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NINE CHARACTERIZATIONS OF WEIGHTED HOMOGENEOUS ISOLATED HYPERSURFACE SINGULARITIES*

STEPHEN S.-T. YAU† AND HUAI QING ZUO‡

Dedicated to Dr. Henry Laufer on the occasion of his 70th birthday

Abstract. This survey paper discusses many different ways of characterizations of weighted homogeneous (quasi-homogeneity) isolated hypersurface singularities.

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1. Introduction. Recall that a polynomial $f \in \mathbb{C}[z_1, z_2, \cdots, z_n]$ is called weighted homogeneous if there exist positive rational numbers $w_1, \cdots, w_n$ (weights of $z_1, \cdots, z_n$) and d such that, for each monomial $\prod z_i^{a_i}$ appearing in f with nonzero coefficient, one has $\sum a_i w_i = d$. The number d is called the weighted homogeneous degree (w-degree) of f with respect to weights w_j. The collection $(w_1, w_2, \cdots, w_n; d)$ is called the weight type of f. Obviously, without loss of generality one can assume that w-deg$f = 1$.

Question: How many different ways can the weighted homogeneous isolated hypersurface singularities of a polynomial be recognized? As we shall see in this paper, there are at least nine different ways to characterize weighted homogeneous hypersurface singularities.

This richness of characterizations reflects that weighted homogeneous isolated hypersurface singularities play an important role in singularity theory.

DEFINITION 1.1. *A polynomial $f(z_1, \cdots, z_n)$ is called quasi-homogeneous if f is in the Jacobian ideal of f i.e., $f \in (\partial f/\partial z_1, \cdots, \partial f/\partial z_n)$.*

By a theorem of Saito (see Characterization A), if f is quasi-homogeneous with an isolated critical point at 0, then after a biholomorphic change of coordinates, f becomes a weighted homogeneous polynomial.

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a complex analytic function with an isolated critical point at the origin. Let $V = \{z \in \mathbb{C}^n : f(z) = 0\}$. The Milnor number μ and the Tjurina number τ of the singularity $(V, 0)$ are defined respectively by

$$\mu = \dim \mathbb{C}\{z_1, z_2, \ldots, z_n\}/(f_{z_1}, \ldots, f_{z_n}),$$
$$\tau = \dim \mathbb{C}\{z_1, z_2, \ldots, z_n\}/(f, f_{z_1}, \ldots, f_{z_n}),$$

where $f_{z_i} := \partial f/\partial z_i, 1 \le i \le n$. They are numerical invariants of $(V, 0)$.

It is well known that the Milnor number of a weighted homogeneous isolated hypersurface singularities can be calculated by only using the weight type.

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†Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, P. R. China. (yau@uic.edu).

‡Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, P. R. China (hqzuo@math.tsinghua.edu.cn).

THEOREM 1.1 (Milnor-Orlik [MO70]). *Let $f(z_1, z_2, \cdots, z_n)$ be a weight homogeneous polynomial of type $(w_1, w_2, \cdots, w_n; 1)$ with isolated singularity at the origin. Then the Milnor number $\mu = (\frac{1}{w_1} - 1)(\frac{1}{w_2} - 1) \cdots (\frac{1}{w_n} - 1)$.*

In 1971, Saito proved the following theorem which gives a necessary and sufficient condition for V to be defined by a weighted homogeneous polynomial.

CHARACTERIZATION A (Saito [Sa71]). Let $f = f(z)$ be a holomorphic function in a neighborhood of the origin in $\mathbb{C}^n$ defining an isolated singularity at the origin 0. The following conditions are equivalent;

(1)There is a holomorphic coordinate transformation ϕ such that $\phi(f)$ is a weighted-homogeneous polynomial.

(2) $f(z)$ is quasi-homogeneous, i.e.,there exist holomorphic functions $a_j(z) \in \mathcal{O}_{\mathbb{C}^n,0}, j = 1, \cdots, n$ such that

$$f(z) = a_1(z) f_{z_1} + \cdots + a_n(z) f_{z_n}.$$

(3) $\mu = \tau$, where μ and τ are Milnor and Tjurina number of the isolated singularity defined by f.

2. A criterion for quasihomogeneity in terms of Lie algebra. A moduli algebra $A(V) := \mathbb{C}\{z_1, \cdots, z_n\}/(f, f_{z_1}, \cdots, f_{z_n})$ of an isolated hypersurface singularity $(V, 0)$ is a finite dimensional $\mathbb{C}-$algebra. In 1982, Mather and Yau [MY82] proved that two germs of complex analytic hypersurfaces of the same dimension with isolated singularities are biholomorphically equivalent if and only if their moduli algebra are isomorphic. The quasi-homogeneity of a complex analytic isolated hypersurface singularity can be characterized in terms of its moduli algebra.

CHARACTERIZATION B ([XY96]). Let $(V, 0) = \{(z_1, \cdots, z_n) : f(z_1, \cdots, z_n) = 0\} \subseteq \mathbb{C}^n$ be an isolated hypersurface singularity. Then $(V, 0)$ admits a quasi-homogeneous structure if and only if its moduli algebra $A(V)$ is isomorphic to a finite dimensional nonnegatively graded algebra $\oplus_{i \geq 0} A_i$, with $A_0 = \mathbb{C}$.

In 1983 [Ya83.1], Yau introduced a finite dimensional Lie algebra $L(V)$ to an isolated hypersurface singularity $(V, 0)$. $L(V)$ is defined as the algebra of derivations of the moduli algebra $A(V)$. It was proved in [Ya86.1], [Ya86.2] and [Ya91] that $L(V)$ is actually a solvable Lie algebra. In [SY90], Seeley and Yau showed how to use the Lie algebra structures to distinguish complex analytic structures of isolated hypersurface singularities. In fact, they have constructed continuous numerical invariants out of these Lie algebras. It is natural to ask the possibility of characterizing the quasi-homogeneity of an isolated hypersurface singularity in terms of its Lie algebra. Let A be a graded algebra $\oplus_{i=0}^{\infty} A_i$. A derivation D of A is said to have weight k if D sends A_i to A_{i+k} for all i. The Lie algebra $L(V)$ of a quasi-homogeneous isolated hypersurface singularity is obviously a graded Lie algebra over $\mathbb{C}$. It also has the left module structure over the moduli algebra $A(V)$. With this in mind, we have the following micro-local characterization of quasi-homogeneity of isolated hypersurface singularity.

CHARACTERIZATION C ([XY96]). Let $(V, 0) = \{(z_1, \cdots, z_n) : f(z_1, \cdots, z_n) = 0\} \subseteq \mathbb{C}^n$ be an isolated hypersurface singularity. Let $A(V) := \mathbb{C}\{z_1, \cdots, z_n\}/(f, f_{z_1}, \cdots, f_{z_n})$ be the moduli algebra and $L(V)$ be the Lie algebra of derivations of $A(V)$. Then $(V, 0)$ is a quasi-homogeneous singularity, if

(1) $L(V)$ is isomorphic to a nonnegatively graded Lie algebra $\oplus_i^k L_i$ without center.

(2) There exists $E \in L_0$ such that $[E, D_i] = iD_i$ for any $D_i \in L_i$.

(3) For any element $\alpha \in m - m^2$ where m is the maximal ideal of $A(V)$, E is not in L_0.

It is easy to see that the conditions (2) and (3) above are necessary for $(V, 0)$ to be quasi-homogeneous. One expects that (1) above is also a necessary condition. In [CXY95], the authors proved that this is indeed the case for $n = 2$, i.e., $L(V)$ has no negatively graded part if $n = 2$. The claim about that (1) above is also a necessary condition, is called Yau Conjecture, which is a special case of the Halperin conjecture. The Halperin conjecture is one of the most important open problems in rational homotopy theory.

Yau Conjecture. Let $(V, 0) = \{(z_1, \cdots, z_n) : f(z_1, \cdots, z_n) = 0\} \subseteq \mathbb{C}^n$ be an isolated hypersurface singularity. Then there are no non-zero negative weight derivations on the moduli algebra (= Milnor algebra here)

$$A(V) = \mathbb{C}[z_1, z_2, \ldots, z_n]/(\partial f/\partial z_1, \partial f/\partial z_2, \ldots, \partial f/\partial z_n).$$

i.e., the Yau algebra $L(V) := Der(A(V), A(V))$ of V is non-negatively graded algebra.

Halperin Conjecture ([FHT01] p. 516). Suppose that F is a rationally elliptic space with non-zero Euler-Poincaré characteristic, and $F \to E \to B$ is a Serre fibration of simply-connected spaces. Then the (rational) Serre spectral sequence for this fibration collapses at E_2.

Actually the above conjecture is equivalent to the following conjecture about the nonexistence of negative weight derivation ([Ha94], [Me82] (Theorem A on p. 329)).

Halperin Conjecture (equivalent form). Let $P = \mathbb{C}[z_1, \cdots, z_n]$ be the polynomial ring of n weighted variables $z_1, \ldots, z_n$ with positive integer weights $w_1, w_2, \ldots, w_n$ and $f_1, f_2, \ldots, f_n$ be weighted homogeneous polynomials in P. Suppose that R is an Artinian algebra of the form

$$\mathbb{C}[z_1, z_2, \ldots, z_n]/(f_1, f_2, \ldots, f_n).$$

Then there is no non-zero negative weight derivation on R.

For recent progress on the Halperin Conjecture and Yau Conjecture, please see [CYZ15] and [YZ15.2].

3. A criterion for quasi-homogeneity in terms of geometric genus and irregularity. In this section, we consider characterization of quasi-homogeneity in a more general situation. Let $(V, 0)$ be a Stein germ of an analytic space with an isolated singularity at 0. $(V, 0)$ is a singularity with a (good) $\mathbb{C}^*$-action if the complete local ring of V at 0 is the completion of a (positively) graded ring. $(V, 0)$ is a quasi-homogeneous singularity if there exists an analytic isomorphism type of $(V, 0)$ which is defined by weighted homogeneous polynomials. It is easy to see that if $(V, 0)$ is a singularity with a (good) $\mathbb{C}^*$-action then it is a quasi-homogeneous singularity.

Let $f \in \mathbb{C}\{z_0, z_1, \cdots, z_n\}$ be a holomorphic function germ with an isolated singularity at the origin. It is well known that Milnor number μ and Tjurina number τ are two very important invariants for hypersurface singularities. Clearly, $\mu \geq \tau$, and the equality holds if and only if f is quasi-homogeneous singularity by a well known theorem of Saito (Characterization A). Both μ and τ can also be defined for n-dimensional isolated complete intersection singularity (ICIS) with $n \geq 1$ in the following manner:

$$\mu = \mathrm{rk} H_n(F)$$

and

$$\tau = \dim T^1_{V,0},$$

where F is the Milnor fibre of a Milnor fibration of $(V, 0)$, and τ is the dimension of the base space of a semi-universal deformation of $(V, 0)$. From the defining equations of $(V, 0)$, one can give formulas for μ and τ as dimensions of certain finite length modules, but it is no longer clear what the relation between these invariants is. This problem was first considered by Greuel [Gr80], who conjectured $\mu \geq \tau$, and proved the inequality in two cases: $n = 1$ or the link of V is a rational homotopy sphere. Greuel also proved that (in every dimension) $\mu = \tau$ if $(V, 0)$ is quasi-homogeneous. Looijenga [Lo82] proved that for ICIS of dimension $n = 2$, $\mu \geq \tau + b$ where $b =$ number of loops in the resolution dual graph of $(V, 0)$. Then Looijenga and Steenbrink [LS85] generalized this result for all $n \geq 2$. In [Wa85], Wahl proved that for two-dimensional isolated complete intersection singularity $(V, 0)$, $\mu \geq \tau + b$ and $\mu = \tau + b$ if and only if $(V, 0)$ is quasi-homogeneous (for $b = 0$) or $(V, 0)$ is cusp ($b = 1$). More recently, Vosegaard [Vo02] generalized this result for general n. If $(V, 0)$ is an isolated complete intersection singularity of any dimension, he proved that $(V, 0)$ is quasi-homogeneous if and only if $\mu = \tau$.

Let $(V, 0)$ be a normal surface singularity. Wagreich [Wa70] first defined an invariant geometric genus p_g for the singularity $(V, 0)$. It turns out that this is an important invariant for the theory of normal surface singularities. In [Ya83.2], Yau introduced another invariant called irregularity q of the singularity $(V, 0)$. In fact, both geometric genus and irregularity can also be defined for general n-dimensional isolated singularities. Let $(V, 0)$ be a normal isolated singularity of dimension $n(\geq 2)$. Let $\pi : \tilde{V} \to V$ be a resolution of the singularity of V with exceptional set $E = \pi^{-1}(0)_{\mathrm{red}}$. Then $p_g := \dim R^{n-1}\pi_* \mathcal{O}_{\tilde{V}}$, and $q := \dim H^0(\Omega^{n-1}_{\tilde{V}-E})/H^0(\Omega^{n-1}_{\tilde{V}})$.

Theorem 3.1 ([YZ13]). *Let $(V, 0)$ be a normal isolated singularity of dimension $n(\geq 2)$ with $\mathbb{C}^*$-action. Let $\pi : \tilde{V} \to V$ be a good resolution of the singularity $(V, 0)$ with $E = f^{-1}(0)_{\mathrm{red}}$. Then $q = p_g - h^{n-1}(\mathscr{O}_E)$.*

A natural question is how to use p_g and q to characterize quasi-homogeneous singularities. In fact the converse of the above theorem is also correct for non-Du Bois isolated complete intersection singularities.

Definition 3.1. *If $(V, 0)$ is a two-dimensional isolated Gorenstein singularity, then $(V, 0)$ is a Du Bois singularity if and only if $(V, 0)$ is either rational, simple elliptic or cusp (see [Is85]).*

Characterization D ([YZ13]). Let $(V, 0)$ be a normal isolated complete intersection singularity of dimension $n(\geq 2)$, and $\pi : \tilde{V} \to V$ be a good resolution of the singularity $(V, 0)$ with $E = f^{-1}(0)_{\mathrm{red}}$. If $q = p_g - h^{n-1}(\mathscr{O}_E)$, then either $(V, 0)$ has a $\mathbb{C}^*$-action or $(V, 0)$ is a Du Bois singularity.

Characterization D is a generalization of Wahl's well-known theorem in two-dimensional case (Theorem 1.9 [Wa85]). Let $(V, 0)$ be a normal surface singularity,

$\pi : \tilde{V} \to V$ a good resolution, and $E \subset \tilde{V}$ the (reduced) exceptional fibre. E is a union of smooth curves E_i, $i = 1, \cdots, k$. Let g_i be genus of E_i, $g = \sum_{i=1}^{k} g_i$. Also define b = first Betti number of the dual graph of E (= number of loops). Then $h^1(\mathscr{O}_E) = g + b$, $\dim H^1(E, \mathbb{C}) = 2g + b$. Denote the geometric genus by $p_g = h^1(\mathscr{O}_{\tilde{V}})$ and irregularity by $q = \dim H^0(\Omega^1_{\tilde{V}-E})/H^0(\Omega^1_{\tilde{V}})$.

Steenbrink introduced three other invariants α, β, γ which are non-negative integers (see (1.8) [Wa85]).

THEOREM 3.2 (Steenbrink). *Let $E \subset \tilde{V} \to V$ be a good resolution of a normal surface singularity, with p_g, g, b and $\alpha, \beta, \gamma \geq 0$ as above. Then the irregularity q is given by*

$$q = p_g - g - b - \alpha - \beta - \gamma.$$

THEOREM 3.3 (Wahl). *Let $(V, 0)$ be a two-dimensional Gorenstein surface singularity. Then $\alpha = \beta = \gamma = 0$ iff either $(V, 0)$ is quasi-homogeneous (so $b = 0$), or $(V, 0)$ is a cusp (so $b = 1$).*

The Characterization D is a generalization of Theorem 3.3 to ICIS in arbitray dimension. Moreover, for recent progress on the characterization of isolated homogeneous singularities, please see [YZ12] and [YZ15.1].

4. A criterion for quasihomogeneity in terms of D-module theory. In this section, we consider isolated hypersurface singularities and give in particular characterization of quasi-homogeneity of these singularities from the viewpoint of the theory of D-modules.

Let X be a Stein neighborhood of the origin 0 of $\mathbb{C}^n$ and $\mathcal{O}_X$ the sheaf of germs of holomorphic functions in X. Let $f = f(z_1, \cdots, z_n) \in \mathcal{O}_{X,0}$ be a germ of a holomorphic function defining an isolated singularity at the origin 0. Let $\mathcal{J}_f$ be the ideal in $\mathcal{O}_{X,0}$ generated by partial derivatives $f_{z_j} = \frac{\partial f}{\partial z_j}$ $(j = 1, \cdots, n)$ of f:

$$\mathcal{J}_f = (f_{z_1}, \cdots, f_{z_n}).$$

Let Σ_f denote the space consisting of algebraic local cohomology classes annihilated by the Jacobi ideal $\mathcal{J}_f$:

$$\Sigma_f = \{\eta \in \mathcal{H}^n_{[0]}(\mathcal{O}_X) \mid g\eta = 0, \forall g \in \mathcal{J}_f\}. \tag{4.1}$$

REMARK 4.1. *$\mathcal{H}^n_{[0]}(\mathcal{O}_X) \cong H^{n-1}(X - \{0\}, \mathcal{O}_X)$ is an infinite dimensional vector space.*

Σ_f can be identified with $\mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}_f, \mathcal{O}_X)$. We can also identify the Milnor algebra $\mathcal{O}_X/\mathcal{J}_f$ with $\Omega^n_X/\mathcal{J}_f\Omega^n_X$ where Ω^n_X is the sheaf of holomorphic differential n-forms. Then, by the non-degeneracy of the Grothendieck local duality

$$\Omega^n_X/\mathcal{J}_f\Omega^n_X \times \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}_f, \mathcal{O}_X) \to \mathbb{C},$$

Σ_f can be considered as the dual space of the Milnor algebra $\mathcal{O}_X/\mathcal{J}_f$ by treating them as finite dimensional vector spaces.

The dual space Σ_f can be generated by a single algebraic local cohomology class, denoted by σ, over $\mathcal{O}_{X,0}$ (see [TN05]):

$$\Sigma_f = \mathcal{O}_{X,0}\,\sigma.$$

One considers first order differential operators that annihilate σ in the sheaf $\mathcal{D}_{X,0}$ of linear partial differential operators. We have the following fundamental property;

LEMMA 4.1 ([TN05]). *Let σ be an algebraic local cohomology class which generates Σ_f over $\mathcal{O}_{X,0}$. Annihilating differential operators of order one for the cohomology class σ act on the space Σ_f.*

Proof. Let $P = \sum_{j=1}^n a_j(z)\frac{\partial}{\partial z_j} + a_0(z)$ be an annihilator of σ where $a_j(z) = a_j(z_1, \cdots, z_n) \in \mathcal{O}_{X,0}$ $(j = 0, 1, \cdots, n)$. Put $v_p = \sum_{j=1}^n a_j(z)\frac{\partial}{\partial z_j}$. Since any class η in Σ_f can be written as $\eta = h(z)\sigma$ with some holomorphic function $h(z) = h(z_1, \cdots, z_n) \in \mathcal{O}_{X,0}$, we have

$$\begin{aligned} P\eta &= P(h(z)\sigma) \\ &= (PQ - QP)\sigma + h(z)P\sigma \\ &= (v_P h(z))\sigma \in \Sigma_f \end{aligned} \tag{4.2}$$

where Q is the multiplication operator in $\mathcal{D}_{X,0}$ defined by $Q = h(z)$. □

Let $\mathcal{L}_f$ be the set of linear partial differential operators of order at most 1 which annihilate σ:

$$\mathcal{L}_f = \{P = \sum_{j=1}^n a_j(z)\frac{\partial}{\partial z_j} + a_0(z) \mid P\sigma = 0, a_j(z) \in \mathcal{O}_{X,0}, j = 0, 1, \cdots, n\}.$$

The condition whether a given first order differential operator P acts on Σ_f or not depends only on the first order part v_P of P. We denote by Θ_f the set of differential operators of the form $\sum_{j=1}^n a_j(z)\partial/\partial z_j$ with $a_j(z) \in \mathcal{O}_{X,0}, j = 1, \cdots, n$ acting on Σ_f. Then, an operator v is in Θ_f if and only if v satisfies the condition $vg(z) \in \mathcal{J}_f$ for all $g(z) = g(z_1, \cdots, z_n) \in \mathcal{J}_f$, i.e.,

$$\Theta_f = \{v = \sum_{j=1}^n a_j(z)\frac{\partial}{\partial z_j} \mid vg(z) \in \mathcal{J}_f, \forall g(z) \in \mathcal{J}_f, a_j(z) \in \mathcal{O}_{X,0}, j = 1, \cdots, n\}.$$

LEMMA 4.2 ([TN05]). *The mapping, from $\mathcal{L}_f$ to Θ_f, which sends first order differential operator $P \in \mathcal{L}_f$ to its first order part $v_P \in \Theta_f$, is surjective.*

Let $P \in \mathcal{L}_f$ be an annihilator of σ of the form $P = \sum_{j=1}^n a_j(z)\frac{\partial}{\partial z_j} + a_0(z)$. If an algebraic local cohomology class $\eta = h(z)\sigma \in \Sigma_f$ is a solution of the homogeneous differential equation $P\eta = 0$, we have

$$v_p h(z) = \sum_{j=1}^n a_j(z)\frac{\partial h(z)}{\partial z_j} \in \mathcal{J}_f$$

where $v_P \in \Theta_f$ is the first order part of the operator P. It is obvious that, in order to represent $\eta \in \Sigma_f$ in the form $\eta = h(z)\sigma$, it suffices to take the modulo class in $\mathcal{O}_X/\mathcal{J}_f$ of the holomorphic function $h(z) \in \mathcal{O}_{X,0}$. Furthermore any element v in Θ_f induces a linear operator acting on $\mathcal{O}_X/\mathcal{J}_f$ which is also denoted by v:

$$v : \mathcal{O}_{X,0}/\mathcal{J}_f \to \mathcal{O}_{X,0}/\mathcal{J}_f.$$

Now we make the following definition:

Definition 4.1. *A solution space* $\mathcal{H}_f$ *is the set of solutions in* $\mathcal{O}_{X,0}/\mathcal{J}_f$ *of differential equations* $vh(z) = 0$ *for all* $v \in \Theta_f$*:*

$$\mathcal{H}_f = \{h(z) \in \mathcal{O}_{X,0}/\mathcal{J}_f \mid vh(z) = 0, \forall\, v \in \Theta_f\}.$$

Then, by Lemma 4.2, we have the following result:

Lemma 4.3 ([TN05]).

$$\mathcal{H}_f = \{h(z) \in \mathcal{O}_{X,0}/\mathcal{J}_f \mid P(h(z)\sigma) = 0, \forall P \in \mathcal{L}_f\}.$$

From the above definition, $\mathcal{H}_f$ *does not depend on the choice of a generator* σ*.*

Proof. "$\subseteq$", for any $h(z) \in \mathcal{H}_f$, we have $h(z) \in \mathcal{O}_{X,0}/\mathcal{J}_f$, such that $\forall\, v \in \Theta_f, vh(z) = 0$ by Definition 4.1. Since $v \in \Theta_f$, so by Lemma 4.2, there is a corresponding $P_v \in \mathcal{L}_f$ such that its first order part is v. We have $P_v(h(z)\sigma) = (vh(z))\sigma$ by (4.2). It follows that $vh(z) \in \mathcal{J}_f$ from $vh(z) = 0$ in $\mathcal{O}_{X,0}/\mathcal{J}_f$. Combining with (4.1), we get $P_v(h(z)\sigma) = (vh(z))\sigma = 0$ which implies "$\subseteq$".

"$\supseteq$", we assume $h(z) \in \mathcal{O}_{X,0}/\mathcal{J}_f$ such that $P(h(z)\sigma) = 0, \forall P \in \mathcal{L}_f$. It follows that $P(h(z)\sigma) = (v_P h(z))\sigma = 0$ from (4.2). Since P is arbitrary, so v_P is also arbitrary by Lemma 4.2. Thus, $v_P h(z) = 0$ which implies $h(z) \in \mathcal{H}_f$. ☐

Let $\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,0}}(\sigma)$ be a left ideal in $\mathcal{D}_{X,0}$ defined to be $\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,0}}(\sigma) = \mathcal{D}_{X,0}\mathcal{L}_f$. By the above Lemma 4.3, we have the following result;

Theorem 4.1 ([TN05]). *Let* $f \in \mathcal{O}_{X,0}$ *define an isolated singularity at the origin. Let* σ *be a generator of* Σ_f *over* $\mathcal{O}_{X,0}$*. Then*

$$Hom_{\mathcal{D}_{X,0}}(\mathcal{D}_{X,0}/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,0}}(\sigma), \mathcal{H}^n_{[0]}(\mathcal{O}_X)) = \{h(z)\sigma \mid h(z) \in \mathcal{H}_f\}.$$

Let $f \in \mathcal{O}_{X,0}$ be a function which defines an isolated singularity at the origin and $\mathcal{J}_f$ the Jacobi ideal of f. Let σ be a generator of Σ_f over $\mathcal{O}_{X,0}$.

Proposition 4.1. *Assume that a function* f *is quasi-homogeneous. Then the set* $\mathcal{H}_f$ *is an one-dimensional vector space* $\mathrm{Span}_{\mathbb{C}}\{1\}$.

Let $\mathcal{A}nn_{\mathcal{D}_{X,0}}(\sigma)$ be a left ideal in $\mathcal{D}_{X,0}$ consisting of all annihilators of the algebraic local cohomology class σ.

Characterization E ([TN05]). Let $f \in \mathcal{O}_{X,0}$ define an isolated hypersurface singularity at the origin. The following three conditions are equivalent;

(1) $(f, \mathcal{J}_f) = \mathcal{J}_f$, i.e., f is quasi-homogeneous.

(2) $\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,0}}(\sigma) = \mathcal{A}nn_{\mathcal{D}_{X,0}}(\sigma)$.

(3)$Hom_{\mathcal{D}_{X,0}}(\mathcal{D}_{X,0}/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,0}}(\sigma), \mathcal{H}^n_{[0]}(\mathcal{O}_{\mathcal{X}}) = \mathrm{Span}_{\mathbb{C}}\{\sigma\}$.

5. A criterion for quasihomogeneity in terms of b-function theory. Let $f(z)$ be a germ of holomorphic function of n variables, and $b_f(s)$ the b-function (i.e. Bernstein polynomial) associate with $f(z)$ is defined to be a monic generator of the ideal consisting of polynomials $b(s)$ which satisfy the relation

$$P(s)f(z)^{s+1} = b(s)f(z)^s,$$

where $P(s) \in \mathfrak{D}_X[s]$, $\mathfrak{D}_X$ denotes the germs of holomorphic differential operators on $X := (\mathbb{C}^n, 0)$, and $\mathfrak{D}_X[s] = \mathfrak{D}_X \otimes_{\mathbb{C}} \mathbb{C}[s]$.

The following is a example of the equality of this type for a quadratic form $Q(z) = \sum_{i=1}^{n} z_i^2$,

$$\Delta Q(z)^{s+1} = 4(s+1)(s+\frac{n}{2})Q(z)^s,$$

where

$$\Delta = \sum_{i=1}^{n} \partial_i^2.$$

It is well-known that b-function theory plays important roles in singularity theory. In fact, $b_f(s)$ is an subtle invariant of the hypersurface $f^{-1}(0)$ finer than local monodromy (cf.[Ya78], section 16).

In b-function theory, the following ideal in $\mathfrak{D}_X[s]$ and $\mathfrak{D}_X[s]$-Module are very interesting and important objects.

Definition 5.1. $\mathfrak{I}(s) := \{P(s) \in \mathfrak{D}_X[s] \mid P(s)f(z)^s = 0\}$, $\mathfrak{N} := \mathfrak{D}_X[s]/\mathfrak{I}(s)$.

We shall use the ideal $\mathfrak{I}(s)$ to introduce a number $L(f)$, which measures the non-quasi-homogeneity of f.

We can define the notion of order of an element of $\mathfrak{D}_X[s]$ by regarding s as element of order 1. To be more precise, we define

Definition 5.2. *Given* $P(s) = \sum s^j P_j(z, D) \in \mathfrak{D}_X[s]$, $\max_j(j + \mathrm{ord}P_j)$ *is called the total order of* P *and denoted by* $\mathrm{ord}^T(P(s))$. *Let* $\ell = \mathrm{ord}^T(P(s))$. *Then we call*

$$\sigma^T(P) := \sum s^j \sigma_{\ell-j}(P_j),$$

the total symbol of P, *where the* $\sigma_{\ell-j}(P_j)$ *is the* $(\ell-j)$*-order part of* P_j. *For an ideal* $\mathfrak{I}(s) \in \mathfrak{D}_X[s]$, *we define its total symbol ideal by*

$$\sigma^T(\mathfrak{I}(s)) := \{\sigma^T(P) \mid P \in \mathfrak{I}(s)\}.$$

Let T^*X denote the cotangent bundle of X and $\mathfrak{I}$ be an ideal in $\mathcal{O}_{T^*X}[s]$ and S be a subset of $\mathbb{C} \times T^*X$. Then we denote by $V(\mathfrak{I})$ and $\mathfrak{I}(S)$ the zero set of $\mathfrak{I}$ and the ideal of functions that vanish on S, respectively.

Definition 5.3. *i) We define*

$$\overset{\vee}{S}S[s](\mathcal{L}) := V(\sigma^T(\mathfrak{I}(s))),$$

for a $\mathfrak{D}[s]$*-Module* $\mathcal{L} = \mathfrak{D}[s]/\mathfrak{I}(s)$. *More generally we define*

$$\overset{\vee}{S}S[s](\mathcal{L}) := \bigcup_{i=1}^{l} \overset{\vee}{S}S[s](\mathfrak{D}[s]u_i),$$

for finitely generated $\mathfrak{D}[s]$*-Module* $\mathcal{L} = \mathfrak{D}[s]u_1 + \cdots + \mathfrak{D}[s]u_l$.

ii)Let f *be a holomorphic function. The subset* $W[s]$ *in* $\mathbb{C} \times T^*X$ *is defined by*

$$W[s] = \{(s, x, s\,\mathrm{grad}\,\log f) \mid f \neq 0, s \in \mathbb{C}\}^{closure}.$$

With the same notation in Definition 5.3, we have

PROPOSITION 5.1 (Proposition 2.3, [Ya78]).

$$\overset{\vee}{S}S[s](\mathfrak{N}) = W[s].$$

Let $f \in \mathcal{O}_X$ be a function and $\mathcal{J}_f$ the Jacobi ideal of f. Furthermore we assume that $V(\mathcal{J}_f) \subset V(f)$ (note: if f define an isolated singularity at the origin, then f satisfy this condition obviously). We denote by $l(f)$ the degree of integral dependence of f over $\mathcal{J}_f$, whose existence is assured by the presence of

THEOREM 5.1 ([Hi64]). *f is integral over $\mathcal{J}_f$ (i.e., there exists $p(s,x,\xi) \in \mathcal{O}_X[s,\xi]$, homogeneous in (s,ξ) and $p(f,x,df) = 0$, $p(s,0,0) = s^{\deg p}$).*

COROLLARY 5.1 (Corollary 2.7, [Ya78]). *Let $f \in \mathcal{O}_X$ be a function and $V(\mathcal{J}_f) \subset V(f)$. There exists $P(s) = \sum_{j=0}^{l} s^{l-j} P_j(x,D)$ in $\mathfrak{I}(s)$ such that $\operatorname{ord}^T P = l$ (here l denotes $l(f)$ which is defined above), $P_0(x,D) = 1$.*

Proof. The $p(s,x,\xi)$ in Theorem 5.1 belongs to $\mathfrak{I}(W[s])$. It follows then from Proposition 5.1 that there exist k and $P(s) \in \mathfrak{I}[s]$ satisfying

$$\sigma^T(P(s)) = p^k.$$

Obviously, this $P(s)$ is an announced one. □

REMARK 5.1. *The existence of $P(s)$ with $P_0(x,D) = 1$ follows from Theorem 5.1 and some other deep results of D-module theory.*

We write $L(f)$ for the minimum of $\operatorname{ord}^T P$ where $P(s) \in \mathfrak{I}(s)$ which is of the form specified in Corollary 5.1.

PROPOSITION 5.2 (Corollary 3.8, [Sa74]). *When f has an isolated singularity, the condition $L(f) \geq 2$ is equivalent to the condition*

$$\det\left(\frac{\partial^2 f}{\partial z_i \partial z_j}\right) \in (f, \mathcal{J}_f)$$

REMARK 5.2. *In fact, Saito's original statement ([Sa74], Corollary 3.8) is as follows: Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ define an isolated singularity at 0, f is not quasi-homogeneous precisely when*

$$\det\left(\frac{\partial^2 f}{\partial z_i \partial z_j}\right) \in (f, \mathcal{J}_f).$$

In the case of $L(f) = 1$, We have the following theorem:

CHARACTERIZATION F ([Ya78]). Let $f \in \mathcal{O}_X$ such that $V(\mathcal{J}_f) \subset V(f)$, then f is quasi-homogeneous if and only if $L(f) = 1$.

6. A criterion for quasi-homogeneity in terms of module-theory. In this section, we shall give module-theoretic conditions guaranteeing the quasihomogeneity of an isolated Cohen-Macaulay singularity. From a broader perspective, the existence of such conditions demonstrates the close ties between "representation"theory and "structure" theory ([Ma91]).

Our rings are local analytic k-algebras, understood here as quotients of formal power series rings over a field k. An local analytic k-algebra (R, m) is called graded if there exist a system of generators $z_1, \cdots, z_n$ of the maximal ideal, positive integers $d_1, \cdots, d_n$, and a k-derivation $\delta : R \to R$ such that $\delta z_i = d_i z_i, i = 1, \cdots, n$. The elements of the eigenspace V_l are said to be homogeneous of degree l. (Unlike the case of affine algebras, the direct sum of V_l does not span the whole ring R.) The words "graded" and "quasi-homogeneous" are synonymous. If $d_i = 1, i = 1, \cdots, n$, the singularity is called homogeneous.

Definition 6.1. *Let (R, m, k) be a local ring. An R-module M is Cohen-Macaulay (CM) provided M is finitely generated and depth(M) = dim(M). The ring R is CM provided R is CM as an R-module. The module M is maximal Cohen-Macaulay (MCM) provided M is CM and depth(M) = dim(R).*

Definition 6.2. *Let X be an R-module. An MCM approximation of X is a short exact sequence $0 \to V \to M \xrightarrow{p} X \to 0$ of R-modules where M is MCM and V has finite injective dimension.*

Characterization G ([Ma91]). Let $P := k[[z_1, \cdots, z_n]]$ be the formal power series ring in n variables over a field k of characteristic 0, $f \in \underline{m}_P := (z_1, \cdots, z_n)$ a formal power series whose Jacobi ideal $\mathcal{J}(f)$ is $\underline{m}_P$-primary and $R := P/(f)$. Then the following are equivalent.

(1) The moduli algebra $R/\overline{\mathcal{J}(f)}$ (the overbar denotes the image in R) is Gorenstein.

(2) $f \in \mathcal{J}(f)$.

(3) $f \in \underline{m}_P \mathcal{J}(f)$.

(4) there exist an MCM module N without free summands and a surjection $\alpha : N \to R/\overline{\mathcal{J}(f)}$.

(5) The MCM approximation of the moduli algebra $R/\overline{\mathcal{J}(f)}$ has no free summands.

We shall give another characterization of quasi-homogeneity in dimension two by means of module theory. We firstly recall the definition of an almost split sequence.

Definition 6.3. *Let R be a ring. An exact sequence $0 \to A \to B \to C \to 0$ of R-modules is called almost split if*

1) it is not split

2) if $h : X \to C$ is not a split epimorphism then h can be lifted to B

3) if $t : A \to Y$ is not a split monomorphism then t can be extended to B.

Let (R, m, k) be a complete integrally closed nonregular local domain with dualizing module ω and let Ref(R) denote the category of finite generated reflexive R-modules. As was shown by Auslander [Au86], the category Ref(R) has almost split sequences and they can all be obtained, by means of a simple procedure, from the fundamental exact sequence

$$0 \to \omega \xrightarrow{q} A \xrightarrow{p} R \to k \to 0 \tag{6.1}$$

which is nothing but a representative of any nonzero element of $\mathrm{Ext}^2(k, \omega) \cong k$. The module A nowadays called the Auslander module, turns out to be reflexive. In fact all exact sequences representing nonzero elements in $\mathrm{Ext}^2(k, \omega) \cong k$ are isomorphic. The above sequence (6.1) has properties completely analogous to those of an almost split sequence. Namely, for any reflexive module X any map $g : X \to R$ which is not a split epimorphism can be lifted to A via p. Similarly, for any reflexive module Y, any map $g : \omega \to Y$ which is not a split monomorphism can be extended to A via q.

The module of Kähler differentials of R over k will be denoted by $D_k(R)$. This is a finitely generated R-module together with a k-derivation $d_{R/k} : R \to D_k(R)$ (whose image generates $D_k(R)$) having the universal factorization property with respect to the k-derivation of R into finitely generated R-modules. Such a module exists for any analytic k-algebra ([Sc68]). The bidual $D_k(R)^{**}$ of the module of Kähler differentials of R is called Zariski differentials, where "$*$" stands for $\mathrm{Hom}_R(-, R)$.

Characterization H ([He92]). Let k be an algebraically closed field of characteristic zero, (R, m) a two-dimensional analytic Gorenstein k-algebra with isolated singularity, $D_k(R)$ the module of Kähler differentials of R over k, and A the Auslander module of R. The following two conditions are equivalent:

(1) R is quasi-homogeneous.

(2) The natural homomorphism $D_k(R) \otimes_R k \to D_k(R)^{**} \otimes_R k$ is injective and $D_k(R)^{**} \cong A$.

7. A criterion for quasihomogeneity in terms of Jacobian syzygies. Let $V = V(f) = \{f = 0\}$ be a projective hypersurface having only isolated singularities. In this section, we shall give a characterization of quasi-homogeneity of these singularities in terms of the syzygies of the Jacobian ideal of f.

Let X be the complex projective space $\mathbb{P}^n$ and consider the associated graded polynomial algebra $S = \oplus_k S_k$ with $S_k = H^0(X, \mathcal{O}_X(k))$. For a nonzero section $f \in S_N$ with $N > 1$, we consider the hypersurface $V = V(f)$ in X given by the zero locus of f and let Y denote the singular locus of V, endowed with its natural scheme structure [Di13]. We assume in this section that V has isolated singularities.

Let $\mathcal{I}_Y \subset \mathcal{O}_X$ be the ideal sheaf defining this subscheme $Y \subset X$ and consider the graded ideal $I = \oplus_k I_k$ in S with $I_k = H^0(X, \mathcal{I}_Y(k))$. Let $Z = \mathrm{Spec}(S)$ be the corresponding affine space $\mathbb{A}^{n+1}$ and denote by $\Omega^k = H^0(Z, \Omega^k_Z)$ the S-module of global, regular k-forms on Z. Using a linear coordinate system $z = (z_0, \cdots, z_n)$ on X, one sees that there is a natural isomorphism of S-modules

$$\Omega^j = S^{\binom{n+1}{j}}$$

which is used to put a grading (independent of the choice of z) on the modules Ω^j, i.e. a differential form

$$\omega = \sum_K \omega_K(z) dz_K$$

is homogeneous of degree m if all the coefficients $\omega_K(z)$ are in S_m for all multi-indices $K = (k_1 < \cdots < k_j)$ and $dz_K = dz_{k_1} \wedge \cdots \wedge dz_{k_j}$.

Since f can be thought of as a homogeneous polynomial of degree N on Z, it follows that there is a well defined differential 1-form $df \in \Omega^1$. Using this we define two graded S-submodules in Ω^n, namely

$$AR(f) = \ker\{df\wedge : \Omega^n \to \Omega^{n+1}\}$$

and

$$KR(f) = \mathrm{im}\{df\wedge : \Omega^{n-1} \to \Omega^n\}.$$

If one computes in a coordinate system z, then $AR(f)$ is the module of all relations of the type

$$R : a_0 f_{z_0} + \cdots + a_n f_{z_n} = 0,$$

with f_{z_j} being the partial derivative of the polynomial f with respect to z_j. Moreover, $KR(f)$ is the module of Koszul relations spanned by obvious relations of the type $f_{z_j} f_{z_i} + (-f_{z_i}) f_{z_j} = 0$ and the quotient

$$ER(f) = AR(f)/KR(f)$$

is the module of essential relations (which is of course nothing else but the n-th cohomology group of the Koszul complex of $f_{z_0}, \cdots, f_{z_n}$), see [Di13]. We have the following characterization of the fact that the singularities of V are all weighted homogeneous.

Characterization I ([DS15]). Assume that the coordinates z have been chosen such that the hyperplane $H_0 : z_0 = 0$ is transversal to V. Consider the projection on the first factor

$$p_0 : AR(f)_k \to S_k/I_k, (a_0, \cdots, a_n) \to [a_0].$$

Then the equality

$$KR(f)_k = \ker p_0 \tag{7.1}$$

holds for any integer k if and only if all of the singularities of V are weighted homogeneous.

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