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Recent Results on Rigidity of CR Morphisms Between Compact Strongly Pseudoconvex CR Manifolds

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Dedicated to Professor George Daniel Mostow on the occasion of his 90th birthday

Abstract

The purpose of this paper is to summarize the results on rigidity of CR morphisms between compact strongly pseudoconvex CR manifolds developed by the first author and his coauthors. This theory is related to the rigidity of etale covering between resolutions of isolated singularities.

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1 Introduction

Rigidity phenomena in complex geometry have received a lot of attention historically. Borel and Narasimhan [Bo-Na67] obtained some general results on rigidity of morphisms with domain space carrying no non-constant pseudoconvex function which is bounded above and target space being covered by analytic subset of a bounded domain in $\mathbb{C}^n$. In 1975, Kobayashi and Ochiai [Ko-Oc75] proved that there are only finitely many surjective morphisms between two fixed projective manifolds of general type. In 1981, Kalka, Shiffman and Wong [KSW81] developed a general theory to study the finiteness and rigidity theorems for holomorphic mappings. Let $\mathrm{Hol}_k(X, Y)$ denote the complex space of holomorphic maps of rank $\geqslant k$

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from the compact complex space X into the complex manifold Y. They proved that if Y satisfies certain convexity or cohomological conditions, then for suitable k, $\mathrm{Hol}_k(X, Y)$ is either discrete or finite. They also showed that if the tangent bundle of Y satisfies k-pseudoconvexity condition, then $\mathrm{Hol}_k(X, Y)$ is discrete. As a corollary they asserted that if Y is a compact Hermitian manifold with negative holomorphic sectional curvature, then the set of surjective holomorphic maps from X onto Y is finite. On the other hand, if Y is a compact Kähler manifold with $c_1(Y)$ represented by a negative semi-definite form and either $c_n(Y) \neq 0$ or $c_1^n(Y) \neq 0$, Kalka, Shiffman and Wong showed that $\mathrm{Hol}_n(X, Y)$ is discrete.

Inspiring by the Grauert-Oka principle, Mok [Mok84] formulated the following problem on n-dimensional Stein spaces.

Problem. Suppose V is an n-dimensional Stein space which has the homotopy type of a real n-dimensional CW-complex. Suppose $f : V \to V$ is a holomorphic homotopy equivalence. Does it follow that f is a biholomorphism?

Mok [Mok84] pointed out that obvious counter-example on the punctured disc show that the answer is in general negative even in the case of bounded domains and injective holomorphic self-mappings. In [Mok84] , instead of assuming that f is a homotopy equivalence, Mok assumes that V is hyperbolic in the sense of Caratheodory and f induces an isomorphism on the n-th integral homology group $H_n(V, \mathbb{Z})$, which is assumed to be nonzero and finitely generated. Then he proved that f is a biholomorphism.

Recently Huang, Kebekus and Peternell [H-K-P06] have proved the following beautiful result. Let Y be a projective n-dimensional manifold which is not uniruled. If either $\pi_1(Y)$ is finite or $c_n(Y) \neq 0$, then for each connected normal compact complex variety X, the space of surjective morphism from X to Y is discrete.

In 1977, Wong [Wo77] proved an important result in complex geometry that any strongly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$ with noncompact automorphism group must be biholomorphically equivalent to the unit ball. In 1978, Yau [Ya78] proved the Calabi Conjecture. As a consequence, he [Ya77] proved that $3c_2 \geqslant c_1^2$ for any Kähler surface with ample canonical bundle, and the equality holds if and only if the surface is covered by the ball in $\mathbb{C}^2$. Using this results, he [Ya77] proved the Severi Conjecture that every complex surface which is homotopic to the complex projective plane $\mathbb{CP}^2$ is biholomorphic to $\mathbb{CP}^2$.

As a consequence of his famous strong rigidity theorem [Mo83] Mostow showed that two compact quotients of the ball of complex dimension $\geqslant 2$ with isomorphic fundamental groups are either biholomorphic or conjugate biholomorphic. S.-T. Yau conjectured that this phenomenon of strong rigidity should hold also for compact Kähler manifolds of complex dimension $\geqslant 2$ with same homotopy type and negative sectional curvature. In his paper [Siu80], Siu proved that Yau's conjecture is true when the curvature tensor of one of the two compact Kähler manifolds is strongly negative with no curvature assumption on the other manifold.

CR manifolds are abstract models of boundary of complex manifolds. Strongly pseudoconvex CR manifolds have rich geometric and analytic structures. The harmonic theory for the $\overline{\partial}_b$ complex on compact strongly pseudoconvex CR manifolds was developed by Kohn [Kohn65]. Using this theory, Boutet de Monvel [BM75] proved that if X is a compact strongly pseudoconvex CR manifold of real dimension $2n-1$, $n \geqslant 3$ then there exist C^∞ functions $f_1, \ldots, f_N$ on X such that each $\overline{\partial}_b f_j = 0$ and $f = (f_1, \ldots, f_N)$ defines an embedding of X in $\mathbb{C}^N$. Thus, any compact strongly pseudoconvex CR manifold of dimension $\geqslant 5$ can be CR embedded in some complex Euclidean space. On the other hand, 3-dimensional strongly pseudoconvex compact orientable CR manifolds are not necessarily embeddable. Throughout this paper, our strongly pseudoconvex CR manifolds are always assumed to be compact orientable and embedded in some $\mathbb{C}^N$. By a beautiful theorem of Harvey and Lawson [Ha-La00, Ha-La75], these CR manifolds are the boundaries of subvarieties with only isolated normal singularities.

Rigidity problems of CR immersions into spheres and hyperquadrics were studied by Ebenfelt, Huang and Zaitsev [EHZ04, EHZ05]. Let X be a strongly pseudoconvex CR manifold of dimension $2n-1$. For $p \in X$, let $f\colon (X, p) \longrightarrow S^{2n+2d-1}$ be a local CR immersion of X near p into unit sphere $S^{2n+2d-1}$ in $\mathbb{C}^{n+d}$. The result of [EHZ04] states that if $d < \frac{n}{2} - 1$, then f is rigid in the sense that any other immersion of (X, p) into $S^{2n+2d-1}$ is of the form $\phi \circ f$, where ϕ is biholomorphic automorphism of the unit ball $B \subseteq \mathbb{C}^{n+d}$. As a striking corollary, they show that if X and X' are two strongly pseudoconvex CR manifolds of dimension $2n-1$ in $S^{2n+2d-1}$ with $d < \frac{n}{2} - 1$ and if X and X' are locally CR equivalent at some points $p \in X$ and $p' \in X'$, then there exists a unitary linear transformation which maps X to X'.

In a remarkable paper [Pin74], Pinchuk showed that a proper holomorphic mapping between strongly pseudoconvex domains in $\mathbb{C}^n$ is locally biholomorphic. In fact he proved that proper holomorphic self-maps of strongly pseudoconvex domains are necessarily biholomorphic. It was proved in [Be-Ca82] and [Di-Fo82] that proper holomorphic maps extend smoothly to the boundaries and hence induce CR morphisms between the boundaries.

In [Yau11], the first author of this paper took another point of view. In view of Fornaess theory on strongly pseudoconvex domains [For76], he investigated the rigidity property of CR morphisms between strongly pseudoconvex CR manifolds by means of the singularities theory. Later in his joint work with Tu and Zuo [TYZ13], the rigidity of CR morphisms between CR manifolds lying in the same variety is proved. The purpose of this paper is to summarize the results on rigidity of CR morphisms between strongly pseudoconvex CR manifolds by means of the singularity theory.

In Section 2, we shall recall some basic notions and theorems of CR manifolds and the interplay between CR manifold and singularities. In Section 3, we shall discuss the rigidity of CR morphisms between any two compact strongly pseudoconvex CR manifolds. In Section 4, we will show that the rigidity of CR morphisms between two compact strongly pseudoconvex CR manifolds lying in a same variety with isolated singularities is related to the rigidity of etale covering between resolutions of isolated singularities. In Section 5, we give examples of strongly pseudoconvex 3-dimensional CR manifolds and computes their non-constant CR endomorphisms.

2 Preliminaries on CR manifolds

Definition 2.1. *Let X be a connected oriented orientable manifold of real dimension $2n-1$. A CR structure on X is an $(n-1)$-dimensional subbundle S of the complexified tangent bundle $\mathbb{C}TX$ such that*

(1) $S \cap \overline{S} = \{0\}$;

(2) *If L, L' are local sections of X, then so is $[L, L']$.*

A manifold with a CR structure is called a CR manifold. There is a unique subbundle $\mathcal{H}$ of the tangent bundle $T(X)$ such that $\mathbb{C}\mathcal{H} = S \oplus \overline{S}$. Furthermore, there is a unique homomorphism $J : \mathcal{H} \to \mathcal{H}$ such that $J^2 = -1$ and $S = \{v - iJv : v \in \mathcal{H}\}$. The pair $(\mathcal{H}, J)$ is called the real expression of the CR structure.

Definition 2.2. *Let $L_1, \ldots, L_{n-1}$ be a local frame of S. Then $\overline{L}_1, \ldots, \overline{L}_{n-1}$ is a local frame of $\overline{S}$ and one may choose a local section N of $T(X)$ which is purely imaginary, such that $L_1, \ldots, L_{n-1}, \overline{L}_1, \ldots, \overline{L}_{n-1}, N$ is a local frame of $\mathbb{C}T(X)$. The matrix (c_{ij}) defined by*

$$[L_i, \overline{L}_j] = \sum a_{ij}^k L_k + \sum b_{ij}^k \overline{L}_k + \sqrt{-1} c_{ij} N$$

is Hermitian and called the Levi form of X.

Proposition 2.1. *The number of nonzero eigenvalues and the absolute value of the signature of the Levi form (c_{ij}) at each point are independent of the choice of $L_1, \ldots, L_{n-1}, N$.*

Definition 2.3. *The CR manifold X is called strongly pseudoconvex if the Levi form is definite at each point of X.*

Theorem 2.1. (Boutet de Monvel [BM75]) *If X is a compact strongly pseudoconvex CR manifold of dimension $(2n-1)$ and $n \geq 3$, then X is CR embeddable in $\mathbb{C}^N$.*

Although there are non-embeddable compact 3-dimensional CR manifolds, in this paper all CR manifolds are assumed to be embeddable in complex Euclidean space.

Theorem 2.2. (Harvey-Lawson [Ha-La75, Ha-La00]) *For any compact connected embeddable CR manifold X, there is a unique complex variety V in $\mathbb{C}^N$ for some N such that the boundary of V is X and V has only normal isolated singularities.*

With the notation in Definition 2.1, for any C^∞ function u, there is a section $\overline{\partial}_b u \in \Gamma(\overline{S}^*)$ defined by $(\overline{\partial}_b u)(\overline{L}) = \overline{L}u$ for any $L \in \Gamma(S)$. This can be generalized as follows:

Definition 2.4. *A complex vector bundle E over X is said to be holomorphic if there is a differential operator $\overline{\partial}_E : \Gamma(E) \to \Gamma(E \otimes \overline{S}^*)$ such that if $\overline{L}u$ denotes $(\overline{\partial}_E u)(\overline{L})$ for $u \in \Gamma(E)$ and $L \in \Gamma(S)$, then for any L_1, $L_2 \in \Gamma(S)$ and any C^∞ function f on X*

(1) $\overline{L}(fu) = (\overline{L}f)u + f(\overline{L}u)$;

(2) $[\overline{L}_1, \overline{L}_2]u = \overline{L}_1\overline{L}_2 u - \overline{L}_2\overline{L}_1 u$.

A solution u of the equation $\overline{\partial}_E u = 0$ is called a holomorphic section.

The vector bundle $\widehat{T}(X) = \mathbb{C}T(X)/\overline{S}$ is holomorphic with respect to the following $\overline{\partial} = \overline{\partial}_{\widehat{T}(X)}$. Let ω be the projection from $\mathbb{C}T(X)$ to $\widehat{T}(X)$. Take any $u \in \Gamma(\widehat{T}(X))$ and express it as $u = \omega(Z)$, $Z \in \Gamma(\mathbb{C}T(X))$. For any $L \in \Gamma(S)$, define $(\overline{\partial}u)(\overline{L}) = \omega([\overline{L}, Z])$. The section $(\overline{\partial}u)(\overline{L})$ of $\widehat{T}(X)$ does not depend on the choice of Z and $\overline{\partial}u$ gives a section of $\widehat{T}(X) \otimes \overline{S}^*$. Further the operator $\overline{\partial}$ satisfies the conditions in Definition 2.4. The resulting holomorphic vector bundle $\widehat{T}(X)$ is called the holomorphic tangent bundle of X.

Lemma 2.1. *If X is a real hypersurface in a complex manifold M, then the holomorphic tangent bundle $\widehat{T}(X)$ is naturally isomorphic to the restriction to X of the bundle $T^{1,0}(M)$ of all $(1,0)$ tangent vectors to M.*

Definition 2.5. *Let X be a compact CR manifold of real dimension $2n-1$. X is said to be a Calabi-Yau CR manifold if there exists a nowhere vanishing holomorphic section in $\Gamma(\Lambda^n\widehat{T}(X)^*)$ where $\widehat{T}(X) = \mathbb{C}T(X)/S^*$ is the holomorphic tangent bundle of X.*

Remark 2.1. (a) *Let X be a compact CR manifold of real dimension $2n-1$ which is a boundary of domain in $\mathbb{C}^n$. Then X is a Calabi-Yau CR manifold.*

(b) *Let X be a strongly pseudoconvex CR manifold of real dimension $2n-1$ contained in the boundary of a bounded strongly pseudoconvex domain in $\mathbb{C}^{n+1}$. Then X is a Calabi-Yau CR manifold.*

(c) *More generally, let V be a Stein variety V with singular set S such that the canonical bundle of $V - S$ is trivial. If X is a real hypersurface in $V - S$, then X is a Calabi-Yau CR manifold.*

Definition 2.6. *Let X be a compact connected strongly pseudoconvex embeddable CR manifold of real dimension $2n-1$. Let V be the normal subvariety in $\mathbb{C}^N$ such that the boundary of V is X. Let $\pi : M \to V$ be a resolution of singularities of V. The geometric genus of X denoted by $p_g(X)$ is defined to be $\dim H^{n-1}(M, \mathcal{O})$.*

Proposition 2.2. ([Ya-Yu02]) *Let X be a connected compact strongly pseudoconvex CR manifold of real dimension $2n-1$ and $n \geq 2$. Suppose that X bounds a normal variety $V \subseteq \mathbb{C}^N$ with isolated singularities $Y = \{q_1, \ldots, q_m\}$. Let $\pi : M \to V$ be a resolution of singularities of V. Then the geometric genus $p_g(X) := \dim H^{n-1}(M, \mathcal{O})$ is a CR invariant of X. In fact, let U be any small strongly pseudoconvex neighborhood of Y. Then*

$$p_g(X) = \dim H^0(U - Y, \Omega^n)/L^2(U - Y, \Omega^n)$$

where Ω^n is the sheaf of germs of holomorphic n-forms and $L^2(U - Y, \Omega^n)$ is the space of holomorphic n-forms ω on $U - Y$ which are L^2-integrable for some small neighborhood $U' \subset\subset U$, i.e., $\int_{U'-Y} \omega \wedge \overline{\omega} < \infty$.

Definition 2.7. *Let (V, q) be a complex analytic with isolated normal singularity at q of dimension n. The geometric genus $p_g(V, q)$ of the singularity (V, q) is defined to be* $\dim H^0(U - \{q\}, \Omega^n)/L^2(U - \{q\}, \Omega^n)$ *where U is any strongly pseudoconvex neighborhood of q.*

Remark 2.2. (a) *Let $\pi : (M, A) \to (U, q)$ be a resolution of singularity. By a theorem of* [Yau77], *$H^{n-1}(M, \mathcal{O}) = \dim H^0(V - \{q\}, \Omega^n)/L^2(V - \{q\}, \Omega^n)$. It follows easily that $p_g(V, q)$ is independent of U in the above definition.*

(b) *With the notation in Proposition* 2.2, *we have $p_g(X) = \sum_{i=1}^{m} p_g(V, q_i)$.*

3 General theory on rigidity of CR morphism

The starting point of our investigation on CR morphisms between two embeddable strongly pseudoconvex compact CR manifolds is the following result.

Theorem 3.1. *Let X_1 and X_2 be two compact connected strongly pseudoconvex CR manifolds of dimension $2n - 1 \geq 3$ which bound complex varieties V_1 and V_2 in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively. Suppose the singular set S_i of V_i, $i = 1, 2$ is either an empty set or a set consisting of only isolated normal singularities. If $\phi\colon X_1 \to X_2$ is a non-constant CR morphism, then ϕ is surjective and ϕ can be extended to a proper surjective holomorphic map from V_1 to V_2 such that $\phi(S_1) \subseteq S_2$, $\phi^{-1}(X_2) = X_1$ and $\phi\colon V_1 - \phi^{-1}(S_2) \to V_2 - S_2$ is a covering map. Moreover, if S_2 does not have quotient singularity, then $\phi^{-1}(S_2) = S_1$.*

Proof. Let $\phi_1, \ldots, \phi_{N_2}$ be the component functions of ϕ. Then ϕ_i as CR holomorphic function on X_1 can be extended onto a one-sided neighborhood of X_1 in V_1. By Andreotti and Grauert ([An-Gr62], Théorème 15), ϕ_i can be holomorphically extended to $V_1 - S_1$ where S_1 is the singular set of V_1. Since S_1 is either an empty set or a set consisting of only isolated normal singularities, ϕ_i can be holomorphically extended onto V_1.

We claim that $\phi(V_1) \subseteq V_2$. To see this, let $f_1, \ldots, f_k$ be the defining equations of V_2, i.e. $V_2 = \{y \in \mathbb{C}^{N_2} : f_1(y) = \cdots = f_k(y) = 0\}$. Clearly $\phi^*(f_i) = f_i \circ \phi$ is a holomorphic function on V_1 which vanishes on X_1 for $1 \leq i \leq k$. Since X_1 is of real codimension one in V_1, $\phi^*(f_i)$ is identically zero on V_1 for $1 \leq i \leq k$. This implies that $\phi(V_1) \subseteq V_2$. By the maximum principle, $\phi(X_1) \cap \phi(V_1 - X_1) = \emptyset$. It follows that $\phi^{-1}(X_2) = X_1$ and ϕ is a proper map from V_1 to V_2. By the proper mapping theorem, $\phi(V_1)$ is a complex variety.

We next claim that $\dim \phi(V_1) = n$. If $\dim \phi(V_1) < n$, then for some q in $\phi(V_1)$, $\phi^{-1}(q)$ is a compact variety of dimension at least one sitting inside V_1. This gives a contradiction since V_1 is Stein. As $\phi(V_1) \subseteq V_2$ and $\dim V_1 = n = \dim V_2$, we have $\phi(V_1) = V_2$. It follows that $\phi(X_1) = X_2$. A local computation of Fornaess ([For76], Proposition 12) would apply to show that ϕ is a local biholomorphism near X_1. (This was observed independently by Pinchuk [Pin74].) In particular, $\phi : V_1 - \phi^{-1}(S_2 \cup \phi(S_1)) \to V_2 - (S_2 \cup \phi(S_1))$ is locally biholomorphic and hence is a finite covering.

Let $p \in S_1$ and $q = \phi(p)$. We claim that $q \in S_2$. Suppose on the contrary that q is a smooth point in V_2; then ϕ maps a neighborhood U_1 of p to a neighborhood U_2 of q as a branch covering. Since p is a normal singularity, the punctured neighborhood $U_1 - \{p\}$ of p is connected. On the other hand, the punctured neighborhood $U_2 - \{q\}$ of q is simply connected because q is a smooth point. We conclude that $\phi|_{U_1} : U_1 \to U_2$ is one-to-one and onto. By the Hartogs extension theorem, the inverse map $\phi^{-1}|_{U_2-\{q\}} : U_2 - \{q\} \to U_1 - \{p\}$ can be holomorphically extended onto U_2. It follows that $\phi|_{U_1} : U_1 \to U_2$ is a biholomorphic map. This leads to a contradiction. Therefore $\phi(S_1) \subseteq S_2$ and $\phi : V_1 - \phi^{-1}(S_2) \to V_2 - S_2$ is a covering map.

Now assume that S_2 does not have a quotient singularity. Let q be any point in S_2. We need to show that $\phi^{-1}(q) \subseteq S_1$. If $\phi^{-1}(q)$ is not contained in S_1, then there exists a smooth point q' of V_1 in $\phi^{-1}(q)$. Recall that $\phi^{-1}(q)$ is a finite set. We can find an open neighborhood U of q' which is biholomorphic to a domain in $\mathbb{C}^n$ such that $\phi|_U$ from U to the germ of (V_2, q) is a branch covering with ramification locus $\{q'\}$. By Theorem 1 of [Pr67], we conclude that (V_2, q) is a quotient singularity. This is a contradiction. $\square$

As a corollary of Theorem 3.1, we have the following super-rigidity results of CR morphisms between strongly pseudoconvex manifolds.

Corollary 3.1. *Let X_1 be a compact strongly pseudoconvex CR manifold of dimension $2n-1 \geq 3$ which bounds a complex variety V_1 in $\mathbb{C}^{N_1}$ with isolated normal singularities. Let X_2 be a compact strongly pseudoconvex CR manifold of dimension $2n-1$ which bounds a complex submanifold V_2 in $\mathbb{C}^{N_2}$. Then there is no non-constant CR morphism from X_1 to X_2.*

It is a natural question to ask what happens if we interchange the roles of X_1 and X_2 in Corollary 3.1.

Proposition 3.1. *Let X_1 be a compact strongly pseudoconvex CR manifold of dimension $2n-1 \geq 3$ which bounds a complex submanifold V_1 in $\mathbb{C}^{N_1}$. Let X_2 be a compact strongly pseudoconvex CR manifold of dimension $2n-1$ with either* (i) *geometric genus $p_g(X_2) > 0$, or* (ii) *$p_g(X_2) \geq 0$ and X_2 bounds a complex variety V_2 in $\mathbb{C}^{N_2}$ with a non-quotient singularity. Then there is no non-constant CR morphism from X_1 to X_2.*

Proof. By Theorem 3.1, if there exists a non-constant CR morphism $\phi : X_1 \to X_2$, then ϕ can be extended as a ramified covering map from V_1 to V_2 with ramification locus S_2. Since V_1 is smooth, by the proof of Theorem 3.1, S_2 consists of only quotient singularities and hence the geometric genus of these singularities are zero. It follows that $p_g(X_2) = 0$ in view of Remark 2.2(b). This leads to a contradiction. $\square$

Remark 3.1. *Proposition* 3.1 *is false if $p_g(X_2) = 0$ and V_2 has only quotient singularities in the interior, as we can see from the following example.*

Example 3.1. *Let $B = \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 < 1\}$ and $S = \partial B$. In the notation of Proposition* 3.1*, let $X_1 = S$ be the standard sphere and $V_1 = B$. Let $\sigma : B \to B$*

be the map given by $\sigma(x,y) = (-x,-y)$. Let V_2 be the quotient of V_1 by the cyclic group of order 2 generated by σ. Then V_2 is a strongly pseudoconvex variety with A_1 singularity $\{(z_1,z_2,z_3) \in \mathbb{C}^3 : z_1z_2 = z_3^2\}$. The quotient map $\phi : V_1 \to V_2$ is given by $(z_1,z_2,z_3) = (x^2,y^2,xy)$. Clearly ϕ sends X_1 surjectively onto $X_2 = \partial V_2$ and ϕ is a non-constant CR morphism.

Proposition 3.2. *Let X_1 and X_2 be two compact strongly pseudoconvex embeddable CR manifolds of dimension $2n-1 \geq 3$. If there is a non-constant CR morphism from X_1 to X_2, then $p_g(X_1) \geq p_g(X_2)$.*

Proof. Let V_i be a normal variety in $\mathbb{C}^{N_i}$ with only isolated singularities such that $\partial V_i = X_i, i = 1,2$. Let S_1 and S_2 be the singular set of V_1 and V_2 respectively. Let $\phi : X_1 \to X_2$ be a non-constant CR morphism. In view of Theorem 3.1, ϕ can be extended to a proper surjective holomorphic map from V_1 and V_2 such that $\phi(S_1) = S_2$, and $\phi : V_1 - \phi^{-1}(S_2) \to V_2 - S_2$ is a covering map. There is a natural map

$$\phi^* : \frac{\Gamma(V_2 - S_2, \Omega^n)}{L^2(V_2 - S_2, \Omega^n)} \to \frac{\Gamma(V_1 - \phi^{-1}(S_2), \Omega^n)}{L^2(V_1 - \phi^{-1}(S_2), \Omega^n)}.$$

Since $\phi : V_1 - \phi^{-1}(S_2) \to V_2 - S_2$ is a finite covering map, a form $w \in \Gamma(V_2 - S_2, \Omega^n)$ is L^2-integrable if and only if $\phi^*(w)$ is L^2-integrable. Thus ϕ^* is injective. Observe that $\phi^{-1}(S_2) - S_1$ is a discrete subset in the smooth part of V_1. By Hartog's theorem, $\Gamma(V_1 - \phi^{-1}(S_2), \Omega^n) = \Gamma(V_1 - S_1, \Omega^n)$ and $L^2(V_1 - \phi^{-1}(S_2), \Omega^n) = L^2(V_1 - S_1, \Omega^n)$. It follows that $p_g(X_2) \leq p_g(X_1)$. □

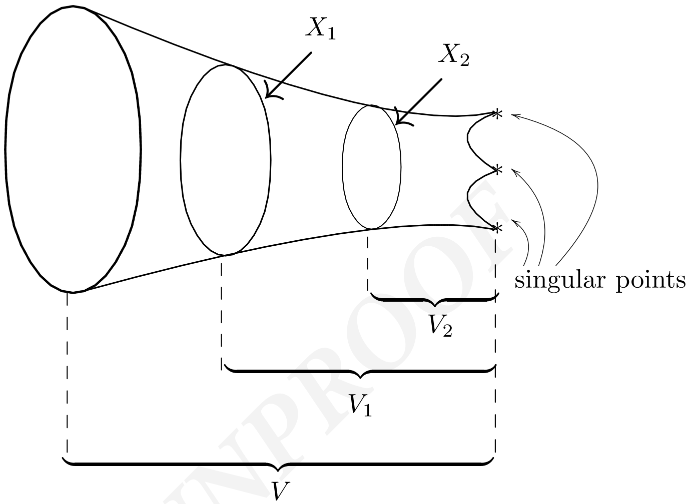
Corollary 3.2. *Let X_1, X_2 be two compact strongly pseudoconvex embeddable CR manifolds of dimension $2n-1 \geq 3$. If $p_g(X_1) < p_g(X_2)$, then there is no non-constant CR morphism from X_1 to X_2.*

The following theorem says that if the codimension of X_2 is small and $\dim X_2 \geq 5$, then there is no non-constant CR morphism from X_1 to X_2 except CR biholomorphic maps. This rigidity phenomenon does not require any curvature assumption on X_1 or X_2.

Theorem 3.2. *Let X_1 and X_2 be two compact strongly pseudoconvex CR manifolds of dimension $2n-1 \geq 5$ which bound complex varieties V_1 and V_2 with only isolated normal singularities in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively. Let S_1 and S_2 be the singular sets of V_1 and V_2 respectively and S_2 is non-empty. Suppose $2n - N_2 - 1 \geq 1$. Then there exists no non-constant CR morphism from X_1 to X_2 if $|S_1|$ is not divisible by $|S_2|$. If $|S_1| = |S_2|$, then X_1 is CR biholomorphic to X_2.*

Proof. Let $\phi : X_1 \to X_2$ be a non-constant CR morphism. Theorem 3.1 says that ϕ can be extended to a proper surjective holomorphic map from V_1 to V_2 such that $\phi(S_1) \subseteq S_2$ and $\phi : V_1 - \phi^{-1}(S_2) \to V_2 - S_2$ is a covering map of degree d. For any $q \in S_2$, we know that the punctured neighborhood of q in V_2 is $(2n - N_2 - 1)$-connected in view of a theorem of Hamm [Ham81]. Since $2n - N_2 - 1 \geq 1$ by assumption, the punctured neighborhood of q is simply connected. We claim that $\phi^{-1}(q) \subseteq S_1$. If $\phi^{-1}(q)$ is not contained in S_1, then there exists a smooth point q' of V_1 in $\phi^{-1}(q)$. Recall that $\phi^{-1}(q)$ is a finite set. We can find an open neighborhood

U of q' which is biholomorphic to a domain in $\mathbb{C}^n$ such that $\phi|_U$ from U to the germ of (V_2, q) is a branch covering with ramification locus $\{q'\}$. Since the punctured neighborhood of q in V_2 is simply connected, this implies $\phi|_U$ is injective and hence $\phi|_U$ is a biholomorphism. This leads to a contradiction because q is a singular point. We have shown that $\phi^{-1}(q) = \{q'_1, \ldots, q'_d\} \subseteq S_1$. There are exactly d points in $\phi^{-1}(q)$ because the punctured neighborhood of q is simply connected. Therefore $|S_1|$ is divisible by $|S_2|$. On the other hand if $|S_1| = |S_2|$, then $d = 1$ and hence ϕ is a biholomorphism. □



4 CR morphisms between two compact strongly pseudoconvex CR manifolds lying in a same variety

The main purpose of this section is to prove the following theorem.

Theorem 4.1. *Let X_1 and X_2 be two $(2n-1)$-dimensional compact strongly pseudoconvex CR manifolds lying in a Stein variety V of dimension n in $\mathbb{C}^N$. Let $V_1 \subseteq V$, $V_2 \subseteq V$ such that $\partial V_1 = X_1$ and $\partial V_2 = X_2$. Assume that the singular set S of V is nonempty and is equal to the singular set of V_i, $i = 1, 2$. Then nontrivial CR morphisms from X_1 to X_2 are necessarily CR biholomorphisms.*

Proof. Let $\Phi\colon X_1 \longrightarrow X_2$ be a non-constant CR morphism. In view of Theorem 3.1, Φ can be extended to a proper holomorphic map from V_1 to V_2 such that $\Phi\colon V_1 - \Phi^{-1}(S) \longrightarrow V_2 - S$ is a covering map of degree d and $\Phi(S) = S$. Let $S = \{q_1, \ldots, q_m\}$. Then $\Phi^{-1}(S) = \{q_1, \ldots, q_m, p_1, \ldots, p_k\}$. We shall prove that $\Phi^{-1}(S) = S$. Let $\pi\colon M \longrightarrow V_2$ be a resolution of singularities of V_2 such that the exceptional sets

$$E_1 = \pi^{-1}(q_1) = \bigcup_{i=1}^{\ell_1} A_i^1, \ \ldots, \ E_m = \pi^{-1}(q_m) = \bigcup_{i=1}^{\ell_m} A_i^m$$

are normal crossing divisors.

$$\begin{tikzcd}
\widetilde{M} \arrow[dr, "\tau"] \arrow[drr, "\widetilde{\Phi}"] \arrow[ddr, "\widetilde{\pi}"'] & & \\
& V_1 \underset{V_2}{\times} M \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & M \arrow[d, "\pi"] \\
& V_1 \arrow[r, "\Phi"] & V_2
\end{tikzcd}$$

Consider the fiber product $V_1 \underset{V_2}{\times} M$ of the maps $\Phi\colon V_1 \longrightarrow V_2$ and $\pi\colon M \longrightarrow V_2$. Let $\tau\colon \widetilde{M} \longrightarrow V_1 \underset{V_2}{\times} M$ be the normalization map. Then we have the following commutative diagram where π_1 and π_2 are natural projections. Notice that $\pi_1: V_1 \underset{V_2}{\times} M \longrightarrow V_1$ is a biholomorphism outside $\pi_1^{-1}(\Phi^{-1}(S))$ and $\pi_2\colon V_1 \underset{V_2}{\times} M \to M$ is a covering map outside $\bigcup_{i=1}^{m} E_i$. Thus

$$\widetilde{\Phi} := \pi_2 \circ \tau\colon \widetilde{M} \longrightarrow M$$

is a d-fold branch covering. For each $A_i^j \subseteq E_j$, and any point $q_i^j \in A_i^j$ which is a smooth point in $\bigcup_{i=1}^{m} E_i$, we choose a germ of a curve Γ_i^j at the point q_i^j which intersects with $\bigcup_{i=1}^{m} E_i$ only at q_i^j and the intersection of A_i^j and Γ_i^j is transversal at q_i^j. Let $\Gamma = \bigcup \Gamma_i^j$, $1 \leqslant j \leqslant m$, $1 \leqslant i \leqslant \ell_j$. Notice that $\widetilde{\pi} := \pi_1 \circ \tau$ is a proper map which is a biholomorphism outside $\widetilde{E} := \widetilde{\pi}^{-1}(\Phi^{-1}(S)) = \widetilde{\Phi}^{-1}(E)$ where $E = E_1 \cup \cdots \cup E_m$. Observe that $\widetilde{E}$ has exactly $m+k$ connected components $\widetilde{E} = \widetilde{E}_1 \cup \cdots \cup \widetilde{E}_{m+k}$. Clearly

$$\widetilde{\Phi}_*(\widetilde{E}) = \sum_{i,j} d_i^j A_i^j, \text{ where } d_i^j \leqslant d.$$

By the projection formula (cf. p. 34 of [Ful98], or p. 426 of [Har77]),

$$\begin{aligned}
\sum_{j=1}^{m} \sum_{i=1}^{\ell_j} d_i^j &= \Gamma \cdot \widetilde{\Phi}_*(\widetilde{E}) = \widetilde{\Phi}^*(\Gamma) \cdot \widetilde{E} \\
&\geqslant (\ell_1 + \cdots + \ell_m) d.
\end{aligned}$$

The last inequality comes from the fact that $\widetilde{\Phi}^*(\Gamma_i^j)$ has d distinct branches because $\widetilde{\Phi}\colon \widetilde{M} - \widetilde{\Phi}^{-1}(E) \longrightarrow M - E$ is a d-fold covering space. Since $d_i^j \leqslant d$, we conclude that $d_i^j = d$ for all i, j. It follows that the branch locus of $\widetilde{\Phi}$ is contained in the singular locus of $\bigcup_{i=1}^{m} E_i$ which is of dimension $n-2$. As $\widetilde{M}$ is normal and M is smooth, $\widetilde{\Phi}\colon \widetilde{M} \longrightarrow M$ is a covering map by purity of branch locus. In particular, $\widetilde{M}$ is smooth.

Now we are ready to prove that $\Phi^{-1}(S) = S$, i.e., there are no $p_1, \ldots, p_k$ points in $\Phi^{-1}(S)$. Observe that $\widetilde{\pi}^{-1}(p_i)$, $1 \leqslant i \leqslant k$ and E_j, $1 \leqslant j \leqslant m$ are maximal compact connected analytic subsets in $\widetilde{M}$ and M respectively. Since $\widetilde{\Phi}$ is a covering map, there is a neighborhood $\widetilde{U}_i$ of $\widetilde{\pi}^{-1}(p_i)$ which maps biholomorphically to a neighborhood U_j of E_j for some j via $\widetilde{\Phi}$. As $\widetilde{\pi}\colon \widetilde{M} \longrightarrow V_1$ is a point modification in a neighborhood of p_i, there is a neighborhood D_i of p_i such that

$$\widetilde{\pi}\colon \widetilde{\pi}^{-1}(D_i) - \widetilde{\pi}^{-1}(p_i) \longrightarrow D_i - \{p_i\}$$

is a biholomorphism. Similarly, there is a neighborhood O_j of q_j such that

$$\pi\colon \pi^{-1}(O_j) - E_j \longrightarrow O_j - \{q_j\}$$

is a biholomorphism. Therefore

$$\pi \circ \widetilde{\Phi} \circ \widetilde{\pi}^{-1}\colon D_i - \{p_i\} \longrightarrow O_j - \{q_j\}$$

is a biholomorphism. Observe that p_i is a smooth point of D_i and q_j is an isolated normal singularity. It follows that $\pi \circ \widetilde{\Phi} \circ \widetilde{\pi}^{-1}$ extends to a biholomorphism from D_i to O_j. In particular q_j is not a singular point. This contradiction shows that $\Phi^{-1}(S) = S$ and hence $\widetilde{\pi}\colon \widetilde{M} \longrightarrow V_1$ is also a resolution of singularities of V_1.

If $\Phi(q_i) = q_j$, then (V, q_i) is isomorphic to (V, q_j) as germs of singularities. This is because the resolution of (V, q_j) is a resolution of (V, q_i). The proof of Theorem 4.1 is completed in view of Theorem 4.3 below. $\square$

Let (V, x) be a germ of complex analytic space with only one isolated singularity x. By Hironaka's paper [Hir63], it is biholomorphic equivalent to a germ of a complex algebraic singularity. Now let V be a complex analytic variety with only finitely many isolated singularities. By the equivalence and resolution theorems of algebraic varieties over field of characteristic 0, we can construct a resolution $\pi : \widetilde{V} \to V$ of V such that $\widetilde{V}$ is smooth and π is a bimeromorphic proper morphism.

The key point in the proof of Theorem C of [Yau11] for surface case is applying the minimal resolution. But in higher dimensional cases, there is no minimal resolution in general. Fortunately, by [BCHM10], there is a unique partial resolution $f : V^{can} \to V$ called the relative canonical model of V such that V^{can} has canonical singularity and the canonical divisor $K_{V^{can}}$ is f-ample. For surface, the relative canonical model is obtained by contracting all (-2)-rational curves in the minimal resolution of V. In general, the relative canonical model is isomorphic to

$$Proj \bigoplus_{m \geq 0} g_* \mathcal{O}(mK_Z)$$

where $g : Z \to V$ is any resolution of V.

Definition 4.1. *Let (V, x) be a germ such that x is the only isolated singularity. Take the relative canonical model $f : V^{can} \to V$ of V and denote E to be the exceptional set. Define*

$$cv_V(x) = (K_{V^{can}})^{\dim E} \cdot E,$$

which is called the canonical volume *of x.*

If V^{can} is not isomorphic to V, we have E being nonempty and $cv_V(x) = (K_{V^{can}})^{\dim E} \cdot E > 0$ by the f-ampleness. If V^{can} is isomorphic to V and x is a singular point, we set $cv_V(x) = (K_{V^{can}})^{\dim E} \cdot E = (K_{V^{can}})^0 \cdot x = 1$. Finally, if x is a smooth point, we set $cv_V(x) = 0$.

In general, if V has finitely many isolated normal singularities $x_i, i = 1, ..., m$, then we consider the sum of canonical volume

$$\sum_{i=1}^{m} cv_V(x_i) = \sum_{i=1}^{m} (K_{V^{can}})^{\dim E_{x_i}} \cdot E_{x_i}$$

where $f : V^{can} \to V$ is the relative canonical model of V and E_{x_i} is the exceptional set over x_i. From the definition, we see that $\sum\limits_{i=1}^{m} cv_V(x_i) > 0$ for nonempty isolated normal singularities $x_i, i = 1, ..., m$, on V.

Theorem 4.2. *Let V be an algebraic variety or a complex space with finitely many normal isolated singularities $x_i, i = 1, ..., m$, on V. Let $cv_V(x_i)$ be the canonical volume of x_i. Then $\sum\limits_{i=1}^{m} cv_V(x_i)$ is multiplicative in étale covering maps between resolutions. That is, if W is another algebraic variety or complex space with normal isolated singularities $y_j, j = 1, ..., s$, $p_1 : \widetilde{W} \to W$ and $p_2 : \widetilde{V} \to V$ are resolutions of W and V respectively, $\Phi : \widetilde{W} \to \widetilde{V}$ is an étale covering map, then we have $\sum\limits_{j=1}^{s} cv_W(y_j) = d \sum\limits_{i=1}^{m} cv_V(x_i)$ where d is the degree of Φ.*

Proof. Suppose we have resolutions $p_1 : \widetilde{W} \to W$ and $p_2 : \widetilde{V} \to V$, and we consider relative canonical models $p_1' : W^{can} \to W$ and $p_2' : V^{can} \to V$. We start from the following claim:

Claim: If $\Phi : W^{can} \to V^{can}$ is an étale covering map, then $\sum\limits_{j=1}^{s} cv_W(y_j) = d \sum\limits_{i=1}^{m} cv_V(x_i)$ where d is the degree of Φ.

Let E_1 be the exceptional set of p_1' and E_2 the exceptional set of p_2'. $E_1 = \sum\limits_{j=1}^{s} E_{y_j}$ where E_{y_j} is the exceptional set over y_j, and similarly $E_2 = \sum\limits_{i=1}^{m} E_{x_i}$ where E_{x_i} is the exceptional set over x_i. Since the canonical divisors $K_{W^{can}}$ and $K_{V^{can}}$ are p_1'-ample and p_2'-ample respectively, if E_1 is not empty, we have $\sum\limits_{j=1}^{s} cv_W(y_j) =$

$\sum\limits_{j=1}^{s}(K_{W^{can}})^{\dim E_{y_j}} \cdot E_{y_j} > 0$. Similarly, $\sum\limits_{i=1}^{m} cv_V(x_i) = \sum\limits_{i=1}^{m}(K_{V^{can}})^{\dim E_{x_i}} \cdot E_{x_i} > 0$ if E_2 is not empty.

By Φ being an étale covering map, we have the pullback $\Phi^* K_{V^{can}} = K_{W^{can}}$. Also, $\Phi^* E_2 = E_1$ since E_1 and E_2 are the only proper sets in W^{can} and in V^{can} respectively if we shrink V and W. Therefore, from

$$\sum_{j=1}^{s}(K_{W^{can}})^{\dim E_{y_j}} \cdot E_{y_j} = \sum_{i=1}^{m}(\Phi^* K_{V^{can}})^{\dim E_{x_i}} \cdot \Phi^* E_{x_i} = d\sum_{i=1}^{m}(K_{V^{can}})^{\dim E_{x_i}} \cdot E_{x_i}$$

where d is the degree of Φ, we have $\sum\limits_{j=1}^{s} cv_W(y_j) = d\sum\limits_{i=1}^{m} cv_V(x_i)$.

If $V^{can} \cong V$, E_2 has dimension 0 and $E_2 = \sum\limits_{i=1}^{m} x_i$. Since étale morphisms are locally isomorphisms, Φ sends singular points to singular points, and we have $E_1 = \sum\limits_{j=1}^{s} y_j$ is also 0-dimensional and $W^{can} \cong W$. The intersection $(K_{V^{can}})^0 \cdot x_i$ is just 1 by definition. By counting the singular points, we have $\sum\limits_{j=1}^{s} cv_W(y_j) = s = dm = d\sum\limits_{i=1}^{m} cv_V(x_i) > 0$. The claim is proved.

Now, for two resolutions $p_1 : \widetilde{W} \to W$ and $p_2 : \widetilde{V} \to V$, we have birational map $\phi_2 : \widetilde{V} \dashrightarrow V^{can}$ over V (see the diagram below).

$$\begin{tikzcd}
\widetilde{W}' \arrow[r, "\Phi'"] \arrow[d, "f"] & \widetilde{V}' \arrow[d, "g"] \arrow[dr, "\phi_2'"] & \\
\widetilde{W} \arrow[r, "\Phi"] \arrow[d, "p_1"] & \widetilde{V} \arrow[r, dashed, "\phi_2"] \arrow[d, "p_2"'] & V^{can} \arrow[dl, "p_2'"] \\
W & V &
\end{tikzcd}$$

Take a common resolution $\widetilde{V}'$ of $\widetilde{V}$ and V^{can} with birational morphisms $g : \widetilde{V}' \to \widetilde{V}$ and $\phi_2' : \widetilde{V}' \to V^{can}$, $\Phi' : \widetilde{W}' \to \widetilde{V}'$ is the base-change of $\Phi : \widetilde{W} \to \widetilde{V}$. Then Φ' is also an étale covering map with the same degree of Φ. After replacing Φ and ϕ_2 by Φ' and ϕ_2' respectively, we can assume $\phi_2 : \widetilde{V} \to V^{can}$ a birational morphism.

By the property of canonical model, V^{can} and W^{can} have canonical singularities. We use a theorem in [Tak03]:

Theorem. ([Tak03], Theorem 1.1) Let V be a normal analytic space and let $f : \widetilde{V} \to V$ be a resolution of singularities. Then the induced homomorphism $f_* : \pi_1(\widetilde{V}) \to \pi_1(V)$ is an isomorphism if (V, Δ) is Kawamata log-terminal (klt) for some divisor Δ.

Definition 4.2. *A pair (X, Δ) of a normal variety and an effective $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and let $\Delta = \sum d_i \Delta_i$ be the prime decomposition. We say that (X, Δ) is* (1) *Kawamata log-terminal (klt) iff $d_i < 1$ for all i and there exists a projective birational morphism $\mu : Y \to X$ from a smooth variety Y with a normal crossing divisor E_i such that $K_Y \equiv \mu^*(K_X + \Delta) + \sum e_i E_i$ holds with $e_i > -1$;* (2) *canonical iff there exists a projective birational morphism $\mu : Y \to X$ from a smooth variety Y with a normal crossing divisor E_i such that $K_Y \equiv \mu^*(K_X + \Delta) + \sum e_i E_i$ holds with $e_i \geq 0$ for all i.*

Continuation of the proof of Theorem 4.2.

We see that $\pi_1(V^{can}) \cong \pi_1(\widetilde{V})$ by the above theorem since V^{can} has canonical singularities, $(V^{can}, 0)$ is *klt*. Now we take the étale cover $\beta : W' \to V^{can}$ which gives the subgroup $\beta_* \pi_1(W') \subseteq \pi_1(V^{can})$ isomorphic to the subgroup $\Phi_* \pi_1(\widetilde{W}) \subseteq \pi_1(\widetilde{V})$.

$$\begin{tikzcd}
\widetilde{W} \arrow[r, "\Phi"] \arrow[d, "h"'] & \widetilde{V} \arrow[d, "\phi_2"] \\
W' \arrow[r, "\beta"'] & V^{can}
\end{tikzcd}$$

We see that β is an étale covering map with the same degree as Φ. Note that W' has canonical singularities since étale morphisms are locally isomorphisms. Because $\phi_{2*} \Phi_* \pi_1(\widetilde{W}) = \beta_* \pi_1(W')$, there is a morphism $h : \widetilde{W} \to W'$ coming from the morphism $\phi_2 : \widetilde{V} \to V^{can}$ extending to the étale covers $\widetilde{W}$ and W' of $\widetilde{V}$ and V^{can} respectively, and h is birational since ϕ_2 is. In fact, h is a resolution morphism from $\widetilde{W}$ to W'. We want to construct a morphism $q : W' \to W$ such that $p_1 = q \circ h$. Let z_i be a coordinate function defined on W. Since h is proper with connected fiber, $p_1^*(z_i)$ is a function on $\widetilde{W}$ which descends to W' as a continuous function $h_* p_1^*(z_i)$ which is holomorphic outside codimension one subvariety of W'. $h_* p_1^*(z_i)$ is actually holomorphic on the smooth part of W' because it is a continuous function on W'. Recall that the singular set of W' consists of isolated normal singularities. So $h_* p_1^*(z_i)$ is actually holomorphic on W'. This gives a morphism $q : W' \to W$ such that $p_1 = q \circ h$. As p_1 and q are birational, q is also birational. If $E \subseteq W'$ is an exceptional curve over W, by projection formula, we have

$$K_{W'} \cdot E = \beta^* K_{V^{can}} \cdot E = K_{V^{can}} \cdot \beta_* E > 0$$

since $\beta_* E$ is a sum of exceptional curves in V^{can} over V and $K_{V^{can}}$ is relative ample over V. So $K_{W'}$ is relatively ample over W. Then, by the uniqueness of relative canonical model, we have W' isomorphic to W^{can}. Replace W' by W^{can}, we have an étale covering map $\beta : W^{can} \to V^{can}$ and this is the claim above, which gives $\sum_{j=1}^{s} cv_W(y_j) = d \sum_{i=1}^{m} cv_V(x_i)$ where $d = \deg\beta = \deg\Phi$. □

Theorem 4.3. *Let V be a normal Stein space whose singular set is nonempty and finite. Let $f_1 : \widetilde{V}_1 \to V$ and $f_2 : \widetilde{V}_2 \to V$ be two resolutions of V. If $\widetilde{\Phi} : \widetilde{V}_1 \to \widetilde{V}_2$ is a finite étale covering map, $\widetilde{\Phi}$ must be an isomorphism.*

Proof. We take $W = V$ in Theorem 4.2. Since the isolated singular points are nonempty, we have $\sum_{i=1}^{m} cv_V(x_i) > 0$. The equation $\sum_{i=1}^{m} cv_V(x_i) = d \sum_{i=1}^{m} cv_V(x_i)$, where d is the degree of $\widetilde{\Phi}$, gives $d = 1$. Hence $\widetilde{\Phi}$ is an isomorphism. □

Remark 4.1. *Note that even if V is a smooth projective variety, an étale endomorphism $\Phi : V \to V$ is not necessary an isomorphism. For example, let V be an abelian variety, $\hat{n} : V \to V$ is the morphism sending point $x \in V$ to its n times, nx, then $\hat{n}$ is an étale covering map which is not an isomorphism if $n > 1$. Therefore, we need some restraints on V to force the degree to be 1.*

In [BFF12], a notion of volume for an isolated singular point of a normal variety is defined and the volume is also multiplicative in étale covering maps. By theorem A of [BFF12], if K_V is $\mathbb{Q}$-Cartier and V is not log canonical, the volume of the singularity is nonzero and we can determine the degree of an étale morphism. Our definition of canonical volume is like another multiplicative number between resolutions, and it determines the degree of étale coverings. Our method of proving the multiplicativity of canonical volume by taking the étale cover corresponding to a subgroup of the fundamental group is like the proof in discussion of nearly étale map in [NZ09].

The following example shows that strongly pseudoconvexity plays an important role in the above theory.

Example 4.1. *Let $X_1 = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2, a|x|^4 + |y|^4 + |z|^4 = \epsilon_0\}$ and $X_2 = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2, a|x|^2 + |y|^2 + |z|^2 = \epsilon_0\}$ where a is a positive real number. Let $\psi : X_1 \to X_2$ be given by $\psi(x, y, z) = (x^2, y^2, z^2)$. Then ψ is a surjective CR morphism from X_1 to X_2, but ψ is not a CR biholomorphism. Note that X_2 is strongly pseudoconvex, but X_1 is only weakly pseudoconvex.*

5 Explicit computation of CR automorphisms of strongly pseudoconvex 3-dimensional CR manifolds

The purpose of this section is to prove the following results by direct computation.

Theorem 5.1. *Let a be a positive real number and $X_a = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2, a|x|^2 + |y|^2 + |z|^2 = \epsilon_0\}$.*

(1) *If there exists a non-constant CR morphism from X_a to X_b, then $a = b$.*

(2) *For $a \neq \frac{1}{4}$, any non-constant CR morphism ψ from X_a to itself must be a CR biholomorphism and ψ must be one of the following forms:*

(a) $\psi(x,y,z)=(e^{i\theta_1}x, e^{i\theta_2}y, e^{i\frac{\theta_1+\theta_2}{2}}z)$, $0\le\theta_1,\ \theta_2\le 2\pi$.

(b) $\psi(x,y,z)=(e^{i\theta_1}y/\sqrt{a}, e^{i\theta_2}\sqrt{a}x, e^{i\frac{\theta_1+\theta_2}{2}}z)$, $0\le\theta_1,\ \theta_2\le 2\pi$.

(3) *For $a=\frac{1}{4}$, any non-constant CR morphism ψ from $X_{\frac{1}{4}}$ to itself must be a CR biholomorphism and ψ must be one of the following form:*

(a) $\psi(x,y,z)=(e^{i\theta_1}x, e^{i\theta_2}y, e^{i\frac{\theta_1+\theta_2}{2}}z)$, $0\le\theta_1,\ \theta_2\le 2\pi$.

(b) $\psi(x,y,z)=(2e^{i\theta_1}y, \frac{1}{2}e^{i\theta_2}x, e^{i\frac{\theta_1+\theta_2}{2}}z)$, $0\le\theta_1,\ \theta_2\le 2\pi$.

(c) $$\psi(x,y,z)=\begin{pmatrix} \frac{-2r^2}{2r^2+1}e^{i(\theta_{31}-\theta)} & \frac{2}{2r^2+1}e^{i(\theta_{32}-\theta)} & \frac{4r}{2r^2+1}e^{i\theta_{13}} \\ \frac{-1}{2(2r^2+1)}e^{i(\theta_{31}-\theta)} & \frac{2r^2}{2r^2+1}e^{i(\theta_{32}+\theta)} & \frac{-2r}{2r^2+1}e^{i(\theta_{13}+2\theta)} \\ \frac{r}{2r^2+1}e^{i\theta_{31}} & \frac{2r}{2r^2+1}e^{i\theta_{32}} & \frac{2r^2-1}{2r^2+1}e^{i(\theta_{13}+\theta)} \end{pmatrix}$$
$$\times\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \text{higher order terms in } x, y \text{ and } z,$$

where $0\le\theta,\ \theta_{31},\ \theta_{32},\ \theta_{13}\le 2\pi$, $\theta_{13}=\frac{\pi}{2}+\frac{\theta_{31}}{2}+\frac{\theta_{32}}{2}-\theta$ *and* $r>0$.

Corollary 5.1. *The automorphism group of the compact strongly pseudoconvex CR manifold* $X_{\frac{1}{4}}=\{(x,y,z)\in\mathbb{C}^3 : xy=z^2,\ \frac{1}{4}|x|^2+|y|^2+|z|^2=\epsilon_0\}$ *consists of the following mappings:*

(a) $\psi(x,y,z)=(e^{i\theta_1}x, e^{i\theta_2}y, e^{i\frac{\theta_1+\theta_2}{2}}z)$, $0\le\theta_1,\ \theta_2\le 2\pi$.

(b) $\psi(x,y,z)=(2e^{i\theta_2}y, \frac{1}{2}e^{i\theta_2}x, e^{i\frac{\theta_1+\theta_2}{2}}z)$, $0\le\theta_1,\theta_2\le 2\pi$.

(c) $$\psi(x,y,z)=\begin{pmatrix} \frac{-2r^2}{2r^2+1}e^{i(\theta_{31}-\theta)} & \frac{2e^{i(\theta_{32}-\theta)}}{2r^2+1} & \frac{4re^{i\theta_{13}}}{2r^2+1} \\ \frac{-e^{i(\theta_{31}-\theta)}}{2(2r^2+1)} & \frac{2r^2e^{i(\theta_{32}+\theta)}}{2r^2+1} & \frac{-2re^{i(\theta_{13}+2\theta)}}{2r^2+1} \\ \frac{re^{i\theta_{31}}}{2r^2+1} & \frac{2re^{i\theta_{32}}}{2r^2+1} & \frac{2r^2-1}{2r^2+1}e^{i(\theta_{13}+\theta)} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Proof of Theorem 5.1. Firstly, we prove the theorem for $b\neq\frac{1}{4}$.

Let $V_a=\{(x,y,z)\in\mathbb{C}^3 : xy=z^2$ and $a|x|^2+|y|^2+|z|^2\le\epsilon_0\}$. Let $\psi : X_a\to X_b$ be a non-constant CR morphism. Then Theorem 3.1 says that ψ can be extended to a proper surjective holomorphic map $\psi : V_a\to V_b$ such that $\psi(0)=0$, $\psi(X_a)=X_b$ and $X_a=\Psi^{-1}(X_b)$. Write ψ in the following form:

$$\begin{pmatrix} \psi_1(x,y,z) \\ \psi_2(x,y,z) \\ \psi_3(x,y,z) \end{pmatrix}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}+\text{higher order terms in } x, y \text{ and } z.$$

Then the constants a_{ij} satisfy the following equations:

$$a_{11}a_{21}-a_{31}^2=0, \tag{5.1}$$
$$a_{12}a_{22}-a_{32}^2=0, \tag{5.2}$$
$$a_{13}a_{23}-a_{33}^2+a_{11}a_{22}+a_{12}a_{21}-2a_{31}a_{32}=0, \tag{5.3}$$
$$a_{11}a_{23}+a_{13}a_{21}-2a_{31}a_{33}=0, \tag{5.4}$$
$$a_{12}a_{23}+a_{13}a_{22}-2a_{32}a_{33}=0. \tag{5.5}$$

For any $(x, y, z) \in X_a$, we have the following equation:

$$
\begin{aligned}
& b|\psi_1(x,y,z)|^2 + |\psi_2(x,y,z)|^2 + |\psi_3(x,y,z)|^2 = a|x|^2 + |y|^2 + |z|^2 \\
\Longrightarrow\ & (b|a_{11}|^2 + |a_{21}|^2 + |a_{31}|^2)|x|^2 + (b|a_{12}|^2 + |a_{22}|^2 + |a_{32}|^2)|y|^2 \\
& +(b|a_{13}|^2 + |a_{23}|^2 + |a_{33}|^2)|z|^2 + (ba_{11}\overline{a}_{12} + a_{21}\overline{a}_{22} + a_{31}\overline{a}_{32})x\overline{y} \\
& +(ba_{12}\overline{a}_{11} + a_{22}\overline{a}_{21} + a_{32}\overline{a}_{31})\overline{x}y + (ba_{11}\overline{a}_{13} + a_{21}\overline{a}_{23} + a_{21}\overline{a}_{33})x\overline{z} \\
& +(ba_{13}\overline{a}_{11} + a_{23}\overline{a}_{21} + a_{33}\overline{a}_{31})\overline{x}z + (ba_{12}\overline{a}_{13} + a_{22}\overline{a}_{23} + a_{32}\overline{a}_{33})y\overline{z} \\
& +(ba_{13}\overline{a}_{12} + a_{23}\overline{a}_{22} + a_{33}\overline{a}_{32})\overline{y}z = a|x|^2 + |y|^2 + |z|^2.
\end{aligned}
$$

It follows that

$$b|a_{11}|^2 + |a_{21}|^2 + |a_{31}|^2 = a, \tag{5.6}$$
$$b|a_{12}|^2 + |a_{22}|^2 + |a_{32}|^2 = 1, \tag{5.7}$$
$$b|a_{13}|^2 + |a_{23}|^2 + |a_{33}|^2 = 1, \tag{5.8}$$
$$ba_{11}\overline{a}_{12} + a_{21}\overline{a}_{22} + a_{31}\overline{a}_{32} = 0, \tag{5.9}$$
$$ba_{11}\overline{a}_{13} + a_{21}\overline{a}_{23} + a_{31}\overline{a}_{33} = 0, \tag{5.10}$$
$$ba_{12}\overline{a}_{13} + a_{22}\overline{a}_{23} + a_{32}\overline{a}_{33} = 0. \tag{5.11}$$

Case 1. $a_{31} \neq 0$ and $a_{32} \neq 0$. In view of (5.1) and (5.2), we have $a_{11} \neq 0$, $a_{21} \neq 0$, $a_{12} \neq 0$ and $a_{22} \neq 0$ in this case.

$$(5.1) \Rightarrow \frac{a_{11}}{a_{31}} = \frac{a_{31}}{a_{21}} := r_1 \neq 0$$
$$\Rightarrow a_{11} = r_1 a_{31}, a_{21} = \frac{1}{r_1} a_{31}, \tag{5.12}$$
$$(5.2) \Rightarrow \frac{a_{22}}{a_{32}} = \frac{a_{32}}{a_{12}} := r_2 \neq 0$$
$$\Rightarrow a_{22} = r_2 a_{32}, a_{12} = \frac{1}{r_2} a_{32}, \tag{5.13}$$
$$(5.4) \text{ and } (5.12) \Rightarrow r_1 a_{23} + \frac{1}{r_1} a_{13} - 2a_{33} = 0, \tag{5.14}$$
$$(5.5) \text{ and } (5.13) \Rightarrow \frac{1}{r_2} a_{23} + r_2 a_{13} - 2a_{33} = 0, \tag{5.15}$$
$$(5.14) \text{ and } (5.15) \Rightarrow \left(r_1 - \frac{1}{r_2}\right) a_{23} + \left(\frac{1}{r_1} - r_2\right) a_{13} = 0. \tag{5.16}$$

There are two cases to be considered.

Case 1 (a): $r_1 - \dfrac{1}{r_2} = 0$, i.e. $r_2 = \dfrac{1}{r_1}$.

$$(5.13) \Rightarrow a_{22} = \frac{1}{r_1} a_{32}, a_{12} = r_1 a_{32}, \tag{5.17}$$
$$(5.14) \Rightarrow a_{33} = \frac{1}{2} r_1 a_{23} + \frac{1}{2r_1} a_{13}, \tag{5.18}$$
$$(5.3), (5.12) \text{ and } (5.17) \Rightarrow a_{13}a_{23} - a_{33}^2 = 0, \tag{5.19}$$
$$(5.19) \text{ and } (5.18) \Rightarrow a_{13} = r_1^2 a_{23}, \tag{5.20}$$

$$(5.18) \text{ and } (5.20) \Rightarrow a_{33} = r_1 a_{23} = \frac{1}{r_1} a_{13},$$
$$(5.9), (5.12) \text{ and } (5.17) \Rightarrow b|r_1|^2 + \frac{1}{|r_1|^2} + 1 = 0,$$

which is a contradiction. Hence Case 1 (a) cannot happen.

Case 1 (b): $r_1 - \dfrac{1}{r_2} \neq 0$.

$$(5.16) \Rightarrow a_{23} = \frac{r_2}{r_1} a_{13}, \tag{5.21}$$
$$(5.14) \text{ and } (5.21) \Rightarrow a_{33} = \left(\frac{r_2}{2} + \frac{1}{2r_1}\right) a_{13}. \tag{5.22}$$

In view of (5.21) and (5.22), we have $a_{13} \neq 0$ because of (5.8).

$$(5.9), (5.12) \text{ and } (5.13) \Rightarrow b\frac{r_1}{\overline{r}_2} + \frac{\overline{r}_2}{r_1} + 1 = 0. \tag{5.23}$$

(5.10), (5.12), (5.21), (5.22) and the fact that $a_{31} \neq 0$, $a_{13} \neq 0$ imply

$$br_1 + \frac{\overline{r}_2}{|r_1|^2} + \frac{\overline{r}_2}{2} + \frac{1}{2\overline{r}_1} = 0. \tag{5.24}$$

(5.11), (5.13), (5.21) and (5.22) and the fact that $a_{32} \neq 0$ and $a_{13} \neq 0$ imply

$$\frac{b}{r_2} + \frac{|r_2|^2}{\overline{r}_1} + \frac{\overline{r}_2}{2} + \frac{1}{2\overline{r}_1} = 0. \tag{5.25}$$

Clearly (5.23), (5.24) and (5.25) imply

$$\det \begin{pmatrix} \dfrac{r_1}{\overline{r}_2} & \dfrac{\overline{r}_2}{r_1} & 1 \\ r_1 & \dfrac{\overline{r}_2}{|r_1|^2} & \dfrac{\overline{r}_2}{2} + \dfrac{1}{2\overline{r}_1} \\ \dfrac{1}{r_2} & \dfrac{|r_2|^2}{\overline{r}_1} & \dfrac{\overline{r}_2}{2} + \dfrac{1}{2\overline{r}_1} \end{pmatrix} = 0$$
$$\Rightarrow (\overline{r_1 r_2} + 1)(r_1 r_2 + \overline{r_1 r_2})(r_1 r_2 - 1) + 2\overline{r}_1 \overline{r}_2 (1 - r_1 r_2)(1 + r_1 r_2) = 0.$$

Since $r_1 - \frac{1}{r_2} \neq 0$, i.e. $r_1 r_2 - 1 \neq 0$, we have

$$(\overline{r}_1 \overline{r}_2 + 1)(r_1 r_2 + \overline{r}_1 \overline{r}_2) - 2\overline{r}_1 \overline{r}_2 (1 + r_1 r_2) = 0$$
$$\Rightarrow (r_1 r_2 - \overline{r}_1 \overline{r}_2)(1 - \overline{r}_1 \overline{r}_2) = 0.$$

Since $\overline{r}_1 \overline{r}_2 - 1 \neq 0$, we have

$$r_1 r_2 = \overline{r}_1 \overline{r}_2. \tag{5.26}$$

Let $\alpha = \frac{r_1}{\overline{r}_2}$. Then $\alpha = \overline{\alpha}$, $r_1 = \alpha \overline{r}_2$ and $\overline{r}_1 = \alpha r_2$. (5.23), (5.24) and (5.25) can be rewritten as

$$b\alpha^2 + \alpha + 1 = 0, \tag{5.27}$$

$$\frac{b}{r_2} + \frac{\overline{r}_2}{\alpha} + \frac{\overline{r}_2}{2} + \frac{1}{2\alpha r_2} = 0, \tag{5.28}$$

$$b\alpha + \frac{\alpha |r_2|^2}{2} + \frac{1}{2} + |r_2|^2 = 0. \tag{5.29}$$

(5.28) $-\alpha$ (5.29) $\Rightarrow$

$$(b\alpha^2 - 1)(\alpha |r_2|^2 - 1) = 0,$$

$$\text{i.e. } b\alpha^2 = 1 \text{ or } \alpha |r_2|^2 = 1. \tag{5.30}$$

If $\alpha |r_2|^2 = 1$, then (5.29) implies

$$\frac{b}{|r_2|^2} + 1 + |r_2|^2 = 0,$$

which is absurd since the left hand side is positive. Therefore we conclude that $\alpha^2 = \frac{1}{b}$. Then (5.27) and (5.29) imply

$$\alpha = -2 \text{ and } b = \frac{1}{4}. \tag{5.31}$$

Case 2. $a_{31} = 0$. By (5.1), we have either $a_{11} = 0$ or $a_{21} = 0$.
Case 2 (a): $a_{31} = 0$ and $a_{11} = 0$. By (5.6), we have $a_{21} \neq 0$.

$$(5.9) \Rightarrow a_{21}\overline{a}_{22} = 0 \Rightarrow a_{22} = 0, \tag{5.32}$$

$$(5.4) \Rightarrow a_{13}a_{21} = 0 \Rightarrow a_{13} = 0, \tag{5.33}$$

$$(5.2) \text{ and } (5.32) \Rightarrow a_{32} = 0. \tag{5.34}$$

By (5.7), (5.32) and (5.34), we have $a_{12} \neq 0$

$$(5.5), (5.32) \text{ and } (5.34) \Rightarrow a_{12}a_{23} = 0 \Rightarrow a_{23} = 0, \tag{5.35}$$

$$(5.3) \text{ and } (5.35) \Rightarrow -a_{33}^2 + a_{12}a_{21} = 0, \tag{5.36}$$

$$(5.6) \Rightarrow |a_{21}|^2 = a, \tag{5.37}$$

$$(5.7), (5.32) \text{ and } (5.34) \Rightarrow |a_{12}|^2 = \frac{1}{b}, \tag{5.38}$$

$$(5.8), (5.33) \text{ and } (5.35) \Rightarrow |a_{33}|^2 = 1. \tag{5.39}$$

(5.36), (5.37), (5.38) and (5.39) imply $a = b$ and

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & e^{i\theta_1}/\sqrt{a} & 0 \\ e^{i\theta_2}\sqrt{a} & 0 & 0 \\ 0 & 0 & e^{i(\theta_1+\theta_2)} \end{pmatrix}.$$

Case 2 (b): $a_{31} = 0$ and $a_{21} = 0$. By (5.6), we have $a_{11} \neq 0$.

$$\begin{aligned}
(5.9) &\Rightarrow a_{12} = 0, && (5.40)\\
(5.4) &\Rightarrow a_{23} = 0, && (5.41)\\
(5.2) \text{ and } (5.40) &\Rightarrow a_{32} = 0, && (5.42)\\
(5.42), (5.40) \text{ and } (5.7) &\Rightarrow a_{22} \neq 0, &&\\
(5.42), (5.40), (5.5) \text{ and } a_{22} \neq 0 &\Rightarrow a_{13} = 0, && (5.43)\\
(5.3), (5.43) \text{ and } (5.40) &\Rightarrow -a_{33}^2 + a_{11}a_{22} = 0, && (5.44)\\
(5.6) &\Rightarrow b|a_{11}|^2 = a, && (5.45)\\
(5.7) &\Rightarrow |a_{22}|^2 = 1, && (5.46)\\
(5.8) &\Rightarrow |a_{33}|^2 = 1. && (5.47)
\end{aligned}$$

(5.44), (5.45), (5.46) and (5.47) $\Rightarrow a = b$ and

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i(\theta_1+\theta_2)} \end{pmatrix}.$$

Case 3. $a_{32} = 0$. By (5.2), we have either $a_{12} = 0$ or $a_{22} = 0$.
Case 3 (a): $a_{32} = 0$ and $a_{12} = 0$. By (5.7), we have $a_{22} \neq 0$.

$$\begin{aligned}
(5.9) &\Rightarrow a_{21} = 0, && (5.48)\\
(5.5) &\Rightarrow a_{13} = 0, && (5.49)\\
(5.1) \text{ and } (5.48) &\Rightarrow a_{31} = 0, && (5.50)\\
(5.6), (5.48) \text{ and } (5.50) &\Rightarrow b|a_{11}|^2 = a \text{ and } a_{11} \neq 0, && (5.51)\\
(5.4), (5.48) \text{ and } (5.5) &\Rightarrow a_{23} = 0, && (5.52)\\
(5.7) &\Rightarrow |a_{22}|^2 = 1, && (5.53)\\
(5.8) &\Rightarrow |a_{33}|^2 = 1, && (5.54)\\
(5.3) &\Rightarrow a_{33}^2 = a_{11}a_{22}. && (5.55)
\end{aligned}$$

(5.51), (5.53), (5.54) and (5.55) $\Rightarrow a = b$ and

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\frac{\theta_1+\theta_2}{2}} \end{pmatrix}.$$

Case 3 (b): $a_{32} = 0 = a_{22}$. By (5.7), we have $a_{12} \neq 0$.

$$\begin{aligned}
(5.9) &\Rightarrow a_{11} = 0, && (5.56)\\
(5.1) \text{ and } (5.50) &\Rightarrow a_{31} = 0, && (5.57)\\
(5.6), (5.56) \text{ and } (5.57) &\Rightarrow a_{21} \neq 0, && (5.58)\\
(5.10), (5.56), (5.58) \text{ and } (5.57) &\Rightarrow a_{23} = 0, && (5.59)\\
(5.11) &\Rightarrow a_{13} = 0, && (5.60)\\
(5.6), (5.56) \text{ and } (5.57) &\Rightarrow |a_{21}|^2 = a, && (5.61)
\end{aligned}$$

$$(5.7) \Rightarrow b|a_{12}|^2 = 1, \tag{5.62}$$
$$(5.8), (5.59) \text{ and } (5.60) \Rightarrow |a_{33}|^2 = 1, \tag{5.63}$$
$$(5.3), (5.56), (5.57) \text{ and } (5.59) \Rightarrow a_{33}^2 = a_{12}a_{21}. \tag{5.64}$$

(5.61), (5.62), (5.63) and (5.64) $\Rightarrow a = b$ and

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{33} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & \frac{e^{i\theta_1}}{\sqrt{a}} & 0 \\ e^{i\theta_2}\sqrt{a} & 0 & 0 \\ 0 & 0 & e^{\frac{i(\theta_1+\theta_2)}{2}} \end{pmatrix}.$$

We have shown that if there exists a non-constant CR morphism $\psi : X_a \longrightarrow X_b$, then $a = b$ and ψ must be one of the following forms.

(i) $\psi_1 = a_{11}x + \psi_1^{(2)} + \psi_1^{(3)} + \cdots,$

$\psi_2 = a_{22}y + \psi_2^{(2)} + \psi_2^{(3)} + \cdots,$

$\psi_3 = a_{33}z + \psi_3^{(2)} + \psi_3^{(3)} + \cdots,$

where $a_{11} = e^{i\theta_1}$, $a_{22} = e^{i\theta_2}$ and $a_{33} = e^{i\frac{\theta_1+\theta_2}{2}}$, $\psi_i^{(j)}$ = homogeneous polynomial of degree j.

(ii) $\psi_1 = a_{12}y + \psi_1^{(2)} + \psi_1^{(3)} + \cdots,$

$\psi_2 = a_{21}x + \psi_2^{(2)} + \psi_1^{(3)} + \cdots,$

$\psi_3 = a_{33}z + \psi_3^{(2)} + \psi_3^{(3)} + \cdots,$

where $a_{12} = e^{i\theta_1}/\sqrt{a}$, $a_{21} = e^{i\theta_2}\sqrt{a}$ and $a_{33} = e^{i\frac{\theta_1+\theta_2}{2}}$. In both case (i) and case (ii), we have

$$a\psi_1\overline{\psi}_1 + \psi_2\overline{\psi}_2 + \psi_3\overline{\psi}_3 = a|x|^2 + |y|^2 + |z|^2. \tag{5.65}$$

By comparing the 3rd order terms in (5.65), we see easily that the 2nd order terms of (ψ_1, ψ_2, ψ_3) are zero. Repeating this argument, we see that (ψ_1, ψ_2, ψ_3) has only linear terms.

Secondly, we prove the theorem for $b = \frac{1}{4}$.

The proof is the same as those for $b \neq \frac{1}{4}$ above except in Case 1 (b). Here we shall follow our previous notations. Let us summarize what we have proved in this situation:

$$a_{31} \neq 0, a_{32} \neq 0, a_{11} \neq 0, a_{21} \neq 0, a_{12} \neq 0 \text{ and } a_{22} \neq 0, \tag{5.66}$$
$$r_1r_2 \neq 1, r_1 \neq 0, r_2 \neq 0, \tag{5.67}$$
$$r_1 = -2\overline{r}_2. \tag{5.68}$$

(5.1) implies

$$a_{11} = r_1a_{31}, \ \ a_{21} = \frac{1}{r_1}a_{31}, \tag{5.69}$$

(5.2) implies

$$a_{22} = r_2 a_{32},\ a_{12} = \frac{1}{r_2} a_{32},$$

(5.4) and (5.5) imply

$$a_{23} = \frac{r_2}{r_1} a_{13},\ a_{33} = \left(\frac{r_2}{2} + \frac{1}{2r_1}\right) a_{13}. \tag{5.70}$$

Notice that (5.9), (5.10) and (5.11) are equivalent to (5.68) in this situation

$$(5.6) \Rightarrow |a_{31}|^2 = \frac{4ar^2}{(2r^2+1)^2}, \tag{5.71}$$

$$(5.7) \Rightarrow |a_{32}|^2 = \frac{4r^2}{(2r^2+1)^2}, \tag{5.72}$$

$$(5.8) \Rightarrow |a_{13}|^2 = \frac{16r^2}{(2r^2+1)^2}, \tag{5.73}$$

$$(5.3) \Rightarrow \frac{a_{13}^2}{4r_1} = \frac{a_{31}a_{32}}{r_2}. \tag{5.74}$$

Let

$$r_2 = re^{i\theta}, \tag{5.75}$$

$$(5.68) \Rightarrow r_1 = -2re^{-i\theta}, \tag{5.76}$$

$$(5.74), (5.75) \text{ and } (5.76) \Rightarrow a_{13}^2 = -8e^{-2i\theta} a_{31}a_{32}, \tag{5.77}$$

$$(5.77), (5.71), (5.72) \text{ and } (5.73) \Rightarrow a = \frac{1}{4}. \tag{5.78}$$

$$a_{31} = \frac{r}{2r^2+1} e^{i\theta_{31}}, \tag{5.79}$$

$$a_{32} = \frac{2r}{2r^2+1} e^{i\theta_{32}}, \tag{5.80}$$

$$a_{13} = \frac{4r}{2r^2+1} e^{i\theta_{13}}, \tag{5.81}$$

$$\theta_{13} = \frac{\pi}{2} + \frac{\theta_{31}}{2} + \frac{\theta_{32}}{2} - \theta. \tag{5.82}$$

It follows that the automorphism group of $X_{\frac{1}{4}}$ contains a linear subgroup of dimension 4 in the following form

$$\begin{aligned}
\psi(x,y,z) &= \begin{pmatrix} -2re^{-i\theta}a_{31} & \frac{1}{r}e^{-i\theta}a_{32} & a_{13} \\ -\frac{1}{2r}e^{i\theta}a_{31} & re^{i\theta}a_{32} & -\frac{1}{2}e^{2i\theta}a_{13} \\ a_{31} & a_{32} & \frac{2r^2-1}{4r}e^{i\theta}a_{13} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= \begin{pmatrix} \frac{-2r^2}{2r^2+1}e^{i(\theta_{31}-\theta)} & \frac{2}{2r^2+1}e^{i(\theta_{32}-\theta)} & \frac{4r}{2r^2+1}e^{i\theta_{13}} \\ \frac{-1}{2(2r^2+1)}e^{i(\theta_{31}+\theta)} & \frac{2r^2}{2r^2+1}e^{i(\theta_{32}+\theta)} & \frac{-2r}{2r^2+1}e^{i(\theta_{13}+2\theta)} \\ \frac{r}{2r^2+1}e^{i\theta_{31}} & \frac{2r}{2r^2+1}e^{i\theta_{32}} & \frac{2r^2-1}{2r^2+1}e^{i(\theta_{13}+\theta)} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\end{aligned}$$

It can be shown that

$$\det\begin{pmatrix} -2re^{i\theta}a_{31} & \frac{1}{r}e^{-i\theta}a_{32} & a_{13} \\ -\frac{1}{2r}e^{i\theta}a_{31} & re^{i\theta}a_{32} & -\frac{1}{2}e^{2i\theta}a_{13} \\ a_{31} & a_{32} & \frac{2r^2-1}{4r}e^{i\theta}a_{13} \end{pmatrix}$$

$$= a_{31}a_{32}a_{33}\left(-r^3 - \frac{3r}{2} - \frac{6r^2+1}{8r^3}\right) \neq 0.$$

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