ is the cardinality of this set. It turns out that the computation of $\psi(x, y)$ is equivalent to compute the number of integral points in an n-dimensional tetrahedron $\triangle(a_1, a_2, \ldots, a_n)$ with real vertices $(a_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, a_n)$. Let $p_1 < p_2 < \cdots < p_n$ denotes the primes up to y. It is clear that $p_1^{l_1} p_2^{l_2} \cdots p_n^{l_n} \leq x$ if and only if $l_1 \log p_1 + l_2 \log p_2 + \cdots + l_n \log p_n \leq \log x$. Therefore, $\psi(x, y)$ is precisely the number Q_n of (integer) lattice points inside the n-dimensional tetrahedron (1.1) with $a_i = \frac{\log x}{\log p_i}$, $1 \leq i \leq n$.

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The general problem of counting the number Q_n has been a challenging problem for many years. Tremendous researches have been putting to develop an exact formula when $a_1, \ldots, a_n$ are positive integers (see [1], [2], [7], [14]). Mordell gave a formula for Q_3, expressed in terms of three Dedekind sums, in the case that a_1, a_2, and a_3 are pairwise relatively prime [20]; Pommersheim extended the formula for Q_3 to arbitrary a_1, a_2, and a_3 using toric varieties [21] and so forth. Meanwhile, the problem of counting the number of integral points in an n-dimensional tetrahedron with real vertices is a classical subject which has attracted a lot of famous mathematicians. Also from the view of estimating the Dickman-de Bruijn function, the a_i's, $1 \leq i \leq n$, are not always integers. Hardy and Littlewood wrote several papers that have been applied on Diophantine approximation ([11], [12], [13]). A more general approximation of Q_n was obtained by D. C. Spencer [22], [23] via complex function-theoretic methods.

According to Granville [9], an upper polynomial estimate of P_n is a key topic in number theory. Such an estimate could be applied to finding large gaps between primes, to Waring's problem, to primality testing and factoring algorithms, and to bounds for the least prime k-th power residues and non-residues (mod n). Granville ([9]) obtained the following estimate

$$P_n \leq \frac{1}{n!} a_1 a_2 \ldots a_n. \tag{1.3}$$

This estimate of $P_{(a_1, a_2, \ldots, a_n)}$ given by (1.3) is interesting, but not strong enough to be useful, particularly when many of the a_i's are small [9]. In geometry and singularity theory, estimating P_n for real right-angled simplices is related to the Durfee Conjecture [26]. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a complex analytic function with an isolated critical point at the origin. Let $V = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : f(z_1, \ldots, z_n) = 0\}$. The Milnor number of the singularity $(V, 0)$ is defined as

$$\mu = \dim \mathbb{C}\{z_1, \ldots, z_n\}/(f_{z_1}, \ldots, f_{z_n})$$

the geometric genus p_g of $(V, 0)$ is defined as

$$p_g = \dim H^{n-2}\big(M, \Omega^{n-1}\big)$$

where M is a resolution of V and Ω^{n-1} is the sheaf of germs of holomorphic $n-1$ forms on M. In 1978, Durfee [8] made the following conjecture:

Durfee Conjecture. $n! p_g \leq \mu$ with equality only when $\mu = 0$.

If $f(z_1, \ldots, z_n)$ is a weighted homogeneous polynomial of type $(a_1, a_2, \ldots, a_n)$ with an isolated singularity at the origin, Milnor and Orlik [19] proved that $\mu = (a_1 - 1)(a_2 - 1)\ldots(a_n - 1)$. On the other hand, Merle and Teissier [18] showed that $p_g = P_n$. Finding a sharp estimate of P_n will lead to a resolution of the Durfee Conjecture.

Starting from early 1990's, the authors of [16], [25] and [27] tried to get sharp upper estimates of P_n where the a_i's are positive real numbers. They were successful for $n = 3,\ 4$, and 5:

$$\begin{aligned}
3!P_3 \leq f_3 &= a_1a_2a_3 - (a_1a_2 + a_1a_3 + a_2a_3) + a_1 + a_2, \\
4!P_4 \leq f_4 &= a_1a_2a_3a_4 - \frac{3}{2}(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4) \\
&\quad + \frac{11}{3}(a_1a_2 + a_1a_3 + a_2a_3) - 2(a_1 + a_2 + a_3), \\
5!P_5 \leq f_5 &= a_1a_2a_3a_4a_5 - 2(a_1a_2a_3a_4 + a_1a_2a_3a_5 + a_1a_2a_4a_5 + a_1a_3a_4a_5 + a_2a_3a_4a_5) \\
&\quad + \frac{35}{4}(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4) \\
&\quad - \frac{50}{6}(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_2a_5) + 6(a_1 + a_2 + a_3 + a_4).
\end{aligned}$$

They then proposed a general conjecture:

Conjecture 1.1 (Granville-Lin-Yau (GLY) Conjecture) Let P_n = number of element of set $\big\{(x_1, x_2, \ldots, x_n) \in \mathbf{Z}^n_+ ; \frac{x_1}{a_1} + \frac{x_2}{a_2} + \ldots, + \frac{x_n}{a_n} \leq 1\big\}$. Let $n \geq 3$,

(1) Sharp Estimate: if $a_1 \geq a_2 \geq \cdots \geq a_n \geq n-1$, then

$$n!P_n \leq f_n := A_0^n + \frac{s(n,n-1)}{n} A_1^n + \sum_{l=1}^{n-2} \frac{s(n,n-1-l)}{\binom{n-1}{l}} A_l^{n-1}, \tag{1.4}$$

where $s(n,k)$ is the Stirling number of the first kind defined by generating function:

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k,$$

and A_k^n is defined as

$$A_k^n = \left(\prod_{i=1}^{n} a_i\right)\left(\sum_{1\leq i_1<i_2<\cdots<i_k\leq n} \frac{1}{a_{i_1}a_{i_2}\ldots a_{i_k}}\right),$$

for $k = 1, 2, \ldots, n-1$. Equality holds if and only if $a_1 = a_2 = \cdots = a_n =$ integer.

(2) Weak Estimate: If $a_1 \geq a_2 \geq \cdots \geq a_n > 1$

$$n!P_n < q_n := \prod_{i=1}^{n}(a_i - 1). \tag{1.5}$$

These estimates are all polynomials of a_i. They are sharp because the equality holds true if and only if all a_i take the same integer. The weak estimate in (1.5) has recently been proven true by the authors of [28]. Before that, [15], [16], [25], [27] showed that (1.5) holds for $3 \leq n \leq 5$. The sharp estimate conjecture was first formulated in [17]. In private communication to the second author, Granville formulated this sharp estimated conjecture independently after reading [15]. Again, the sharp GLY Conjecture has been proven individually for $n = 3, 4, 5$ by [16], [26], and [27] respectively. It has also been proven generally for $n \leq 6$ in [24]. However, for $n = 7$, a counterexample to the conjecture has been given. The YZZ conjecture, an improved version of GLY conjecture, has been studied in [29].

Counterexample to the GLY Conjecture. Take $n = 7$. Let $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 2000$ and $a_7 = 6.09$. Consider the following 7-dimensional tetrahedron: $x_i > 0$, $1 \leq i \leq 7$,

$$\frac{x_1}{2000} + \frac{x_2}{2000} + \frac{x_3}{2000} + \frac{x_4}{2000} + \frac{x_5}{2000} + \frac{x_6}{2000} + \frac{x_7}{6.09} \leq 1.$$

P_7 has been computed to be $3.9656226290532420 \times 10^{16}$. Meanwhile, $f_7 = 1.99840413 \times 10^{20}$ when $a_1 = a_2 = \cdots = a_6 = 2000$, $a_7 = 6.09$. Thus,

$$f_7 - 7!P_7 = -2.69675 \times 10^{16}.$$

This implies that the sharp estimate of GLY Conjecture fails in the case $n = 7$.

In order to characterize the homogeneous polynomial with isolated singularity, the second author made the following conjecture in 1995.

Conjecture 1.2 (Yau Geometric Conjecture) Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a weighted homogeneous polynomial with isolated critical points at the origin. Let μ, P_g and ν be the Milnor number, geometric genus and multiplicity of the singularity $V = \{z : f(z) = 0\}$. Then

$$\mu - h(\nu) \geq (n+1)!P_g, \tag{1.6}$$

where $h(\nu) = (\nu-1)^{n+1} - \nu(\nu-1)\cdots(\nu-n)$, and equality holds if and only if f is a homogeneous polynomial.

The Yau Geometric Conjecture was answered affirmatively for $n = 3, 4, 5$ by [3], [16] and [26] respectively.

In order to overcome the difficulty that the GLY sharp estimate conjecture is only true if a_n is larger than $y(n)$, a positive integer depending on n, the second author propose to prove a new sharp polynomial estimate conjecture which is motivated from the Yau Geometric Conjecture. The importance of this conjecture is that we only need $a_n > 1$ and hence the conjecture will give a sharp upper estimate of the Dickman-de Bruijn function $\psi(x, y)$.

Conjecture 1.3 Assume that $a_1 \geq a_2 \geq \cdots \geq a_n > 1$, $n \geq 3$ and let P_n = number of element of set $\left\{(x_1, x_2, \ldots, x_n) \in \mathbf{Z}_+^n; \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1\right\}$. If $P_n > 0$, then

$$n!P_n \leq (a_1 - 1)(a_2 - 1)\cdots(a_n - 1) - (a_n - 1)^n + a_n(a_n - 1)\cdots(a_n - (n - 1)) \tag{1.7}$$

and equality holds if and only if $a_1 = a_2 = \cdots = a_n =$ integer.

Obviously, there is an intimate relation between the Yau Geometric Conjecture (1.6) and the number theoretic conjecture (1.7). Recall that if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a weighted homogeneous polynomial with isolated singularity at the origin, then the multiplicity ν of f at the origin is given by $\inf\{n \in \mathbb{Z}_+ : n \geq \inf\{w_1, \ldots, w_n\}\}$, where w_i is the weight of x_i. Notice that in general, w_i is only a rational number. In case the minimal weight is an integer, then the Yau Geometric Conjecture (1.6) and the number theoretic conjecture (1.7) are the same. In general, these two conjectures do not imply each other, although they are intimately related.

The number theoretic conjecture (1.7) is much sharper than the weak GLY conjecture (1.5). The estimate in (1.7) is optimal in the sense that the equality occurs precisely when $a_1 = a_2 = \cdots = a_n =$ integer. Moreover, the sharp GLY conjecture (1.4) does not hold for $n = 7$ as the counterexample shows. However, the number theoretic conjecture (1.7) does hold for this example.

In this paper, two different kinds of unexpected important results are given, i.e., Theorem 1.4 and Theorem 1.5. We show the number theoretic conjecture is true for $n = 4$ in Theorem 1.4. By the previous works of Xu and Yau [25], [27], it was shown that the number theoretic conjecture is true for $n = 3$. However, the method used in [25], [27] cannot be generalized in higher dimension. The estimates there are totally different from the current one. The main difficulties are the cases when a_4 are lying between 1 and 3. In our previous papers [25] and [27], the estimate cannot cover these cases. This is the reason why all our previous results cannot be used to give estimate the Dickman-de Bruijn function. The major breakthrough of this paper is that we discover some inequalities which can be used to simplify our calculation of estimations. Then we are able to prove Theorem 1.4 for $n = 4$. Our method is new, complete, and it shed a light for the number theory conjecture in higher dimension. Furthermore, in Theorem 1.5, we give an explicit formula for the estimate of Dickman-de Bruijn function $\psi(x, y)$, when $y < 11$. Mathematica 4.0 is adopted to do some involved computations. The following are our main theorems.

Theorem 1.4 (Number theoretic conjecture for $n = 4$) *Let $a_1 \geq a_2 \geq a_3 \geq a_4 > 1$ be real numbers. Let P_4 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} \leq 1$, i.e.*

$$P_4 = \sharp\left\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}_+^4 : \frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} \leq 1\right\},$$

where $\mathbb{Z}_+$ is the set of positive integers. If $P_4 > 0$, then

$$24P_4 \leq (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1) - (a_4 - 1)^4 + a_4(a_4 - 1)(a_4 - 2)(a_4 - 3)$$

and the equality holds if and only if $a_1 = a_2 = a_3 = a_4 =$ integer. This can also be expressed as

$$\begin{aligned} 24P_4 \leq\ & a_1a_2a_3a_4 - (a_1a_2a_3 + a_1a_2a_4 + a_2a_3a_4 + a_1a_3a_4) - 2a_4^3 \\ & + (a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4) + 5a_4^2 - (a_1 + a_2 + a_3) - 3a_4. \end{aligned} \tag{1.8}$$

Theorem 1.5 (Estimate of the Dickman-de Bruijn function) *Let $\psi(x, y)$ be the Dickman-de Bruijn function. We have the following upper estimate for $5 \leq y < 11$:*

(i) *when $5 \leq y < 7$ and $x > 5$, we have*

$$\begin{aligned} \psi(x, y) \leq \frac{1}{6}\Bigg\{ & \frac{1}{\log 2 \log 3 \log 5} \log^3 x + \frac{\log 900}{\log 2 \log 3 \log 5} \log^2 x \\ & + \left[\frac{1}{\log 2 \log 3 \log 5}(\log 15 \log 10 + \ln 10 \log 6 + \log 15 \log 6) - \frac{1}{\log 5}\right] \log x \\ & + \frac{\log 6(\log 15 \log 10 - \log 2 \log 3)}{\log 2 \log 3 \log 5}\Bigg\}; \end{aligned}$$

(ii) *when* $7 \le y < 11$ *and* $x > 11$, *we have*

$$\psi(x,y) \le \frac{1}{24}\Bigg\{\frac{1}{\log 2\log 3\log 5\log 7}\log^4 x + \left[\frac{\log 105+\log 70+\log 42+\log 30}{\log 2\log 3\log 5\log 7} - \frac{2}{\log^2 7}\right]\log^3 x$$
$$+ \Bigg[\frac{1}{\log 2\log 3\log 5\log 7}(\log 42\log 30+\log 70\log 30+\log 105\log 30$$
$$+\log 42\log 70+\log 42\log 105+\log 70\log 105)$$
$$-\frac{1}{\log^3 7}(\log 7+6\log 30)\Bigg]\log^2 x$$
$$+\Bigg[\frac{1}{\log 2\log 3\log 5\log 7}(\log 70\log 42\log 30+\log 105\log 42\log 30$$
$$+\log 105\log 70\log 30+\log 105\log 70\log 42)$$
$$-\frac{2}{\log^3 7}\big(-\log^2 7+\log 7\log 30+3\log^2 30\big)\Bigg]\log x$$
$$+\frac{\log 105\log 70\log 42\log 30}{\log 2\log 3\log 5\log 7} - \frac{1}{\log^3 7}\big(-2\log^2 7\log 30+\log 7\log^2 30+2\log^3 30]\big)\Bigg\}.$$

2 Proof of theorems

2.1 Proof of Theorem 1.1

Our strategy is to divide our proof of the main theorem into three cases:

(1) $a_4 > 3$;
(2) $3 \ge a_4 > 2$;
(3) $2 \ge a_4 > 1$.

To prove case (1), we only need to notice the main theorem in [27].

Theorem 2.1 ([27]) *Let* $a_1 \ge a_2 \ge a_3 \ge a_4 \ge 2$ *be real numbers. Let* P_4 *be the number of positive integral points satisfying*

$$\frac{x_1}{a_1}+\frac{x_2}{a_2}+\frac{x_3}{a_3}+\frac{x_4}{a_4} \le 1.$$

Then

$$24P_4 \le a_1a_2a_3a_4 - \frac{3}{2}(a_1a_2a_3+a_1a_2a_4+a_2a_3a_4+a_1a_3a_4)$$
$$+\frac{11}{3}(a_1a_2+a_1a_3+a_2a_3) - 2(a_1+a_2+a_3), \tag{2.1}$$

and the equality is attained if and only if $a_1 = a_2 = a_3 = a_4 =$ *integer.*

Case (1) is solved by showing that our sharp upper bound is larger than or equal to theirs, and the equality holds if and only if $a_1 = a_2 = a_3 = a_4$.

Lemma 2.2 *When* $a_4 > 3$, *R.H.S. of* (1.8) $\ge$ *R.H.S. of* (2.1).

P r o o f . We first subtract R.H.S. of (1.8) by R.H.S. of (2.1):
R.H.S. of (1.8) -R.H.S. of (2.1)

$$= (a_1+a_2+a_3-3a_4)+(a_1+a_2+a_3)a_4+5a_4^2-\frac{8}{3}(a_1a_2+a_1a_3+a_2a_3)$$
$$+\frac{a_1a_2a_3}{2}+\frac{a_1a_2+a_1a_3+a_2a_3}{2}a_4-2a_4^3$$

$$
\begin{aligned}
&= a_1 + a_2 + a_3 - 3a_4 + \frac{a_1a_2a_3}{2} - \frac{a_4^3}{2} + \frac{a_4}{2}\big(a_1a_2 + a_1a_3 + a_2a_3 - 3a_4^2\big) \\
&\quad + (a_1a_4 + a_2a_4 + a_3a_4 - a_1a_2 - a_1a_3 - a_2a_3) + \frac{5}{3}\big(3a_4^2 - a_1a_2 - a_1a_3 - a_2a_3\big) \\
&= \left(\frac{a_4}{2} - \frac{5}{3}\right)\big(a_1a_2 + a_1a_3 + a_2a_3 - 3a_4^2\big) + (a_1 + a_2 + a_3 - 3a_4) \\
&\quad + \frac{1}{2}\big[a_1a_2a_3 - a_4^3 + 2a_1a_4 + 2a_2a_4 + 2a_3a_4 - 2a_1a_2 - 2a_1a_3 - 2a_2a_3\big]. \qquad (2.2)
\end{aligned}
$$

Now we consider $a_1a_2a_3$ in the following form:

$$
\begin{aligned}
a_1a_2a_3 &= (a_1 - a_4 + a_4)(a_2 - a_4 + a_4)(a_3 - a_4 + a_4) \\
&= (a_1 - a_4)(a_2 - a_4)(a_3 - a_4) \\
&\quad + (a_1 - a_4)(a_2 - a_4)a_4 + (a_1 - a_4)(a_3 - a_4)a_4 + (a_2 - a_4)(a_3 - a_4)a_4 \\
&\quad + (a_1 - a_4)a_4^2 + (a_2 - a_4)a_4^2 + (a_3 - a_4)a_4^2 + a_4^3 \\
&\geq (a_1 - a_4)(a_2 - a_4)(a_3 - a_4) \\
&\quad + \frac{7}{3}[(a_1 - a_4)(a_2 - a_4) + (a_1 - a_4)(a_3 - a_4) + (a_2 - a_4)(a_3 - a_4)] \\
&\quad + \frac{8}{3}[(a_1 - a_4)a_4 + (a_2 - a_4)a_4 + (a_3 - a_4)a_4] + a_4^3 \\
&= (a_1 - a_4)(a_2 - a_4)(a_3 - a_4) \\
&\quad + \frac{7}{3}(a_1a_2 + a_1a_3 + a_2a_3) - \frac{14}{3}(a_1a_4 + a_2a_4 + a_3a_4) + 7a_4^2 \\
&\quad + \frac{8}{3}(a_1a_4 + a_2a_4 + a_3a_4) - 8a_4^2 + a_4^3 \\
&= (a_1 - a_4)(a_2 - a_4)(a_3 - a_4) \\
&\quad + \frac{7}{3}(a_1a_2 + a_1a_3 + a_2a_3) - 2(a_1a_4 + a_2a_4 + a_3a_4) - a_4^2 + a_4^3, \qquad (2.3)
\end{aligned}
$$

since $a_1 \geq a_2 \geq a_3 \geq a_4 > 3$. Substitute (2.3) back to (2.2):

R.H.S. of (1.8) −R.H.S. of (2.1)

$$
\begin{aligned}
&\geq \left(\frac{3}{2} - \frac{5}{3}\right)\big(a_1a_2 + a_1a_3 + a_2a_3 - 3a_4^2\big) + (a_1 + a_2 + a_3 - 3a_4) \\
&\quad + \frac{1}{2}\Big[(a_1 - a_4)(a_2 - a_4)(a_3 - a_4) + \frac{1}{3}(a_1a_2 + a_1a_3 + a_2a_3) - a_4^2\Big] \\
&= \frac{1}{2}(a_1 - a_4)(a_2 - a_4)(a_3 - a_4) + (a_1 + a_2 + a_3 - 3a_4) \geq 0,
\end{aligned}
$$

since $a_1 \geq a_2 \geq a_3 \geq a_4 > 3$.

It is easy to see that the equalities can be attained if and only if $a_1 = a_2 = a_3 = a_4$. So R.H.S. of (1.8) = R.H.S. of (2.1). By Theorem 2.1, the equality in (2.1) holds if and only if $a_1 = a_2 = a_3 = a_4 =$ integer. Therefore, the equality in (1.8) is achieved if and only if $a_1 = a_2 = a_3 = a_4 =$ integer. □

For case (2) and (3), we adopt the similar strategy: basically, we partition the 4th tetrahedron into 3th tetrahedra [24]. We have:

$$\begin{aligned}&\frac{x_1}{a_1}+\frac{x_2}{a_2}+\frac{x_3}{a_3}+\frac{k}{a_4}\le 1,\\&\frac{x_1}{a_1(1-\frac{k}{a_4})}+\frac{x_2}{a_2(1-\frac{k}{a_4})}+\frac{x_3}{a_3(1-\frac{k}{a_4})}\le 1,\end{aligned}\tag{2.4}$$

for $k=1,\dots,[a_4]$, where $[a_4]$ is the largest integer less than a_4. Let $P_3(k)$ be the number of positive integral solutions of (2.4). Then

$$P_4=\sum_{k=1}^{[a_4]}P_3(k).\tag{2.5}$$

According to Theorem 2.1 in [27], if $P_3(k)>0$, then we have

$$24P_3(k)\le 4\left[\left(a_1\left(1-\frac{k}{a_4}\right)-1\right)\left(a_2\left(1-\frac{k}{a_4}\right)-1\right)\left(a_3\left(1-\frac{k}{a_4}\right)-1\right)-a_3\left(1-\frac{k}{a_4}\right)+1\right].$$

By (2.5), we have

$$\begin{aligned}24P_4&=24\sum_{k=1}^{[a_4]}P_3(k)\\&\le 4\sum_{k=1}^{[a_4]}\left[\left(a_1\left(1-\frac{k}{a_4}\right)-1\right)\left(a_2\left(1-\frac{k}{a_4}\right)-1\right)\left(a_3\left(1-\frac{k}{a_4}\right)-1\right)-a_3\left(1-\frac{k}{a_4}\right)+1\right],\end{aligned}\tag{2.6}$$

if $P_3(k)>0$, for $k=1,\dots,[a_4]$. Otherwise, add nothing in the corresponding level, when $P_3(k)=0$, for some k. In order to prove (1.8), it is sufficient to show that R.H.S. of (1.8) $\ge$ R.H.S. of (2.6). For both case (2) and (3), the equality in (1.8) can't be attained by any chance. On the one hand, $P_4>0$ won't be satisfied if $a_1=a_2=a_3=a_4\le 3$. On the other hand, we could show that R.H.S. of (1.8) is strictly larger than R.H.S. of (2.6) in these two cases. Therefore, no such $a_1\ge a_2\ge a_3\ge a_4$ and $a_4\in(1,3]$ could make the equality in (1.8) happen.

Now, for case (3), there are two levels $k=1$ and $k=2$. But it is easy to see that $P_3(2)=0$. And from the condition $P_4>0$, we know that the level $k=1$ can't have no positive integral solution, i.e. $P_3(1)=P_4>0$. It is also implied that the smallest integral solution $(1,1,1,1)$ must be its solution, which gives that $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\le 1-\frac{1}{a_4}$. This condition is very useful in our following proof. For convenience of computation below, let $\beta\triangleq 1-\frac{1}{a_4}\in\left(0,\frac{1}{2}\right]$, since $a_4\in(1,2]$. (2.6) can be rewritten as

$$24P_4=24P_3\le 4[(a_1\beta-1)(a_2\beta-1)(a_3\beta-1)-a_3\beta+1].\tag{2.7}$$

To prove (1.8) in this case, it is sufficient to show that R.H.S. of (1.8) > R.H.S. of (2.7).

Lemma 2.3 *When $1<a_4\le 2$ (i.e. $0<\beta\le\frac{1}{2}$), R.H.S. of* (1.8) *> R.H.S. of* (2.7).

P r o o f. Substitute $a_4=\frac{1}{1-\beta}$ in R.H.S of (1.8), subtract that by R.H.S. of (2.7) and multiply $\frac{(1-\beta)^3}{\beta}$, we get:

$$\begin{aligned}F(\beta)\triangleq{}& a_1a_2a_3\left(4\beta^5-12\beta^4+12\beta^3-3\beta^2-2\beta+1\right)\\&+(a_1a_2+a_1a_3+a_2a_3)\left(-4\beta^4+12\beta^3-13\beta^2+6\beta-1\right)\\&+(a_1+a_2)\left(4\beta^3-11\beta^2+10\beta-3\right)+a_3\left(\beta^2-2\beta+1\right)-3\beta+1\\={}&\left(a_1a_2a_3-\frac{1}{\beta}(a_1a_2+a_1a_3+a_2a_3)\right)\left(4\beta^5-12\beta^4+12\beta^3-3\beta^2-2\beta+1\right)\\&+(a_1a_2+a_1a_3+a_2a_3)\left(-\beta^2+3\beta-3+\frac{1}{\beta}\right)\end{aligned}$$

$$
\begin{aligned}
&+ (a_1 + a_2)\left(4\beta^3 - 11\beta^2 + 10\beta - 3\right) + a_3\left(\beta^2 - 2\beta + 1\right) - 3\beta + 1 \\
\geq\ & (a_1a_2 + a_1a_3 + a_2a_3)\left(-\beta^2 + 3\beta - 3 + \frac{1}{\beta}\right) \\
&+ (a_1 + a_2)\left(4\beta^3 - 11\beta^2 + 10\beta - 3\right) + a_3\left(\beta^2 - 2\beta + 1\right) - 3\beta + 1, \qquad (2.8)
\end{aligned}
$$

since $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \leq \beta$, i.e. $a_1a_2a_3 \geq \frac{1}{\beta}(a_1a_2 + a_1a_3 + a_2a_3)$ and $4\beta^5 - 12\beta^4 + 12\beta^3 - 3\beta^2 - 2\beta + 1 \geq 0$, if $\beta \in \left(0, \frac{1}{2}\right]$. In fact,

$$
\begin{aligned}
P(\beta) &= 4\beta^5 - 12\beta^4 + 11\beta^3 - 3\beta^2 - 2\beta + 1 \\
&= (2\beta - 1)\left(2\beta^4 - 5\beta^3 + 3\beta^2 - 1\right).
\end{aligned}
$$

Let $f(\beta) = 2\beta^4 - 5\beta^3 + 3\beta^2 - 1$. We claim that $f(\beta)$ is an increasing function in $\left(0, \frac{1}{2}\right]$. Consider $f'(\beta) = 8\beta^3 - 15\beta^2 + 6\beta = \beta\left[8(\beta - \frac{15}{16})^2 - \frac{33}{32}\right]$. Thus, $\frac{f'(\beta)}{\beta} \geq \frac{f'(\frac{1}{2})}{\frac{1}{2}} = \frac{1}{2} > 0$, which implies that $f'(\beta) > 0$, for $\beta \in \left(0, \frac{1}{2}\right]$. Moreover, $f(\beta) \leq f(\frac{1}{2}) = -\frac{3}{4} < 0$, for $\beta \in \left(0, \frac{1}{2}\right]$. Therefore, $P(\beta) \geq 0$, for $\beta \in \left(0, \frac{1}{2}\right]$. Back to $F(\beta)$:

$$
\begin{aligned}
F(\beta) \geq\ & \left(a_1a_2 + a_1a_3 + a_2a_3 - \frac{2}{\beta}(a_1 + a_2 + a_3)\right)\left(-\beta^2 + 3\beta - 3 + \frac{1}{\beta}\right) \\
&+ (a_1 + a_2 + a_3)\left(-2\beta + 6 - \frac{6}{\beta} + \frac{2}{\beta^2}\right) \\
&+ (a_1 + a_2)\left(4\beta^3 - 11\beta^2 + 10\beta - 3\right) + a_3\left(\beta^2 - 2\beta + 1\right) - 3\beta + 1 \\
\geq\ & (a_1 + a_2)\left(4\beta^3 - 11\beta^2 + 8\beta + 3 - \frac{6}{\beta} + \frac{2}{\beta^2}\right) \\
&+ a_3\left(\beta^2 - 4\beta + 7 - \frac{6}{\beta} + \frac{2}{\beta^2}\right) - 3\beta + 1,
\end{aligned}
$$

since $a_ia_j \geq \frac{1}{\beta}(a_i + a_j)$, $i, j = 1, 2, 3$, $i \neq j$ and $-\beta^2 + 3\beta - 3 + \frac{1}{\beta} = \frac{1}{\beta}(1 - \beta)^3 > 0$ for $\beta \in \left(0, \frac{1}{2}\right]$. Indeed, $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \leq \beta$ implies that $\frac{1}{a_i} + \frac{1}{a_j} \leq \beta$, for $i, j = 1, 2, 3$, $i \neq j$. That is, $a_ia_j \geq \frac{1}{\beta}(a_i + a_j)$, $i, j = 1, 2, 3$, $i \neq j$. Back to $F(\beta)$ again:

$$
\begin{aligned}
F(\beta) \geq\ & 6(a_1 + a_2)\beta^3 + (a_1 + a_2)\left(-2\beta^3 - 11\beta^2 + 8\beta + 3 - \frac{6}{\beta} + \frac{2}{\beta^2}\right) \\
&+ a_3\left(\beta^2 - 4\beta + 7 - \frac{6}{\beta} + \frac{2}{\beta^2}\right) - 3\beta + 1 \\
>\ & 6(a_1 + a_2)\beta^3 + a_3\left(\beta^2 - 4\beta + 7 - \frac{6}{\beta} + \frac{2}{\beta^2}\right) - 3\beta + 1,
\end{aligned}
$$

since $a_1 \geq a_2 \geq a_4 > 1$ and $-2\beta^3 - 11\beta^2 + 8\beta + 3 - \frac{6}{\beta} + \frac{2}{\beta^2} > 0$, for $\beta \in \left(0, \frac{1}{2}\right]$. In fact, let

$$
\begin{aligned}
P(\beta) &= -2\beta^3 - 11\beta^2 + 8\beta + 3 - \frac{6}{\beta} + \frac{2}{\beta^2} \\
&= \frac{1}{\beta^2}(2\beta - 1)\left(-\beta^4 - 6\beta^3 + \beta^2 + 2\beta - 2\right) > 0,
\end{aligned}
$$

since $2\beta - 1 \leq 0$ and $-\beta^4 - 6\beta^3 + \beta^2 + 2\beta - 2 \leq -\beta^4 + \beta^2 - 1 = -\left(\beta^2 - \frac{1}{2}\right)^2 - \frac{3}{4} < 0$, for $\beta \in \left(0, \frac{1}{2}\right]$. Come back to $F(\beta)$:

$$
F(\beta) > 6(a_1 + a_2)\beta^3 - 3\beta + 1,
$$

since $a_3 \geq a_4 > 1$ and $\beta^2 - 4\beta + 7 - \frac{6}{\beta} + \frac{2}{\beta^2} = \frac{1}{\beta^2}(\beta-1)^2\left(\beta^2 - 2\beta + 2\right) = \frac{1}{\beta^2}(\beta-1)^2\left[(\beta-1)^2+1\right] > 0$, for $\beta \in \left(0, \frac{1}{2}\right]$. Our last step is

$$F(\beta) > 12\beta^3 - 3\beta + 1 > 0,$$

since $a_1 \geq a_2 \geq a_4 > 1$. Indeed, let $f(\beta) = 12\beta^3 - 3\beta + 1$, thus $f'(\beta) = 36\beta^2 - 3$. It is easy to see that $\beta = \frac{\sqrt{3}}{6}$ is the minimum point in $\left(0, \frac{1}{2}\right]$, since $f''\left(\frac{\sqrt{3}}{6}\right) = 12\sqrt{3} > 0$. Therefore, $f(\beta) \geq f\left(\frac{\sqrt{3}}{6}\right) = 1 - \frac{\sqrt{3}}{3} > 0$. □

Last, we treat case (2). In this case, there are three levels $k = 1, 2$ and $k = 3$. It is also easy to see that $P_3(3) = 0$. From $P_4 > 0$, we could only say that $P_3(1) > 0$, but the positivity of $P_3(2)$ is unknown. Thus, we split this case into two subcases:

(i) $P_3(2) = 0$;
(ii) $P_3(2) > 0$.

For subcase (i), $P_4 = P_3(1)$, we still have (2.7), where $\beta \triangleq 1 - \frac{1}{a_4} \in \left(\frac{1}{2}, \frac{2}{3}\right]$, since $a_4 \in (2, 3]$. Moreover, $(1, 1, 1, 1)$ must be the solution, which gives that $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \leq \beta$. To show (1.8), it is sufficient to show the following lemma, which is similar to Lemma 2.3.

Lemma 2.4 *R.H.S. of* (1.8) > *R.H.S. of* (2.6)*, for* $a_4 \in (2, 3]$ *(i.e.* $\beta \in \left(\frac{1}{2}, \frac{2}{3}\right]$*).*

Proof. Again substitute $a_4 \frac{1}{1-\beta}$ in R.H.S. of (1.8), subtract that by R.H.S. of (2.7) and multiply $\frac{(1-\beta)^3}{\beta}$, we can get:

$$\begin{aligned}
F(\beta) &\triangleq a_1a_2a_3\left(4\beta^5 - 12\beta^4 + 12\beta^3 - 3\beta^2 - 2\beta + 1\right)\\
&\quad + (a_1a_2 + a_1a_3 + a_2a_3)\left(-4\beta^4 + 12\beta^3 - 13\beta^2 + 6\beta - 1\right)\\
&\quad + (a_1 + a_2)\left(4\beta^3 - 11\beta^2 + 10\beta - 3\right) + a_3\left(\beta^2 - 2\beta + 1\right) - 3\beta + 1\\
&= \left(a_1a_2a_3 - \frac{1}{\beta}(a_1a_2 + a_1a_3 + a_2a_3)\right)\left(4\beta^5 - 12\beta^4 + 12\beta^3 - 3\beta^2 - 2\beta + 1\right)\\
&\quad + (a_1a_2 + a_1a_3 + a_2a_3)\left(-\beta^2 + 3\beta - 3 + \frac{1}{\beta}\right)\\
&\quad + (a_1 + a_2)\left(4\beta^3 - 11\beta^2 + 10\beta - 3\right) + a_3\left(\beta^2 - 2\beta + 1\right) - 3\beta + 1\\
&\geq (a_1a_2 + a_1a_3 + a_2a_3)\left(-\beta^2 + 3\beta - 3 + \frac{1}{\beta}\right)\\
&\quad + \left(a_1 + a_2\right)(4\beta^3 - 11\beta^2 + 10\beta - 3\right) + a_3\left(\beta^2 - 2\beta + 1\right) - 3\beta + 1,
\end{aligned}$$

since $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \leq \beta$ and $4\beta^5 - 12\beta^4 + 12\beta^3 - 3\beta^2 - 2\beta + 1 \geq 0$, for $\beta \in \left(\frac{1}{2}, \frac{2}{3}\right]$. In fact, let

$$P(\beta) = 4\beta^5 - 12\beta^4 + 12\beta^3 - 3\beta^2 - 2\beta + 1 = (\beta - 1)^2\left(4\beta^3 - 4\beta^2 + 1\right).$$

Claim that $f(\beta) = 4\beta^3 - 4\beta^2 + 1 > 0$, for $\beta \in \left(\frac{1}{2}, \frac{2}{3}\right]$. $f(\beta)$ is decreasing in $\left(\frac{1}{2}, \frac{2}{3}\right]$, since $f'(\beta) = 4\beta(3\beta - 2) \leq 0$. Therefore, $f(\beta) \geq f\left(\frac{2}{3}\right) = \frac{11}{27} > 0$. Thus, $P(\beta) \geq 0$, for $\beta \in \left(\frac{1}{2}, \frac{2}{3}\right]$. Back to $F(\beta)$:

$$\begin{aligned}
F(\beta) &\geq \left(a_1a_2 + a_1a_3 + a_2a_3 - \frac{2}{\beta}(a_1 + a_2 + a_3)\right)\left(-\beta^2 + 3\beta - 3 + \frac{1}{\beta}\right)\\
&\quad + (a_1 + a_2 + a_3)\left(-2\beta + 6 - \frac{6}{\beta} + \frac{2}{\beta^2}\right)\\
&\quad + (a_1 + a_2)\left(4\beta^3 - 11\beta^2 + 10\beta - 3\right) + a_3\left(\beta^2 - 2\beta + 1\right) - 3\beta + 1\\
&\geq (a_1 + a_2)\left(4\beta^3 - 11\beta^2 + 8\beta + 3 - \frac{6}{\beta} + \frac{2}{\beta^2}\right)\\
&\quad + a_3\left(\beta^2 - 4\beta + 7 - \frac{6}{\beta} + \frac{2}{\beta^2}\right) - 3\beta + 1,
\end{aligned}$$

since $a_i a_j \geq \frac{1}{\beta}(a_i + a_j)$, for $i, j = 1, 2, 3$ and $i \neq j$ and $-\beta^2 + 3\beta - 3 + \frac{1}{\beta} = \frac{1}{\beta}(1-\beta)^3 > 0$, for $\beta \in \left(\frac{1}{2}, \frac{2}{3}\right]$. Back to $F(\beta)$ again:

$$\begin{aligned} F(\beta) &\geq (a_1 + a_2 - 4)\left(4\beta^3 - 11\beta^2 + 8\beta + 3 - \frac{6}{\beta} + \frac{2}{\beta^2}\right) + (a_3 - 2)\left(\beta^2 - 4\beta + 7 - \frac{6}{\beta} + \frac{2}{\beta^2}\right) \\ &\quad + 4\left(4\beta^3 - 11\beta^2 + 8\beta + 3 - \frac{6}{\beta} + \frac{2}{\beta^2}\right) + 2\left(\beta^2 - 4\beta + 7 - \frac{6}{\beta} + \frac{2}{\beta^2}\right) - 3\beta + 1 \\ &> (a_3 - 2)\left(\beta^2 - 4\beta + 7 - \frac{6}{\beta} + \frac{2}{\beta^2}\right) + 16\beta^3 - 42\beta^2 + 21\beta + 27 - \frac{36}{\beta} + \frac{12}{\beta^2} \end{aligned}$$

since $a_1 \geq a_2 \geq a_3 \geq a_4 > 2$ and $4\beta^3 - 11\beta^2 + 8\beta + 3 - \frac{6}{\beta} + \frac{2}{\beta^2} = \frac{1}{\beta^2}(\beta - 1)^2(4\beta^3 - 3\beta^2 - 2\beta + 2) > 0$. Indeed, let $f(\beta) = 4\beta^3 - 3\beta^2 - 2\beta + 2 \geq -\beta^2 - 2\beta + 2 = -(\beta + 1)^2 + 3$, for $\beta \in \left(\frac{1}{2}, \frac{2}{3}\right]$. Thus, $f(\beta) \geq -(\frac{2}{3} + 1)^2 + 3 = \frac{2}{9} > 0$. Moreover,

$$F(\beta) > 16\beta^3 - 42\beta^2 + 21\beta + 27 - \frac{36}{\beta} + \frac{12}{\beta^2} > 0,$$

since $a_3 \geq a_4 > 2$, $\beta^2 - 4\beta + 7 - \frac{6}{\beta} + \frac{2}{\beta^2} = \frac{1}{\beta^2}(\beta - 1)^2(\beta^2 - 2\beta + 2) = \frac{1}{\beta^2}(\beta - 1)^2[(\beta - 1)^2 + 1] > 0$ and $16\beta^3 - 42\beta^2 + 21\beta + 27 - \frac{36}{\beta} + \frac{12}{\beta^2} > 0$, for $\beta \in \left(\frac{1}{2}, \frac{2}{3}\right]$. In fact, let

$$f(\beta) = 16\beta^5 - 42\beta^4 + 21\beta^3 + 27\beta^2 - 36\beta + 12,$$

then $f'(\beta) = 80\beta^4 - 168\beta^3 + 63\beta^2 + 54\beta - 36$. Claim that $f'(\beta)$ is increasing in $\left(\frac{1}{2}, \frac{2}{3}\right]$. Consider $f''(\beta) = 320\beta^3 - 504\beta^2 + 126\beta + 54 = 2(4\beta - 3)(40\beta^2 - 33\beta - 9)$. Let $g(\beta) = 40\beta^2 - 33\beta - 9 = 40(\beta - \frac{33}{80})^2 - \frac{2529}{160}$. Thus, $g(\beta) \leq g(\frac{2}{3}) = -\frac{119}{9} < 0$, for $\beta \in \left(\frac{1}{2}, \frac{2}{3}\right]$. With $4\beta - 3 < 0$, $f''(\beta) > 0$, for $\beta \in \left(\frac{1}{2}, \frac{2}{3}\right]$, which implies $f'(\beta)$ is increasing in $\left(\frac{1}{2}, \frac{2}{3}\right]$. Moreover, $f'(\beta) \leq f'(\frac{2}{3}) = -\frac{484}{81} < 0$, which implies that $f(\beta)$ is decreasing in $\left(\frac{1}{2}, \frac{2}{3}\right]$. Therefore, $f(\beta) \geq f(\frac{2}{3}) = \frac{8}{243} > 0$. □

We come to our last step, subcase (ii). $P_3(2) > 0$ implies that $(1, 1, 1, 2)$ must be the solution, which gives $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \leq 1 - \frac{2}{a_4} \triangleq \gamma$, where $\gamma \in \left(0, \frac{1}{3}\right]$, since $a_4 \in (2, 3]$. By (2.6), we have

$$\begin{aligned} 24P_4 &= 24[P_3(1) + P_3(2)] \\ &= 4\Bigg[\left(a_1\left(1 - \frac{1-\gamma}{2}\right) - 1\right)\left(a_2\left(1 - \frac{1-\gamma}{2}\right) - 1\right)\left(a_3\left(1 - \frac{1-\gamma}{2}\right) - 1\right) \\ &\quad - a_3\left(1 - \frac{1-\gamma}{2}\right) + 1 + (a_1\gamma - 1)(a_2\gamma - 1)(a_3\gamma - 1) - a_3\gamma + 1\Bigg]. \end{aligned} \tag{2.9}$$

Therefore, if we could show that R.H.S. of (1.8) is strictly larger than R.H.S. of (2.9), we've done.

Lemma 2.5 *When* $2 < a_4 \leq 3$ *(i.e.* $\gamma \in \left(0, \frac{1}{3}\right]$*), R.H.S. of* (1.8) $>$ *R.H.S. of* (2.9).

P r o o f. Substitute $a_4 = \frac{2}{1-\gamma}$ in R.H.S. of (1.8), subtract by R.H.S. of (2.9) and multiply $(1-\gamma)^3$, we get

$$\begin{aligned} G(\gamma) &\triangleq a_1a_2a_3\left(\frac{9}{2}\gamma^6 - 12\gamma^5 + \frac{21}{2}\gamma^4 - 3\gamma^3 + \frac{1}{2}\gamma^2 - \gamma + \frac{1}{2}\right) \\ &\quad + (a_1a_2 + a_2a_3 + a_1a_3)\left(-5\gamma^5 + 13\gamma^4 - 11\gamma^3 + 3\gamma^2\right) \\ &\quad + (a_1 + a_2)\left(6\gamma^4 - 15\gamma^3 + 11\gamma^2 - \gamma - 1\right) + a_3\left(\gamma^3 - \gamma^2 - \gamma + 1\right) + \left(-6\gamma^2 - 8\gamma - 2\right) \\ &= \left[a_1a_2a_3 - \frac{1}{\gamma}(a_1a_2 + a_1a_3 + a_2a_3)\right]\left(\frac{9}{2}\gamma^6 - 12\gamma^5 + \frac{21}{2}\gamma^4 - 3\gamma^3 + \frac{1}{2}\gamma^2 - \gamma + \frac{1}{2}\right) \\ &\quad + (a_1a_2 + a_2a_3 + a_1a_3)\left(-\frac{1}{2}\gamma^5 + \gamma^4 - \frac{1}{2}\gamma^3 + \frac{1}{2}\gamma - 1 + \frac{1}{2\gamma}\right) \end{aligned}$$

$$
\begin{aligned}
&+ (a_1+a_2)\left(6\gamma^4-15\gamma^3+11\gamma^2-\gamma-1\right)+a_3(\gamma^3-\gamma^2-\gamma+1\big)+\left(-6\gamma^2-8\gamma-2\right)\\
&\ge (a_1a_2+a_2a_3+a_1a_3)\left(-\frac{1}{2}\gamma^5+\gamma^4-\frac{1}{2}\gamma^3+\frac{1}{2}\gamma-1+\frac{1}{2\gamma}\right)\\
&\quad+(a_1+a_2)\left(6\gamma^4-15\gamma^3+11\gamma^2-\gamma-1\right)+a_3\left(\gamma^3-\gamma^2-\gamma+1\right)+\left(-6\gamma^2-8\gamma-2\right),
\end{aligned}
$$

since $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\le\gamma$ and $\frac{9}{2}\gamma^6-12\gamma^5+\frac{21}{2}\gamma^4-3\gamma^3+\frac{1}{2}\gamma^2-\gamma+\frac{1}{2}>0$, for $\gamma\in\left(0,\frac{1}{3}\right]$. Indeed, let

$$P(\gamma)=\frac{9}{2}\gamma^6-12\gamma^5+\frac{21}{2}\gamma^4-3\gamma^3+\frac{1}{2}\gamma^2-\gamma+\frac{1}{2}=\frac{1}{2}(\gamma-1)^2\left(9\gamma^4-6\gamma^3+1\right),$$

where $9\gamma^4-6\gamma^3+1\ge 9\gamma^4-2\gamma^2+1=9(\gamma^2-\frac{1}{9})^2+\frac{8}{9}>0$. Therefore, $P(\gamma)>0$, for $\gamma\in\left(0,\frac{1}{3}\right]$. Back to $G(\gamma)$:

$$
\begin{aligned}
G(\gamma)&\ge\left[(a_1a_2+a_1a_3+a_2a_3)-\frac{2}{\gamma}(a_1+a_2+a_3)\right]\left(-\frac{1}{2}\gamma^5+\gamma^4-\frac{1}{2}\gamma^3+\frac{1}{2}\gamma-1+\frac{1}{2\gamma}\right)\\
&\quad+2(a_1+a_2+a_3)\left(-\frac{1}{2}\gamma^4+\gamma^3-\frac{1}{2}\gamma^2+\frac{1}{2}-\frac{1}{\gamma}+\frac{1}{2\gamma^2}\right)\\
&\quad+(a_1+a_2)\left(6\gamma^4-15\gamma^3+11\gamma^2-\gamma-1\right)+a_3\left(\gamma^3-\gamma^2-\gamma+1\right)+\left(-6\gamma^2-8\gamma-2\right)\\
&\ge(a_1+a_2)\left(5\gamma^4-13\gamma^3+10\gamma^2-\gamma-\frac{2}{\gamma}+\frac{1}{\gamma^2}\right)\\
&\quad+a_3\left(-\gamma^4+3\gamma^3-2\gamma^2-\gamma+2-\frac{2}{\gamma}+\frac{1}{\gamma^2}\right)+\left(-6\gamma^2-8\gamma-2\right),
\end{aligned}
$$

since $a_ia_j\ge\frac{1}{\gamma}(a_i+a_j), i,j=1,2,3, i\ne j$ and $-\frac{1}{2}\gamma^5+\gamma^4-\frac{1}{2}\gamma^3+\frac{1}{2}\gamma-1+\frac{1}{2\gamma}>0$, for $\gamma\in\left(0,\frac{1}{3}\right]$. In fact, let

$$P(\gamma)=-\frac{1}{2}\gamma^5+\gamma^4-\frac{1}{2}\gamma^3+\frac{1}{2}\gamma-1+\frac{1}{2\gamma}>\frac{1}{\gamma}\left[-\frac{1}{2}\gamma^6-\frac{1}{6}\gamma^3+\frac{1}{6}\right].$$

And let $f(\gamma)=-\frac{1}{2}\gamma^6-\frac{1}{6}\gamma^3+\frac{1}{6}=-\frac{1}{2}(\gamma^3+\frac{1}{6})^2+\frac{13}{72}$, thus, $f(\gamma)\ge f(\frac{1}{3})=\frac{233}{1458}>0$. Therefore, $P(\gamma)>0$, for $\gamma\in\left(0,\frac{1}{3}\right]$. Back to $G(\gamma)$ again:

$$
\begin{aligned}
G(\gamma)&\ge(a_1+a_2)\left(5\gamma^4-13\gamma^3+4\gamma^2-\gamma-\frac{2}{\gamma}+\frac{1}{\gamma^2}\right)+6(a_1+a_2)\gamma^2\\
&\quad+a_3\left(-\gamma^4+3\gamma^3-2\gamma^2-\gamma-\frac{2}{\gamma}+\frac{1}{\gamma^2}\right)+2a_3\\
&\quad+\left(-6\gamma^2-8\gamma-2\right)\\
&\ge 6(a_1+a_2)\gamma^2+2a_3+\left(-6\gamma^2-8\gamma-2\right),
\end{aligned}
$$

since $a_1\ge a_2\ge a_3\ge a_4>2$, $5\gamma^4-13\gamma^3+4\gamma^2-\gamma-\frac{2}{\gamma}+\frac{1}{\gamma^2}>0$ and $-\gamma^4+3\gamma^3-2\gamma^2-\gamma-\frac{2}{\gamma}+\frac{1}{\gamma^2}>0$, for $\gamma\in\left(0,\frac{1}{3}\right]$. Indeed, on one hand, let

$$P(\gamma)=5\gamma^6-13\gamma^5+4\gamma^4-\gamma^3-2\gamma+1.$$

Claim that $P(\gamma)$ is decreasing in $\left(0,\frac{1}{3}\right]$. Consider $P'(\gamma)=30\gamma^5-65\gamma^4+16\gamma^3-3\gamma^2-2\le-55\gamma^4+\frac{7}{3}\gamma^2-2=-55(\gamma^2-\frac{7}{330})^2-\frac{3911}{1980}<0$. Therefore, $P(\gamma)>P(\frac{1}{3})=\frac{218}{729}>0$. On the other hand, let

$$
\begin{aligned}
Q(\gamma)&=-\gamma^4+3\gamma^3-2\gamma^2-\gamma-\frac{2}{\gamma}+\frac{1}{\gamma^2}\\
&=\frac{1}{\gamma^2}\left(-\gamma^6+3\gamma^5-2\gamma^4-\gamma^3-2\gamma+1\right)\\
&\ge\frac{1}{\gamma^2}\left(-\gamma^6-\frac{5}{3}\gamma^3+\frac{1}{3}\right).
\end{aligned}
$$

Let $f(\gamma) = -\gamma^6 - \frac{5}{3}\gamma^3 + \frac{1}{3} = -(\gamma^3 + \frac{5}{6})^2 + \frac{37}{36}$. Thus, $f(\gamma) > f(\frac{1}{3}) = \frac{197}{729} > 0$. Therefore, $Q(\gamma) > 0$, for $\gamma \in \left(0, \frac{1}{3}\right]$. Last,

$$G(\gamma) \geq 18\gamma^2 - 8\gamma + 2 > 0,$$

since $a_1 \geq a_2 \geq a_3 \geq a_4 > 2$ and $18\gamma^2 - 8\gamma + 2 > 0$, for $\gamma \in \left(0, \frac{1}{3}\right]$. In fact, $18\gamma^2 - 8\gamma + 2 > 6\gamma^2 - 8\gamma + 2 = 2(\gamma - 1)(3\gamma - 1) > 0$, for $\gamma \in \left(0, \frac{1}{3}\right]$. □

2.2 Proof of Theorem 1.2

As we state in introduction, the estimate of Dickman-de Bruijn function $\psi(x, y)$ is equivalent to a sharp estimate of Q_n (or P_n by (1.2)). We've already got the estimate of P_n, $n \leq 4$, thus, apply this to our estimate of $\psi(x, y)$. In detail, Let $p_1 < p_2 < \cdots < p_n$ be four prime numbers up to y. It is clear that $p_1^{l_1} p_2^{l_2} \cdots p_4^{l_4} \leq x$ if and only if $\frac{l_1}{\frac{\log x}{\log p_1}} + \frac{l_2}{\frac{\log x}{\log p_2}} \cdots \frac{l_n}{\frac{\log x}{\log p_n}} \leq 1$. Therefore, $\psi(x, y)$ is precisely the number Q_n of (1.1) with $a_i = \frac{\log x}{\log p_i}$, $1 \leq i \leq n$. Moreover, by (1.2), $\psi(x, y)$ is also precisely the number $P(a_1(1 + a), a_2(1 + a), \ldots, a_n(1 + a))$, where $a = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$.

According to the number of prime numbers up to y, we split the proof of Theorem 1.2 into four cases:

Case (i): $5 \leq y < 7$;

Case (ii): $7 \leq y < 11$.

For Case (i), we have three prime numbers $p_1 = 2$, $p_2 = 3$ and $p_3 = 5$, thus $a = \frac{\log 2 + \log 3 + \log 5}{\log x}$. Therefore,

$$\begin{aligned}
\psi(x, y) &= Q_3 \\
&= P\left(\frac{\log x}{\log 2}\left(1 + \frac{\log(2 \times 3 \times 5)}{\log x}\right), \frac{\log x}{\log 3}\left(1 + \frac{\log(2 \times 3 \times 5)}{\log x}\right),\right. \\
&\quad \left.\frac{\log x}{\log 5}\left(1 + \frac{\log(2 \times 3 \times 5)}{\log x}\right)\right) \\
&\leq \frac{1}{3!}\left\{\left(\frac{\log x}{\log 2} + \frac{\log 15}{\log 2}\right)\left(\frac{\log x}{\log 3} + \frac{\log 10}{\log 3}\right)\left(\frac{\log x}{\log 5} + \frac{\log 6}{\log 5}\right)\right. \\
&\quad \left. -\left[\left(\frac{\log x}{\log 5} + \frac{\log 6}{\log 5}\right)^3 - \left(\frac{\log x}{\log 5} + \frac{\log 6}{\log 5} + 1\right)\left(\frac{\log x}{\log 5} + \frac{\log 6}{\log 5}\right)\left(\frac{\log x}{\log 5} + \frac{\log 6}{\log 5} - 1\right)\right]\right\} \\
&= \frac{1}{6}\left\{\frac{1}{\log 2 \log 3 \log 5}(\log x + \log 15)(\log x + \log 10)(\log x + \log 6)\right. \\
&\quad \left. - \frac{1}{\log^3 5}\left[(\log x + \log 6)^3 - (\log x + \log 6 + \log 5)(\log x + \log 6)(\log x + \log 6 - \log 5)\right]\right\} \\
&= \frac{1}{6}\left\{\frac{1}{\log 2 \log 3 \log 5}\log^3 x + \frac{\log 6 + \log 10 + \log 15}{\log 2 \log 3 \log 5}\log^2 x\right. \\
&\quad + \left[\frac{1}{\log 2 \log 3 \log 5}(\log 15 \log 10 + \log 15 \log 6 + \log 10 \log 6) - \frac{1}{\log 5}\right]\log x \\
&\quad \left. + \frac{\log 6(\log 15 \log 10 - \log 2 \log 3)}{\log 2 \log 3 \log 5}\right\}.
\end{aligned}$$

For Case (ii), we have four prime numbers $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and $p_4 = 7$, thus $a = \frac{\log 2 + \log 3 + \log 5 + \log 7}{\log x}$. Therefore,

$$\psi(x, y) = Q_4$$

$$= P\left(\frac{\log x}{\log 2}\left(1+\frac{\log(2\times 3\times 5\times 7)}{\log x}\right), \frac{\log x}{\log 3}\left(1+\frac{\log(2\times 3\times 5\times 7)}{\log x}\right),\right.$$

$$\left.\frac{\log x}{\log 5}\left(1+\frac{\log(2\times 3\times 5\times 7)}{\log x}\right), \frac{\log x}{\log 7}\left(1+\frac{\log(2\times 3\times 5\times 7)}{\log x}\right)\right)$$

$$\leq \frac{1}{4!}\left\{\left(\frac{\log x}{\log 2}+\frac{3\times 5\times 7}{\log 2}\right)\left(\frac{\log x}{\log 3}+\frac{2\times 5\times 7}{\log 3}\right)\left(\frac{\log x}{\log 5}+\frac{2\times 3\times 7}{\log 5}\right)\right.$$

$$\times\left(\frac{\log x}{\log 7}+\frac{2\times 3\times 5}{\log 7}\right)$$

$$-\left[\left(\frac{\log x}{\log 7}+\frac{2\times 3\times 5}{\log 7}\right)^4-\left(\frac{\log x}{\log 7}+\frac{2\times 3\times 5}{\log 7}+1\right)\left(\frac{\log x}{\log 7}+\frac{2\times 3\times 5}{\log 7}\right)\right.$$

$$\left.\left.\left(\frac{\log x}{\log 7}+\frac{2\times 3\times 5}{\log 7}-1\right)\left(\frac{\log x}{\log 7}+\frac{2\times 3\times 5}{\log 7}-2\right)\right]\right\}$$

$$= \frac{1}{24}\left\{\frac{1}{\log 2\log 3\log 5\log 7}(\log x+\log(3\times 5\times 7))(\log x+\log(2\times 5\times 7))\right.$$

$$(\log x+\log(2\times 3\times 7))(\log x+\log(2\times 3\times 5))$$

$$-\frac{1}{\log^4 7}\left[(\log x+\log(2\times 3\times 5))^4\right.$$

$$-(\log x+\log(2\times 3\times 5)+\log 7)(\log x+\log(2\times 3\times 5))$$

$$\left.\left.(\log x+\log(2\times 3\times 5)-\log 7)(\log x+\log(2\times 3\times 5)-2\log 7)\right]\right\}$$

$$= \frac{1}{24}\left\{\frac{1}{\log 2\log 3\log 5\log 7}(\log x+\log 105)(\log x+\log 70)\right.$$

$$(\log x+\log 42)(\log x+\log 30)$$

$$-\frac{1}{\log^4 7}\left[(\log x+\log 30)^4-(\log x+\log 30+\log 7)(\log x+\log 30)\right.$$

$$\left.\left.(\log x+\log 30-\log 7)(\log x+\log 30-2\log 7)\right]\right\}$$

$$= \frac{1}{24}\left\{\frac{1}{\log 2\log 3\log 5\log 7}\log^4 x+\left[\frac{\log 105+\log 70+\log 42+\log 30}{\log 2\log 3\log 5\log 7}-\frac{2}{\log^2 7}\right]\log^3 x\right.$$

$$+\left[\frac{1}{\log 2\log 3\log 5\log 7}(\log 42\log 30+\log 70\log 30+\log 105\log 30\right.$$

$$+\log 42\log 70+\log 42\log 105+\log 70\log 105)$$

$$\left.-\frac{1}{\log^3 7}(\log 7+6\log 30)\right]\log^2 x$$

$$+\left[\frac{1}{\log 2\log 3\log 5\log 7}(\log 70\log 42\log 30+\log 105\log 42\log 30\right.$$

$$
\begin{aligned}
&\left(+\log 105 \log 70 \log 30+\log 105 \log 70 \log 42\right) \\
&\left.-\frac{2}{\log^3 7}\left(-\log^2 7+\log 7 \log 30+3 \log^2 30\right)\right] \log x \\
&\left.+\frac{\log 105 \log 70 \log 42 \log 30}{\log 2 \log 3 \log 5 \log 7}-\frac{1}{\log^3 7}\left(-2 \log^2 7 \log 30+\log 7 \log^2 30+2 \log^3 30\right)\right\}.
\end{aligned}
$$

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