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A sharp estimate of positive integral points in 6-dimensional polyhedra and a sharp estimate of smooth numbers

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Abstract Inspired by Durfee Conjecture in singularity theory, Yau formulated the Yau number theoretic conjecture (see Conjecture 1.3) which gives a sharp polynomial upper bound of the number of positive integral points in an n-dimensional ($n \geqslant 3$) polyhedron. It is well known that getting the estimate of integral points in the polyhedron is equivalent to getting the estimate of the de Bruijn function $\psi(x, y)$, which is important and has a number of applications to analytic number theory and cryptography. We prove the Yau number theoretic conjecture for $n = 6$. As an application, we give a sharper estimate of function $\psi(x, y)$ for $5 \leqslant y < 17$, compared with the result obtained by Ennola.

Keywords integral points, tetrahedron, sharp estimate

MSC(2010) 11P21, 11Y99

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1 Introduction

Let $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant 1$ be positive real numbers. An n-dimensional polyhedron $\Delta(a_1, a_2, \dots, a_n)$ is defined by

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leqslant 1, \quad x_1, x_2, \dots, x_n \geqslant 0. \tag{1.1}$$

Let

$$Q_n = Q(a_1, a_2, \dots, a_n) := \sharp\left\{(x_1, \dots, x_n) \in \mathbb{Z}^n_{\geqslant 0} : \sum_{i=1}^n \frac{x_i}{a_i} \leqslant 1\right\},$$

$$P_n = P(a_1, a_2, \dots, a_n) := \sharp\left\{(x_1, \dots, x_n) \in \mathbb{Z}^n_{+} : \sum_{i=1}^n \frac{x_i}{a_i} \leqslant 1\right\}.$$

Then they are related by the following formula:

$$Q(a_1, a_2, \dots, a_n) = P(a_1(1+a), a_2(1+a), \dots, a_n(1+a)),$$

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where

$$a = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.$$

So the study of P_n and Q_n is equivalent. In this paper, for the sake of applications in number theory and geometry, we are interested in the problem of estimating the number $P_n = P(a_1, a_2, \ldots, a_n)$ of positive integral points satisfying (1.1), where $a_1, a_2, \ldots, a_n$ are positive real numbers.

The estimate of integral points has many applications in number theory. According to Granville[1)], finding an upper polynomial estimate of $P(a_1, a_2, \ldots, a_n)$ is an extremely important subject in number theory. It could be applied to finding large gaps between primes, to Waring's problem, to primality testing and factoring algorithms, and to bounds for the least prime k-th power residues and non-residues (mod n). For more information about applications of $P(a_1, a_2, \ldots, a_n)$ and $Q(a_1, a_2, \ldots, a_n)$, see Pomerance's ICM 1994 lecture at Zurich [29] and his lecture notes [28].

In analytic number theory, a smooth number is a number with only small prime factors. In particular, a positive integer is y-smooth if it has no prime factor exceeding y. According to Pomerance [29], smooth numbers are a useful tool in number theory because they not only have a simple multiplicative structure, but are also fairly numerous. These properties of smooth numbers are the main reason they play a key role in almost every modern integer factorization algorithm. Smooth numbers play a similar essential role in discrete logarithm algorithms (methods to represent some group element as a power of another), and a lesser, but still important, role in primality tests. Recall that the Dickman-de Bruijn function $\rho(u)$ is a special continuous function that satisfies the delay differential equation $u\rho'(u) + \rho(u-1) = 0$ with initial conditions $\rho(u) = 1$ for $0 \leqslant u \leqslant 1$ and is used to estimate the proportion of smooth numbers up to a given bound. It was first studied by the actuary Dickman, who defined it in his only mathematical publication [7] and later studied by the Dutch mathematician de Bruijn [4, 5]. Dickman [7] showed heuristically that $\psi(x, x^{\frac{1}{a}}) \sim x\rho(a)$, where $\psi(x, y)$ is the number of y-smooth integers below x.

One of the central topics in computational number theory is the estimate of $\psi(x, y)$ (see [4, 5, 7, 10]). It turns out that the computation of $\psi(x, y)$ is equivalent to computing the number of integral points in a k-dimensional tetrahedron $\Delta(a_1, a_2, \ldots, a_k)$ with real vertices $(a_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, a_k)$. Let $p_1 < p_2 < \cdots < p_k$ denote the primes up to y. It is clear that $p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k} \leqslant x$ which is also equivalent to counting the number of $(l_1, l_2, \ldots, l_k) \in \mathbb{Z}^n_{\geqslant 0}$ such that

$$\frac{l_1}{a_1} + \frac{l_2}{a_2} + \cdots + \frac{l_k}{a_k} \leqslant 1, \quad \text{where} \ \ a_i = \frac{\log x}{\log p_i}.$$

Therefore, $\psi(x, y)$ is precisely the number Q_k of (integer) lattice points inside the n-dimensional tetrahedron (1.1) with $a_i = \frac{\log x}{\log p_i}$, $n = k$, and $1 \leqslant i \leqslant k$. In [9], Ennola gave both lower and upper bounds for the $\psi(x, y)$,

$$\frac{(\log x)^k}{k! \prod_{i=1}^k \log p_i} < \psi(x, y) \leqslant \frac{(\log x + \sum_{i=1}^k \log p_i)^k}{k! \prod_{i=1}^k \log p_i}, \tag{1.2}$$

which yields the following result.

Theorem 1.1 (See [9]). *Uniformly for* $2 \leqslant y \leqslant \sqrt{\log x \log_2 x}$, *we have*

$$\psi(x, y) = \frac{1}{k!} \prod_{p \leqslant y} \left(\frac{\log x}{\log p}\right) \left[1 + O\left(\frac{y^2}{\log x \log y}\right)\right].$$

In fact, there are some other results about the asymptotic formula for $\psi(x, y)$. The interested reader may also refer to the theorems by Saias [31], Hildebrand [14–16], and Hildebrand and Tenenbaum [17, 18]. All these can be found in an excellent book by Tenenbaum [34, Theorem 9 on p. 380, Corollary 9.3 on p. 381, Theorem 10 on p. 385 and Theorem 11 on p. 386].

The general problem of counting the number Q_n has been a challenging problem for many years. In 1951, Mordell [27] gave a formula for Q_3, expressed in terms of three Dedekind sums, in the case that a_1, a_2

1) Granville A. A letter to Y.-J. Xu. 1992.

and a_3 are pairwise relatively prime. In 1993, Pommersheim [30], using toric varieties, gave a formula for Q_3 for arbitrary positive integers a_1, a_2, a_3 and so forth. Meanwhile, the problem of counting the number of integral points in an n-dimensional tetrahedron with real vertices is a classical subject which has attracted a lot of famous mathematicians. Also from the view of estimating the de Bruijn function $\psi(x,y)$, a_i, $1 \leqslant i \leqslant n$, are not always integers. For $n=2$, Hardy and Littlewood [12,13] wrote two famous papers on the lattice points of a right-angled triangle because of its relations to their International Congress of Mathematics lecture in 1912 on Diophantine approximation [11]. A more general approximation of Q_n was obtained by Spencer [32,33] via complex function-theoretic methods. In recent years, there are tremendous activities in finding the exact formula for $P(a_1,\dots,a_n)$ or $Q(a_1,\dots,a_n)$ for positive integers $a_1,\dots,a_n$, see [1,2,6,19]. The exact formula is complicated. It involves the generalized Dedekind sum. It is difficult to tell how large $P(a_1,\dots,a_n)$ is from the exact formula. Therefore one would like to get a sharp upper polynomial estimate of $P(a_1,\dots,a_n)$ in terms of a polynomial in $a_1,\dots,a_n$. Such a sharp upper polynomial estimate of $P(a_1,\dots,a_n)$ is important because it would have application in the following Durfee Conjecture which is one of the long-standing open problems in singularity theory.

Granville[2)] obtained the following estimate:

$$P_n \leqslant \frac{1}{n!} a_1 a_2 \cdots a_n. \tag{1.3}$$

This estimate of P_n given by (1.3) is interesting, but not strong enough to be useful, particularly when many of the a_i's are small. In geometry and singularity theory, estimating P_n for real right-angled simplices is related to [37, Durfee Conjecture]. Let $f : (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of a complex analytic function with an isolated critical point at the origin. Let

$$V = \{(z_0,z_1,\dots,z_n) \in \mathbb{C}^{n+1} : f(z_0,z_1,\dots,z_n)=0\}.$$

The Milnor number of the singularity $(V,0)$ is defined as

$$\mu = \dim \mathbb{C}\{z_0,z_1,\dots,z_n\}/(f_{z_0}, f_{z_1},\dots,f_{z_n}).$$

The geometric genus p_g of $(V,0)$ is defined as $p_g = \dim H^{n-1}(M,\mathscr{O})$, where M is a resolution of V and $\mathscr{O}$ is the structure sheaf on M. In 1978, Durfee [8] made the following conjecture:

Durfee Conjecture. Let $(V,0)$ be an isolated hypersurface singularity defined by a holomorphic function $f : (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$. Let μ and p_g be the Milnor number and geometric genus of $(V,0)$, respectively. Then $n! p_g \leqslant \mu$ with equality only when $\mu = 0$.

We say that $f(z_1,\dots,z_n)$ is weighted homogeneous of type $(w_1,\dots,w_n)$, where $w_1,\dots,w_n$ are fixed positive rational numbers, if f can be expressed as a linear combination of monomials $z_1^{i_1}\cdots z_n^{i_n}$ for which $i_1/w_1 + \cdots + i_n/w_n = 1$. If $f(z_1,\dots,z_n)$ is a weighted homogeneous polynomial of type $(a_1,a_2,\dots,a_n)$ with an isolated singularity at the origin, Milnor and Orlik [26] proved that $\mu = (a_1-1)(a_2-1)\cdots(a_n-1)$. On the other hand, Merle and Teissier [25] showed that $p_g = P_n = P(a_1,\dots,a_n)$. Finding a sharp estimate of P_n will lead to a resolution to Durfee Conjecture.

Starting from early 1990's, the authors of [22,36,38] tried to get sharp upper estimates of P_n where a_i's are positive real numbers. They were successful for $n = 3, 4$ and 5. They then proposed a general conjecture:

Conjecture 1.1 (Granville-Lin-Yau (GLY) conjecture). Let

$$P_n = \sharp\left\{(x_1,x_2,\dots,x_n) \in \mathbb{Z}_+^n; \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leqslant 1\right\}$$

and $n \geqslant 3$.

(1) Sharp estimate: If $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant n-1$, then

$$n! P_n \leqslant f_n := A_0^n + \frac{s(n,n-1)}{n} A_1^n + \sum_{l=1}^{n-2} \frac{s(n,n-1-l)}{\binom{n-1}{l}} A_l^{n-1}, \tag{1.4}$$

2) Granville A. A letter to Y.-J. Xu. 1992.

where $s(n,k)$ is the Stirling number of the first kind defined by the generating function:

$$x(x-1)\cdots(x-n+1)=\sum_{k=0}^{n}s(n,k)x^k,$$

and A_k^n is defined as

$$A_k^n=\Big(\prod_{i=1}^{n}a_i\Big)\Big(\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n}\frac{1}{a_{i_1}a_{i_2}\cdots a_{i_k}}\Big),$$

for $k=1,2,\ldots,n-1$. Equality holds if and only if $a_1=a_2=\cdots=a_n=$ integer.

(2) Weak estimate: If $a_1\geqslant a_2\geqslant\cdots\geqslant a_n>1$, then

$$n!P_n<q_n:=\prod_{i=1}^{n}(a_i-1). \tag{1.5}$$

These estimates are all polynomials of a_i. They are sharp because the equality holds true if and only if all a_i's take the same integer. In [21,22,36,38], the authors showed that (1.5) holds for $3\leqslant n\leqslant 5$. The sharp estimate conjecture was first formulated in [23]. In private communication to the second author, Granville formulated this sharp estimated conjecture independently after reading [21]. Again, the sharp GLY conjecture has been proven individually for $n=3,4,5$ by [22,37,38], respectively. It has also been proven generally for $n\leqslant 6$. However, for $n=7$, a counterexample to the conjecture has been given in [35]. In [40], a revised verision of GLY conjecture, i.e., Yau-Zhao-Zuo (YZZ) conjecture was proposed and proved to be ture in low dimensions.

The breakthrough in the subject is the following theorem by Yau and Zhang [39] which states that the weak GLY conjecture holds for all $n\geqslant 3$.

Theorem 1.2 (See [39])**.** *For $n\geqslant 3$, let $a_1\geqslant a_2\geqslant\cdots\geqslant a_n\geqslant 1$ be real numbers. Let P_n be the number of positive integral solutions to $\frac{x_1}{a_1}+\frac{x_2}{a_2}+\cdots+\frac{x_n}{a_n}\leqslant 1$, i.e.,*

$$P_n=\sharp\Big\{(x_1,x_2,\ldots,x_n)\in\mathbb{Z}_+^n:\frac{x_1}{a_1}+\frac{x_2}{a_2}+\cdots+\frac{x_n}{a_n}\leqslant 1\Big\},$$

where $\mathbb{Z}_+$ is the set of positive integers. Then $n!P_n\leqslant(a_1-1)(a_2-1)\cdots(a_n-1)$ and the equality holds if and only if $a_n=1$.

Theorem 1.2 above implies that Durfee Conjecture holds true for weighted homogeneous singularities. However, the Yau-Zhang estimate is not sharp. It is not good enough to characterize the homogeneous polynomial with an isolated singularity. In order to do that, the second author made the following conjecture in 1995.

Conjecture 1.2 (Yau geometric conjecture)**.** Let $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ be a germ of a weighted homogeneous polynomial with isolated critical points at the origin. Let μ, P_g and ν be the Milnor number, geometric genus and multiplicity of the singularity $V=\{z:f(z)=0\}$. Then

$$\mu-h(\nu)\geqslant(n+1)!P_g, \tag{1.6}$$

where $h(\nu)=(\nu-1)^{n+1}-\nu(\nu-1)\cdots(\nu-n)$, and equality holds if and only if f is a homogeneous polynomial.

The Yau geometric conjecture was answered affirmatively for $n=3,4,5$ by [3,22,37], respectively.

In order to overcome the difficulty that the GLY sharp estimate conjecture is only true if a_n is larger than $y(n)$, a positive integer depending on n, Yau proposes to prove a new sharp polynomial estimate conjecture which is motivated from the Yau geometric conjecture. The importance of this conjecture is that we only need $a_n>1$ and hence the conjecture will give a sharp upper estimate of the de Bruijn function $\psi(x,y)$.

Conjecture 1.3 (Yau number theoretic conjecture). Assume that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n > 1$, $n \geqslant 3$ and let P_n be the number of elements of the set

$$\left\{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_+^n; \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leqslant 1\right\}.$$

If $P_n > 0$, then

$$n!P_n \leqslant (a_1 - 1)(a_2 - 1) \cdots (a_n - 1) - (a_n - 1)^n + a_n(a_n - 1) \cdots (a_n - (n - 1)) \tag{1.7}$$

and equality holds if and only if $a_1 = a_2 = \cdots = a_n =$ integer.

Obviously, there is an intimate relation between the Yau geometric conjecture (1.6) and the number theoretic conjecture (1.7). Recall that if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a weighted homogeneous polynomial with isolated singularity at the origin, then the multiplicity ν of f at the origin is given by $\inf\{n \in \mathbb{Z}_+ : n \geqslant \inf\{w_1, \ldots, w_n\}\}$, where w_i is the weight of x_i. Notice that in general, w_i is only a rational number. In the case that the minimal weight is an integer, the Yau geometric conjecture (1.6) and the Yau number theoretic conjecture (1.7) are the same. In general, these two conjectures do not imply each other, although they are intimately related.

The number theoretic conjecture (1.7) is much sharper than the weak GLY conjecture (1.5). The estimate in (1.7) is optimal in the sense that the equality occurs precisely when $a_1 = a_2 = \cdots = a_n$ $=$ integer. Moreover, the sharp GLY conjecture (1.4) does not hold for $n = 7$ as the counterexample shows. However, the number theoretic conjecture (1.7) does hold for this example.

By the previous works of Xu and Yau [36, 38], it was shown that the number theoretic conjecture is true for $n = 3$. For $n = 4, 5$, the conjecture has been shown in our previous work [20, 24]. The purpose of this paper is to prove that the number theoretic conjecture is true for $n = 6$. The strategy of this paper is different from our previous papers [20, 24]. Here for the case $n = 6$ the techniques used are more complicated than those in the case $n \leqslant 5$ and the feasibility of the strategy has been challenged, even if the dimension has only been increased by 1. As we will see in our proof, the number of subcases has been increased from 5 (when $n = 5$) to 21 (when $n = 6$). Showing subcases one by one will absolutely cause tremendous involved computations, and it is tedious to our readers. In this paper, based on the intrinsic observation, we simplify 21 subcases into 6 major classes ($k = 1, 2, 3, 4, 5$ and $a_6 \geqslant 5$), and modify the former 5 classes with delicate analysis of A_i's domain, where $A_i = a_i(1 - \frac{k}{a_6})$, $i = 1, 2, 3, 4, 5$ to deal with the subcases one by one. This paper may shed a new light on the conjecture for the case of arbitrary dimension number theoretic conjecture.

Furthermore, the de Bruijn function $\psi(x, y)$ is important and has a number of applications to analytic number theory and cryptography. For example, to optimize the complexity of steps in several cryptographic algorithms, one often needs more precise information about $\psi(x, y)$ than current estimates and asymptotic formulae can provide. In this paper, we give an explicit formula for the estimate of $\psi(x, y)$, when $5 \leqslant y < 13$, and our upper bound of $\psi(x, y)$ is better than the one obtained by Ennola (see (1.2)). Mathematica 4.0 is adopted to do some involved computations. The following are our main theorems.

Theorem 1.3 (Number theoretic conjecture for $n = 6$). *Let $a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 \geqslant a_5 \geqslant a_6 > 1$ be real numbers. Let P_6 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} \leqslant 1$, i.e.,*

$$P_6 = \sharp\left\{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}_+^6 : \frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} \leqslant 1\right\},$$

where $\mathbb{Z}_+$ is the set of positive integers. If $P_6 > 0$, then

$$\begin{aligned} 720P_6 \leqslant \mathrm{NTC6} := {}& (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1) - (a_6 - 1)^6 \\ & + a_6(a_6 - 1)(a_6 - 2)(a_6 - 3)(a_6 - 4)(a_6 - 5) \end{aligned}$$

and the equality holds if and only if $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 =$ integer.

Theorem 1.4 (Estimate of $\psi(x,y)$). *Let $\psi(x,y)$ be the function as before. We have the following upper estimates for $5 \leqslant y < 17$.*

(I) *When $5 \leqslant y < 7$ and $x > 5$, we have*

$$\psi(x,y) \leqslant \frac{1}{6}\left\{\frac{1}{\log 2 \log 3 \log 5}(\log x + \log 15)(\log x + \log 10)(\log x + \log 6) - \frac{1}{\log^3 5}[(\log x + \log 6)^3 - (\log x + \log 6 + \log 5)(\log x + \log 6)(\log x + \log 6 - \log 5)]\right\}.$$

(II) *When $7 \leqslant y < 11$ and $x > 7$, we have*

$$\psi(x,y) \leqslant \frac{1}{24}\left\{\frac{1}{\log 2 \log 3 \log 5 \log 7}(\log x + \log 105)(\log x + \log 70)(\log x + \log 42)(\log x + \log 30) - \frac{1}{\log^4 7}[(\log x + \log 30)^4 - (\log x + \log 7 + \log 30)(\log x + \log 30) \times (\log x + \log 30 - \log 7)(\log x + \log 30 - 2\log 7)]\right\}.$$

(III) *When $11 \leqslant y < 13$ and $x > 11$, we have*

$$\psi(x,y) \leqslant \frac{1}{120}\left\{\frac{1}{\log 2 \log 3 \log 5 \log 7 \log 11}(\log x + \log 1155)(\log x + \log 770)(\log x + \log 462) \times (\log x + \log 330)(\log x + \log 210) - \frac{1}{\log^5 11}[(\log x + \log 210)^5 - (\log x + \log 11 + \log 210)(\log x + \log 210)(\log x + \log 210 - \log 11) \times (\log x + \log 210 - 2\log 11)(\log x + \log 210 - 3\log 11)]\right\}.$$

(IV) *When $13 \leqslant y < 17$ and $x > 13$, we have*

$$\psi(x,y) \leqslant \frac{1}{720}\left\{\frac{1}{\log 2 \log 3 \log 5 \log 7 \log 11 \log 13}(\log x + \log 15015) \times (\log x + \log 10010)(\log x + \log 6006)(\log x + \log 4290)(\log x + \log 2730) \times (\log x + \log 2310) - \frac{1}{\log^6 13}[(\log x + \log 2310)^6 - (\log x + \log 13 + \log 2310)(\log x + \log 2310)(\log x + \log 2310 - \log 13) \times (\log x + \log 2310 - 2\log 13)(\log x + \log 2310 - 3\log 13)(\log x + \log 2310 - 4\log 13)]\right\}.$$

Remark 1.1. For comparison, we list the Ennola's upper bounds (see (1.2)) for $5 \leqslant y < 17$ as follows:

(1) $5 \leqslant y < 7$ and $x > 5$,

$$\psi(x,y) \leqslant \frac{(\log x + \log 30)^3}{6 \log 2 \log 3 \log 5}.$$

(2) $7 \leqslant y < 11$ and $x > 7$,

$$\psi(x,y) \leqslant \frac{(\log x + \log 210)^4}{24 \log 2 \log 3 \log 5 \log 7}.$$

(3) $11 \leqslant y < 13$ and $x > 11$,

$$\psi(x,y) \leqslant \frac{(\log x + \log 2310)^5}{120 \log 2 \log 3 \log 5 \log 7 \log 11}.$$

(4) $13 \leqslant y < 17$ and $x > 13$,

$$\psi(x, y) \leqslant \frac{(\log x + \log 30030)^6}{720 \log 2 \log 3 \log 5 \log 7 \log 11 \log 13}.$$

It is easy to see that our upper bound of $\psi(x, y)$ is substantially better than the one obtained by Ennola. For example, in the case that $13 \leqslant y < 17$ and $x > 13$, though the coefficient of $(\log x)^6$ in our estimate is same as Ennola's, but our coefficient of $(\log x)^5$ is

$$\frac{1}{720}\left(\frac{\log 15015 + \log 10010 + \log 6006 + \log 4290 + \log 2730 + \log 2310}{\log 2 \log 3 \log 5 \log 7 \log 11 \log 13} - \frac{9}{\log^5 13}\right),$$

which is smaller than Ennola's

$$\frac{1}{720}\frac{6 \log 30030}{\log 2 \log 3 \log 5 \log 7 \log 11 \log 13}.$$

2 Proofs of the theorems

2.1 Proof of Theorem 1.3

The proof is divided into six cases:

(1) $a_6 \in (1, 2]$;

(2) $a_6 \in (2, 3]$;

(3) $a_6 \in (3, 4]$;

(4) $a_6 \in (4, 5]$;

(5) $a_6 \in (5, 6)$;

(6) $a_6 \in [6, \infty)$.

The proof will begin with Case (1) and solve the rest of the cases in numerical order. For conciseness, we shall give the detailed proofs of Cases (1), (2) and (6). Since the proofs of Cases (3)–(5) are akin to that of Case (2), we shall only list the subcases in each cases.

For Cases (1)–(6), the plan is to partition the 6-dimensional polyhedron into 5-dimensional polyhedra of several levels. For each level, we can use the estimate for 5-dimensional polyhedra. Then we only need to show our estimate in Theorem 1.3 is greater than the sum of the estimates of all levels. Basically, for some level k, we have

$$\begin{aligned} &\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{k}{a_6} \leqslant 1, \\ &\frac{x_1}{a_1(1-\frac{k}{a_6})} + \frac{x_2}{a_2(1-\frac{k}{a_6})} + \frac{x_3}{a_3(1-\frac{k}{a_6})} + \frac{x_4}{a_4(1-\frac{k}{a_6})} + \frac{x_5}{a_5(1-\frac{k}{a_6})} \leqslant 1, \end{aligned} \tag{2.1}$$

for $k = 1, 2, \ldots, [a_6]$, where $[a_6] + \beta = a_6$, $0 \leqslant \beta < 1$. For this proof, let $k = 1, 2, \ldots, [a_6]$, and $P_5(k)$ be the number of positive integral solutions to (2.1). Then $P_6 = \sum_{k=1}^{[a_6]} P_5(k)$. Assume that $P_5(k) > 0$. Then incorporating the 5-dimensional version of Theorem 1.3, we have

$$\begin{aligned} 6!P_5(k) \leqslant 6\Bigg(&\left(a_1\left(1-\frac{k}{a_6}\right)-1\right)\left(a_2\left(1-\frac{k}{a_6}\right)-1\right)\left(a_3\left(1-\frac{k}{a_6}\right)-1\right) \\ &\times \left(a_4\left(1-\frac{k}{a_6}\right)-1\right)\left(a_5\left(1-\frac{k}{a_6}\right)-1\right) - \left(a_5\left(1-\frac{k}{a_6}\right)-1\right)^5 \\ &+ a_5\left(1-\frac{k}{a_6}\right)\left(a_5\left(1-\frac{k}{a_6}\right)-1\right)\left(a_5\left(1-\frac{k}{a_6}\right)-2\right)\left(a_5\left(1-\frac{k}{a_6}\right)-3\right) \\ &\times \left(a_5\left(1-\frac{k}{a_6}\right)-4\right)\Bigg). \end{aligned}$$

Let $k_0 \in Z$ such that $1 \leqslant k_0 \leqslant [a_6]$ and k_0 is the largest integer so that $P_5(k_0) > 0$ and $P_5(k) = 0$ for all $k_0 < k \leqslant a_6$. By combining the above two equations, we have

$$\begin{aligned}6!P_6 &= 6!\sum_{k=1}^{k_0} P_5(k)\\ &\leqslant 6\sum_{k=1}^{k_0}\bigg[\bigg(a_1\bigg(1-\frac{k}{a_6}\bigg)-1\bigg)\bigg(a_2\bigg(1-\frac{k}{a_6}\bigg)-1\bigg)\bigg(a_3\bigg(1-\frac{k}{a_6}\bigg)-1\bigg)\\ &\quad\times\bigg(a_4\bigg(1-\frac{k}{a_6}\bigg)-1\bigg)\bigg(a_5\bigg(1-\frac{k}{a_6}\bigg)-1\bigg)-\bigg(a_5\bigg(1-\frac{k}{a_6}\bigg)-1\bigg)^5\\ &\quad+a_5\bigg(1-\frac{k}{a_6}\bigg)\bigg(a_5\bigg(1-\frac{k}{a_6}\bigg)-1\bigg)\bigg(a_5\bigg(1-\frac{k}{a_6}\bigg)-2\bigg)\bigg(a_5\bigg(1-\frac{k}{a_6}\bigg)-3\bigg)\\ &\quad\times\bigg(a_5\bigg(1-\frac{k}{a_6}\bigg)-4\bigg)\bigg].\end{aligned}$$

In order to prove Theorem 1.3 for Subcases (1)–(6), it is sufficient to show that NTC6 $\geqslant 6!\sum P_5(k)$.

Case (1) $a_6 \in (1,2]$. In this case, we know there are two levels: $k=1$ and $k=2$. It is easy to see that $P_5(2)=0$. From the condition $P_6>0$, we also know that the level $k=1$ must have a positive integral solution, i.e., $P_5(1)>0$. This implies that $(1,1,1,1,1,1)$ is the smallest positive integral solution to the level $k=1$. Hence, we have $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{1}{a_5} \leqslant 1-\frac{1}{a_6}=\beta_1, \beta_1 \in (0,\frac{1}{2}]$, since $a_6 \in (1,2]$. Let $A_i = a_i\beta_1, i=1,2,3,4,5$. Also notice that $A_1 \geqslant 5, A_2 \geqslant 4, A_3 \geqslant 3, A_4 \geqslant 2, A_5 \geqslant 1$, since $\frac{1}{A_5} \leqslant 1, \frac{2}{A_4} \leqslant \frac{1}{A_4}+\frac{1}{A_5} \leqslant 1, \frac{3}{A_3} \leqslant \frac{1}{A_3}+\frac{1}{A_4}+\frac{1}{A_5} \leqslant 1, \frac{4}{A_2} \leqslant \frac{1}{A_2}+\frac{1}{A_3}+\frac{1}{A_4}+\frac{1}{A_5} \leqslant 1$, and $\frac{5}{A_1} \leqslant \frac{1}{A_1}+\frac{1}{A_2}+\frac{1}{A_3}+\frac{1}{A_4}+\frac{1}{A_5} \leqslant 1$. $6!P_6$ can now be rewritten as follows,

$$\begin{aligned}6!P_6 &= 6!(P_5(1))\\ &\leqslant 6((A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)-(A_5-1)^5\\ &\quad+A_5(A_5-1)(A_5-2)(A_5-3)(A_5-4)).\end{aligned}$$

It is sufficient to prove that NTC6 $\geqslant$ the right-hand side (R.H.S.) of the above inequality. We will first subtract the R.H.S. from NTC6, then substitute $a_i=\frac{A_i}{\beta_1}, i=1,2,3,4,5, a_6=\frac{1}{1-\beta_1}$, and multiply the difference by $\beta_1^5(1-\beta_1)^5$,

$$\begin{aligned}\Phi_1 := {}&(A_1+A_2+A_3+A_4)(-5\beta_1^5+26\beta_1^6-54\beta_1^7+56\beta_1^8-29\beta_1^9+6\beta_1^{10})\\ &+(A_1A_2+A_1A_3+A_2A_3+A_1A_4+A_2A_4+A_3A_4+A_1A_5+A_2A_5+A_3A_5+A_4A_5)\\ &\times(-\beta_1^4+10\beta_1^5-36\beta_1^6+64\beta_1^7-61\beta_1^8+30\beta_1^9-6\beta_1^{10})\\ &+(A_1A_2A_3+A_1A_2A_4+A_1A_3A_4+A_2A_3A_4+A_1A_2A_5+A_1A_3A_5+A_2A_3A_5\\ &+A_1A_4A_5+A_2A_4A_5+A_3A_4A_5)(\beta_1^3-4\beta_1^4+26\beta_1^6-59\beta_1^7+60\beta_1^8-30\beta_1^9+6\beta_1^9-6\beta_1^{10})\\ &+(A_1A_2A_3A_4+A_1A_2A_3A_5+A_1A_2A_4A_5+A_1A_3A_4A_5+A_2A_3A_4A_5)\\ &\times(-\beta_1^2+4\beta_1^3-6\beta_1^4+10\beta_1^5-31\beta_1^6+60\beta_1^7-60\beta_1^8+30\beta_1^9-6\beta_1^{10})\\ &+A_1A_2A_3A_4A_5(\beta_1-4\beta_1^2+6\beta_1^3-4\beta_1^4-5\beta_1^5+30\beta_1^6-60\beta_1^7+60\beta_1^8-30\beta_1^9+6\beta_1^{10})\\ &+A_5(-119\beta_1^5+596\beta_1^6-1194\beta_1^7+1196\beta_1^8-599\beta_1^9+120\beta_1^{10})\\ &+A_5^2(240\beta_1^5-1200\beta_1^6+2400\beta_1^7-2400\beta_1^8+1200\beta_1^9-240\beta_1^{10})\\ &+A_5^3(-150\beta_1^5+750\beta_1^6-1500\beta_1^7+1500\beta_1^8-750\beta_1^9+150\beta_1^{10})\\ &+A_5^4(30\beta_1^5-150\beta_1^6+300\beta_1^7-300\beta_1^8+150\beta_1^9-30\beta_1^{10})+(23\beta_1^6-118\beta_1^7+201\beta_1^8-115\beta_1^9).\end{aligned}$$

The idea is to show that for all $\beta_1 \in (0,\frac{1}{2}]$, the minimum of Φ_1 for $A_1 \geqslant 5, A_2 \geqslant 4, A_3 \geqslant 3, A_4 \geqslant 2$ and $A_5 \geqslant 1$ occurs at $A_1=5, A_2=4, A_3=3, A_4=2, A_5=1$, and $\Phi_1\big|_{A_1=5,A_2=4,A_3=3,A_4=2,A_5=1} > 0$,

$$\frac{\partial^5\Phi_1}{\partial A_1\partial A_2\partial A_3\partial A_4\partial A_5} = (-1+\beta_1)^4\beta_1(1-6\beta_1^4+6\beta_1^5) > 0,$$

for $\beta_1 \in (0,1)$. It follows that $\frac{\partial^4\Phi_1}{\partial A_1\partial A_2\partial A_3\partial A_4}$ is an increasing function with respect to A_5 for $A_5 \geqslant 1$ and $\beta_1 \in (0,1)$. Hence the minimum of $\frac{\partial^4\Phi_1}{\partial A_1\partial A_2\partial A_3\partial A_4}$ occurs at $A_5 = 1$,

$$\begin{aligned}\left.\frac{\partial^4\Phi_1}{\partial A_1\partial A_2\partial A_3\partial A_4}\right|_{A_5=1} &= [(-1+\beta_1)^4\beta_1\{-\beta_1+6\beta_1^4-6\beta_1^5+A_5(1-6\beta_1^4+6\beta_1^5)\}]|_{A_5=1}\\ &= -(-1+\beta_1)^5\beta_1 > 0,\end{aligned}$$

for $\beta_1 \in (0,1)$. It follows that $\frac{\partial^4\Phi_1}{\partial A_1\partial A_2\partial A_3\partial A_4} > 0$ for $A_5 \geqslant 1$ and $\beta_1 \in (0,1)$. Note that $\frac{\partial^3\Phi_1}{\partial A_1\partial A_2\partial A_3}$ is symmetric with respect to A_4 and A_5. Therefore, $\frac{\partial^4\Phi_1}{\partial A_1\partial A_2\partial A_3\partial A_5} > 0$ for $A_4 \geqslant 1$ and $\beta_1 \in (0,1)$ and $\frac{\partial^3\Phi_1}{\partial A_1\partial A_2\partial A_3}$ is an increasing function with respect to A_4 and A_5 for $A_4 \geqslant A_5 \geqslant 1$ and $\beta_1 \in (0,1)$. The minimum of $\frac{\partial^3\Phi_1}{\partial A_1\partial A_2\partial A_3}$ occurs at $A_4 = A_5 = 1$,

$$\begin{aligned}\left.\frac{\partial^3\Phi_1}{\partial A_1\partial A_2\partial A_3}\right|_{A_4=A_5=1} &= [(-1+\beta_1)^4\beta_1\{\beta_1(\beta_1-6\beta_1^3+6\beta_1^4+A_5(-1+6\beta_1^3-6\beta_1^4))\\ &\quad + A_4(-\beta_1+6\beta_1^4-6\beta_1^5+A_5(1-6\beta_1^4+6\beta_1^5))\}]|_{A_4=A_5=1}\\ &= -(-1+\beta_1)^6\beta_1 > 0,\end{aligned}$$

for $\beta_1 \in (0,1)$. It follows that $\frac{\partial^3\Phi_1}{\partial A_1\partial A_2\partial A_3} > 0$ for $A_4 \geqslant A_5 \geqslant 1$ and $\beta_1 \in (0,1)$. Since $\frac{\partial^2\Phi_1}{\partial A_1\partial A_2}$ is symmetric with respect to A_3, A_4 and A_5, $\frac{\partial^3\Phi_1}{\partial A_1\partial A_2\partial A_4} > 0$ for $A_3 \geqslant A_5 \geqslant 1$ and $\beta_1 \in (0,1)$, $\frac{\partial^3\Phi_1}{\partial A_1\partial A_2\partial A_5} > 0$ for $A_3 \geqslant A_4 \geqslant 1$ and $\beta_1 \in (0,1)$, and $\frac{\partial^2\Phi_1}{\partial A_1\partial A_2}$ is an increasing function with respect to A_3, A_4 and A_5 for $A_3 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $\beta_1 \in (0,1)$. Thus the minimum of $\frac{\partial^2\Phi_1}{\partial A_1\partial A_2}$ occurs at $A_3 = A_4 = A_5 = 1$ and we have

$$\begin{aligned}\left.\frac{\partial^2\Phi_1}{\partial A_1\partial A_2}\right|_{A_3=A_4=A_5=1} &= [(-1+\beta_1)^4\beta_1\{-\beta_1^3(1-6\beta_1+6\beta_1^2)+A_5\beta_1^2(1-6\beta_1^2+6\beta_1^3)\\ &\quad + A_4\beta_1(\beta_1-6\beta_1^3+6\beta_1^4)+A_4A_5\beta_1(-1+6\beta_1^3-6\beta_1^4)\\ &\quad + A_3\beta_1(\beta_1-6\beta_1^3+6\beta_1^4)+A_3A_5\beta_1(-1+6\beta_1^3-6\beta_1^4)\\ &\quad + A_3A_4(-\beta_1+6\beta_1^4-6\beta_1^5)+A_3A_4A_5(1-6\beta_1^4+6\beta_1^5)\}]|_{A_3=A_4=A_5=1}\\ &= -(-1+\beta_1)^7\beta_1 > 0,\end{aligned}$$

for $\beta_1 \in (0,1)$. It follows that $\frac{\partial^2\Phi_1}{\partial A_1\partial A_2} > 0$ for $A_3 \geqslant A_4 \geqslant A_5$ and $\beta_1 \in (0,1)$. Note that $\frac{\partial\Phi_1}{\partial A_1}$ is symmetric with respect to A_2, A_3, A_4 and A_5. Therefore, $\frac{\partial^2\Phi_1}{\partial A_1\partial A_3} > 0$ for $A_2 \geqslant A_4 \geqslant A_5$ and $\beta_1 \in (0,1)$, $\frac{\partial^2\Phi_1}{\partial A_1\partial A_4} > 0$ for $A_2 \geqslant A_3 \geqslant A_5$ and $\beta_1 \in (0,1)$, $\frac{\partial^2\Phi}{\partial A_1\partial A_5} > 0$ for $A_2 \geqslant A_3 \geqslant A_4$ and $\beta \in (0,1)$, and $\frac{\partial\Phi_1}{\partial A_1}$ is an increasing function with respect to A_2, A_3, A_4 and A_5 for $A_2 \geqslant A_3 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $\beta_1 \in (0,1)$. Thus the minimum of $\frac{\partial\Phi_1}{\partial A_1}$ occurs at $A_2 = A_3 = A_4 = A_5 = 1$ and we have

$$\begin{aligned}\left.\frac{\partial\Phi_1}{\partial A_1}\right|_{A_2=A_3=A_4=A_5=1} &= [(-1+\beta_1)^4\beta_1\{-\beta_1^3(5-6\beta_1)-A_5\beta_1^2(1-6\beta_1+6\beta_1^2)\\ &\quad - A_4\beta_1^2(1-6\beta_1+6\beta_1^2)-A_4A_5\beta_1(-1+6\beta_1^2-6\beta_1^3)\\ &\quad + A_3\beta_1^2(1-6\beta_1+6\beta_1^2)+A_3A_5\beta_1(-1+6\beta_1^2-6\beta_1^3)\\ &\quad + A_3A_4(-\beta_1+6\beta_1^3-6\beta_1^4)+A_3A_4A_5(1-6\beta_1^3+6\beta_1^4)\\ &\quad + A_2\beta_1^3(-1+6\beta_1-6\beta_1^2)+A_2A_5\beta_1^2(1-6\beta^2+6\beta_1^3)\\ &\quad + A_2A_4\beta_1(\beta_1-6\beta_1^3+6\beta_1^4)+A_2A_4A_5\beta_1(-1+6\beta_1^3-6\beta_1^4)\\ &\quad + A_2A_3\beta_1(\beta_1-6\beta_1^3+6\beta_1^4)+A_2A_3A_5\beta_1(-1+6\beta^3-6\beta_1^4)\\ &\quad + A_2A_3A_4(-\beta_1+6\beta_1^4-6\beta_1^5)\end{aligned}$$

$$+ A_2A_3A_4A_5(1-6\beta_1^4+6\beta_1^5)\}]|_{A_2=A_3=A_4=A_5=1}$$
$$= (-1+\beta_1)^8\beta_1 > 0,$$

for $\beta_1 \in (0,1)$. It follows that $\frac{\partial\Phi_1}{\partial A_1} > 0$ for $A_2 \geqslant A_3 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $\beta_1 \in (0,1)$. Since Φ_1 is symmetric with respect to A_2, A_3 and A_4, $\frac{\partial\Phi_1}{\partial A_2} > 0$ for $A_1 \geqslant A_3 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $\beta_1 \in (0,1)$, so $\frac{\partial\Phi_1}{\partial A_3} > 0$ for $A_1 \geqslant A_2 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $\beta_1 \in (0,1)$, and $\frac{\partial\Phi_1}{\partial A_4} > 0$ for $A_1 \geqslant A_2 \geqslant A_3 \geqslant A_5 \geqslant 1$ and $\beta_1 \in (0,1)$.

Meanwhile,

$$\frac{\partial^4\Phi_1}{\partial A_5^4} = -720(-1+\beta_1)^5\beta_1^5 > 0,$$

for $\beta_1 \in (0,1)$. It follows that $\frac{\partial^4\Phi_1}{\partial A_5^4} > 0$ for $A_5 > 1$ and $\beta_1 \in (0,1)$; therefore, $\frac{\partial^3\Phi_1}{\partial A_5^3}$ is an increasing function with respect to A_5 for $A_5 \geqslant 1$ and $\beta_1 \in (0,1)$. The minimum of $\frac{\partial^3\Phi_1}{\partial A_5^3}$ occurs at $A_5 = 1$ and we have

$$\left.\frac{\partial^3\Phi_1}{\partial A_5^3}\right|_{A_5=1} = [-180(-5+4A_5)(-1+\beta_1)^5\beta_1^5]_{A_5=1} = 180(-1+\beta_1)^5\beta_1^5 < 0,$$

for $\beta_1 \in (0,1)$. This presents a problem in the proof; however, we know that the possible roots of $\frac{\partial^3\Phi_1}{\partial A_5^3}$ are $A_5 = \frac{5}{4}$ and $\beta_1 = 0, 1$. Since neither of the roots for β_1 is within the domain, $A_5 = \frac{5}{4}$ is the only option. We also still know that $\frac{\partial^3\Phi_1}{\partial A_5^3}$ is an increasing function with respect to A_5 for $A_5 \geqslant 1$ and $\beta_1 \in (0,1)$. Therefore, $\frac{\partial^2\Phi_1}{\partial A_5^2}$ is an increasing function (i.e., $\frac{\partial^3\Phi_1}{\partial A_5^3} > 0$) with respect to A_5 for $A_5 > \frac{5}{4}$ and $\beta_1 \in (0,1)$, and $\frac{\partial^2\Phi_1}{\partial A_5^2}$ is a decreasing function (i.e., $\frac{\partial^3\Phi_1}{\partial A_5^3} < 0$) with respect to A_5 for $1 \geqslant A_5 < \frac{5}{4}$ and $\beta_1 \in (0,1)$. In addition, because $\frac{\partial^3\Phi_1}{\partial A_5^3} = 0$ when $A_5 = \frac{5}{4}$, the minimum of $\frac{\partial^2\Phi_1}{\partial A_5^2}$ occurs at $A_5 = \frac{5}{4}$ and we have

$$\left.\frac{\partial^2\Phi_1}{\partial A_5^2}\right|_{A_5=\frac{5}{4}} = [-60(8-15A_5+6A_5^2)(-1+\beta_1)^5\beta_1^5]|_{A_5=\frac{5}{4}} = \frac{165}{2}(-1+\beta_1)^5\beta_1^5 < 0,$$

for $\beta_1 \in (0,1)$. The possible roots for $\frac{\partial^2\Phi_1}{\partial A_5^2}$ are $A_5 = \frac{1}{12}(15-\sqrt{33}) \approx 0.771286$, $\frac{1}{12}(15+\sqrt{33}) \approx 1.72871$ and $\beta_1 = 0, 1$. Due to the fact that $A_5 = \frac{1}{12}(15-\sqrt{33})$ and $\beta_1 = 0, 1$ is not within the domain, $A_5 = \frac{1}{12}(15+\sqrt{33})$ is our only option. Note that $\frac{\partial^2\Phi_1}{\partial A_5^2}$ is increasing for $A_5 > \frac{5}{4}$ and decreasing for $A_5 < \frac{5}{4}$. Because of these properties, we know that $\frac{\partial\Phi_1}{\partial A_5}$ is an increasing function (i.e., $\frac{\partial^2\Phi_1}{\partial A_5^2} > 0$) with respect to A_5 for $1 \leqslant A_5 < \frac{1}{12}(15+\sqrt{33})$ and $\beta_1 \in (0,1)$. Notice that $\frac{\partial\Phi_1}{\partial A_5}$ is increasing with respect to A_1, A_2, A_3 and A_4 (due to the symmetric properties of $\frac{\partial\Phi_1}{\partial A_1}$ and Φ_1) for $A_1 \geqslant A_2 \geqslant A_3 \geqslant A_4 \geqslant 1$ and $\beta_1 \in (0,1)$, and $\frac{\partial^2\Phi_1}{\partial A_5^2} = 0$ when $A_5 = \frac{1}{12}(15+\sqrt{33})$. Since Φ_1 will eventually be evaluated for $A_1 \geqslant 5, A_2 \geqslant 4, A_3 \geqslant 3, A_4 \geqslant 2$ and $A_5 \geqslant 1$, we can take the minimum of $\frac{\partial\Phi_1}{\partial A_5}$ at $A_1 = A_2 = A_3 = A_4 = A_5 = \frac{1}{12}(15+\sqrt{33})$; A_5 can be evaluated at $\frac{1}{12}(15+\sqrt{33})$ since it is the minimum of $\frac{\partial\Phi_1}{\partial A_1}$, and if $\frac{\partial\Phi_1}{\partial A_1} > 0$ at $\frac{1}{12}(15+\sqrt{33})$, then $\frac{\partial\Phi_1}{\partial A_1} > 0$ for $A_5 \geqslant 1$ and we have

$$\begin{aligned}
&\left.\frac{\partial\Phi_1}{\partial A_5}\right|_{A_1=A_2=A_3=A_4=A_5=\frac{1}{12}(15+\sqrt{33})}\\
&= [(-1+\beta_1)^4\beta_1\{-\beta_1[\beta_1(\beta_1(\beta_1(119+480A_5(-1+\beta_1)-450A_5^2(-1+\beta_1)\\
&\quad +120A_5^3(-1+\beta_1)-120\beta_1)+A_4(1-6\beta_1+6\beta_1^2))+A_3(\beta_1(1-6\beta_1+6\beta_1^2)\\
&\quad +A_4(-1+6\beta_1^2-6\beta_1^3)))+A_2(\beta_1(\beta_1(1-6\beta_1+6\beta_1^2)+A_4(-1+6\beta_1^2-6\beta_1^3))\\
&\quad +A_3(-\beta_1+6\beta_1^3-6\beta_1^4+A_4(1-6\beta_1^3+6\beta_1^4)))]\\
&\quad +A_1[\beta_1(\beta_1(\beta_1(-1+6\beta_1-6\beta_1^2)+A_4(1-6\beta_1^2+6\beta_1^3))\\
&\quad +A_3(\beta_1-6\beta_1^3+6\beta_1^4+A_4(-1+6\beta_1^3-6\beta_1^4)))
\end{aligned}$$

$$
\begin{aligned}
&+ A_2(\beta_1(\beta_1 - 6\beta_1^3 + 6\beta_1^4 + A_4(-1 + 6\beta_1^3 - 6\beta_1^4)) + A_3(-\beta_1 + 6\beta_1^4 - 6\beta_1^5 \\
&+ A_4(1 - 6\beta_1^4 + 6\beta_1^5)))]\}]|_{A_1=A_2=A_3=A_4=A_5=\frac{1}{12}(15+\sqrt{33})} \\
= &\frac{1}{288}(-1+\beta_1)^4\beta_1(1337 + 215\sqrt{33} - 8(405 + 59\sqrt{33})\beta_1 + 72(43 + 5\sqrt{33})\beta_1^2 \\
&- 96(15 + 33\sqrt{33})\beta_1^3 - 6(-835 + 227\sqrt{33})\beta_1^4 + 6(-787 + 227)\beta_1^5) \\
> &\, 0,
\end{aligned}
$$

for $\beta_1 \in (0, 0.76402)$. It follows that $\frac{\partial \Phi_1}{\partial A_5} > 0$ for $A_1 \geqslant A_2 \geqslant A_3 \geqslant A_4 \geqslant \frac{1}{12}(15 + \sqrt{33}), A_5 \geqslant 1$ and $\beta_1 \in (0, 0.76402)$. Therefore, Φ_1 is an increasing function with respect to A_1, A_2, A_3, A_4 and A_5, $A_1 \geqslant A_2 \geqslant A_3 \geqslant A_4 \geqslant \frac{1}{12}(15 + \sqrt{33}), A_5 \geqslant 1$ and $\beta_1 \in (0, 0.76402)$. The minimum of Φ_1 occurs at $A_1 = A_2 = A_3 = A_4 = A_5 = \frac{1}{12}(15 + \sqrt{33})$. In addition, we take the parameters for A_1, A_2, A_3, A_4 and A_5 into consideration, then

$$
\begin{aligned}
&\frac{\partial \Phi_1}{\partial A_5}\bigg|_{A_1=5,A_2=4,A_3=3,A_4=2,A_5=1} \\
&\quad = \beta_1(120 - 754\beta_1 + 2041\beta_1^2 - 3109\beta_1^3 + 2921\beta_1^4 - 1721\beta_1^6 + 56\beta_1^7 - 100\beta_1^8) \\
&\quad > 0,
\end{aligned}
$$

for $\beta_1 \in (0, \frac{1}{2}]$. It follows that $\Phi_1 > 0$ for $A_1 \geqslant 5, A_2 \geqslant 4, A_3 \geqslant 3, A_4 \geqslant 2, A_5 \geqslant 1$ and $\beta_1 \in (0, \frac{1}{2}]$.

Case (2) $a_6 \in (2, 3]$. Case (2) contains three levels: $k = 1, k = 2$, and $k = 3$. Obviously, $P_5(3) = 0$. We know that $P_5(1)$ must contain solutions due to the fact that $P_6 > 0$, but it is unknown whether $P_5(2) > 0$ or $P_5(2) = 0$. Therefore, we split the proof into the following cases.

(a) $P_5(2) = 0$. In this case only level $k = 1$ has positive integral solutions.

(b) $P_5(2) > 0$. In this case levels $k = 1, 2$ have positive integral solutions.

Case (2a) For this case, the proof is almost the same as the one in Case (1), and only $P_5(1) > 0$. This implies that $(1, 1, 1, 1, 1, 1)$ is the smallest positive integral solution to level $k = 1$. Thus we have $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leqslant 1 - \frac{1}{a_6} = \beta_1$, $\beta_1 \in (\frac{1}{2}, \frac{2}{3}]$ since $a_6 \in (2, 3]$. We can also improve the parameters for A_1, A_2, A_3, A_4 and A_5 to the following:

$$
A_1 \geqslant 5, \quad A_2 \geqslant 4, \quad A_3 \geqslant 3, \quad A_4 \geqslant 2, \quad A_5 \geqslant \frac{\beta_1}{1 - \beta_1},
$$

since $A_5 = a_5\beta_1 \geqslant a_6\beta_1 = \frac{\beta_1}{1-\beta_1}$. Because $\beta_1 \in (\frac{1}{2}, \frac{2}{3}]$, it is easy to see that $\frac{\beta_1}{1-\beta_1} \in (1, 2]$. Therefore, it is sufficient to prove that $\Phi_1 > 0$ for $A_1 \geqslant 5, A_2 \geqslant 4, A_3 \geqslant 3, A_4 \geqslant 2, A_5 \geqslant \frac{\beta_1}{1-\beta_1}$ and $\beta_1 \in (\frac{1}{2}, \frac{2}{3}]$,

$$
\begin{aligned}
&\Phi_1|_{A_1=5,A_2=4,A_3=3,A_4=2,A_5=\frac{\beta_1}{1-\beta_1}} \\
&\quad = \beta_1^3(120 - 514\beta_1 + 1031\beta_1^2 - 1733\beta_1^3 + 2940\beta_1^4 - 3707\beta_1^5 + 2670\beta_1^6 - 816\beta_1^7) > 0,
\end{aligned}
$$

for $\beta_1 \in (\frac{1}{2}, \frac{2}{3}]$. It follows that $\Phi_1 > 0$ for $A_1 \geqslant 5, A_2 \geqslant 4, A_3 \geqslant 3, A_4 \geqslant 2, A_5 \geqslant \frac{\beta_1}{1-\beta_1}$ and $\beta_1 \in (\frac{1}{2}, \frac{2}{3}]$.

Case (2b) In this case, $P_5(3) = 0$ and $P_5(2) > 0$. This implies that $(1, 1, 1, 1, 1, 2)$ is the smallest positive integral solution to the level $k = 2$. Hence, we have $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leqslant 1 - \frac{2}{a_6} = \beta_2, \beta_2 \in (0, \frac{1}{3}]$ since $a_6 \in (2, 3]$. Let $A_i = a_i\beta_2, i = 1, 2, 3, 4, 5$. Also notice that

$$
A_1 \geqslant 5, \quad A_2 \geqslant 4, \quad A_3 \geqslant 3, \quad A_4 \geqslant 2, \quad A_5 \geqslant 1,
$$

because $\frac{1}{A_5} \leqslant 1, \frac{2}{A_4} \leqslant \frac{1}{A_4} + \frac{1}{A_5} \leqslant 1, \frac{3}{A_3} \leqslant \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leqslant 1, \frac{4}{A_2} \leqslant \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leqslant 1$, and $\frac{5}{A_1} \leqslant \frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leqslant 1$. $6!P_6$ can now be rewritten as follows,

$$
\begin{aligned}
6!P_6 &= 6!(P_5(1) + P_5(2)) \\
&\leqslant 6\bigg(\bigg(A_1\frac{1+\beta_2}{2\beta_2} - 1\bigg)\bigg(A_2\frac{1+\beta_2}{2\beta_2} - 1\bigg)\bigg(A_3\frac{1+\beta_2}{2\beta_2} - 1\bigg)
\end{aligned}
$$

$$\times\left(A_4\frac{1+\beta_2}{2\beta_2}-1\right)\left(A_5\frac{1+\beta_2}{2\beta_2}-1\right)$$
$$-\left(A_5\frac{1+\beta_2}{2\beta_2}-1\right)^5+A_5\frac{1+\beta_2}{2\beta_2}\left(A_5\frac{1+\beta_2}{2\beta_2}-1\right)\left(A_5\frac{1+\beta_2}{2\beta_2}-2\right)\left(A_5\frac{1+\beta_2}{2\beta_2}-3\right)$$
$$\times\left(A_5\frac{1+\beta_2}{2\beta_2}-4\right)+(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)$$
$$\left.-(A_5-1)^5+A_5(A_5-1)(A_5-2)(A_5-3)(A_5-4)\right).$$

It is sufficient to prove that NTC6 $\geqslant$ the R.H.S. of the above inequality. We will first subtract the R.H.S. from NTC6, then substitute $a_i=\frac{A_i}{\beta_2}$, $i=1,2,3,4,5, a_6=\frac{2}{1-\beta_2}$, and multiply the difference by $\beta_2^5(1-\beta_2)^5$,

$$\begin{aligned}\Phi_2 := {}&(A_1+A_2+A_3+A_4)(-2\beta_2^4+3\beta_2^5+17\beta_2^6-58\beta_2^7+72\beta_2^8-41\beta_2^9+9\beta_2^{10})\\
&+(A_1A_2+A_1A_3+A_2A_3+A_1A_4+A_2A_4+A_3A_4+A_1A_5+A_2A_5+A_3A_5+A_4A_5)\\
&\times\left(\frac{1}{2}\beta_2^3-\frac{3}{2}\beta_2^4+\frac{11}{2}\beta_2^5-\frac{49}{2}\beta_2^6+\frac{111}{2}\beta_2^7-\frac{125}{2}\beta_2^8+\frac{69}{2}\beta_2^9-\frac{15}{2}\beta_2^{10}\right)\\
&+(A_1A_2A_3+A_1A_2A_4+A_1A_3A_4+A_2A_3A_4+A_1A_2A_5\\
&+A_1A_3A_5+A_1A_4A_5+A_2A_3A_5+A_2A_4A_5+A_3A_4A_5)\\
&\times\left(\frac{1}{4}\beta_2^2-\frac{3}{2}\beta_2^3+\frac{7}{2}\beta_2^4-\frac{17}{2}\beta_2^5+27\beta_2^6-\frac{109}{2}\beta_2^7+\frac{117}{2}\beta_2^8-\frac{63}{2}\beta_2^9+\frac{27}{4}\beta_2^{10}\right)\\
&+(A_1A_2A_3A_4+A_1A_2A_3A_5+A_1A_2A_4A_5+A_1A_3A_4A_5+A_2A_3A_4A_5)\\
&\times\left(-\frac{5}{8}\beta_2+\frac{21}{8}\beta_2^2-\frac{7}{2}\beta_2^3-\frac{1}{2}\beta_2^4+\frac{45}{4}\beta_2^5-\frac{133}{4}\beta_2^6+\frac{117}{2}\beta_2^7-\frac{117}{2}\beta_2^8+\frac{243}{8}\beta_2^9-\frac{51}{8}\beta_2^{10}\right)\\
&+A_1A_2A_3A_4A_5\left(\frac{13}{16}-3\beta_2+\frac{47}{16}\beta_2^2+2\beta_2^3-\frac{39}{8}\beta_2^4-5\beta_2^5+\frac{255}{8}\beta_2^6\right.\\
&\left.-60\beta_2^7+\frac{945}{16}\beta_2^8-30\beta_2^9+\frac{99}{16}\beta_2^{10}\right)\\
&+A_5(-59\beta_2^4+117\beta_2^5+302\beta_2^6-1198\beta_2^7+1497\beta_2^8-839\beta_2^9+180\beta_2^{10})\\
&+A_5^2(60\beta_2^3-180\beta_2^4+300\beta_2^5-900\beta_2^6+2100\beta_2^7-2460\beta_2^8+1380\beta_2^9-300\beta_2^{10})\\
&+A_5^3\left(-\frac{75}{4}\beta_2^2+\frac{75}{2}\beta_2^3+\frac{75}{2}\beta_2^4-\frac{525}{2}\beta_2^5+750\beta_2^6-\frac{2775}{2}\beta_2^7+\frac{2925}{2}\beta_2^8-\frac{1575}{2}\beta_2^9+\frac{675}{4}\beta_2^{10}\right)\\
&+A_5^4\left(\frac{15}{8}\beta_2-\frac{15}{8}\beta_2^2-\frac{15}{2}\beta_2^3+\frac{15}{2}\beta_2^4+\frac{165}{4}\beta_2^5-\frac{645}{4}\beta_2^6+\frac{585}{2}\beta_2^7-\frac{585}{2}\beta_2^8\right.\\
&\left.+\frac{1215}{8}\beta_2^9-\frac{255}{8}\beta_2^{10}\right)-(2\beta_2^5+28\beta_2^6-88\beta_2^7+116\beta_2^8+230\beta_2^9).\end{aligned}$$

The idea is to show that for all $\beta_2\in(0,\frac{1}{3}]$, the minimum of Φ_2 for $A_1\geqslant 5, A_2\geqslant 4, A_3\geqslant 3, A_4\geqslant 2$ and $A_5\geqslant 1$ occurs at $A_1=5, A_2=4, A_3=3, A_4=2, A_5=1$, and $\Phi_2|_{A_1=5,A_2=4,A_3=3,A_4=2,A_5=1}>0$,

$$\frac{\partial^5\Phi_2}{\partial A_1\partial A_2\partial A_3\partial A_4\partial A_5}=\frac{1}{16}(-1+\beta_2)^4(13+4\beta_2-15\beta_2^2+15\beta_2^4-84\beta_2^5+99\beta_2^6)>0,$$

for $\beta_2\in(0,1)$. It follows that $\frac{\partial^4\Phi_2}{\partial A_1\partial A_2\partial A_3\partial A_4}$ is an increasing function with respect to $A_5\geqslant 1$ and $\beta_2\in(0,1)$. The minimum of $\frac{\partial^4\Phi_2}{\partial A_1\partial A_2\partial A_3\partial A_4}$ occurs at $A_5=1$ and we have

$$\begin{aligned}\left.\frac{\partial^4\Phi_2}{\partial A_1\partial A_2\partial A_3\partial A_4}\right|_{A_5=1}={}&\left[\frac{1}{16}(-1+\beta_2)^4(-2\beta_2(5-\beta_2-6\beta_2^2+6\beta_2^3-39\beta_2^4+51\beta_2^5)\right.\\
&\left.\left.+A_5(13+4\beta_2-15\beta_2^2+15\beta_2^4-84\beta_2^5+99\beta_2^6))\right]\right|_{A_5=1}\end{aligned}$$

$$= \frac{1}{16}(-1+\beta_2)^4(-13+6\beta_2+13\beta_2^2-12\beta_2^3-3\beta_2^4+6\beta_2^5+3\beta_2^6)$$
$$> 0,$$

for $\beta_2 \in (0,1)$. It follows that $\frac{\partial^4\Phi_2}{\partial A_1\partial A_2\partial A_3\partial A_4} > 0$ for $A_5 \geqslant 1$ and $\beta_2 \in (0,1)$. Since $\frac{\partial^3\Phi_2}{\partial A_1\partial A_2\partial A_3}$ is symmetric with respect to A_4 and A_5, $\frac{\partial^4\Phi_2}{\partial A_1\partial A_2\partial A_3\partial A_5} > 0$ for $A_4 \geqslant 1$ and $\beta_2 \in (0,1)$ and $\frac{\partial^3\Phi_2}{\partial A_1\partial A_2\partial A_3}$ is an increasing function with respect to A_4 and A_5 for $A_4 \geqslant A_5 \geqslant 1$ and $\beta_2 \in (0,1)$. Hence the minimum of $\frac{\partial^3\Phi_2}{\partial A_1\partial A_2\partial A_3}$ occurs at $A_4 = A_5 = 1$ and we have

$$\begin{aligned}\left.\frac{\partial^3\Phi_2}{\partial A_1\partial A_2\partial A_3}\right|_{A_4=A_5=1} &= \Big[\frac{1}{16}(-1+\beta_2)^4\{-2\beta_2(2\beta_2(-1+2\beta_2+18\beta_2^3-27\beta_2^4)\\ &\quad + A_5(5-\beta_2-6\beta_2^2+6\beta_2^3-39\beta_2^4+51\beta_2^5))\\ &\quad + A_4(-2\beta_2(5-\beta_2-6\beta_2^2+6\beta_2^3-39\beta_2^4+51\beta_2^5)\\ &\quad + A_5(13+4\beta_2-15\beta_2^2+15\beta_2^4-84\beta_2^5+99\beta_2^6))\}\Big]\Big|_{A_4=A_5=1}\\ &= \frac{1}{16}(-1+\beta_2)^6(13+10\beta_2+6\beta_2^3+3\beta_2^4) > 0,\end{aligned}$$

for $\beta_2 \in (0,1)$. It follows that $\frac{\partial^3\Phi_2}{\partial A_1\partial A_2\partial A_3} > 0$ for $A_4 \geqslant A_5 \geqslant 1$ and $\beta_2 \in (0,1)$. Note that $\frac{\partial^2\Phi_2}{\partial A_1\partial A_2}$ is symmetric with respect to A_3, A_4 and A_5. Therefore, $\frac{\partial^3\Phi_2}{\partial A_1\partial A_2\partial A_4} > 0$ for $A_3 \geqslant A_5 \geqslant 1$ and $\beta_2 \in (0,1)$, $\frac{\partial^3\Phi_2}{\partial A_1\partial A_2\partial A_5} > 0$ for $A_3 \geqslant A_4 \geqslant 1$ and $\beta_2 \in (0,1)$, and $\frac{\partial^2\Phi_2}{\partial A_1\partial A_2}$ is an increasing function with respect to A_3, A_4 and A_5 for $A_3 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $\beta_2 \in (0,1)$. The minimum of $\frac{\partial^2\Phi_2}{\partial A_1\partial A_2}$ occurs at $A_3 = A_4 = A_5 = 1$.

$$\begin{aligned}&\left.\frac{\partial^2\Phi_2}{\partial A_1\partial A_2}\right|_{A_3=A_4=A_5=1}\\ &= \Big[\frac{1}{16}(-1+\beta_2)^4\{-2\beta_2(-2\beta_2(2\beta_2(1+\beta_2+9\beta_2^2-15\beta_2^3)+A_5(1-2\beta_2-18\beta_2^3+27\beta_2^4)\\ &\quad + A_4(2\beta_2(-1+2\beta_2+18\beta_2^3-27\beta_2^4)+A_5(5-\beta_2-6\beta_2^2+6\beta_2^3-39\beta_2^4+51\beta_2^5))))\\ &\quad + A_3(-2\beta_2(2\beta_2(-1+2\beta_2+18\beta_2^3-27\beta_2^4)+A_5(5-\beta_2-6\beta_2^2+6\beta_2^3-39\beta_2^4+51\beta_2^5)\\ &\quad + A_4(-2\beta_2(5-\beta_2-6\beta_2^2+6\beta_2^3-39\beta_2^4+51\beta_2^5)\\ &\quad + A_5(13+4\beta_2-15\beta_2^2+15\beta_2^4-84\beta_2^5+99\beta_2^6))))\}\Big]\Big|_{A_3=A_4=A_5=1}\\ &= -\frac{1}{16}(-1+\beta_2)^7(13+13\beta_2+3\beta_2^2+3\beta_2^3) > 0,\end{aligned}$$

for $\beta_2 \in (0,1)$. It follows that $\frac{\partial^2\Phi_2}{\partial A_1\partial A_2} > 0$ for $A_3 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $\beta_2 \in (0,1)$. Since $\frac{\partial\Phi_2}{\partial A_1}$ is symmetric with respect to A_2, A_3, A_4 and A_5, $\frac{\partial^2\Phi_2}{\partial A_1\partial A_3} > 0$ for $A_2 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $\beta_2 \in (0,1)$, $\frac{\partial^2\Phi_2}{\partial A_1\partial A_4} > 0$ for $A_2 \geqslant A_3 \geqslant A_5 \geqslant 1$ and $\beta_2 \in (0,1)$, $\frac{\partial^2\Phi_2}{\partial A_1\partial A_5} > 0$ for $A_2 \geqslant A_3 \geqslant A_4 \geqslant 1$ and $\beta_2 \in (0,1)$, and $\frac{\partial\Phi_2}{\partial A_1}$ is an increasing function with respect to A_2, A_3, A_4 and A_5 for $A_2 \geqslant A_3 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $\beta_2 \in (0,1)$. Thus the minimum of $\frac{\partial\Phi_2}{\partial A_1}$ occurs at $A_2 = A_3 = A_4 = A_5 = 1$ and we have

$$\begin{aligned}\left.\frac{\partial\Phi_2}{\partial A_1}\right|_{A_2=A_3=A_4=A_5=1} &= \Big[\frac{1}{16}(-1+\beta_2)^8(13+16\beta_2+3\beta_2^2)\Big]\Big|_{A_2=A_3=A_4=A_5=1}\\ &= \frac{1}{16}(-1+\beta_2)^8(13+16\beta_2+3\beta_2^2) > 0,\end{aligned}$$

for $\beta_2 \in (0,1)$. It follows that $\frac{\partial\Phi_2}{\partial A_1} > 0$ for $A_2 \geqslant A_3 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $\beta_2 \in (0,1)$. Since Φ_2 is symmetric with respect to A_1, A_2, A_3 and A_4, $\frac{\partial\Phi_2}{\partial A_2} > 0$ for $A_1 \geqslant A_3 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $\beta_2 \in (0,1)$, $\frac{\partial\Phi_2}{\partial A_3} > 0$ for $A_1 \geqslant A_2 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $\beta_2 \in (0,1)$, and $\frac{\partial\Phi_2}{\partial A_4} > 0$ for $A_1 \geqslant A_2 \geqslant A_3 \geqslant A_5 \geqslant 1$ and $\beta_2 \in (0,1)$.

Meanwhile,

$$\frac{\partial^4\Phi_2}{\partial A_5^4} = -45(-1+\beta_2)^5\beta_2(1+4\beta_2+6\beta_2^2+4\beta_2^3+17\beta_2^4) > 0,$$

for $\beta_2 \in (0,1)$. It follows that $\frac{\partial^3\Phi_2}{\partial A_5^3}$ is an increasing function with respect to A_5 for $A_5 \geqslant 1$ and $\beta_2 \in (0,1)$. Also, the minimum of $\frac{\partial^3\Phi_2}{\partial A_5^3}$ occurs at $A_5 = 1$ and we have

$$\begin{aligned}\left.\frac{\partial^3\Phi_2}{\partial A_5^3}\right|_{A_5=1} &= [-45(-1+\beta_2)^5\beta_2(-5\beta_2(1+3\beta_2+3\beta_2^2+9\beta_2^3)\\ &\quad + 2A_5(1+4\beta_2+6\beta_2^2+4\beta_2^3+17\beta_2^4))]|_{A_5=1}\\ &= \frac{45}{2}(-1+\beta_2)^5\beta_2(-2-3\beta_2+3\beta_2^2+7\beta_2^3+11\beta_2^4) > 0,\end{aligned}$$

for $\beta_2 \in (0, 0.587327)$. It follows that $\frac{\partial^3\Phi_2}{\partial A_5^3} > 0$ for $A_5 \geqslant 1$ and $\beta_2 \in (0, 0.587327)$. Also, $\frac{\partial^2\Phi_2}{\partial A_2^2}$ is an increasing function with respect to $A_5 \geqslant 1$ and $\beta_2 \in (0, 0.587327)$. The minimum of $\frac{\partial^2\Phi_2}{\partial A_2^2}$ occurs at $A_5 = 1$ and we have

$$\begin{aligned}\left.\frac{\partial^2\Phi_2}{\partial A_5^2}\right|_{A_5=1} &= \bigg[-\frac{15}{2}(-1+\beta_2)^5\beta_2(16\beta_2^2(1+2\beta_2+5\beta_2^2)\\ &\quad \left.\left. - 15A_5\beta_2(1+3\beta_2+3\beta_2^2+9\beta_2^3) + 3A_5^2(1+4\beta_2+6\beta_2^2+4\beta_2^3+17\beta_2^4))\right]\right|_{A_5=1}\\ &= \frac{15}{2}(-1+\beta_2)^5\beta_2(-3+3\beta_2+11\beta_2^2+\beta_2^3+4\beta_2^4) > 0,\end{aligned}$$

for $\beta_2 \in (0, 0.390388)$. It follows that $\frac{\partial^2\Phi_2}{\partial A_5^2} > 0$ for $A_5 \geqslant 1$ and $\beta_2 \in (0, 0.390388)$. In addition, $\frac{\partial\Phi_2}{\partial A_5}$ is an increasing function with respect to A_1, A_2, A_3, A_4 (due to the symmetric properties of Φ_2 and $\frac{\partial\Phi_2}{\partial A_1}$) and A_5 for $A_1 \geqslant A_2 \geqslant A_3 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $\beta_2 \in (0, 0.390388)$. The minimum of $\frac{\partial\Phi_2}{\partial A_5}$ occurs at $A_1 = A_2 = A_3 = A_4 = A_5 = 1$ and we have

$$\begin{aligned}&\left.\frac{\partial\Phi_2}{\partial A_5}\right|_{A_1=A_2=A_3=A_4=A_5=1}\\ &= \bigg[\frac{1}{16}(-1+\beta_2)^4\{8A_1\beta_2^3(1+\beta_2+9\beta_2^2-15\beta_2^3) + 4A_1A_4\beta_2^2(1-2\beta_2-18\beta_2^3+27\beta_2^4)\\ &\quad - A_1A_3\beta_2^2(-1+2\beta_2+18\beta_2^3-27\beta_2^4) - 2A_1A_3A_4\beta_2(5-\beta_2-6\beta_2^2+6\beta_2^3-39\beta_2^4+51\beta_2^5)\\ &\quad - 4A_1A_2\beta_2^2(-1+2\beta_2+18\beta_2^3-27\beta_2^4) - 2A_1A_2A_4\beta_2(5-\beta_2-6\beta_2^2+6\beta_2^3-39\beta_2^4+51\beta_2^5)\\ &\quad - 2A_1A_2A_3\beta_2(5-\beta_2-6\beta_2^2+6\beta_2^3-39\beta_2^4+51\beta_2^5)\\ &\quad + A_1A_2A_3A_4(13+4\beta_2-15\beta_2^2+15\beta_2^4-84\beta_2^5+99\beta_2^6)\\ &\quad - 2\beta_2(-4A_2\beta_2^2(1+\beta_2+9\beta_2^2-15\beta_2^3) - 2A_2A_4\beta_2(1-2\beta_2-18\beta_2^3+27\beta_2^4)\\ &\quad + 2A_2A_3\beta_2(-1+2\beta_2+18\beta_2^3-27\beta_2^4) + A_2A_3A_4(5-\beta_2-6\beta_2+6\beta_2^3-39\beta_2^4+51\beta_2^5)\\ &\quad - 960A_5\beta_2^2(-1-\beta_2-3\beta_2^2+5\beta_2^3) + 450A_5^2\beta_2(-1-2\beta_2-6\beta_2^3+9\beta_2^4)\\ &\quad - 60A_5^3(-1-3\beta_2-2\beta_2^2+2\beta_2^3-13\beta_2^4+17\beta_2^5)\\ &\quad + 8\beta_2^3(-59-119\beta_2+180\beta_2^2) + 4A_4\beta_2^2(1+\beta_2+9\beta_2^2-15\beta_2^3)\\ &\quad \left.\left. + 4A_3\beta_2^2(1+\beta_2+9\beta_2^2-15\beta_2^3) + 2\beta_2A_3A_4(1-2\beta_2-18\beta_2^3+27\beta_2^4))\}\right]\right|_{A_1=A_2=A_3=A_4=A_5=1}\\ &= -\frac{1}{16}(-1+\beta_2)^4(-13-84\beta_2+523\beta_2^2-392\beta_2^3-735\beta_2^4-100\beta_2^5+801\beta_2^6) > 0,\end{aligned}$$

for $\beta_2 \in (0, 0.390388)$. It follows that $\frac{\partial\Phi_2}{\partial A_5} > 0$ for $A_1 \geqslant A_2 \geqslant A_3 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $\beta_2 \in (0, 0.390388)$. Therefore, Φ_2 is an increasing function with respect to A_1, A_2, A_3, A_4 and A_5 for $A_1 \geqslant A_2 \geqslant A_3 \geqslant A_4$

$\geqslant A_5 \geqslant 1$ and $\beta_2 \in (0, 0.390388)$. The minimum of Φ_2 occurs at $A_1 = A_2 = A_3 = A_4 = A_5 = 1$. In addition, we take the parameters for A_1, A_2, A_3, A_4 and A_5 into consideration, then

$$\Phi_2|_{A_1=A_2=A_3=A_4=A_5=1} = \frac{1}{8}(780 - 4235\beta_2 + 8859\beta_2^2 - 7392\beta_2^3 - 2272\beta_2^4 + 10186\beta_2^5 - 8738\beta_2^6 + 2936\beta_2^7 + 60\beta_2^8 - 2647\beta_2^9 + 159\beta_2^{10}) > 0,$$

for $\beta_2 \in (0, \frac{1}{3})$. It follows that $\Phi_2 > 0$ for $A_1 \geqslant 5, A_2 \geqslant 4, A_3 \geqslant 3, A_4 \geqslant 2, A_5 \geqslant 1$ and $\beta_2 \in (0, \frac{1}{3}]$.

Case (3) $a_6 \in (3, 4]$. The four levels of Case (3) are $k = 1, k = 2, k = 3$ and $k = 4$. Notice that $P_5(4) = 0$. Because positivity for levels above 1 is unknown, the proof for the above case is split into the following cases:

(a) $P_5(2) = 0$. In this case only level $k = 1$ has positive integral solutions.

(b) $P_5(3) = 0, P_5(2) > 0$. In this case levels $k = 1, 2$ have positive integral solutions.

(c) $P_5(3) > 0$. In this case levels $k = 1, 2, 3$ have positive integral solutions.

Case (4) $a_6 \in (4, 5]$. The 5 levels of Case (4) are $k = 1, k = 2, k = 3, k = 4$ and $k = 5$. It is easy to see that $P_5(5) = 0$. Since the positivity of levels above 1 is unknown, the proof is split into the following cases:

(a) $P_5(2) = 0$. In this case only level $k = 1$ has positive integral solutions.

(b) $P_5(3) = 0, P_5(2) > 0$. In this case levels $k = 1, 2$ have positive integral solutions.

(c) $P_5(4) = 0, P_5(3) > 0$. In this case levels $k = 1, 2, 3$ have positive integral solutions.

(d) $P_5(4) > 0$. In this case levels $k = 1, 2, 3, 4$ have positive integral solutions.

Case (5) $a_6 \in (5, 6)$. The 5 levels of Case (5) are $k = 1, k = 2, k = 3, k = 4$ and $k = 5$. Since the positivity of levels above 1 is unknown, the proof is split into the following cases:

(a) $P_5(2) = 0$. In this case only level $k = 1$ has positive integral solutions.

(b) $P_5(3) = 0, P_5(2) > 0$. In this case levels $k = 1, 2$ have positive integral solutions.

(c) $P_5(4) = 0, P_5(3) > 0$. In this case levels $k = 1, 2, 3$ have positive integral solutions.

(d) $P_5(5) = 0, P_5(4) > 0$. In this case levels $k = 1, 2, 3, 4$ have positive integral solutions.

(e) $P_5(5) > 0$. In this case levels $k = 1, 2, 3, 4, 5$ have positive integral solutions.

Case (6) $a_6 \in [6, \infty)$. The proof of Case (6) is a little bit different from the previous cases, due to the unclarity of the positivity of some newly appeared expressions using the similar method as before. The positivity depends on which interval a_6 resides in. For this reason, we have to split Case (6) into two cases: (6a), $\frac{1}{9}(31 + \sqrt{574}) \geqslant a_6 \geqslant 6$, and (6b), $a_6 \geqslant \frac{1}{9}(31 + \sqrt{574}) \approx 6.10648$.

Case (6a) $\frac{1}{9}(31 + \sqrt{574}) \approx 6.10648 > a_6 \geqslant 6$. In this case, there are six subcases to consider:

(i) $P_5(6) = 0, P_5(5) = 0, P_5(4) = 0, P_5(3) = 0, P_5(2) = 0, P_5(1) > 0$. In this case levels $k = 6, 5, 4, 3, 2$ have no positive integral solutions.

(ii) $P_5(6) = 0, P_5(5) = 0, P_5(4) = 0, P_5(3) = 0, P_5(2) > 0$. In this case levels $k = 6, 5, 4, 3$ have no positive integral solutions.

(iii) $P_5(6) = 0, P_5(5) = 0, P_5(4) = 0, P_5(3) > 0$. In this case levels $k = 6, 5, 4$ have no positive integral solutions.

(iv) $P_5(6) = 0, P_5(5) = 0, P_5(4) > 0$. In this case levels $k = 6, 5$ have no positive integral solutions.

(v) $P_5(6) = 0, P_5(5) > 0$. In this case only level $k = 6$ has no positive integral solutions.

(vi) $P_5(6) > 0$. In this case the level $k = 6$ has positive integral solutions.

Since the proofs of Subcases (i)–(vi) are analogous, we only provide the detailed proof of Subcase (i).

Subcase (i) We will use the proof of (1) for this case. In this case only $P_5(1) > 0$. This implies that $(1, 1, 1, 1, 1, 1)$ is the smallest positive integral solution to level $k = 1$. It follows that

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leqslant 1 - \frac{1}{a_6} = \beta_1, \quad \beta_1 \in \left[\frac{5}{6}, 0.836239\right),$$

since $a_6 \in [6, 6.10648)$. In addition, the parameters for A_1, A_2, A_3, A_4 and A_5 can be improved to the following:

$$A_1 \geqslant A_2 \geqslant A_3 \geqslant A_4 \geqslant A_5 \geqslant \frac{\beta_1}{1 - \beta_1}$$

since $A_5 = a_5\beta_1 \geqslant a_6\beta_1 = \frac{\beta_1}{1-\beta_1}$. Because $\beta_1 \in [\frac{5}{6}, 0.836239)$, $\frac{\beta_1}{1-\beta_1} \in [5, 5.10648)$. Therefore, it is sufficient to prove that $\Phi_1 > 0$ for $A_1 \geqslant A_2 \geqslant A_3 \geqslant A_4 \geqslant A_5 \geqslant \frac{\beta_1}{1-\beta_1}$ and $\beta_1 \in [\frac{5}{6}, 0.836239)$,

$$\Phi_1|_{A_1=A_2=A_3=A_4=A_5=\frac{\beta_1}{1-\beta_1}} = \frac{\beta_1^6(120 - 847\beta_1 + 2521\beta_1^2 - 3604\beta_1^3 + 2556\beta_1^4 - 720\beta_1^5)}{-1+\beta_1} \geqslant 0,$$

for $\beta_1 \in [\frac{5}{6}, 0.836239)$, because the roots of the numerator are $\beta_1 = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$. It follows that $\Phi_1 > 0$ for $A_1 \geqslant A_2 \geqslant A_3 \geqslant A_4 \geqslant A_5 \geqslant \frac{\beta_1}{1-\beta_1}$ and $\beta_1 \in [\frac{5}{6}, 0.836239)$.

Case (6b) $a_6 \geqslant \frac{1}{9}(31 + \sqrt{5}74) \approx 6.10648$. In order to solve ths case we will need to use the sharp estimate of the GLY conjecture for $n = 6$, which has already been proven by Wang and Yau [35].

GLY Conjecture (Sharp Estimate), $n = 6$ (GLY6). Let $a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 \geqslant a_5 \geqslant a_6 \geqslant 5$ be real numbers and P_6 be the number of positive integral points satisfying

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} \leqslant 1.$$

Then

$$\begin{aligned}720P_6 \leqslant{} & a_1a_2a_3a_4a_5a_6 - \frac{5}{2}(a_1a_2a_3a_4a_5 + a_1a_2a_3a_4a_6\\
& + a_1a_2a_3a_5a_6 + a_1a_2a_4a_5a_6 + a_1a_3a_4a_5a_6 + a_2a_3a_4a_5a_6)\\
& + 17(a_1a_2a_3a_4 + a_1a_2a_3a_5 + a_1a_2a_4a_5 + a_1a_3a_4a_5 + a_2a_3a_4a_5)\\
& - \frac{45}{2}(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4 + a_1a_2a_5\\
& + a_1a_3a_5 + a_2a_3a_5 + a_1a_4a_5 + a_2a_4a_5 + a_3a_4a_5)\\
& + \frac{137}{5}(a_1a_2 + a_1a_3 + a_2a_3 + a_1a_4 + a_2a_4 + a_3a_4 + a_1a_5 + a_2a_5 + a_3a_5 + a_4a_5)\\
& - 24(a_1 + a_2 + a_3 + a_4 + a_5),\end{aligned}$$

and the equality is attained if and only if $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = \mathbb{Z}$. Since GLY6 has already been shown to be true, it is sufficient to prove that NTC6 $\geqslant$ GLY6, and that equality holds if and only if $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = \mathbb{Z}$. To prove this, we will first subtract GLY6 from NTC6 and let $A_i = \frac{a_i}{a_6}, i = 1, \ldots, 5$. Then, we will show that the difference is greater than or equal to 0:

$$\begin{aligned}\Phi_7 :={} & \frac{3}{2}a_6^5(A_1A_2A_3A_4A_5 + A_2A_3A_4A_5 + A_1A_3A_4A_5 + A_1A_2A_4A_5 + A_1A_2A_3A_5 + A_1A_2A_3A_4)\\
& - 16a_6^4(A_2A_3A_4A_5 + A_1A_3A_4A_5 + A_1A_2A_4A_5 + A_1A_2A_3A_5 + A_1A_2A_3A_4)\\
& + a_6^4(A_3A_4A_5 + A_2A_4A_5 + A_1A_4A_5 + A_2A_3A_5 + A_1A_3A_5)\\
& + (A_1A_2A_5 + A_2A_3A_4 + A_1A_3A_4 + A_1A_2A_4 + A_1A_2A_3)\\
& + \frac{43}{2}a_6^3(A_3A_4A_5 + A_2A_4A_5 + A_1A_4A_5 + A_2A_3A_5 + A_1A_3A_5 + A_1A_2A_5\\
& + A_2A_3A_4 + A_1A_3A_4 + A_1A_2A_4 + A_1A_2A_3)\\
& - a_6^3(A_4A_5 + A_3A_5 + A_2A_5 + A_1A_5 + A_3A_4 + A_2A_4 + A_1A_4 + A_2A_3 + A_1A_3 + A_1A_2)\\
& - \frac{132}{5}a_6^2(A_4A_5 + A_3A_5 + A_2A_5 + A_1A_5 + A_3A_4 + A_2A_4 + A_1A_4 + A_2A_3 + A_1A_3 + A_1A_2)\\
& + a_6^2(A_5 + A_4 + A_3 + A_2 + A_1) + 23\beta_6(A_5 + A_4 + A_3 + A_2 + A_1)\\
& - (115a_6 - 259a_6^2 + 205a_6^3 - 70a_6^4 + 9a_6^5).\end{aligned}$$

We will show that for all $a_6 \geqslant 6$, the minimum of Φ_7 for $A_1 \geqslant A_2 \geqslant A_3 \geqslant A_4 \geqslant A_5 \geqslant 1$ occurs at $A_1 = A_2 = A_3 = A_4 = A_5 = 1$ and $\Phi_7|_{A_1=A_2=A_3=A_4=A_5=1} = 0$. Notice that Φ_7 is symmetric with respect to A_1, A_2, A_3, A_4 and A_5, then we have

$$\frac{\partial^5\Phi_7}{\partial A_1\partial A_2\partial A_3\partial A_4\partial A_5} = \frac{3a_6^5}{2} > 0,$$

for $a_6>0$. It follows that $\frac{\partial^4\Phi_7}{\partial A_1\partial A_2\partial A_3\partial A_4}$ is an increasing function of A_5 for $A_5\geqslant 1$ and $a_6>0$. Therefore, the minimum of $\frac{\partial^4\Phi_7}{\partial A_1\partial A_2\partial A_3\partial A_4}$ occurs at $A_5=1$ and we have

$$\frac{\partial^4\Phi_7}{\partial A_1\partial A_2\partial A_3\partial A_4}\bigg|_{A_5=1}=\bigg[-16a_6^4+\frac{3a_6^5}{2}+\frac{3}{2}a_6^5A_5\bigg]\bigg|_{A_5=1}=-16a_6^4+3a_6^5>0,$$

for $a_6>\frac{16}{3}$. Hence $\frac{\partial^4\Phi_7}{\partial A_1\partial A_2\partial A_3\partial A_4}>0$ for $A_5\geqslant 1$ and $a_6>\frac{16}{3}$. Since $\frac{\partial^3\Phi_7}{\partial A_1\partial A_2\partial A_3}$ is symmetric with respect to A_4 and A_5, $\frac{\partial^4\Phi_7}{\partial A_1\partial A_2\partial A_3\partial A_5}>0$ for $A_4\geqslant 1$ and $a_6>\frac{16}{3}$ and $\frac{\partial^3\Phi_7}{\partial A_1\partial A_2\partial A_3}$ is an increasing function with respect to A_4 and A_5 for $A_4\geqslant A_5\geqslant 1$ and $a_6>\frac{16}{3}$. The minimum of $\frac{\partial^3\Phi_7}{\partial A_1\partial A_2\partial A_3}$ occurs at $A_4=A_5=1$ and we have

$$\begin{aligned}\frac{\partial^3\Phi_7}{\partial A_1\partial A_2\partial A_3}\bigg|_{A_4=A_5=1}&=\bigg[\frac{43a_6^3}{2}+a_6^4-16a_6^4(A_4+A_5)+\frac{3}{2}a_6^5(A_4+A_5+A_4A_5)\bigg]\bigg|_{A_4=A_5=1}\\&=\frac{43a_6^3}{2}-31a_6^4+\frac{9a_6^5}{2}>0,\end{aligned}$$

for $a_6\geqslant\frac{1}{9}(31+\sqrt{574})\approx 6.10648$. It follows that $\frac{\partial^3\Phi_7}{\partial A_1\partial A_2\partial A_3}>0$ for $A_4\geqslant A_5\geqslant 1$ and $a_6>6.10648$. Because $\frac{\partial^2\Phi_7}{\partial A_1\partial A_2}$ is symmetric with respect to A_3, A_4 and A_5, $\frac{\partial^3\Phi_7}{\partial A_1\partial A_2\partial A_4}>0$ for $A_3\geqslant A_5\geqslant 1$ and $a_6>6.10648$, $\frac{\partial^3\Phi_7}{\partial A_1\partial A_2\partial A_5}>0$ for $A_3\geqslant A_4\geqslant 1$ and $a_6>6.10648$, and $\frac{\partial^2\Phi_7}{\partial A_1\partial A_2}$ is an increasing function with respect to A_3, A_4, A_5 for $A_3\geqslant A_4\geqslant A_5\geqslant 1$ and $a_6>6.10648$. Therefore, the minimum of $\frac{\partial^2\Phi_7}{\partial A_1\partial A_2}$ occurs at $A_3=A_4=A_5=1$ and we have

$$\begin{aligned}&\frac{\partial^2\Phi_7}{\partial A_1\partial A_2}\bigg|_{A_3=A_4=A_5=1}\\&=\bigg[-\frac{132a_6^2}{5}-a_6^3+\frac{43}{2}a_6^3A_3+a_6^4A_3+\frac{43}{2}a_6^3A_4+a_6^4A_4-16a_6^4A_3A_4\\&\quad+\frac{3}{2}a_6^5A_3A_4+\frac{43}{2}a_6^3A_5+a_6^4A_5-16a_6^4A_3A_5+\frac{3}{2}a_6^5A_3A_5-16a_6^4A_4A_5\\&\quad+\frac{3}{2}a_6^5A_4A_5+\frac{3}{2}a_6^5A_3A_4A_5\bigg]\bigg|_{A_3=A_4=A_5=1}\\&=-\frac{132a_6^2}{5}+\frac{127a_6^3}{2}-45a_6^4+6a_6^5>0,\end{aligned}$$

for $a_6>6.10648$. It follows that $\frac{\partial^2\Phi_7}{\partial A_1\partial A_2}>0$ for $A_3\geqslant A_4\geqslant A_5\geqslant 1$ and $a_6>6.10648$. Since $\frac{\partial\Phi_7}{\partial A_1}$ is symmetric with respect to A_2, A_3, A_4 and A_5, $\frac{\partial^2\Phi_7}{\partial A_1\partial A_3}>0$ for $A_2\geqslant A_4\geqslant A_5\geqslant 1$ and $a_6>6.10648$, $\frac{\partial^2\Phi_7}{\partial A_1\partial A_4}>0$ for $A_2\geqslant A_3\geqslant A_5\geqslant 1$ and $a_6>6.10648$, $\frac{\partial^2\Phi_7}{\partial A_1\partial A_5}>0$ for $A_2\geqslant A_3\geqslant A_4\geqslant 1$ and $a_6>6.10648$, and $\frac{\partial\Phi_7}{\partial A_1}$ is an increasing function with respect to A_2, A_3, A_4 and A_5 for $A_2\geqslant A_3\geqslant A_4\geqslant A_5\geqslant 1$ and $a_6>6.10648$. Hence the minimum of $\frac{\partial\Phi_7}{\partial A_1}$ occurs at $A_2=A_3=A_4=A_5=1$ and we have

$$\begin{aligned}&\frac{\partial\Phi_7}{\partial A_1}\bigg|_{A_2=A_3=A_4=A_5=1}\\&=23a_6+a_6^2-\frac{132}{5}a_6^2A_2-a_6^3A_2-\frac{132}{5}a_6^2A_3-a_6^3A_3+\frac{43}{2}a_6^3A_2A_3\\&\quad+a_6^4A_2A_3-\frac{132}{5}a_6^2A_4-a_6^3A_4+\frac{43}{2}a_6^3A_2A_4+a_6^4A_2A_4+\frac{43}{2}a_6^3A_3A_4+a_6^3A_3A_4\\&\quad-16a_6^4A_2A_3A_4+\frac{3}{2}a_6^5A_2A_3A_4-\frac{132}{5}a_6^2A_5-a_6^3A_5+\frac{43}{2}a_6^3A_2A_5+a_6^4A_4A_5\\&\quad+\frac{43}{2}a_6^3A_3A_5+a_6^4A_3A_5-16a_6^4A_2A_3A_5+\frac{3}{2}a_6^5A_2A_3A_5+\frac{43}{2}a_6^3A_4A_5+a_6^4A_4A_5\\&\quad-16a_6^4A_2A_4A_5+\frac{3}{2}a_6^5A_2A_4A_5-16a_6^4A_3A_4A_5+\frac{3}{2}a_6^5A_3A_4A_5+\frac{3}{2}a_6^5A_2A_3A_4A_5>0,\end{aligned}$$

for $a_6 > 6.10648$. It follows that $\frac{\partial \Phi_7}{\partial A_1} > 0$ for $A_2 \geqslant A_3 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $a_6 > 6.10648$. Notice that Φ_7 is symmetric with respect to A_1, A_2, A_3, A_4 and A_5. Therefore, $\frac{\partial \Phi_7}{\partial A_2} > 0$ for $A_1 \geqslant A_3 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $a_6 > 6.10648$, $\frac{\partial \Phi_7}{\partial A_3} > 0$ for $A_1 \geqslant A_2 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $a_6 > 6.10648$, $\frac{\partial \Phi_7}{\partial A_4} > 0$ for $A_1 \geqslant A_2 \geqslant A_3 \geqslant A_5 \geqslant 1$ and $a_6 > 6.10648$, $\frac{\partial \Phi_7}{\partial A_5} > 0$ for $A_1 \geqslant A_2 \geqslant A_3 \geqslant A_4 \geqslant 1$ and $a_6 > 6.10648$, and Φ_7 is an increasing function with respect to A_1, A_2, A_3, A_4 and A_5 for $A_1 \geqslant A_2 \geqslant A_3 \geqslant A_4 \geqslant A_5 \geqslant 1$ and $a_6 > 6.10648$. The minimum occurs at $A_1 = A_2 = A_3 = A_4 = A_5 = 1$ and we have

$$\Phi_7|_{A_1=A_2=A_3=A_4=A_5=1} = 0,$$

for $a_6 > 6.10648$. Therefore, $\Phi_7 \geqslant 0$ for $a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 \geqslant a_5 \geqslant a_6 \geqslant 1$ and $\Phi_7 = 0$ if and only if $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = Z$.

2.2 Proof of Theorem 1.4

Due to the fact that $\psi(x, y) = Q_n$, we can apply our sharp estimate of P_6 to the function in order to obtain an estimate. Let $p_1 < p_2 < p_3 < p_4 < p_5 < p_6$ be the first six prime numbers up to y. If $p_1^{l_1} p_2^{l_2} p_3^{l_3} p_4^{l_4} p_5^{l_5} p_6^{l_6} \leqslant x$, then

$$\frac{l_1}{\frac{\log x}{\log p_1}} + \frac{l_2}{\frac{\log x}{\log p_2}} + \frac{l_3}{\frac{\log x}{\log p_3}} + \frac{l_4}{\frac{\log x}{\log p_4}} + \frac{l_5}{\frac{\log x}{\log p_5}} + \frac{l_6}{\frac{\log x}{\log p_6}} \leqslant 1.$$

It follows that $a_i = \frac{\log x}{\log p_i}$ and $x_i = l_i$, $1 \leqslant i \leqslant 6$. Note that $Q_6 = P(a_1(1+a), a_2(1+a), a_3(1+a), a_4(1+a), a_5(1+a), a_6(1+a))$, where

$$a = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6}.$$

We split the estimate into four cases:

(I) $5 \leqslant y < 7$;

(II) $7 \leqslant y < 11$;

(III) $11 \leqslant y < 13$;

(IV) $13 \leqslant y < 17$.

Cases (I) through (III) have been proven through the estimates of P_3, P_4 and P_5, respectively. Case (IV) involves the first six prime numbers: $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, and $p_6 = 13$. Consequently,

$$a = \frac{\log 2 + \log 3 + \log 5 + \log 7 + \log 11 + \log 13}{\log x}$$

and

$$\begin{aligned}
\psi(x, y) &= Q_6 \\
&= P\bigg(\frac{\log x}{\log 2}\bigg(1 + \frac{\log 2 \times 3 \times 5 \times 7 \times 11 \times 13}{\log x}\bigg), \\
&\quad \frac{\log x}{\log 3}\bigg(1 + \frac{\log 2 \times 3 \times 5 \times 7 \times 11 \times 13}{\log x}\bigg), \frac{\log x}{\log 5}\bigg(1 + \frac{\log 2 \times 3 \times 5 \times 7 \times 11 \times 13}{\log x}\bigg), \\
&\quad \frac{\log x}{\log 7}\bigg(1 + \frac{\log 2 \times 3 \times 5 \times 7 \times 11 \times 13}{\log x}\bigg), \frac{\log x}{\log 11}\bigg(1 + \frac{\log 2 \times 3 \times 5 \times 7 \times 11 \times 13}{\log x}\bigg), \\
&\quad \frac{\log x}{\log 13}\bigg(1 + \frac{\log 2 \times 3 \times 5 \times 7 \times 11 \times 13}{\log x}\bigg)\bigg) \\
&\leqslant \frac{1}{6!}\bigg(\bigg(\frac{\log x}{\log 2} + \frac{\log 3 \times 5 \times 7 \times 11 \times 13}{\log 2}\bigg)\bigg(\frac{\log x}{\log 3} + \frac{\log 3 \times 5 \times 7 \times 11 \times 13}{\log 3}\bigg) \\
&\quad \times \bigg(\frac{\log x}{\log 5} + \frac{\log 3 \times 5 \times 7 \times 11 \times 13}{\log 5}\bigg)\bigg(\frac{\log x}{\log 7} + \frac{\log 3 \times 5 \times 7 \times 11 \times 13}{\log 7}\bigg)
\end{aligned}$$

$$
\begin{aligned}
&\times\left(\frac{\log x}{\log 11}+\frac{\log 3\times 5\times 7\times 11\times 13}{\log 11}\right)\left(\frac{\log x}{\log 13}+\frac{\log 3\times 5\times 7\times 11\times 13}{\log 13}\right)\\
&-\left(\left(\frac{\log x}{\log 13}+\frac{\log 3\times 5\times 7\times 11\times 13}{\log 13}\right)^6-\left(\frac{\log x}{\log 13}+\frac{\log 3\times 5\times 7\times 11\times 13}{\log 13}+1\right)\right.\\
&\times\left(\frac{\log x}{\log 13}+\frac{\log 3\times 5\times 7\times 11\times 13}{\log 13}\right)\left(\frac{\log x}{\log 13}+\frac{\log 3\times 5\times 7\times 11\times 13}{\log 13}-1\right)\\
&\times\left(\frac{\log x}{\log 13}+\frac{\log 3\times 5\times 7\times 11\times 13}{\log 13}-2\right)\left(\frac{\log x}{\log 13}+\frac{\log 3\times 5\times 7\times 11\times 13}{\log 13}-3\right)\\
&\left.\times\left(\frac{\log x}{\log 13}+\frac{\log 3\times 5\times 7\times 11\times 13}{\log 13}-4\right)\right)\Bigg)\\
=&\frac{1}{720}\bigg\{\frac{1}{\log 2\log 3\log 5\log 7\log 11\log 13}(\log x+\log 15015)(\log x+\log 10010)(\log x+\log 6006)\\
&\times(\log x+\log 4290)(\log x+\log 2730)(\log x+\log 2310)-\frac{1}{\log^6 13}[(\log x+\log 2310)^6\\
&-(\log x+\log 13+\log 2310)(\log x+\log 2310)(\log x+\log 2310-\log 13)\\
&\times(\log x+\log 2310-2\log 13)(\log x+\log 2310-3\log 13)\\
&\times(\log x+\log 2310-4\log 13)]\bigg\}.
\end{aligned}
$$

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