On the polynomial sharp upper estimate conjecture in 8-dimensional simplex

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Abstract

Because of its importance in number theory and singularity theory, the problem of finding a polynomial sharp upper estimate of the number of positive integral points in an n-dimensional ($n \geq 3$) polyhedron has received attention by a lot of mathematicians. S. S.-T. Yau proposed the upper estimate, so-called the Yau Number Theoretic Conjecture. The previous results on the Yau Number Theoretic Conjecture in low dimension cases ($n \leq 6$) have been proved by using the sharp GLY conjecture. Unfortunately, it is only valid in low dimension. The Yau Number Theoretic Conjecture for $n = 7$ has been shown with a completely new method in [19]. In this paper, the similar method has been applied to prove the Yau Number Theoretic Conjecture for $n = 8$, but with more meticulous analyses. The main method of proof is summing existing sharp upper bounds for the number of points in 7-dimensional simplexes over the cross sections of eight-dimensional simplex. This reasearch project paves the way for the proof of a fully general sharp upper bound for the number of lattice points in a simplex. It also moves the mathematical community one step closer towards proving the Yau Number Theoretic Conjecture in full generality. As an application, we give a sharper estimate of the Dickman-De Bruijn function $\psi(x, y)$ for $5 \leq y < 23$, compared with the result obtained by Ennola.

1 Introduction

Let $T(a_1, a_2, \ldots, a_n)$ be an n-dimensional simplex described by

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1, x_1, x_2, \ldots, x_n \geq 0 \tag{1}$$

where $a_1 \geq a_2 \geq \ldots \geq a_n \geq 1$ are real numbers. Let $P_n = P(a_1, a_2, \ldots, a_n)$ and $Q_n = Q(a_1, a_2, \ldots, a_n)$ be the number of positive integer solutions and nonnegative integer solutions of (1), respectively. We can see there is a relation

$$Q(a_1, \ldots, a_n) = P(a_1(1+a), \ldots, a_n(1+a))$$

where $a = \frac{1}{a_1} + \cdots + \frac{1}{a_n}$.

The estimate of P_n and Q_n can be applied in number theory. A *smooth number* is a number with only small prime factors. Smooth numbers play important roles in factoring and primality testing[12]. Given an integer y, the number $m = p_1^{l_1} p_2^{l_2} \ldots p_n^{l_n}$ is called y-smooth if all its prime factor $p_i \leq y$ for $i = 1, \ldots, n$. Number theorists want to know the number of y-smooth integers less than or equal to x, which is denoted by $\psi(x, y)$, called the Dickman-De Bruijn function. One of the central topics in computational number theory is the estimate of $\psi(x, y)$, (see [7]). It turns out that the computation of $\psi(x, y)$ is equivalent to compute the number of integral points in an k-dimensional tetrahedron $\triangle(a_1, a_2, \cdots, a_k)$ with real vertices $(a_1, 0, \cdots, 0), \cdots, (0, \cdots, 0, a_k)$. Let $p_1 < p_2 < \cdots < p_k$ denotes the primes up to y. It is clear that $p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k} \leq x$ which is also equivalent to counting the number of $(l_1, l_2, \cdots, l_k) \in \mathbb{Z}_{\geq 0}^n$ such that

$$\frac{l_1}{a_1} + \frac{l_2}{a_2} + \cdots + \frac{l_k}{a_k} \leq 1, \text{ where } a_i = \frac{\log x}{\log p_i}.$$

Therefore, $\psi(x, y)$ is precisely the number Q_k of (integer) lattice points inside the n-dimensional tetrahedron (1) with $a_i = \frac{\log x}{\log p_i}$, $n = k$, and $1 \leq i \leq k$. In [3], Ennola gave both lower and upper bounds for the $\psi(x, y)$:

$$\frac{(\log x)^k}{k! \prod_{i=1}^{k} \log p_i} < \psi(x, y) \leq \frac{(\log x + \sum_{i=1}^{k} \log p_i)^k}{k! \prod_{i=1}^{k} \log p_i} \tag{2}$$

which yields the following result.

Theorem 1.1. *(Ennola, [3]) Uniformly for* $2 \leq y \leq \sqrt{\log x \log_2 x}$, *we have that*

$$\psi(x, y) = \frac{1}{k!} \prod_{p \leq y} (\frac{\log x}{\log p})[1 + O(\frac{y^2}{\log x \log y})].$$

Numbers P_n and Q_n also have applications in geometry and singularity theory. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with isolated critical point at the origin and $V = \{z \in \mathbb{C}^n : f(z) = 0\}$. The geometric genus p_g is defined to be $\dim \Gamma(V - \{0\}, \Omega^{n-1})/L^2(V - \{0\}, \Omega^{n-1})$, where Ω^{n-1} is the sheaf of germs of holomorphic $(n-1)$-forms on $V - \{0\}$. If $f(z_1, \ldots, z_n)$ is weighted homogeneous of type $(w_1, \ldots, w_n)$, where $w_1, \ldots, w_n$ are fixed positive rational numbers, i.e., f can be expressed as a linear combination of monomials $z_1^{i_1} \ldots z_n^{i_n}$ for which $i_1/w_1 + \cdots + i_n/w_n = 1$, then Merle and Teissier [10] showed that p_g is exactly the number $P(w_1, \ldots, w_n)$.

There are a lot of papers on finding the exact formula for P_n or Q_n, in case $a_1, \ldots, a_n$ are integers. For example, Mordell [9] gave an exact formula for Q_3 with a_1, a_2 and a_3 relatively prime. Pommersheim [11] extended this result to arbitrary integers a_1, a_2 and a_3 using toric variety techniques and a result of Ehrhart [2]. The exact formula is complicated, it involves generalized Dedekind sum. It is hard to figure out how large the sum is from the exact formula. Therefore, sometimes we want to get a sharp upper estimate of P_n in terms of a polynomial in $a_1, \ldots, a_n$. Such a polynomial upper estimate have many important applications. For example, it can be used in the following Durfee Conjecture:

Conjecture (Durfee (1978))**.** *Let $(V,0)$ be an isolated hypersurface singularity defined by a holomorphic function $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$. Let*

$$\mu = \dim \mathbb{C}\{z_1,\dots,z_n\}/(f_{z_1},\dots,f_{z_n})$$

be the Milnor number of the singularity. Then $n!\,p_g \le \mu$ where p_g is the geometric genus of $(V,0)$.

If f is weighted homogeneous of type $(w_1,\dots,w_n)$, Milnor and Orlik [8] proved that $\mu = (w_1-1)\dots(w_n-1)$. Therefore Durfee conjecture is a special case of the following theorem, which was proved by Yau and Zhang [18]:

Theorem 1.2 (GLY rough estimate)**.** *Let $a_1,\dots,a_n$ be positive real numbers greater than or equal to 1 and $n\ge 3$. Then*

$$n!\,P(a_1,\dots,a_n) < (a_1-1)(a_2-1)\dots(a_n-1). \tag{3}$$

The estimate in the above theorem is nice. However, it is not sharp enough to provide a solution of the following problem:

Problem. *([21], [22]) Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a holomorphic function with an isolated critical point at the origin. Find an intrinsic characterization for f to be a homogeneous polynomial.*

In 1971, Saito [13] gave an intrinsic characterization for f to be a weighted homogeneous polynomial

Theorem 1.3 (Saito)**.** *Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a holomorphic function with isolated critical point at the origin. Then f is a weighted homogeneous polynomial after a biholomorphic* *change of coordinates if and only if* $\mu = \tau$, *where*

$$\mu = \dim \mathbb{C}\{z_1, \ldots, z_n\}/(f_{z_1}, \ldots, f_{z_n})$$

and

$$\tau = \dim \mathbb{C}\{z_1, \ldots, z_n\}/(f, f_{z_1}, \ldots, f_{z_n})$$

.

To find a necessary and sufficient condition for f to be a homogeneous polynomial, Yau made the following conjecture in 1995:

Conjecture (Yau Geometric Conjecture). *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be a weighted homogeneous polynomial with an isolated singularity at the origin. Let* μ, p_g *and* ν *be the Milnor number, geometric genus and multiplicity of singularity* $V = \{z : f(z) = 0\}$, *respectively. Then*

$$\mu - h(\nu) \geq n!\, p_g \tag{4}$$

where $h(\nu) = (\nu - 1)^n - \nu(\nu - 1) \ldots (\nu - n + 1)$. *The equality holds if and only if* f *is a homogeneous polynomial after a biholomorphic change of coordinates.*

The Yau Geometric conjecture together with Theorem 1.3 will give an intrinsic characterization for a holomorphic function f to be a homogeneous polynomial after a biholomorphic change of coordinates. In order to prove Yau Geometric conjecture, Lin, Yau [5] and Granville have formulated GLY Rough Estimate and the following GLY Sharp Conjecture:

Conjecture (GLY Sharp Estimate). *Let* $n \geq 3$. *If* $a_1 \geq a_2 \geq \ldots \geq a_n \geq n - 1$. *Then*

$$n!\, P_n \leq f_n := A_0^n + \frac{s(n, n-1)}{n} A_1^n + \sum_{l=1}^{n-2} \frac{s(n, n-1-l)}{\binom{n-1}{l}} A_l^{n-1} \tag{5}$$

where $s(n,k)$ is the Stirling number of the first kind defined by the following generating function:

$$x(x-1)\dots(x-n+1)=\sum_{k=0}^{n}s(n,k)x^k$$

and A_k^n is defined as

$$A_k^n=(\prod_{i=1}^{n}a_i)(\sum_{1\le i_1\le i_2\le\ldots\le i_k\le n}\frac{1}{a_{i_1}a_{i_2}\dots a_{i_k}})$$

for $k=1,2,\dots,n-1$. Equality in (5) holds if and only if $a_1=\cdots=a_n$ are integers.

The above GLY Sharp Estimate is true for $n=4,5,6$ (cf. [20], [1]) and there is a counterexample for $n=7$ (cf. [15]). In [15], Wang and Yau also modify GLY Conjecture as follows:

Conjecture (Modified GLY Conjecture)**.** *There exists an integer $y(n)$ which depends only on n such that the sharp estimate (5) holds when $a_1\ge a_2\ge\ldots\ge a_n\ge\ y(n)$.*

In order to overcome the difficulty that GLY sharp estimate is only true when a_n is larger than $y(n)$, Yau proposed a new sharp upper estimate which is motived for the Yau Geometric conjecture:

Conjecture (Yau Number Theoretic Conjecture)**.** *Let*

$$P_n=P_n(a_1,a_2,\dots,a_n)=\#\{(x_1,\dots,x_n)\in\mathbb{Z}_+^n:\frac{x_1}{a_1}+\frac{x_2}{a_2}+\cdots+\frac{x_n}{a_n}\le 1\},$$

where $n\ge 3$, $a_1\ge a_2\ge\ldots\ge a_n>1$ are real numbers. If $P_n>0$, then

$$n!\,P_n\le\prod_{i=1}^{n}(a_i-1)-(a_n-1)^n+\prod_{i=0}^{n-1}(n-i) \tag{6}$$

and equality holds if and only if $a_1=a_2=\cdots=a_n$ are integers.

There is an intimate relation between the Yau Geometric Conjecture and the Yau Number Theoretic Conjecture. Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin, then the multiplicity ν of f at the origin is given by $\inf\{n \in \mathbb{Z}_+ : n \geq \inf\{w_1, \ldots, w_n\}\}$, where w_i is the weight of x_i. In general, the weight w_i is a rational number. In case the minimal weight is an integer, the Yau Geometric Conjecture and Yau Number Theoretic Conjecture are the same. However, in general, these two conjectures do not imply each other.

The Yau Number Theoretic Conjecture has already been verified for $n = 3$ by Xu and Yau ([17], [16]) , for $n = 4, 5$ by Lin, Luo, Yau and Zuo ([4], [6]) and for $n = 6$ by Liang,Yau and Zuo [7]. Yau, Yuan and Zuo [19] gave the following result for $n = 7$:

Theorem 1.4. *Let $a_1 \geq a_2 \geq \ldots \geq a_7 > 1$ be real numbers. Let P_7 be the number of positive integral solutions of $\frac{x_1}{a_1} + \cdots + \frac{x_7}{a_7} \leq 1$. If $P_7 > 0$, then*

$$7!\, P_7 \leq g_7 := \prod_{i=1}^{7}(a_i - 1) - (a_7 - 1)^7 + \prod_{j=0}^{6}(a_7 - j) \tag{7}$$

and equality holds if and only if $a_1 = a_2 = \cdots = a_7 \in \mathbb{Z}$.

In this paper, we will prove the Yau Number Theoretic Conjecture for $n = 8$:

Theorem 1.5 (Main Theorem A)**.** *Let $P_8 = P_8(a_1, a_2, \ldots, a_8) = \#\{(x_1, \ldots, x_8) \in \mathbb{Z}_+^8 : \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_8}{a_8} \leq 1\}$, where $a_1 \geq a_2 \geq \ldots \geq a_8 > 1$ are real numbers. If $P_8 > 0$, then*

$$8!\, P_8 \leq (a_1 - 1)(a_2 - 1) \ldots (a_8 - 1) - (a_8 - 1)^8 + a_8(a_8 - 1) \ldots (a_8 - 7) \tag{8}$$

and equality holds if and only if $a_1 = a_2 = \cdots = a_8$ are integers.

Let

$$g_n(a_1, \ldots, a_n) \;\; := \;\; \prod_{i=1}^{n}(a_i - 1) - (a_n - 1)^n + \prod_{i=0}^{n-1}(n - i)$$

be the right hand of (6). In case $n = 8$,

$$g_8(a_1, \ldots, a_n) \quad := \quad (a_1 - 1) \ldots (a_8 - 1) - (a_8 - 1)^8 + a_8(a_8 - 1) \ldots (a_8 - 7).$$

In [4], we can see that, when $n = 5$, the number of subcases increase from 4 (when $n = 4$) to 11. The authors of [4] ([7] resp.) simplify those 11 (21 resp.) subcases into 5 (6 resp.) major classes. They divide the whole range into five intervals and classify those subcases by which interval the last variable a_n is in. The benefit of this classification is that the number of classes will increase only by 1 as the dimension increase by 1. However, the proofs of $n \leq 6$ relied on the GLY sharp estimate, which is only true for $n \leq 6$. Therefore the proof cannot be generalized to higher dimension. The Yau Number Theoretic Conjecture for $n = 7$ has been shown with a completely new method in [19]. In this paper, the similar method has been applied to prove the Yau Number Theoretic Conjecture for $n = 8$, but with more meticulous analyses. We avoid entirely the GLY sharp estimate, and we will prove our main theorem purely by induction. This is a significant improvement since it suggest a way to prove the general case.

As an application, we will also prove that

Theorem 1.6 (Main Theorem B, Estimate of $\psi(x, y)$)**.** *Let $\psi(x, y)$ be the function as before. We have the following upper estimate for $5 \leq y < 23$:*

(I) when $5 \leq y < 7$ and $x > 5$, we have

$$\begin{aligned}\psi(x, y) \leq & \frac{1}{6}\{\frac{1}{\log 2 \log 3 \log 5}(\log x + \log 15)(\log x + \log 10)(\log x + \log 6) \\ & - \frac{1}{\log^3 5}[(\log x + \log 6)^3 \\ & - (\log x + \log 6 + \log 5)(\log x + \log 6)(\log x + \log 6 - \log 5)]\};\end{aligned}$$

(II) when $7 \le y < 11$ *and* $x > 7$*, we have*

$$\psi(x,y) \le \frac{1}{24}\{\frac{1}{\log 2 \log 3 \log 5 \log 7}(\log x + \log 105)(\log x + \log 70)$$
$$\cdot (\log x + \log 42)(\log x + \log 30)$$
$$- \frac{1}{\log^4 7}[(\log x + \log 30)^4$$
$$- (\log x + \log 7 + \log 30)(\log x + \log 30)$$
$$\cdot (\log x + \log 30 - \log 7)(\log x + \log 30 - 2\log 7)]\};$$

(III) when $11 \le y < 13$ *and* $x > 11$*, we have*

$$\psi(x,y) \le \frac{1}{120}\{\frac{1}{\log 2 \log 3 \log 5 \log 7 \log 11}(\log x + \log 1155)(\log x + \log 770)(\log x + \log 462)$$
$$\cdot (\log x + \log 330)(\log x + \log 210)$$
$$- \frac{1}{\log^5 11}[(\log x + \log 210)^5$$
$$- (\log x + \log 11 + \log 210)(\log x + \log 210)(\log x + \log 210 - \log 11)$$
$$\cdot (\log x + \log 210 - 2\log 11)(\log x + \log 210 - 3\log 11)]\}.$$

(IV) when $13 \le y < 17$ *and* $x > 13$*, we have*

$$\psi(x,y) \le \frac{1}{720}\{\frac{1}{\log 2 \log 3 \log 5 \log 7 \log 11 \log 13}(\log x + \log 15015)(\log x + \log 10010)$$
$$\cdot (\log x + \log 6006)(\log x + \log 4290)(\log x + \log 2730)$$
$$\cdot (\log x + \log 2310) - \frac{1}{\log^6 13}[(\log x + \log 2310)^6$$
$$- (\log x + \log 13 + \log 2310)(\log x + \log 2310)(\log x + \log 2310 - \log 13)$$
$$\cdot (\log x + \log 2310 - 2\log 13)(\log x + \log 2310 - 3\log 13)$$

$$(\log x + \log 2310 - 4\log 13)]\}.$$

(V) when $17 \le y < 19$ *and* $x > 17$*, we have*

$$\psi(x,y) \le \frac{1}{5040}\{\frac{1}{\log 2 \log 3 \log 5 \log 7 \log 11 \log 13 \log 17}(\log x + \log 255255)(\log x + \log 170170)$$
$$\cdot (\log x + \log 102102)(\log x + \log 72930)(\log x + \log 46410)$$
$$\cdot (\log x + \log 39270)(\log x + \log 30030)$$
$$- \frac{1}{\log^7 17}[(\log x + \log 30030)^7 - (\log x + \log 17 + \log 30030)$$
$$\cdot (\log x + \log 30030)(\log x + \log 30030 - \log 17)$$
$$\cdot (\log x + \log 30030 - 2\log 17)(\log x + \log 30030 - 3\log 17)$$
$$(\log x + \log 30030 - 4\log 17)(\log x + \log 30030 - 5\log 17)]\}.$$

(VI) when $19 \le y < 23$ *and* $x > 19$*, we have*

$$\psi(x,y) \le \frac{1}{40320}\{\frac{1}{\log 2 \log 3 \log 5 \log 7 \log 11 \log 13 \log 17 \log 19}(\log x + \log 4849845)(\log x$$
$$\cdot + \log 3233230)(\log x + \log 1939938)(\log x + \log 1385670)(\log x + \log 881790)$$
$$\cdot (\log x + \log 746130)(\log x + \log 570570)(\log x + \log 510510)$$
$$- \frac{1}{\log^8 19}[(\log x + \log 570570)^8 - (\log x + \log 19 + \log 570570)$$
$$\cdot (\log x + \log 570570)(\log x + \log 570570 - \log 19)$$
$$\cdot (\log x + \log 570570 - 2\log 19)(\log x + \log 570570 - 3\log 19)$$
$$(\log x + \log 570570 - 4\log 19)(\log x + \log 570570 - 5\log 19)$$
$$(\log x + \log 570570 - 6\log 19)]\}.$$

Remark. *For comparison, we list the Ennola's upper bounds (see (2)) for* $5 \leq y < 23$ *as follows:*

(1): $5 \leq y < 7$ *and* $x > 5$,

$$\psi(x,y) \leq \frac{(\log x + \log 30)^3}{6\log 2\log 3\log 5}$$

(2): $7 \leq y < 11$ *and* $x > 7$,

$$\psi(x,y) \leq \frac{(\log x + \log 210)^4}{24\log 2\log 3\log 5\log 7}$$

(3): $11 \leq y < 13$ *and* $x > 11$,

$$\psi(x,y) \leq \frac{(\log x + \log 2310)^5}{120\log 2\log 3\log 5\log 7\log 11}$$

(4): $13 \leq y < 17$ *and* $x > 13$,

$$\psi(x,y) \leq \frac{(\log x + \log 30030)^6}{720\log 2\log 3\log 5\log 7\log 11\log 13}$$

(5): $17 \leq y < 19$ *and* $x > 17$,

$$\psi(x,y) \leq \frac{(\log x + \log 510510)^7}{5040\log 2\log 3\log 5\log 7\log 11\log 13\log 17}$$

(6): $19 \leq y < 23$ *and* $x > 19$,

$$\psi(x,y) \leq \frac{(\log x + \log 9699690)^8}{40320\log 2\log 3\log 5\log 7\log 11\log 13\log 17\log 19}.$$

It is easy to see that our upper bound of $\psi(x,y)$ *is substantially better than the one obtained by Ennola. For example, in* $19 \leq y < 23$ *and* $x > 19$ *case, though the coefficient of*

$(\log x)^8$ *in our estimate is same as Ennola's, but our coefficient of* $(\log x)^7$ *is*

$$\frac{1}{40320}[\frac{1}{\log 2\log 3\log 5\log 7\log 11\log 13\log 17\log 19}(\log 4849845+\log 3233230+\log 1939938+ \log 1385670+\log 881790+\log 746130+\log 570570+\log 510510) -\frac{20}{\log^7 19}]\approx 0.007744154691$$

which is smaller than Ennola's

$$\frac{1}{40320}\frac{8\log 9699690}{\log 2\log 3\log 5\log 7\log 11\log 13\log 17\log 19}\approx 0.008950128404.$$

We use the symbolic computation software, Maple 18, to deal with tremendous involved computation. Besides, we have found a quick way to judge the positivity of a polynomial in a restricted domain. We also simplify the process of computation by making use of some characteristics of those polynomials.

2 Some lemmas

We will frequently use the following two lemmas to decide the positivity of polynomials in some restricted domains.

Lemma 2.1 ([15] Lemma 3.1)**.** *Let* $f(\beta)$ *be a polynomial defined by*

$$f(\beta)=\sum_{i=0}^{n}c_i\beta^i \tag{9}$$

where $\beta\in(0,1)$*. If for any* $k=0,1,\ldots,n$

$$\sum_{i=0}^{k}c_i\geq 0 \tag{10}$$

then $f(\beta) \geq 0$ *for* $\beta \in (0, 1)$.

Lemma 2.1 is easy to use. However, the condition of Lemma 2.1 may not be satisfied in some situation. In that case, we shall make use of the following lemma.

Lemma 2.2 (Sturm's Theorem). *Starting from a given polynomial* $X = f(x)$, *let the polynomials* X_1, X_2, ..., X_r *be determined by Euclidean algorithm as follows:*

$$\begin{array}{rcl} X_1 & = & f'(x) \qquad , \\ X & = & Q_1X_1 - X_2, \\ X_1 & = & Q_2X_2 - X_3, \\ \cdots & \cdots & \cdots\cdots \\ X_{r-1} & = & Q_rX_r \end{array} \tag{11}$$

where $\deg X_k > \deg X_{k+1}$ *for* $k = 1, \ldots, r-1$. *For every real number* a *which is not a root of* $f(x)$ *let* $w(a)$ *be the number of variations in sign in the number sequence*

$$X(a), X_1(a), \ldots, X_r(a)$$

in which all zeros are omitted. If b *and* c *are any numbers* ($b < c$) *for which f(x) does not vanish, then the number of the various roots in the interval* $b \leq x \leq c$*(multiple roots to be counted only once) is equal to*

$$w(b) - w(c).$$

Proof. See [14]. □

The condition of Lemma 2.2 is necessary and sufficient, so it can be applied to judge the positivity of any such polynomials in some intervals. The computation in Lemma 2.2 is more complicated than that in Lemma 2.1. Therefore, we prefer Lemma 2.1 when it works.

The following three lemmas come from [18].

Lemma 2.3 ([18] Proposition 3.1). *Given any positive real number β where $0 < \beta < 1$, let $a > 1$ be any number such that $\beta = a - \lfloor a \rfloor$, where $\lfloor a \rfloor$ denotes the greatest positive integer less than or equal to a. If $n \geq 3$, then*

$$a - 1 > (n+1) \sum_{k=0}^{\lfloor a \rfloor - 1} \frac{(k+\beta)^n}{a^n}. \tag{12}$$

Lemma 2.4 ([18] Lemma 3.3). *Let $a_{j-1}, a_j, \ldots, a_{n+1}$ be real numbers and $\beta = a_{n+1} - \lfloor a_{n+1} \rfloor$. Assume that $a_{j-1} > 1$ and $a_j \geq a_{j+1} \geq \ldots \geq a_n \geq a_{n+1} > 1$. If $\frac{a_n}{a_{n+1}}\beta \geq 1$, and*

$$\prod_{i=j}^{n+1}(a_i - 1) > (n+1) \sum_{k=0}^{\lfloor a_{n+1} \rfloor - 1} [\frac{(k+\beta)^{j-1}}{a_{n+1}^{j-1}} \prod_{i=j}^{n}(\frac{a_i}{a_{n+1}}(k+\beta) - 1)] \tag{13}$$

then

$$\prod_{i=j-1}^{n+1}(a_i - 1) > (n+1) \sum_{k=0}^{\lfloor a_{n+1} \rfloor - 1} [\frac{(k+\beta)^{j-2}}{a_{n+1}^{j-2}} \prod_{i=j-1}^{n}(\frac{a_i}{a_{n+1}}(k+\beta) - 1)]. \tag{14}$$

Lemma 2.5 ([18] Lemma 3.4). *Let $a_{j-1}, a_j, \ldots, a_{n+1}$ be real numbers and $\beta = a_{n+1} - \lfloor a_{n+1} \rfloor$. Assume that $a_{j-1} > 1$ and $a_j \geq a_{j+1} \geq \ldots \geq a_n \geq a_{n+1} > 1$. If $\frac{a_n}{a_{n+1}}\beta < 1$, and*

$$\prod_{i=j}^{n+1}(a_i - 1) > (n+1) \sum_{k=1}^{\lfloor a_{n+1} \rfloor - 1} [\frac{(k+\beta)^{j-1}}{a_{n+1}^{j-1}} \prod_{i=j}^{n}(\frac{a_i}{a_{n+1}}(k+\beta) - 1)] \tag{15}$$

then

$$\prod_{i=j-1}^{n+1}(a_i-1) > (n+1)\sum_{k=1}^{\lfloor a_{n+1}\rfloor -1}[\frac{(k+\beta)^{j-2}}{a_{n+1}^{j-2}}\prod_{i=j-1}^{n}(\frac{a_i}{a_{n+1}}(k+\beta)-1)]. \tag{16}$$

3 Proof of the main theorems

We will prove the Main Theorem A (i.e. Theorem 1.5) by induction. Notice that P_n can be obtained by recursion: let k be the possible integer such that $1 \le k \le \lfloor a_n \rfloor$, where $\lfloor a_n \rfloor$ is the biggest integer less than or equal to a_n. For each k, we have an $(n-1)$-dimensional simplex

$$\frac{x_1}{a_1}+\frac{x_2}{a_2}+\cdots+\frac{x_{n-1}}{a_{n-1}}+\frac{k}{a_n} \le 1, x_1, x_2, \ldots, x_{n-1} \ge 0. \tag{17}$$

Let $P_{n-1}^{(k)}$ be the number of positive integer solution of (17). Clearly,

$$P_n = \sum_{k=1}^{\lfloor a_n \rfloor} P_{n-1}^{(k)}. \tag{18}$$

Therefore, since we already know that the Yau Number Theoretic Conjecture is true for $n = 7$ by Theorem 1.4, we want to prove that $g_8(a_1, \cdots, a_8)$ is greater than or equal to the sum of g_7's, the upper estimate of 7-dimensional layers in $T(a_1, a_2, \cdots, a_8)$.

Let m be number of 7-dimensional layers in 8-dimensional simplex, i.e. $P_7^{(m)} > 0$ and $P_7^{(m+1)} = 0$, where $P_7^{(k)} = \#\{(x_1, \ldots, x_7) \in \mathbb{Z}_+^7 : \frac{x_1}{a_1} + \cdots + \frac{x_7}{a_7} + \frac{k}{a_8} \le 1\}$, where $1 \le k \le m$, $a_1 \ge a_2 \ge \ldots \ge a_8 \ge 1$ are real numbers. Let

$$\begin{aligned}\Delta_m &:= g_8(a_1, \ldots, a_8) - 8\sum_{k=1}^{m} g_7(a_1(1-\frac{k}{a_8}), \ldots, a_7(1-\frac{k}{a_8}))\\ &= (a_1-1)\ldots(a_8-1) - (a_8-1)^8 + a_8(a_8-1)\ldots(a_8-7)\end{aligned}$$

$$-8[\sum_{k=1}^{m}(a_1(1-\frac{k}{a_8})-1)\dots(a_7(1-\frac{k}{a_8})-1)-(a_7(1-\frac{k}{a_8})-1)^7$$
$$+a_7(1-\frac{k}{a_8})(a_7(1-\frac{k}{a_8})-1)\dots(a_7(1-\frac{k}{a_8})-6)]$$

be the difference between $g_8(a_1,\dots,a_8)$ and the sum of g_7's. We should prove that $\Delta_m \geq 0$ under the condition of main theorem.

Since $P_7^{(m)} = \#\{(x_1,\dots,x_7) \in \mathbb{Z}_+^7 : \frac{x_1}{a_1}+\cdots+\frac{x_7}{a_7}+\frac{m}{a_8} \leq 1\}$, let $\alpha = 1-\frac{m}{a_8} \in (0,1)$, $A_i = a_i\alpha$, for $i = 1,\dots,7$, then we have

$$\frac{x_1}{A_1}+\frac{x_2}{A_2}+\cdots+\frac{x_7}{A_7} \leq 1 \tag{19}$$

and

$$g_7(m) := \sum_{k=1}^{m} g_7(\frac{m-k+k\alpha}{m\alpha}A_1,\dots,\frac{m-k+k\alpha}{m\alpha}A_7)$$
$$\Delta_m(A_1,\dots,A_7,\alpha) = g_8(\frac{A_1}{\alpha},\dots,\frac{A_7}{\alpha},\frac{m}{1-\alpha})-8g_7(m).$$

Let $B_{7,k}$ be $e_k(A_1,\dots,A_7)$, the elementary symmetric polynomial, for $k = 0,\dots,7$, that is, $B_{7,k} = (A_1A_2\cdots A_7)\sum_{1\leq i_1<\ldots\leq i_k\leq 7}\frac{1}{A_{i_1}\ldots A_{i_k}}$. For example, $B_{7,0} = A_1A_2\cdots A_7$, $B_{7,6} = A_1+A_2+\cdots+A_7$ and $B_{7,7} = 1$. Then

$$\begin{aligned}
g_7(m) &= \sum_{k=1}^{m}(\frac{m-k+k\alpha}{m\alpha}A_1-1)\dots(\frac{m-k+k\alpha}{m\alpha}A_7-1)-\sum_{k=1}^{m}(\frac{m-k+k\alpha}{m\alpha}A_7-1)^7 \\
&\quad +\sum_{k=1}^{m}\frac{m-k+k\alpha}{m\alpha}A_7(\frac{m-k+k\alpha}{m\alpha}A_7-1)\dots(\frac{m-k+k\alpha}{m\alpha}A_7-6) \\
&= \sum_{k=1}^{m}[(\frac{m-k+k\alpha}{m\alpha})^7B_{7,0}-(\frac{m-k+k\alpha}{m\alpha})^6B_{7,1}+(\frac{m-k+k\alpha}{m\alpha})^5B_{7,2} \\
&\quad -(\frac{m-k+k\alpha}{m\alpha})^4B_{7,3}+(\frac{m-k+k\alpha}{m\alpha})^3B_{7,4}-(\frac{m-k+k\alpha}{m\alpha})^2B_{7,5} \\
&\quad +(\frac{m-k+k\alpha}{m\alpha})B_{7,6}+B_{7,7}]
\end{aligned}$$

$$+\sum_{k=1}^{m}[-14(\frac{m-k+k\alpha}{m\alpha})^6 A_7^6 + 154(\frac{m-k+k\alpha}{m\alpha})^5 A_7^5 - 700(\frac{m-k+k\alpha}{m\alpha})^4 A_7^4$$
$$+1589(\frac{m-k+k\alpha}{m\alpha})^3 A_7^3 - 1743(\frac{m-k+k\alpha}{m\alpha})^2 A_7^2 + 713(\frac{m-k+k\alpha}{m\alpha}) A_7 + 1].$$

To make $g_7(m)$ a polynomial of m, we must transform the function to avoid the appearance of m in the sum symbol. Let

$$S_q \;:=\; \sum_{k=1}^{m}(\frac{m-k+k\alpha}{m\alpha})^q, \qquad \text{for } q = 1, \ldots, 7.$$

We will use the first seven S_q in the later computation:

$$\begin{aligned}
S_1 &= \frac{1}{m\alpha}[\frac{1}{2}m(m+1)\alpha + \frac{1}{2}m(m-1)] \\
S_2 &= (\frac{1}{m\alpha})^2[\frac{1}{6}m(m+1)(2m+1)(\alpha-1)^2 + m^2(m+1)(\alpha-1) + m^3] \\
S_3 &= (\frac{1}{m\alpha})^3[\frac{1}{4}m^2(m+1)^2(\alpha-1)^3 + \frac{1}{2}m^2(m+1)(2m+1)(\alpha-1)^2 + \frac{3}{2}m^3(m+1)(\alpha-1) + m^4] \\
S_4 &= (\frac{1}{m\alpha})^4[\frac{1}{30}m(m+1)(2m+1)(3m^2+3m-1)(\alpha-1)^4 + m^3(m+1)^2(\alpha-1)^3 \\
&\quad +m^3(m+1)(2m+1)(\alpha-1)^2 + 2m^4(m+1)(\alpha-1) + m^5] \\
S_5 &= (\frac{1}{m\alpha})^5[\frac{1}{12}m^2(m+1)^2(2m^2+2m-1)(\alpha-1)^5 \\
&\quad +\frac{1}{6}m^2(m+1)(2m+1)(3m^2+3m-1)(\alpha-1)^4 + \frac{5}{2}m^4(m+1)^2(\alpha-1)^3 \\
&\quad +\frac{5}{3}m^4(m+1)(2m+1)(\alpha-1)^2 + \frac{5}{2}m^5(m+1)(\alpha-1) + m^6] \\
S_6 &= (\frac{1}{m\alpha})^6[\frac{1}{42}m(m+1)(2m+1)(3m^4+6m^3-3m+1)(\alpha-1)^6 \\
&\quad +\frac{1}{2}m^3(m+1)^2(2m^2+2m-1)(\alpha-1)^5 \\
&\quad +\frac{1}{2}m^3(m+1)(2m+1)(3m^2+3m-1)(\alpha-1)^4 + 5m^5(m+1)^2(\alpha-1)^3 \\
&\quad +\frac{5}{2}m^5(m+1)(2m+1)(\alpha-1)^2 + 3m^6(m+1)(\alpha-1) + m^7] \\
S_7 &= (\frac{1}{m\alpha})^7[\frac{1}{24}m^2(3m^4+6m^3-m^2-4m+2)(m+1)^2(\alpha-1)^7 \\
&\quad +\frac{1}{6}m^2(2m+1)(m+1)(3m^4+6m^3-3m+1)(\alpha-1)^6
\end{aligned}$$

$$
\begin{aligned}
&+\frac{7}{4}m^4(2m^2+2m-1)(m+1)^2(\alpha-1)^5\\
&+\frac{7}{6}m^4(2m+1)(m+1)(3m^2+3m-1)(\alpha-1)^4\\
&+\frac{35}{4}m^6(m+1)^2(\alpha-1)^3+\frac{7}{2}m^6(2m+1)(m+1)(\alpha-1)^2\\
&+\frac{7}{2}m^7(m+1)(\alpha-1)+m^8].
\end{aligned}
$$

In large part of this paper, we determine the positivity of the polynomial in some restricted domain by using the initial value of all partial derivatives. To make this point clear, we introduce the following lemmas:

Lemma 3.1. *Let $f(m)$ be a polynomial of m, whose degree is s. If*

(1) $\frac{\partial^s f}{\partial m^s} > 0$,

(2) $\frac{\partial^k f}{\partial m^k}|_{m=m_0} > 0$ *for* $k = 0, \ldots, s-1$.

Then $f(m) > 0$ *for* $m \geq m_0$.

Proof. It is trivial. □

Lemma 3.2. *Consider α and m as parameters and let $\Delta_m(A_1, A_2, \ldots, A_7, \alpha)$ be a polynomial of $A_1, \ldots, A_7$. If*

(1) $\Delta_m(A_1^{(0)}, \ldots, A_7^{(0)}, \alpha) \geq 0$,

(2) $\frac{\partial \Delta_m}{\partial A_i} \geq 0$, $\frac{\partial^2 \Delta_m}{\partial A_i \partial A_7} \geq 0$ *and* $\frac{\partial^6 \Delta_m}{\partial A_7^6} \geq 0$ *for all* $1 \leq i \leq 5$, $A_1 \geq A_1^{(0)}, \ldots, A_7 \geq A_7^{(0)}$,

(3) $\frac{\partial^k \Delta_m}{\partial A_7^k}|_{A_1=A_1^{(0)},\ldots,A_7=A_7^{(0)}} \geq 0$ *for all* $1 \leq k \leq 4$.

Then $\Delta_m(A_1, A_2, \ldots, A_7, \alpha) \geq 0$ *for* $A_1 \geq A_1^{(0)}, \ldots, A_7 \geq A_7^{(0)}$.

Proof. Suppose $f(A_1, \ldots, A_7)$ is a polynomial of $A_1, \ldots, A_7$. To prove $f \geq 0$ for $A_1 \geq A_1^{(0)}, \ldots, A_7 \geq A_7^{(0)}$, we only need to show

(1) $f(A_1^{(0)}, \ldots, A_7^{(0)}) \geq 0$ and

(2) $\frac{\partial f}{\partial A_i} \geq 0$, for all $1 \leq i \leq 7$, $A_1 \geq A_1^{(0)}, \ldots, A_7 \geq A_7^{(0)}$.

In particular, we can apply this method to show $\Delta_m \geq 0$ and $\frac{\partial^k \Delta_m}{\partial A_{i_1} \ldots \partial A_{i_k}} \geq 0$, where $1 \leq i_1 \leq \ldots \leq i_k \leq 7$. In order to show $\frac{\partial^k \Delta_m}{\partial A_{i_1} \ldots \partial A_{i_k}} \geq 0$, we only need to show

(1) $\frac{\partial^k \Delta_m}{\partial A_{i_1} \ldots \partial A_{i_k}}|_{A_1=A_1^{(0)}, \ldots, A_7=A_7^{(0)}} \geq 0$ and

(2) $\frac{\partial}{\partial A_j}(\frac{\partial^k \Delta_m}{\partial A_{i_1} \ldots \partial A_{i_k}}) \geq 0$, for all $1 \leq j \leq 7$, $A_1 \geq A_1^{(0)}, \ldots, A_7 \geq A_7^{(0)}$.

Notice that for $k \geq 2$, $\frac{\partial^k \Delta_m}{\partial A_7^k}$ only contains one variable A_7, i.e., $\frac{\partial^{k+1} \Delta_m}{\partial A_i \partial^k A_7} = 0$ for $1 \leq i \leq 6$. Therefore, given the three conditions in the proposition statement, by induction we can prove that $\Delta_m(A_1, A_2, \ldots, A_7, \alpha) \geq 0$ for $A_1 \geq A_1^{(0)}, \ldots, A_7 \geq A_7^{(0)}$. □

So we can use the initial value of all partial derivatives to determine the sign of Δ_m by applying Lemma 3.2. The following proposition gives results about the sign of some partial derivatives of Δ_m in general n-dimensional case. This proposition can save us some labor of computing.

Proposition 3.1. *[19] Let*

$$g_n(a_1, \ldots, a_n) \ := \ (a_1 - 1) \ldots (a_n - 1) - (a_n - 1)^n + a_n(a_n - 1) \ldots (a_n - (n - 1))$$

be the polynomial upper estimate of $P_n(a_1, \ldots, a_n)$ in the Yau Number Theoretic Conjecture. And let m be the number of $(n-1)$-dimensional layers in the n-dimensional simplex, i.e., $P_{n-1}(m) > 0$ and $P_{n-1}(m+1) = 0$. Let $\alpha = 1 - \frac{m}{a_n} \in (0, 1)$, $A_i = a_i \alpha$, for $i = 1, \ldots, n-1$ and

$$g_{n-1}(m) := \sum_{k=1}^{m} g_{n-1}(\frac{m - k + k\alpha}{m\alpha} A_1, \ldots, \frac{m - k + k\alpha}{m\alpha} A_{n-1})$$

$$\Delta_m(A_1,\ldots,A_{n-1},\alpha) = g_n(\frac{A_1}{\alpha},\ldots,\frac{A_{n-1}}{\alpha},\frac{m}{1-\alpha}) - ng_{n-1}(m)$$

then

$$\frac{\partial \Delta_m}{\partial A_i} > 0$$

and

$$\frac{\partial^2 \Delta_m}{\partial A_i \partial A_{n-1}} > 0$$

for all $i = 1,\ldots,n-2$, $A_1 \geq \ldots \geq A_{n-1} \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0,1)$.

Proof. Notice that $A_1,\ldots,A_{n-2}$ are symmetric in the polynomial. Therefore we only need to prove $\frac{\partial \Delta_m}{\partial A_1} > 0$ and $\frac{\partial^2 \Delta_m}{\partial A_1 \partial A_{n-1}} > 0$ for $A_1 \geq \ldots \geq A_{n-1} \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0,1)$. Let $k' = \lfloor a_n \rfloor - k$, $\beta = a_n - \lfloor a_n \rfloor$,

$$\frac{\partial \Delta_m}{\partial A_1} = \frac{1}{\alpha}\{\prod_{i=2}^{n}(a_i - 1) - n\sum_{k'=\lfloor a_n \rfloor - m}^{\lfloor a_n \rfloor - 1}[(\frac{k'+\beta}{a_n})\prod_{i=2}^{n-1}(\frac{a_i}{a_n}(k'+\beta)-1)]\}$$

and

$$\frac{\partial^2 \Delta_m}{\partial A_1 \partial A_{n-1}} = \frac{1}{\alpha^2}\{\prod_{i=2}^{n-2}(a_i - 1)(a_n - 1) - n\sum_{k'=\lfloor a_n \rfloor - m}^{\lfloor a_n \rfloor - 1}[(\frac{k'+\beta}{a_n})^2\prod_{i=2}^{n-2}(\frac{a_i}{a_n}(k'+\beta)-1)]\}.$$

Our goal is to show that

$$\prod_{i=2}^{n}(a_i - 1) - n\sum_{k'=\lfloor a_n \rfloor - m}^{\lfloor a_n \rfloor - 1}[(\frac{k'+\beta}{a_n})\prod_{i=2}^{n-1}(\frac{a_i}{a_n}(k'+\beta)-1)] > 0 \tag{20}$$

and

$$\prod_{i=2}^{n-2}(a_i-1)(a_n-1)-n\sum_{k'=\lfloor a_n\rfloor-m}^{\lfloor a_n\rfloor-1}[(\frac{k'+\beta}{a_n})^2\prod_{i=2}^{n-2}(\frac{a_i}{a_n}(k'+\beta)-1)]>0. \tag{21}$$

We are going to consider two cases.

Case 1: $a_n-1<m<a_n$

In this case, $m=\lfloor a_n\rfloor$. Since $P_n(m)>0$, $\frac{1}{a_1}+\cdots+\frac{1}{a_{n-1}}+\frac{m}{a_n}\leq 1$ must holds. Thus $\frac{1}{a_{n-1}}+\frac{m}{a_n}\leq 1$ and it is equivalent to $\frac{1}{a_{n-1}}+\frac{a_n-\beta}{a_n}\leq 1$, so $\frac{a_{n-1}}{a_n}\beta\geq 1$. By Lemma 2.3,

$$a_n-1>n\sum_{k'=0}^{\lfloor a_n\rfloor-1}\frac{(k'+\beta)^{n-1}}{a_n^{n-1}}. \tag{22}$$

Since we have $a_1\geq a_2\geq\ldots\geq a_{n-2}\geq a_{n-1}\geq a_n>1$,if we repeatedly apply Lemma 2.4 to (22), then after $n-2$ times we will have

$$\prod_{i=2}^{n}(a_i-1)-n\sum_{k'=0}^{\lfloor a_n\rfloor-1}[(\frac{k'+\beta}{a_n})\prod_{i=2}^{n-1}(\frac{a_i}{a_n}(k'+\beta)-1)]>0.$$

Notice that we also have $a_1\geq a_2\geq\ldots\geq a_{n-2}\geq a_n>1$, so if we repeatedly apply Lemma 2.4 to (22) only $n-3$ times we will have

$$\prod_{i=2}^{n-2}(a_i-1)(a_n-1)-n\sum_{k'=0}^{\lfloor a_n\rfloor-1}[(\frac{k'+\beta}{a_n})^2\prod_{i=2}^{n-2}(\frac{a_i}{a_n}(k'+\beta)-1)]>0.$$

Notice that for $k'\geq 0$,

$$(\frac{k'+\beta}{a_n})\prod_{i=2}^{n-1}(\frac{a_i}{a_n}(k'+\beta)-1)\geq 0 \tag{23}$$

$$(\frac{k'+\beta}{a_n})^2\prod_{i=2}^{n-2}(\frac{a_i}{a_n}(k'+\beta)-1)\geq 0. \tag{24}$$

Thus we have (20) and (21)

Case 2: $m \leq a_n - 1$ In this case, $\lfloor a_n \rfloor - m \geq 1$. By Lemma 2.3,

$$\begin{aligned} a_n - 1 &> n \sum_{k'=0}^{\lfloor a_n \rfloor - 1} \frac{(k'+\beta)^{n-1}}{a_n^{n-1}} \\ &> n \sum_{k'=1}^{\lfloor a_n \rfloor - 1} \frac{(k'+\beta)^{n-1}}{a_n^{n-1}}. \end{aligned} \tag{25}$$

Since we have $a_1 \geq a_2 \geq \ldots \geq a_{n-2} \geq a_{n-1} \geq a_n > 1$,if we repeatedly apply Lemma 3.2 to (25), then after $n-2$ times we will have

$$\prod_{i=2}^{n}(a_i - 1) - n \sum_{k'=1}^{\lfloor a_n \rfloor - 1} [(\frac{k'+\beta}{a_n}) \prod_{i=2}^{n-1}(\frac{a_i}{a_n}(k'+\beta) - 1)] > 0.$$

Notice that we also have $a_1 \geq a_2 \geq \ldots \geq a_{n-2} \geq a_n > 1$, so if we repeatedly apply Lemma 3.2 to (25) only $n-3$ times we will have

$$\prod_{i=2}^{n-2}(a_i - 1)(a_n - 1) - n \sum_{k'=1}^{\lfloor a_n \rfloor - 1} [(\frac{k'+\beta}{a_n})^2 \prod_{i=2}^{n-2}((k'+\beta)\frac{a_i}{a_n} - 1)] > 0.$$

Notice that for $k' \geq 0$, (23) and (24) holds. Thus we get (20) and (21) in this case. Therefore, $\frac{\partial \Delta_m}{\partial A_1} > 0$ and $\frac{\partial^2 \Delta_m}{\partial A_1 \partial A_{n-1}} > 0$ for $A_1 \geq \ldots \geq A_{n-1} \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0,1)$. □

Proposition 3.2. *[19] Let*

$$g_n(a_1, \ldots, a_n) \;\; := \;\; (a_1 - 1)\ldots(a_n - 1) - (a_n - 1)^n + a_n(a_n - 1)\ldots(a_n - (n-1))$$

be the polynomial upper estimate of $P_n(a_1, \ldots, a_n)$ in the Yau Number Theoretic Conjecture. And let m be the number of $(n-1)$-dimensional layers in the n-dimensional simplex, i.e., $P_{n-1}(m) > 0$ and $P_{n-1}(m+1) = 0$. Let $\alpha = 1 - \frac{m}{a_n} \in (0,1)$, $A_i = a_i\alpha$, for $i = 1, \ldots, n-1$

and

$$g_{n-1}(m) := \sum_{k=1}^{m} g_{n-1}(\frac{m-k+k\alpha}{m\alpha}A_1, \ldots, \frac{m-k+k\alpha}{m\alpha}A_{n-1})$$
$$\Delta_m(A_1, \ldots, A_{n-1}, \alpha) = g_n(\frac{A_1}{\alpha}, \ldots, \frac{A_{n-1}}{\alpha}, \frac{m}{1-\alpha}) - n g_{n-1}(m)$$

then

$$\frac{\partial^{n-2}\Delta_m}{\partial A_{n-1}^{n-2}} > 0$$

for all $n \geq 5$, $\alpha \in (0,1)$, $m \in \mathbb{Z}^+$.

Proof. In fact, for $n \geq 5$,

$$\frac{\partial^{n-2} g_n(\frac{A_1}{\alpha}, \ldots, \frac{A_{n-1}}{\alpha}, \frac{m}{1-\alpha})}{\partial A_{n-1}^{n-2}} = 0$$

hence,

$$\begin{aligned}
& \frac{\partial^{n-2}\Delta_m}{\partial A_{n-1}^{n-2}} = -n\frac{\partial^{n-2} g_{n-1}(m)}{\partial A_{n-1}^{n-2}} \\
= \;& n\frac{\partial^{n-2}}{\partial A_{n-1}^{n-2}}[\sum_{k=1}^{m}(\frac{m-k+k\alpha}{m\alpha}A_{n-1}-1)^{n-1} \\
& -\sum_{k=1}^{m}\frac{m-k+k\alpha}{m\alpha}A_{n-1}(\frac{m-k+k\alpha}{m\alpha}A_{n-1}-1)\ldots(\frac{m-k+k\alpha}{m\alpha}A_{n-1}-(n-2))] \\
= \;& n\frac{\partial^{n-2}}{\partial A_{n-1}^{n-2}}\sum_{k=1}^{m}[-(n-1)(\frac{m-k+k\alpha}{m\alpha}A_{n-1})^{n-2} + \frac{(n-1)(n-2)}{2}(\frac{m-k+k\alpha}{m\alpha}A_{n-1})^{n-2} \\
& +\text{lower degree terms of } A_{n-1}] \\
= \;& \sum_{k=1}^{m}\frac{(n-4)\cdot n!}{2}(\frac{m-k+k\alpha}{m\alpha})^{n-2} > 0
\end{aligned}$$

for $m \in \mathbb{Z}^+$, $\alpha \in (0,1)$. □

The proof of the Theorem 1.5 (Main Theorem A) is divided into 8 cases:

case 1: $a_8 \in (m, m+1]$;

case 2: $a_8 \in (m+1, m+2]$;

case 3: $a_8 \in (m+2, m+3]$;

case 4: $a_8 \in (m+3, m+4]$;

case 5: $a_8 \in (m+4, m+5]$;

case 6: $a_8 \in (m+5, m+6]$;

case 7: $a_8 \in (m+6, m+7)$;

case 8: $a_8 \geq m+7$.

For case 1 to 7, the equality in (7) cannot be attained by any chance, because in these cases Δ_m is positive. On the other hand, $a_1 = \cdots = a_8$ cannot hold in these cases, it can only hold in case 8.

3.1 Case 1: $a_8 \in (m, m+1]$

For $a_8 \in (m, m+1]$, $\alpha \in (0, \frac{1}{m+1}]$, since $x_1 = \cdots = x_6 = x_7 = 1$, $x_8 = m$ is a solution of the inequality, we know that

$$\frac{1}{A_1} + \cdots + \frac{1}{A_7} \leq 1 \tag{26}$$

and $A_1 \geq A_2 \geq \ldots \geq A_7$. So we just need to show that $\Delta_m \geq 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq 1$. Notice that in this case, $\frac{m\alpha}{1-\alpha} \in (0,1]$, so by Proposition 3.1, $\frac{\partial \Delta_m}{\partial A_i} > 0$ and $\frac{\partial^2 \Delta_m}{\partial A_i \partial A_7} > 0$ for all $i = 1, \ldots, 6$, $A_1 \geq \ldots \geq A_7 \geq 1$, $\alpha \in (0, \frac{1}{m+1}]$.

By Proposition 3.2, $\frac{\partial^6 \Delta_m}{\partial A_7^6} > 0$, for $\alpha \in (0,1)$, $m \geq 2$, m integer.

(i) $\frac{\partial^5 \Delta_m}{\partial A_7^5}|_{A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer.

$$\begin{aligned}
&\frac{\partial^5 \Delta_m}{\partial A_7^5}|_{A_7=1} \\
= \; & \frac{1}{\alpha^6 m^5}[-160(m+1)(85m^5 + 128m^4 + 5m^3 - 5m^2 + 12m - 12)\alpha^6 \\
& -160(m-1)(m+1)(82m^4 - 51m^2 - 72)\alpha^5 \\
& -320(m-1)(m+1)(41m^4 + 41m^2 + 90)\alpha^4 \\
& -320(m-1)(m+1)(41m^4 + 41m^2 - 120)\alpha^3 \\
& -160(m-1)(m+1)(82m^4 - 303m^2 + 180)\alpha^2 \\
& -160(m-1)(82m^5 - 380m^4 + 257m^3 + 257m^2 - 72m - 72)\alpha \\
& +1920(m-1)(2m-1)(3m^4 - 6m^3 + 3m + 1)].
\end{aligned}$$

For $m \geq 4$, the coefficients of $\alpha, \ldots, \alpha^6$ are less than 0, and

$$\begin{aligned}
&\frac{\partial^5 \Delta_m}{\partial A_7^5}|_{A_7=1, \alpha=\frac{1}{m+1}} \\
= \; & 160\frac{(1+m)^7}{m^8}(72m^5 + 26m^4 - 236m^3 - 149m^2 + 65m + 12)\frac{m^9}{(1+m)^6} \\
= \; & 160(1+m)m(72m^5 + 26m^4 - 236m^3 - 149m^2 + 65m + 12) \\
> \; & 0.
\end{aligned}$$

Thus $\frac{\partial^5 \Delta_m}{\partial A_7^5}|_{A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 4$.

For $m = 2$,

$$\begin{aligned}
&\frac{\partial^5 \Delta_2}{\partial A_7^5}|_{A_7=1} \\
= \; & -\frac{420}{\alpha^6}(168\alpha^6 + 37\alpha^5 + 65\alpha^4 + 50\alpha^3 + 10\alpha^2 - 7\alpha - 3)
\end{aligned}$$

$$> \quad 0 \qquad\qquad \text{for } \alpha \in (0, \frac{1}{3}].$$

For $m = 3$,

$$\begin{aligned}
& \frac{\partial^5 \Delta_3}{\partial A_7^5}|_{A_7=1} \\
= \quad & -\frac{4480}{81\alpha^6}(1448\alpha^6 + 582\alpha^5 + 720\alpha^4 + 680\alpha^3 + 390\alpha^2 - 45\alpha - 130) \\
> \quad & 0 \qquad\qquad \text{for } \alpha \in (0, \frac{1}{4}].
\end{aligned}$$

These two ">"s can be proved by Lemma 2.1, you may need to replace α with, for example, $\beta = \alpha/3$, $\beta \in (0, 1]$, for $m = 3$. Thus $\frac{\partial^5 \Delta_m}{\partial A_7^5}|_{A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer.

(ii) $\frac{\partial^4 \Delta_m}{\partial A_7^4}|_{A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer. Let $\beta = (1 + m)\alpha \in (0, 1]$.

$$\begin{aligned}
& \frac{\partial^4 \Delta_m}{\partial A_7^4}|_{A_7=1} \\
= \quad & \frac{160(1+m)}{\beta^6 m^5}[(50m^5 + 34m^4 - 13m^3 + 13m^2 - 6m + 6)\beta^6 \\
& +(50m^6 - 21m^4 + 7m^2 - 36)\beta^5 \\
& +(50m^7 + 50m^6 - 140m^3 - 140m^2 + 90m + 90)\beta^4 \\
& +(50m^8 + 100m^7 - 230m^6 - 560m^5 \\
& +70m^4 + 700m^3 + 230m^2 - 240m - 120)\beta^3 \\
& +(50m^9 - 270m^8 - 445m^7 + 785m^6 \\
& 1190m^5 - 490m^4 - 1065m^3 - 115m^2 + 270m + 90)\beta^2 \\
& +(-118m^{10} - 10m^9 + 629m^8 + 256m^7 - 1133m^6 \\
& -770m^5 + 671m^4 + 668m^3 - 13m^2 - 144m - 36)\beta \\
& +36m^{11} + 54m^{10} - 144m^9 - 270m^8 + 138m^7
\end{aligned}$$

$$+456m^6+90m^5-264m^4-150m^3+18m^2+30m+6].$$

The function Δ_m can be extended to a function of m for $m \in \mathbb{R}^+$. We still denote this extended function by Δ_m.

$$\begin{aligned}
& \frac{\partial^{11}}{\partial m^{11}}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4 \Delta_m}{\partial A_7^4}|_{A_7=1}\right) \\
= \; & 1437004800 > 0 \qquad \text{for } \beta \in (0,1] \\
& \frac{\partial^{10}}{\partial m^{10}}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4 \Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2} \\
= \; & -428198400\beta + 3069964800 > 0 \qquad \text{for } \beta \in (0,1] \\
& \frac{\partial^{9}}{\partial m^{9}}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4 \Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2} \\
= \; & 18144000\beta^2 - 860025600\beta + 3213665280 > 0 \qquad \text{for } \beta \in (0,1] \\
& \frac{\partial^{8}}{\partial m^{8}}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4 \Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2} \\
= \; & 2016000\beta^3 + 25401600\beta^2 - 838293120\beta + 2192520960 > 0 \qquad \text{for } \beta \in (0,1] \\
& \frac{\partial^{7}}{\partial m^{7}}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4 \Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2} \\
= \; & 252000\beta^4 + 4536000\beta^3 + 12272400\beta^2 - 526176000\beta \\
& +1093690080 > 0 \qquad \text{for } \beta \in (0,1] \\
& \frac{\partial^{6}}{\partial m^{6}}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4 \Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2} \\
= \; & 36000\beta^5 + 540000\beta^4 + 4874400\beta^3 - 1501200\beta^2 - 237816720\beta \\
& +424111680 > 0 \qquad \text{for } \beta \in (0,1] \\
& \frac{\partial^{5}}{\partial m^{5}}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4 \Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2} \\
= \; & 6000\beta^6 + 72000\beta^5 + 576000\beta^4 + 3297600\beta^3 - 5631600\beta^2 - 81933840\beta \\
& +132695280 > 0 \qquad \text{for } \beta \in (0,1] \\
& \frac{\partial^{4}}{\partial m^{4}}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4 \Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2}
\end{aligned}$$

$$= 12816\beta^6 + 71496\beta^5 + 408000\beta^4 + 1552080\beta^3 - 4005360\beta^2 - 22202136\beta$$

$$+34320672 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\frac{\partial^3}{\partial m^3}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4 \Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2}$$

$$= 13554\beta^6 + 46992\beta^5 + 215160\beta^4 + 525960\beta^3 - 1776150\beta^2 - 4810536\beta$$

$$+7462044 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\frac{\partial^2}{\partial m^2}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4 \Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2}$$

$$= 9502\beta^6 + 23006\beta^5 + 89240\beta^4 + 125820\beta^3 - 574290\beta^2 - 839322\beta$$

$$+1381212 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\frac{\partial}{\partial m}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4 \Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2}$$

$$= 9502\beta^6 + 23006\beta^5 + 89240\beta^4 + 125820\beta^3 - 574290\beta^2 - 839322\beta$$

$$+1381212 > 0 \qquad \text{for } \beta \in (0,1].$$

The ">"s can be proved by Lemma 2.1 or Lemma 2.2, and this appears frequently throughout the paper, we shall not mention the using of Lemma 2.1 or Lemma 2.2 later. Thus by Lemma 3.1, $\frac{\partial^4 \Delta_m}{\partial A_7^4}|_{A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer.

(iii) $\frac{\partial^3 \Delta_m}{\partial A_7^3}|_{A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer. Let $\beta = (1+m)\alpha \in (0,1]$.

$$\frac{\partial^3 \Delta_m}{\partial A_7^3}|_{A_7=1}$$

$$= \frac{1+m}{\beta^6 m^5}[(-2588m^5 + 1412m^4 + 240m^3 - 240m^2 - 320m + 320)\beta^6$$

$$-(4(m-1))(m+1)(647m^4 + 1060m^2 - 480)\beta^5$$

$$-(4(m-1))(647m^4 - 4120m^2 + 1200)(m+1)^2\beta^4$$

$$-(4(m-1))(647m^5 - 8887m^4 + 7080m^3 + 7080m^2 - 1600m - 1600)(m+1)^2\beta^3$$

$$+(80(m-1))(m-2)(206m^4 - 222m^3 - 133m^2 + 45m + 30)(m+1)^3\beta^2$$

$$-(80(m-1))(130m^5 - 332m^4 + 137m^3 + 137m^2 - 24m - 24)(m+1)^4\beta$$

$$+(320(m-1))(2m-1)(3m^4-6m^3+3m+1)(m+1)^5]$$

$$\begin{aligned}
&\frac{\partial^{11}}{\partial m^{11}}(\frac{\beta^6 m^5}{1+m}\frac{\partial^3\Delta_m}{\partial A_7^3}|_{A_7=1}) \\
= \;& 76640256000 > 0 \qquad \text{for } \beta\in(0,1] \\
&\frac{\partial^{10}}{\partial m^{10}}(\frac{\beta^6 m^5}{1+m}\frac{\partial^3\Delta_m}{\partial A_7^3}|_{A_7=1})|_{m=2} \\
= \;& -37739520000\beta + 163731456000 > 0 \qquad \text{for } \beta\in(0,1] \\
&\frac{\partial^{9}}{\partial m^{9}}(\frac{\beta^6 m^5}{1+m}\frac{\partial^3\Delta_m}{\partial A_7^3}|_{A_7=1})|_{m=2} \\
= \;& 5980262400\beta^2 - 77162803200\beta + 171395481600 > 0 \qquad \text{for } \beta\in(0,1] \\
&\frac{\partial^{8}}{\partial m^{8}}(\frac{\beta^6 m^5}{1+m}\frac{\partial^3\Delta_m}{\partial A_7^3}|_{A_7=1})|_{m=2} \\
= \;& -104348160\beta^3 + 11244441600\beta^2 - 76914432000\beta \\
& +116934451200 > 0 \qquad \text{for } \beta\in(0,1] \\
&\frac{\partial^{7}}{\partial m^{7}}(\frac{\beta^6 m^5}{1+m}\frac{\partial^3\Delta_m}{\partial A_7^3}|_{A_7=1})|_{m=2} \\
= \;& -13043520\beta^4 - 42577920\beta^3 + 10142496000\beta^2 - 49671014400\beta \\
& +58330137600 > 0 \qquad \text{for } \beta\in(0,1] \\
&\frac{\partial^{6}}{\partial m^{6}}(\frac{\beta^6 m^5}{1+m}\frac{\partial^3\Delta_m}{\partial A_7^3}|_{A_7=1})|_{m=2} \\
= \;& -1863360\beta^5 - 27950400\beta^4 + 130608000\beta^3 + 5799801600\beta^2 - 23291078400\beta \\
& +22619289600 > 0 \qquad \text{for } \beta\in(0,1] \\
&\frac{\partial^{5}}{\partial m^{5}}(\frac{\beta^6 m^5}{1+m}\frac{\partial^3\Delta_m}{\partial A_7^3}|_{A_7=1})|_{m=2} \\
= \;& -310560\beta^6 - 3726720\beta^5 - 27525600\beta^4 + 196488960\beta^3 + 2335948800\beta^2 \\
& -8422099200\beta + 7077081600 > 0 \qquad \text{for } \beta\in(0,1] \\
&\frac{\partial^{4}}{\partial m^{4}}(\frac{\beta^6 m^5}{1+m}\frac{\partial^3\Delta_m}{\partial A_7^3}|_{A_7=1})|_{m=2} \\
= \;& -587232\beta^6 - 3766368\beta^5 - 16084128\beta^4 + 143857248\beta^3 + 693742080\beta^2 \\
& -2434988160\beta + 1830435840 > 0 \qquad \text{for } \beta\in(0,1]
\end{aligned}$$

$$
\begin{aligned}
& \frac{\partial^3}{\partial m^3}(\frac{\beta^6 m^5}{1+m}\frac{\partial^3 \Delta_m}{\partial A_7^3}|_{A_7=1})|_{m=2} \\
= \quad & -551904\beta^6 - 2563776\beta^5 - 5816256\beta^4 + 69856320\beta^3 + 153424800\beta^2 \\
& -576339840\beta + 397975680 > 0 \qquad \text{for } \beta \in (0,1] \\
& \frac{\partial^2}{\partial m^2}(\frac{\beta^6 m^5}{1+m}\frac{\partial^3 \Delta_m}{\partial A_7^3}|_{A_7=1})|_{m=2} \\
= \quad & -343904\beta^6 - 1309216\beta^5 - 1052288\beta^4 + 24889344\beta^3 + 24438240\beta^2 \\
& -113620320\beta + 73664640 > 0 \qquad \text{for } \beta \in (0,1] \\
& \frac{\partial}{\partial m}(\frac{\beta^6 m^5}{1+m}\frac{\partial^3 \Delta_m}{\partial A_7^3}|_{A_7=1})|_{m=2} \\
= \quad & -160256\beta^6 - 525120\beta^5 + 143616\beta^4 + 6833664\beta^3 + 2393280\beta^2 \\
& -18947520\beta + 11741760 > 0 \qquad \text{for } \beta \in (0,1]. \\
& (\frac{\beta^6 m^5}{1+m}\frac{\partial^3 \Delta_m}{\partial A_7^3}|_{A_7=1})|_{m=2} \\
= \quad & -59584\beta^6 - 169344\beta^5 + 177408\beta^4 + 1487808\beta^3 - 2721600\beta \\
& +1632960 > 0 \qquad \text{for } \beta \in (0,1].
\end{aligned}
$$

Thus by Lemma 3.1, $\frac{\partial^3 \Delta_m}{\partial A_7^3}|_{A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer.

(iv) $\frac{\partial^2 \Delta_m}{\partial A_7^2}|_{A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer. Let $\beta = (1+m)\alpha \in (0,1]$.

$$
\begin{aligned}
& \frac{\partial^2 \Delta_m}{\partial A_7^2}|_{A_7=1} \\
= \quad & \frac{1+m}{3\beta^6 m^5}[(124m^5 - 3820m^4 + 2000m^3 - 2000m^2 - 240m + 240)\beta^6 \\
& +(4(m-1))(m+1)(31m^4 - 3000m^2 + 360)\beta^5 \\
& +(4(m-1))(m+1)(31m^5 - 10427m^4 + 7360m^3 + 7360m^2 - 900m - 900)\beta^4 \\
& -(4(m-1))(6941m^5 - 21661m^4 + 9440m^3 + 9440m^2 - 1200m - 1200)(m+1)^2\beta^3 \\
& +(80(m-1))(368m^5 - 892m^4 + 333m^3 + 333m^2 - 45m - 45)(m+1)^3\beta^2 \\
& -(160(m-1))(68m^5 - 163m^4 + 61m^3 + 61m^2 - 9m - 9)(m+1)^4\beta
\end{aligned}
$$

$+(240(m-1))(2m-1)(3m^4-6m^3+3m+1)(m+1)^5]$

$$
\begin{aligned}
& \frac{\partial^{11}}{\partial m^{11}}(\frac{3\beta^6 m^5}{1+m}\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=1}) \\
= \quad & 57480192000 > 0 \qquad \text{for } \beta\in(0,1] \\
& \frac{\partial^{10}}{\partial m^{10}}(\frac{3\beta^6 m^5}{1+m}\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=1})|_{m=2} \\
= \quad & -39481344000\beta + 122798592000 > 0 \qquad \text{for } \beta\in(0,1] \\
& \frac{\partial^{9}}{\partial m^{9}}(\frac{3\beta^6 m^5}{1+m}\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=1})|_{m=2} \\
= \quad & 10683187200\beta^2 - 81343180800\beta + 128546611200 > 0 \qquad \text{for } \beta\in(0,1] \\
& \frac{\partial^{8}}{\partial m^{8}}(\frac{3\beta^6 m^5}{1+m}\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=1})|_{m=2} \\
= \quad & -1119444480\beta^3 + 20863180800\beta^2 - 81839923200\beta \\
& +87700838400 > 0 \qquad \text{for } \beta\in(0,1] \\
& \frac{\partial^{7}}{\partial m^{7}}(\frac{3\beta^6 m^5}{1+m}\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=1})|_{m=2} \\
= \quad & 624960\beta^4 - 1942133760\beta^3 + 19774944000\beta^2 - 53459481600\beta \\
& +43747603200 > 0 \qquad \text{for } \beta\in(0,1] \\
& \frac{\partial^{6}}{\partial m^{6}}(\frac{3\beta^6 m^5}{1+m}\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=1})|_{m=2} \\
= \quad & 89280\beta^5 - 28779840\beta^4 - 1590192000\beta^3 + 12082924800\beta^2 \\
& -25423948800\beta + 16964467200 > 0 \qquad \text{for } \beta\in(0,1] \\
& \frac{\partial^{5}}{\partial m^{5}}(\frac{3\beta^6 m^5}{1+m}\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=1})|_{m=2} \\
= \quad & 14880\beta^6 + 178560\beta^5 - 55291680\beta^4 - 804837120\beta^3 + 5330966400\beta^2 \\
& -9355795200\beta + 5307811200 > 0 \qquad \text{for } \beta\in(0,1] \\
& \frac{\partial^{4}}{\partial m^{4}}(\frac{3\beta^6 m^5}{1+m}\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=1})|_{m=2} \\
= \quad & -61920\beta^6 - 112416\beta^5 - 50482848\beta^4 - 274469856\beta^3 + 1802912640\beta^2 \\
& -2764396800\beta + 1372826880 > 0 \qquad \text{for } \beta\in(0,1]
\end{aligned}
$$

$$\begin{aligned}
&\frac{\partial^3}{\partial m^3}\Big(\frac{3\beta^6 m^5}{1+m}\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=1}\Big)|_{m=2}\\
= \quad & -141600\beta^6 - 462912\beta^5 - 29370336\beta^4 - 62773440\beta^3 + 484538400\beta^2\\
& -672088320\beta + 298481760 > 0 \qquad \text{for } \beta\in(0,1]\\
&\frac{\partial^2}{\partial m^2}\Big(\frac{3\beta^6 m^5}{1+m}\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=1}\Big)|_{m=2}\\
= \quad & -143520\beta^6 - 495552\beta^5 - 12210080\beta^4 - 7915968\beta^3 + 106017120\beta^2\\
& -136857600\beta + 55248480 > 0 \qquad \text{for } \beta\in(0,1]\\
&\frac{\partial}{\partial m}\Big(\frac{3\beta^6 m^5}{1+m}\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=1}\Big)|_{m=2}\\
= \quad & -96560\beta^6 - 310400\beta^5 - 3855408\beta^4 + 456192\beta^3 + 19243440\beta^2\\
& -23690880\beta + 8806320 > 0 \qquad \text{for } \beta\in(0,1].\\
&\Big(\frac{3\beta^6 m^5}{1+m}\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=1}\Big)|_{m=2}\\
= \quad & -49392\beta^6 - 133728\beta^5 - 962640\beta^4 + 532224\beta^3 + 2948400\beta^2\\
& -3538080\beta + 1224720 > 0 \qquad \text{for } \beta\in(0,1].
\end{aligned}$$

Thus by Lemma 3.1, $\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=1} > 0$, for $\alpha\in(0,\frac{1}{m+1}]$, $m\geq 2$, m integer.

(v) $\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1} > 0$, for $\alpha\in(0,\frac{1}{m+1}]$, $m\geq 2$, m integer. Let $\beta=(1+m)\alpha\in(0,1]$.

$$\begin{aligned}
&\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\\
= \quad & \frac{1+m}{21\beta^7 m^5(1+m-\beta)}[(1452m^5+48948m^4+38944m^3-38944m^2-38112m+38112)\beta^8\\
& +(-110355m^5-77420m^4+283220m^2-196224)\beta^7\\
& -(7(m+1))(8565m^6-50640m^5+45920m^4+40012m^2-43776)\beta^6\\
& +(7(28023m^6-101934m^5+110516m^4-56840m^2+21120))(m+1)^2\beta^5\\
& -(7(29487m^6-108444m^5+16055m^4-238840m^2+162240))(m+1)^3\beta^4
\end{aligned}$$

$+(7(16437m^6-103761m^5+250544m^4-386372m^2+238464))(m+1)^4\beta^3$

$-(84(682m^6-10703m^5+26999m^4-29435m^2+14464))(m+1)^5\beta^2$

$+(12(2175m^6-69195m^5+139104m^4-101248m^2+37984))(m+1)^6\beta$

$+(35280(m-1))(9m^4-5m^3-5m^2+2m+2)(m+1)^7]$

$$\begin{aligned}
&\frac{\partial^{12}}{\partial m^{12}}\Big(\frac{21\beta^7m^5(1+m-\beta)}{1+m}\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\Big)\\
=\;&12501941760000\beta+152092588032000>0 \qquad \text{for } \beta\in(0,1]\\
&\frac{\partial^{11}}{\partial m^{11}}\Big(\frac{21\beta^7m^5(1+m-\beta)}{1+m}\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\Big)|_{m=2}\\
=\;&-2286753638400\beta^2-1889661312000\beta+373190146560000>0 \qquad \text{for } \beta\in(0,1]\\
&\frac{\partial^{10}}{\partial m^{10}}\Big(\frac{21\beta^7m^5(1+m-\beta)}{1+m}\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\Big)|_{m=2}\\
=\;&417526099200\beta^3-2350460851200\beta^2-39383990553600\beta\\
&+453845306880000>0 \qquad \text{for } \beta\in(0,1]\\
&\frac{\partial^{9}}{\partial m^{9}}\Big(\frac{21\beta^7m^5(1+m-\beta)}{1+m}\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\Big)|_{m=2}\\
=\;&-74901697920\beta^4+738493096320\beta^3+472957470720\beta^2\\
&-59015265845760\beta+364458905395200>0 \qquad \text{for } \beta\in(0,1]\\
&\frac{\partial^{8}}{\partial m^{8}}\Big(\frac{21\beta^7m^5(1+m-\beta)}{1+m}\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\Big)|_{m=2}\\
=\;&7909211520\beta^5-144163393920\beta^4+623340587520\beta^3\\
&+2480019171840\beta^2-49809954378240\beta\\
&+217239766732800>0 \qquad \text{for } \beta\in(0,1]\\
&\frac{\partial^{7}}{\partial m^{7}}\Big(\frac{21\beta^7m^5(1+m-\beta)}{1+m}\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\Big)|_{m=2}\\
=\;&-302173200\beta^6+14199494400\beta^5-135830857680\beta^4\\
&+342108678240\beta^3+2346705224640\beta^2-29438204241600\beta\\
&+102425652403200>0 \qquad \text{for } \beta\in(0,1]
\end{aligned}$$

$$
\begin{aligned}
&\frac{\partial^6}{\partial m^6}\Big(\frac{21\beta^7 m^5(1+m-\beta)}{1+m}\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\Big)|_{m=2}\\
= \;& -392288400\beta^6 + 12251307600\beta^5 - 84140345520\beta^4\\
& +147463066800\beta^3 + 1316922485760\beta^2 - 13301054504640\beta\\
& +39751471872000 > 0 \qquad \text{for } \beta\in(0,1]\\
&\frac{\partial^5}{\partial m^5}\Big(\frac{21\beta^7 m^5(1+m-\beta)}{1+m}\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\Big)|_{m=2}\\
= \;& 174240\beta^8 - 13242600\beta^7 - 176265600\beta^6 + 6749284080\beta^5\\
& -39015241440\beta^4 + 60297581400\beta^3 + 509867588160\beta^2\\
& -4813865964000\beta + 13048166880000 > 0 \qquad \text{for } \beta\in(0,1]\\
&\frac{\partial^4}{\partial m^4}\Big(\frac{21\beta^7 m^5(1+m-\beta)}{1+m}\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\Big)|_{m=2}\\
= \;& 1523232\beta^8 - 28343280\beta^7 + 21433440\beta^6 + 2664822048\beta^5\\
& -14629056288\beta^4 + 25764293520\beta^3 + 138853135008\beta^2\\
& -1434132765504\beta + 3693882890880 > 0 \qquad \text{for } \beta\in(0,1]\\
&\frac{\partial^3}{\partial m^3}\Big(\frac{21\beta^7 m^5(1+m-\beta)}{1+m}\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\Big)|_{m=2}\\
= \;& 2931648\beta^8 - 30201360\beta^7 + 72115176\beta^6 + 804171648\beta^5\\
& -4623215184\beta^4 + 10637709264\beta^3 + 23521477896\beta^2\\
& -357236640576\beta + 915240604320 > 0 \qquad \text{for } \beta\in(0,1]\\
&\frac{\partial^2}{\partial m^2}\Big(\frac{21\beta^7 m^5(1+m-\beta)}{1+m}\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\Big)|_{m=2}\\
= \;& 2971264\beta^8 - 20806520\beta^7 + 46728584\beta^6 + 193196976\beta^5\\
& -1227849840\beta^4 + 3774866760\beta^3 + 243088776\beta^2\\
& -74888478720\beta + 200763450720 > 0 \qquad \text{for } \beta\in(0,1]\\
&\frac{\partial}{\partial m}\Big(\frac{21\beta^7 m^5(1+m-\beta)}{1+m}\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\Big)|_{m=2}\\
= \;& 1955936\beta^8 - 10172960\beta^7 + 17871168\beta^6 + 37626624\beta^5\\
& -257221440\beta^4 + 1079949024\beta^3 - 1443101184\beta^2
\end{aligned}
$$

$$-13160771136\beta + 39350253600 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\left(\frac{21\beta^7 m^5(1+m-\beta)}{1+m}\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\right)|_{m=2}$$

$$= 947296\beta^8 - 3833424\beta^7 + 4647888\beta^6 + 5896800\beta^5$$

$$-36415008\beta^4 + 245678832\beta^3 - 609502320\beta^2$$

$$-1892927232\beta + 6944162400 > 0 \qquad \text{for } \beta \in (0,1].$$

Thus by Lemma 3.1, $\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer. By Lemma 3.2, we conclude that:

Proposition 3.3. $\frac{\partial\Delta_m}{\partial A_7} > 0$*, for* $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq 1$*,* $\alpha \in (0, \frac{1}{m+1}]$*,* $m \geq 2$*,* m *integer.*

(vi) $\Delta_m > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq 1$, $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer. Let $\beta = (1+m)\alpha \in (0,1]$.

$$\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A-7=1}$$

$$= \frac{1+m}{105\beta^7 m^5(1+m-\beta)^7}[(2(m-1))(21459m^4 + 16524m^2 - 45580)\beta^{14}$$

$$+(-785973m^6 - 308280m^5 + 368060m^4 + 1032948m^2 - 832280)\beta^{13}$$

$$-(10(m+1))(136920m^7 - 350628m^6$$

$$-299145m^5 + 333445m^4 + 256802m^2 - 252408)\beta^{12}$$

$$-(5(274596m^8 - 1369200m^7 + 1278253m^6$$

$$+2718618m^5 - 3340232m^4 + 1127532m^2 - 289328))(m+1)^2\beta^{11}$$

$$-(5(140679m^9 - 1098384m^8 + 2738400m^7 - 556608m^6$$

$$-7942557m^5 + 11825856m^4 - 12082532m^2 + 7144280))(m+1)^3\beta^{10}$$

$$-(3(66640m^{10} - 703395m^9 + 2745960m^8 - 4564000m^7 - 3309318m^6$$

$$+29375745m^5 - 54592720m^4 + 72994768m^2 - 44976360))(m+1)^4\beta^9$$

$$-(3(10290m^{11} - 133280m^{10} + 703395m^9 - 1830640m^8 + 2282000m^7$$

$$+8045368m^6 - 55031795m^5 + 120792700m^4 - 165755128m^2 + 99207680))(m+1)^5\beta^8$$

$$-(15(140m^{12} - 2058m^{11} + 13328m^{10} - 46893m^9 + 91532m^8 - 91280m^7$$

$$-2101490m^6 + 17881129m^5 - 42020608m^4 + 53077640m^2 - 29931616))(m+1)^6\beta^7$$

$$-(15(2063454m^6 - 24184461m^5 + 56176610m^4 - 62376776m^2 + 32538792))(m+1)^7\beta^6$$

$$+(5(4906227m^6 - 77088417m^5 + 169624476m^4 - 163110332m^2 + 77774840))(m+1)^8\beta^5$$

$$-(14975472m^6 - 306341385m^5 + 628239290m^4 - 522903612m^2 + 226180240)(m+1)^9\beta^4$$

$$+(6396627m^6 - 173969355m^5 + 331434040m^4 - 240101932m^2 + 93790320)(m+1)^{10}\beta^3$$

$$-(20(83369m^6 - 3319848m^5 + 5887721m^4 - 3740541m^2 + 1316666))(m+1)^{11}\beta^2$$

$$+(20(9885m^6 - 761775m^5 + 1262562m^4 - 708974m^2 + 224762))(m+1)^{12}\beta$$

$$+(176400(m-1))(9m^4 - 5m^3 - 5m^2 + 2m + 2)(m+1)^{13}]$$

$$\frac{\partial^{18}}{\partial m^{18}}\left(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\right)$$

$$= -134449847820288000000\beta^7 + 1265749281622425600000\beta$$

$$+10164408495213772800000 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\frac{\partial^{17}}{\partial m^{17}}\left(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\right)|_{m=2}$$

$$= -2039156025274368000\beta^7 - 5930661038587084800\beta^2 - 2043744393096806400000\beta$$

$$+2679137300898938880000 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\frac{\partial^{16}}{\partial m^{16}}\left(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\right)|_{m=2}$$

$$= -6458865238425600000\beta^8 - 14859783834255360000\beta^7 + 133835282712907776000\beta^3$$

$$-180671219873464320000\beta^2 - 964288619755388928000\beta$$

$$+3517313469839769600000 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\frac{\partial^{15}}{\partial m^{15}}\left(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\right)|_{m=2}$$

$$= -970752067084800000\beta^8 - 6961267192239360000\beta^7 - 19583040883101696000\beta^4$$
$$+123822350770390272000\beta^3 + 150596550614974464 0000\beta^2 - 143724199264457011200000\beta$$
$$+30659541265530777600000 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\frac{\partial^{14}}{\partial m^{14}}(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1})|_{m=2}$$
$$= -17428683976704000\beta^9 - 686318507626752000\beta^8 - 2390512435863552000\beta^7$$
$$+21385824304965120000\beta^5 - 242095878279230976000\beta^4 - 1177014060243141120000\beta^3$$
$$+2763972978903392256000\beta^2 - 1316751289209080524800 0\beta$$
$$+19957155430707855360000 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\frac{\partial^{13}}{\partial m^{13}}(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1})|_{m=2}$$
$$= -26696826059904000\beta^9 - 300841603590528000\beta^8 - 661655115209088000\beta^7$$
$$-1927375646676480000\beta^6 + 30990560836435200000\beta^5 + 64610184799480320 0\beta^4$$
$$-327931157484298368000\beta^3 + 2695427768666400768000\beta^2 - 8759791387794230784000\beta$$
$$+10344660265456957440000 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\frac{\partial^{12}}{\partial m^{12}}(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1})|_{m=2}$$
$$= -336927330432000\beta^{10} - 19013685901056000\beta^9 - 91183831909632000\beta^8$$
$$-145532747340288000\beta^7 - 315490962213504000\beta^6 + 11791929615459840000\beta^5$$
$$+25572322932639436800\beta^4 - 346913834717179699200\beta^3 + 1834373947900741632000\beta^2$$
$$-4576853595753686016000\beta + 4446454187781365760000 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\frac{\partial^{11}}{\partial m^{11}}(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1})|_{m=2}$$
$$= -538866621216000\beta^{10} - 8167273063488000\beta^9 - 21745500044083200\beta^8$$
$$-8804715220224000\beta^7 - 203724633331296000\beta^6 - 1093410114572160000\beta^5$$
$$+30698288791330406400\beta^4 - 241661613378623961600\beta^3 + 963556434202868352000\beta^2$$
$$-1966698879316351488000\beta + 1629598886204889600000 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\frac{\partial^{10}}{\partial m^{10}}(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1})|_{m=2}$$
$$= -4982269824000\beta^{11} - 401434312608000\beta^{10} - 2198508501772800\beta^9$$
$$-6079685525222400\beta^8 + 15142527892800000\beta^7 - 31141945721376000\beta^6$$
$$-182773327099104000\beta^5 + 2198978093378384640\beta^4 - 12708718440828238080\beta^3$$
$$+41216412512011968000\beta^2 - 71649464380135488000\beta$$
$$+51965360483452416000 > 0 \qquad \text{for } \beta \in (0,1]$$
$$\frac{\partial^9}{\partial m^9}(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1})|_{m=2}$$
$$= -847671713280\beta^{11} - 182544096873600\beta^{10} - 262766471627520\beta^9$$
$$-2753587552665600\beta^8 + 9518344856697600\beta^7 + 51134924271715200\beta^6$$
$$-1386562760075808000\beta^5 + 11525079342898517760\beta^4 - 53769417058746144000\beta^3$$
$$+148464893420283686400\beta^2 - 226126230658734067200\beta$$
$$+146413659922193664000 > 0 \qquad \text{for } \beta \in (0,1]$$
$$\frac{\partial^8}{\partial m^8}(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1})|_{m=2}$$
$$= -55206144000\beta^{12} - 6749887536000\beta^{11} - 55587070627200\beta^{10}$$
$$+84308220541440\beta^9 - 1341454193452800\beta^8 + 2811220324819200\beta^7$$
$$+51412180767696000\beta^6 - 724702940280345600\beta^5 + 4779856453326958080\beta^4$$
$$-18994961285538940800\beta^3 + 46078908807724531200\beta^2 - 62821397529421977600\beta$$
$$+36888688393869312000 > 0 \qquad \text{for } \beta \in (0,1]$$
$$\frac{\partial^7}{\partial m^7}(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1})|_{m=2}$$
$$= -99641404800\beta^{12} - 3287796523200\beta^{11} - 11659053999600\beta^{10}$$
$$+68661100841760\beta^9 - 545382672942240\beta^8 + 256119482682000\beta^7$$
$$+27834875028450000\beta^6 - 291053501889393600\beta^5 + 1635161162666512320\beta^4$$
$$-5740514885852898480\beta^3 + 12528992528784547200\beta^2 - 15553513536541142400\beta$$

$$+8390996690862240000 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\frac{\partial^6}{\partial m^6}(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1})|_{m=2}$$

$$= -565900560\beta^{13} - 84192156000\beta^{12} - 1068745194000\beta^{11}$$
$$-1701826297200\beta^{10} + 29135604955680\beta^9 - 190246780280880\beta^8$$
$$-114371373664800\beta^7 + 10618918311524400\beta^6 - 94246288679826000\beta^5$$
$$+473152553158458960\beta^4 - 1509359045660093520\beta^3 + 3021848016486710400\beta^2$$
$$-3465149090870836800\beta + 1736729460910080000 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\frac{\partial^5}{\partial m^5}(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1})|_{m=2}$$

$$= 5150160\beta^{14} - 1168794720\beta^{13} - 42750854400\beta^{12} - 236830813200\beta^{11}$$
$$-221625061800\beta^{10} + 10237097318040\beta^9 - 63791338958280\beta^8$$
$$-28055217234600\beta^7 + 3045346010681400\beta^6 - 25198003866900600\beta^5$$
$$+117666016831846440\beta^4 - 349460020447183080\beta^3$$
$$+652704053515300800\beta^2 - 700098164220794400\beta$$
$$+329190379587216000 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\frac{\partial^4}{\partial m^4}(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1})|_{m=2}$$

$$= 9270288\beta^{14} - 1196954880\beta^{13} - 13248534000\beta^{12}$$
$$-37298861280\beta^{11} - 65681337360\beta^{10} + 3422867875632\beta^9$$
$$-22164308309520\beta^8 + 19124441852400\beta^7 + 652647191460480\beta^6$$
$$-5611236156851760\beta^5 + 25391513083960512\beta^4 - 71864118280754640\beta^3$$
$$+127192185259750080\beta^2 - 129084506776031040\beta$$
$$+57450666580876800 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\frac{\partial^3}{\partial m^3}(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1})|_{m=2}$$

$$= 8438544\beta^{14} - 810854400\beta^{13} - 1561010520\beta^{12} - 7150870320\beta^{11}$$

$$-29879654040\beta^{10} + 1114924380336\beta^9 - 7640924076000\beta^8$$

$$+17398826302320\beta^7 + 92935870233840\beta^6 - 1033879197302640\beta^5$$

$$+4775182879009176\beta^4 - 13204314240914880\beta^3 + 22490401719110040\beta^2$$

$$-21832575435241920\beta + 9273560844592800 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\frac{\partial^2}{\partial m^2}\left(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\right)|_{m=2}$$

$$= 5137296\beta^{14} - 406859064\beta^{13} + 833047240\beta^{12} - 3394003800\beta^{11}$$

$$-9927888120\beta^{10} + 333166341024\beta^9 - 2381023902096\beta^8$$

$$+7710919409520\beta^7 + 2490839385120\beta^6 - 152378031510360\beta^5$$

$$+781442220182184\beta^4 - 2175479058277656\beta^3 + 3624351198784200\beta^2$$

$$-3401660349899280\beta + 1389881048522400 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\frac{\partial}{\partial m}\left(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\right)|_{m=2}$$

$$= 2233288\beta^{14} - 159659504\beta^{13} + 630208560\beta^{12} - 1713065040\beta^{11}$$

$$-1836813240\beta^{10} + 84496668912\beta^9 - 630326514096\beta^8 + 2417300022960\beta^7$$

$$-3376972681560\beta^6 - 15928501867200\beta^5 + 110112142066272\beta^4$$

$$-321715158824736\beta^3 + 533997294982920\beta^2 - 489937583664000\beta$$

$$+194054618268000 > 0 \qquad \text{for } \beta \in (0,1]$$

$$\left(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\right)|_{m=2}$$

$$= 727720\beta^{14} - 50978760\beta^{13} + 251314560\beta^{12} - 657901440\beta^{11}$$

$$+54046440\beta^{10} + 17344960920\beta^9 - 139196776320\beta^8 + 596005089120\beta^7$$

$$-1312716744360\beta^6 - 443054094840\beta^5 + 12953376001440\beta^4 - 42592516092000\beta^3$$

$$+72068650237080\beta^2 - 65412968183640\beta + 25311471948000 > 0 \qquad \text{for } \beta \in (0,1].$$

Thus $\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer. By Lemma 3.2, we know that $\Delta_m > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq$

$2, A_7 \geq 1$, $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer.

(vii) $\Delta_m > 0$, for $m = 1$, $1 < a_8 \leq 2$, $\alpha = 1 - \frac{1}{a_8} \in (0, \frac{1}{2}]$. By Proposition 3.1, $\frac{\partial \Delta_1}{\partial A_i} > 0$, for $1 \leq i \leq 6$, $A_1 \geq \ldots \geq A_7 \geq 1$, $\alpha \in (0, 1)$.

And

$$\frac{\partial^2 \Delta_1}{\partial A_7^2} = 3360A_7^4 - 24640A_7^3 + 67200A_7^2 - 76272A_7 + 27888$$
$$\text{Since } A_7 \geq 1, \text{so set } \frac{\partial^2 \Delta_1}{\partial A_7^2} = 0, \text{we have } A_7 = 1.822613576$$

$$\begin{aligned} & \frac{\partial \Delta_1}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1.822613576} \\ = & \frac{1}{\alpha^6(1-\alpha)}(\alpha^6 - 27\alpha^5 + 295\alpha^4 - 1665\alpha^3 + 5014\alpha^2 - 8028\alpha + 5040) - 6157.912873 \\ > & 0 \qquad \text{for } \alpha \in (0, \frac{1}{2}]. \end{aligned}$$

Thus, we conclude that:

Proposition 3.4. *$\frac{\partial \Delta_1}{\partial A_7} > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq 1$, $\alpha \in (0, 1)$.*

And since

$$\begin{aligned} & \Delta_1|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1} \\ = & \frac{1}{(\alpha-1)^7\alpha^6}(\alpha^{13} - 34\alpha^{12} + 505\alpha^{11} - 4332\alpha^{10} + 23934\alpha^9 - 90012\alpha^8 + 237593\alpha^7 \\ & -442129\alpha^6 + 585344\alpha^5 - 553123\alpha^4 + 366501\alpha^3 - 161776\alpha^2 + 42588\alpha - 5040) \\ > & 0 \qquad \text{for } \alpha \in (0, \frac{1}{2}], \end{aligned}$$

thus by Lemma 3.2, $\Delta_1 > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq 1$, $\alpha \in (0, \frac{1}{2}]$.

Therefore $\Delta_m > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq 1$, $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 1$, m integer.

3.2 Case 2: $a_8 \in (m+1, m+2]$

In this case, $\frac{m\alpha}{1-\alpha} \in (1,2]$, so $A_1 \geq A_2 \geq \ldots \geq A_7 \geq \frac{m\alpha}{1-\alpha}$. By Proposition 3.1, $\frac{\partial \Delta_m}{\partial A_i} > 0, \frac{\partial^2 \Delta_m}{\partial A_j \partial A_7} > 0$, for $1 \leq i, j \leq 6$, $A_1 \geq A_2 \geq \ldots \geq A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0,1)$, $m \geq 1$, m integer.

By Proposition 3.2, $\frac{\partial^6 \Delta_m}{\partial A_7^6} > 0$, for $\alpha \in (0,1)$, $m \geq 2$, m integer.

(i) $\frac{\partial^5 \Delta_m}{\partial A_7^5}|_{A_7=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 2$, m integer.

$$
\begin{aligned}
& \frac{\partial^5 \Delta_m}{\partial A_7^5}|_{A_7=\frac{m\alpha}{1-\alpha}} \\
= & \frac{1}{\alpha^5 m^4 (1-\alpha)}[(160(m+1))(72m^5 + 334m^4 + 380m^3 + 5m^2 - 89m + 12)\alpha^6 \\
& +(320(m+1))(36m^5 - 36m^4 - 321m^3 - 64m^2 + 190m - 36)\alpha^5 \\
& +(160(m-1))(m+1)(72m^4 + 72m^2 + 385m - 180)\alpha^4 \\
& +(3840(m-1))(m+1)(3m^4 + 3m^2 + 10)\alpha^3 \\
& +(160(m-1))(m+1)(72m^4 + 72m^2 - 385m - 180)\alpha^2 \\
& +(320(m-1))(36m^5 + 36m^4 - 321m^3 + 64m^2 + 190m + 36)\alpha \\
& +(160(m-1))(72m^5 - 334m^4 + 380m^3 - 5m^2 - 89m - 12)]
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial^6}{\partial m^6}(m^4 \alpha^5 (1-\alpha) \frac{\partial^5 \Delta_m}{\partial A_7^5}|_{A_7=\frac{m\alpha}{1-\alpha}}) \\
= & 8294400\alpha^6 + 8294400\alpha^5 + 8294400\alpha^4 + 8294400\alpha^3 + 8294400\alpha^2 + 8294400\alpha \\
& +8294400 > 0 \qquad \text{for } \alpha \in (\frac{1}{m+1}, 1]
\end{aligned}
$$

$$\frac{\partial^5}{\partial m^5}(m^4\alpha^5(1-\alpha)\frac{\partial^5\Delta_m}{\partial A_7^5}|_{A_7=\frac{m\alpha}{1-\alpha}})|_{m=2}$$
$$= 24384000\alpha^6+16588800\alpha^5+16588800\alpha^4+16588800\alpha^3+16588800\alpha^2+16588800\alpha$$
$$+8793600>0 \qquad \text{for } \alpha\in(\frac{1}{m+1},1]$$
$$\frac{\partial^4}{\partial m^4}(m^4\alpha^5(1-\alpha)\frac{\partial^5\Delta_m}{\partial A_7^5}|_{A_7=\frac{m\alpha}{1-\alpha}})|_{m=2}$$
$$= 34920960\alpha^6+13847040\alpha^5+16588800\alpha^4+16588800\alpha^3+16588800\alpha^2+13847040\alpha$$
$$+3740160>0 \qquad \text{for } \alpha\in(\frac{1}{m+1},1]$$
$$\frac{\partial^3}{\partial m^3}(m^4\alpha^5(1-\alpha)\frac{\partial^5\Delta_m}{\partial A_7^5}|_{A_7=\frac{m\alpha}{1-\alpha}})|_{m=2}$$
$$= 32502720\alpha^6+4836480\alpha^5+11428800\alpha^4+11059200\alpha^3+10689600\alpha^2+6314880\alpha$$
$$+582720>0 \qquad \text{for } \alpha\in(\frac{1}{m+1},1]$$
$$\frac{\partial^2}{\partial m^2}(m^4\alpha^5(1-\alpha)\frac{\partial^5\Delta_m}{\partial A_7^5}|_{A_7=\frac{m\alpha}{1-\alpha}})|_{m=3}$$
$$= 76491840\alpha^6+13518720\alpha^5+29021760\alpha^4+28047360\alpha^3+26804160\alpha^2+17953920\alpha$$
$$+41174400>0 \qquad \text{for } \alpha\in(\frac{1}{m+1},1]$$
$$\frac{\partial}{\partial m}(m^4\alpha^5(1-\alpha)\frac{\partial^5\Delta_m}{\partial A_7^5}|_{A_7=\frac{m\alpha}{1-\alpha}})|_{m=3}$$
$$= 57013120\alpha^6+1423040\alpha^5+18155840\alpha^4+16957440\alpha^3+14952640\alpha^2+7977280\alpha$$
$$+1093760>0 \qquad \text{for } \alpha\in(\frac{1}{m+1},1]$$
$$(m^4\alpha^5(1-\alpha)\frac{\partial^5\Delta_m}{\partial A_7^5}|_{A_7=\frac{m\alpha}{1-\alpha}})|_{m=3}$$
$$= 34944000\alpha^6-3682560\alpha^5+9542400\alpha^4+8601600\alpha^3+6585600\alpha^2+2674560\alpha$$
$$+120960>0 \qquad \text{for } \alpha\in(\frac{1}{m+1},1].$$

Thus by lemma 3.1, $\frac{\partial^5\Delta_m}{\partial A_7^5}|_{A_7=\frac{m\alpha}{1-\alpha}}>0$, for $\alpha\in(\frac{1}{m+1},1]$, $m\geq 3$. And we can check that

$$(\frac{\partial^5\Delta_m}{\partial A_7^5}|_{A_7=\frac{2\alpha}{1-\alpha}})|_{m=2}$$