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KOHN–ROSSI COHOMOLOGY AND NONEXISTENCE OF CR MORPHISMS BETWEEN COMPACT STRONGLY PSEUDOCONVEX CR MANIFOLDS

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Dedicated to Professor H. Blaine Lawson, Jr. on the occasion of his 75th birthday

Abstract

One of the fundamental questions in CR geometry is: Given two strongly pseudoconvex CR manifolds X_1 and X_2 of dimension $2n-1$, is there a non-constant CR morphism between them? In this paper, we use Kohn–Rossi cohomology to show the non-existence of non-constant CR morphism between such two CR manifolds. Specifically, if $\dim H^{p,q}_{KR}(X_1) < \dim H^{p,q}_{KR}(X_2)$ for any (p,q) with $1 \leq q \leq n-2$, then there is no non-constant CR morphism from X_1 to X_2.

1. Introduction

CR manifolds are abstract models of complex manifolds' boundaries. The harmonic theory for the $\overline{\partial}_b$ complex on compact strongly pseudoconvex CR manifolds was developed by Kohn and Rossi [**Ko-Ro 65**]. Using this theory, on the one hand Boutet de Monvel [**Bo 75**] proved that if X is a compact strongly pseudoconvex CR manifold of real dimension $2n-1$, $n \geq 3$, then there exist C^∞ functions $f_1, \cdots f_N$ on X such that each $\overline{\partial}_b f_j = 0$ and $f = (f_1, \cdots, f_N)$ defines an embedding of X in $\mathbb{C}^N$. Thus, any compact strongly pseudoconvex CR manifold of dimension at least five can be CR embedded in some complex Euclidean space. On the other hand, 3-dimensional strongly pseudoconvex CR manifolds are not necessarily embeddable. Throughout this paper, our strongly pseudoconvex CR manifolds are always assumed to be compact orientable and embeddable in some $\mathbb{C}^N$. A beautiful theorem of Harvey and Lawson ([**Ha-La 75**], [**Ha-La 00**]) says that these CR manifolds are the boundaries of subvarieties with only isolated normal singularities. The ultimate goal in CR geometry is to determine whether any two given strongly pseudoconvex CR manifolds are CR biholomorphically equivalent. This is in general a very difficult problem. In order to prove existence of biholomorphism between two compact strongly pseudoconvex manifolds, one must first establish a non-trivial CR morphism

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from one to the other. In 2011, the first author of this paper [**Ya 11**] investigated the existence of non-trivial CR morphisms between strongly pseudoconvex CR manifolds using the singularity theory. In [**Ya 11**], Yau proved that there is no non-constant CR morphism from X_1 to X_2 if $p_g(X_1) < p_g(X_2)$, where p_g is the so-called geometric genus for any compact strongly pseudoconvex CR manifold [**Ya-Yu 02**]. Recently, Lin, Yau and Zuo [**LYZ 15**] generalized p_g and obtained a series of CR invariants p_m, called plurigenera of compact connected strongly pseudoconvex CR manifolds. The p_1 coincides with the previously defined p_g.

Theorem 1.1. *([**Ya 11**], [**LYZ 15**]) Let X_1 and X_2 be two compact connected $(2n-1)$-dimensional embeddable strongly pseudoconvex CR manifolds. If $p_m(X_1) < p_m(X_2)$ for any positive integer m, then there is no non-constant CR morphism from X_1 to X_2.*

The following theorem states that existence of non-constant CR morphism from X_1 to X_2 is very close to saying that X_1 is CR biholomorphic to X_2.

Theorem 1.2. *([**Ya 11**], [**TYZ 13**]) Let X_1 and X_2 be two compact strongly pseudoconvex CR manifolds of dimension $2n-1 \geq 5$ which bound complex varieties V_1 and V_2 with only isolated normal singularities in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively. Let S_1 and S_2 be the singular sets of V_1 and V_2, respectively, and S_2 is nonempty. Suppose $2n - N_2 - 1 \geq 1$. Then any nonconstant CR morphism from X_1 to X_2 is a covering map. If $|S_1|$ is not divisible by $|S_2|$ or $|S_1| \leq 2|S_2| - 1$, then any nonconstant CR morphism from X_1 to X_2 is necessarily a CR biholomorphism.*

In fact, Tu, Yau and Zuo proved the following surprising theorems.

Theorem 1.3. *([**TYZ 13**]) Let X_1 and X_2 be two $(2n-1)$-dimensional compact strongly pseudoconvex CR manifolds lying in a Stein variety V of dimension n in $\mathbb{C}^N$. Let $V_1 \subset V$, $V_2 \subset V$ with $\partial V_i = X_i$, $i = 1, 2$. Assume that the singular set S of V is nonempty and is equal to the singular set of $V_i, i = 1, 2$. Then nontrivial CR morphisms from X_1 to X_2 are necessarily CR biholomorphisms.*

Theorem 1.4. *([**TYZ 13**]) Let X be a compact strongly pseudoconvex CR manifold of dimension ≥ 3 lying in $\mathbb{C}^N$. Then any nonconstant CR morphism from X to itself must be a CR biholomorphism.*

Though Theorem 1.1 shows that the nonexistence of non-constant CR morphism from X_1 to X_2 can be characterized by the plurigenera of X_1 and X_2, there is no known method to compute them just using the information of the CR manifolds. In order to compute $p_m(X)$, one needs to solve the complex Plateau problem, i.e., find V such that $\partial V = X$. Theorem 1.2 is effective when the co-dimension of X_2 is small. Theorem 1.3 gives the best result, if X_1 and X_2 are in the same variety. In view of this, it is very desirable to have the following theorem.

Main Theorem A. *Let X_1 and X_2 be two compact connected $(2n-1)$-dimensional embeddable strongly pseudoconvex CR manifolds with $2n-1>3$. If $\dim H_{KR}^{p,q}(X_1) < \dim H_{KR}^{p,q}(X_2)$ for any (p,q) with $1 \le q \le n-2$, then there is no non-constant CR morphism from X_1 to X_2.*

When a compact connected strongly pseudoconvex CR manifold X of dimension $2n-1$ is the boundary of a hypersurface in $\mathbb{C}^{n+1}$ with isolated singularities, Yau [**Ya 81**] showed that certain $H_{KR}^{p,q}(X)$ carry the structure of an algebra. Specifically, each of the groups $H_{KR}^{p,q}(X)$, for $p+q=n-1$ or n, and $1 \le q \le n-2$, is isomorphic to the direct sum of the moduli algebras of the singular points of the variety. The Thom-Sebastiani properties of Kohn–Rossi cohomology of compact connected strongly pseudoconvex CR manifolds were investigated in [**YZ 17**]. In [**La-Ya 87**], Lawson and Yau proved the following theorem.

Theorem 1.5. *([**La-Ya 87**]) Let X_1 and X_2 be two compact connected embeddable strongly pseudoconvex CR manifolds of dimension $2n-1>3$ sitting inside $\mathbb{C}^{n+1}$. Suppose X_1 and X_2 admit holomorphic transversal S^1-actions and there exists an algebra isomorphism*

$$H_{KR}^{p,q}(X_1) \cong H_{KR}^{p,q}(X_2),$$

for any (p,q) with $1 \le q \le n-2$. Then there exists a diffeomorphism $f : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ with $f(X_1) = X_2$.

Theorem 1.6. *([**La-Ya 87**]) Let $X \subset \mathbb{C}^{n+1}$ be as above and suppose that $H_{KR}^{*,*}(X) = 0$. Then X is diffeomorphic to the standard sphere. Furthermore, if $X \subset S^{2n+1} = \{z \in \mathbb{C}^{n+1} : \| z \| = 1\}$, then this embedding is isotopic to the standard one.*

It is well known that the vanishing of singular cohomology $H^*(X) = 0$ is not sufficient for the above conclusion. The Brieskorn spheres

$$X_d = \{z \in \mathbb{C}^{2n} : \| z \| = 1 \text{ and } z_1^d + \sum_{k>1} z_k^2 = 0\}$$

are often exotic, and even when they are not, they are knotted in S^{4n-1}.

Main Theorem B. *Let X_1 and X_2 be two compact connected embeddable strongly pseudoconvex CR manifolds of dimension $2n-1>3$ sitting inside $\mathbb{C}^{n+1}$. Suppose X_1 and X_2 admit holomorphic transversal S^1-actions and an algebra isomorphism $H_{KR}^{p,q}(X_1) \cong H_{KR}^{p,q}(X_2)$ for any (p,q) with $1 \le q \le n-2$. If $H_{KR}^{p,q}(X_1) \neq 0$, then any non-constant CR morphism from X_1 to X_2 is necessarily a CR biholomorphism.*

In §2, we shall recall some basic notations and facts about CR manifolds. Also definitions of Kohn–Rossi cohomology groups are stated. In §3, we shall give the proofs of our main theorems.

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2. Kohn–Rossi's $\overline{\partial}_b$-complex

In this section, we recall Kohn–Rossi theory [**Ko-Ro 65**] for the $\overline{\partial}_b$-complex. An excellent reference for this section is [**Fo-Ko 72**]. Let M be a Hermitian complex manifold of complex dimension n with smooth boundary $X = \partial M$ such that $\overline{M} = M \cup X$ is compact. We shall assume, without loss of generality, that M is embedded in a slightly large open manifold M' and that $X = \partial M$ is defined by the equation $r = 0$ where r is a real function with $r < 0$ inside $M, r > 0$ outside $\overline{M}$, and $|dr| = 1$ on $X = \partial M$. Let $\mathfrak{A}^{p,q}(M)$ be the space of $C^\infty(p,q)$-forms on M. $\mathfrak{A}^{p,q}(\overline{M})$ is the subspace of $\mathfrak{A}^{p,q}(M)$ whose elements can be extended smoothly to $\overline{M}$. $\mathfrak{A}^{p,q}_c(M)$ is the subspace of $\mathfrak{A}^{p,q}(M)$ whose elements have compact support disjoint with X. Recall that a Hermitian metric on a complex manifold M is a Hermitian inner product $<,>_x$ on each $\pi_{1,0}(\mathbb{C}T_xM)$ varying smoothly in x, where $\pi_{1,0} : \mathbb{C}T_xM \to T_{1,0}M$ is the natural projection from the complexified tangent bundle to the subbundle consisting of the $(1,0)$ vectors. For $\xi, \eta \in \mathbb{C}T_xM$, we define

$$< \xi, \eta >_x = < \pi_{1,0}\xi, \pi_{1,0}\eta >_x + \overline{< \pi_{1,0}\xi, \pi_{1,0}\eta >_x}.$$

The inner product $<,>_x$ extends naturally to all the spaces $\Lambda^{p,q}\mathbb{C}T^*_xM$. If $\omega_1, \cdots, \omega_n$ is an orthonormal basis for $\Lambda^{1,0}\mathbb{C}T^*_xM$, then $\omega_1 \wedge \overline{\omega}_1 \wedge \cdots \wedge \omega_n \wedge \overline{\omega}_n = \gamma$ is the volume element on M at x. We define global scalar products for the forms by

$$(\phi, \psi) = \int_M < \phi, \psi > \gamma \ \text{ for } \ \phi, \psi \in \mathfrak{A}^{p,q}(M).$$

The formal adjoint of $\overline{\partial}$, denoted as ϑ, is the differential operator from $\mathfrak{A}^{p,q}(M)$ to $\mathfrak{A}^{p,q-1}(M)$ defined by $(\vartheta\phi, \psi) = (\phi, \overline{\partial}\psi)$, for all $\psi \in \mathfrak{A}^{p,q-1}_c(M)$. The operator $\square = \overline{\partial}\vartheta + \vartheta\overline{\partial}$ is called the complex Laplacian. Let $H^{p,q}_0$ be the space of square integrable (p,q)-forms on M. We shall use the symbol $\overline{\partial}$ to mean the closure of $\overline{\partial}\mid_{\mathfrak{A}^{p,q}(\overline{M})}$ with respect to $H^{p,q}_0$; in other words, the operator whose graph is the closure of the graph of $\overline{\partial}\mid_{\mathfrak{A}^{p,q}(\overline{M})}$ in $H^{p,q}_0 \times H^{p,q+1}_0$. The following proposition is obtained by integration by parts.

Proposition 2.1. *For all* $\phi \in \mathfrak{A}^{p,q}(\overline{M}), \theta \in \mathfrak{A}^{p,q+1}(\overline{M}), \psi \in \mathfrak{A}^{p,q-1}(\overline{M})$,

$$(\overline{\partial}\phi, \theta) = (\phi, \vartheta\theta) + \int_{\partial M} < \sigma(\overline{\partial}, dr)\phi, \theta >,$$

and

$$(\vartheta\phi, \psi) = (\phi, \overline{\partial}\psi) + \int_{\partial M} < \sigma(\vartheta, dr)\phi, \psi >,$$

where $\sigma(\overline{\partial}, dr)$ and $\sigma(\vartheta, dr)$ are the symbols of the differential operator $\overline{\partial}$ and ϑ at dr, respectively.

The relation between the Hilbert space adjoint of $\overline{\partial}$, denoted as $\overline{\partial}^*$, and its formal adjoint ϑ is given by the following proposition. Recall that $\overline{\partial}^*$, is defined on the domain $\mathrm{Dom}(\overline{\partial}^*)$ consisting of all $\phi \in H_0^{p,q}$ such that for some constant $c > 0, |(\phi, \overline{\partial}\psi)| \leq c\|\psi\|$ for all $\psi \in \mathfrak{A}_c^{p,q-1}(M)$. For such a ϕ, the $\psi \to (\phi, \overline{\partial}\psi)$ extends to a bounded functional on $H_0^{p,q}$ and $\overline{\partial}^*\phi$ its dual vector.

Proposition 2.2. *Let* $\mathfrak{D}^{p,q} = \mathrm{Dom}(\overline{\partial}^*) \cap \mathfrak{A}^{p,q}(\overline{M})$. *Then*

$$\mathfrak{D}^{p,q} = \{\phi \in \mathfrak{A}^{p,q}(\overline{M}) : \sigma(\vartheta, dr)\phi = 0 \text{ on } \partial M\},$$

and

$$\overline{\partial}^* = \vartheta \text{ on } \mathfrak{D}^{p,q}.$$

Definition 2.1. For each $p \in \partial M$, the Levi form at p is the Hermitian form on the $(n-1)$-dimensional space $(\pi_{1,0}\mathbb{C}T_pM) \cap \mathbb{C}T_p\partial M$ given by

$$(L_1, L_2) \to 2 < \partial\overline{\partial}r, L_1 \wedge \overline{L}_2 > .$$

(It is Hermitian because $\partial\overline{\partial} = -\overline{\partial}\partial = -\overline{\partial\overline{\partial}}$.)

We shall work in special boundary charts U, with the special basis, $\{\omega_i\}, 1 \leq i \leq n$, $\omega_n = \sqrt{2}\partial r$ for $\mathfrak{A}^{1,0}(U)$. Let $L_1, \cdots, L_n$ be the dual vector fields. Then $\{(L_i)_p\}, 1 \leq i \leq n-1$, is an orthonormal basis of the space $(\pi_{1,0}\mathbb{C}T_pM) \cap \mathbb{C}T_p\partial M$ and the Levi form is defined with respect to this basis is given by the matrix coefficients of the Levi form $c_{ij} = 2 < \partial\overline{\partial}r, L_i \wedge \overline{L}_j >$.

Proposition 2.3. *The number of nonzero eigenvalues and the absolute value of the signature of the Levi form (c_{ij}) at each point $p \in \partial M$ are independent of the choice of $L_1, \ldots, L_n$.*

Definition 2.2. (a) M is said to be pseudoconvex (pseudoconcave) if the Levi form is positive (negative) semi-definite at each point of ∂M and strongly pseudoconvex (pseudoconcave) if it is positive (negative) definite at each point of ∂M.

(b) We say that M satisfies the condition $Z(q)$ if the Levi form has at least $n-q$ positive eigenvalues or at least $q+1$ negative eigenvalues at each point of ∂M. (Thus, a strongly pseudoconvex manifold satisfies the condition $Z(q)$ for all $q > 0$.)

Suppose H is a Hilbert space and Q is a Hermitian form defined on a dense subspace D of H satisfying $Q(\phi,\phi)\geq\|\phi\|^2$ for $\phi\in D$. Assume further that D is a Hilbert space under the inner product Q. Then there is a canonical self-adjoint operator F on H associated with Q such that for each $\alpha\in H, \psi\to(\alpha,\psi)$ is a Q-bounded functional on D. Thus, there is a unique $\phi\in D$ such that $Q(\phi,\psi)=(\alpha,\psi)$ for all $\psi\in D$. Define $T: H\to D\subset H$ by $T\alpha=\phi$. Then T is bounded, self-adjoint and injective. By setting $F=T^{-1}$, we have the following famous Friedrichs Extension Theorem.

Theorem 2.1. *F is the unique self-adjoint operator with Dom$(F)\subseteq D$ satisfying $Q(\phi,\psi)=(F\phi,\psi)$ for all $\phi\in\mathrm{Dom}(F)$ and $\psi\in D$.*

In our case, we define the form Q on $\mathfrak{D}^{p,q}$ by

$$Q(\phi,\psi)=(\overline{\partial}\phi,\overline{\partial}\psi)+(\vartheta\phi,\vartheta\psi)+(\phi,\psi),$$

and let $\tilde{\mathfrak{D}}^{p,q}$ be the completion of $\mathfrak{D}^{p,q}$ under Q. The inclusion $\mathfrak{D}^{p,q}\to H_0^{p,q}$ extends uniquely to a norm-decreasing map $\tilde{\mathfrak{D}}^{p,q}\to H_0^{p,q}$. This map is injective. Hence, we can identify $\tilde{\mathfrak{D}}^{p,q}$ with a subspace of $H_0^{p,q}$ and apply the Friedrichs construction. We denote the Friedrichs operator associated to Q by F. Since for $\phi,\psi\in\mathfrak{A}_c^{p,q}(M)$, $Q(\phi,\psi)=((\Box+I)\phi,\psi)$, we see that F is a self-adjoint extension of the Hermitian operator $(\Box+I)\mid_{\mathfrak{A}_c^{p,q}(M)}$. The smooth elements of $\tilde{\mathfrak{D}}^{p,q}$ are described by the boundary condition $\sigma(\vartheta,dr)\phi=0$ on ∂M, while the smooth elements of $\mathrm{Dom}(F)$ are characterized by a further first order boundary condition (the so-called "free boundary condition").

Proposition 2.4. *If $\phi\in\mathfrak{D}^{p,q}$, then $\phi\in\mathrm{Dom}(F)$ if and only if $\overline{\partial}\phi\in\mathfrak{D}^{p,q+1}$ in which case $F\phi=(\Box+I)\phi$.*

Let $\Box_F=F-I$ and let the harmonic space $\mathfrak{H}^{p,q}=\mathcal{N}(\Box_F)$ be the nullspace of the operator $\Box_F$. Kohn [[**Fo-Ko 72**], p. 51] proved that the harmonic space $\mathfrak{H}^{p,q}$ is a finite-dimensional subspace of $\mathfrak{A}^{p,q}(\overline{M})$ provided M satisfies the condition $Z(q)$. As a consequence of his beautiful solution of the $\overline{\partial}$-Neumann problem, Kohn proved the following:

Theorem 2.2. *If M satisfies the condition $Z(q)$, then $H^{p,q}(\overline{M})\cong\tilde{H}^{p,q}(M)\cong\mathfrak{H}^{p,q}$, where*

$$H^{p,q}(\overline{M})=\frac{\{\phi\in\mathfrak{A}^{p,q}(\overline{M}):\overline{\partial}\phi=0\}}{\overline{\partial}\mathfrak{A}^{p,q-1}(\overline{M})},$$

and

$$\tilde{H}^{p,q}(M)=\frac{\{\phi\in H_0^{p,q}\cap\mathrm{Dom}(\overline{\partial}):\overline{\partial}\phi=0\}}{\overline{\partial}(H_0^{p,q-1}\cap\mathrm{Dom}(\overline{\partial}))}.$$

On the other hand, the Dolbeault Theorem asserts that

$$H^{p,q}(M)=\frac{\{\phi\in\mathfrak{A}^{p,q}(M):\overline{\partial}\phi=0\}}{\overline{\partial}\mathfrak{A}^{p,q-1}(M)}$$

is isomorphic to $H^q(M, \Omega^p)$, where Ω^p is the sheaf of germs of holomorphic p-forms. The relationship between these important groups and the previous one is due to Hormander [**Ho 65**].

Theorem 2.3. *If M satisfies the condition $Z(q)$ and $Z(q+1)$, then $H^q(M, \Omega^p) \cong \mathfrak{H}^{p,q}$.*

Definition 2.3. Let $\mathfrak{C}^{p,q} = \{\phi \in \mathfrak{A}^{p,q}(\overline{M}) : \overline{\partial} r \wedge \phi = 0 \ \text{ on } \ \partial M\}$, which can be also written as

$$\mathfrak{C}^{p,q} = \{\phi \in \mathfrak{A}^{p,q}(\overline{M}) : \sigma(\overline{\partial}, dr)\phi = 0 \ \text{ on } \ \partial M\},$$

since $\sigma(\overline{\partial}, dr) = \overline{\partial} r \wedge (\ \cdot\ )$.

Recall that the Hodge star operator $* : \mathfrak{A}^{p,q}(\overline{M}) \to \mathfrak{A}^{n-q,n-p}(\overline{M})$ is defined by $\psi \wedge *\phi =< \psi, \overline{\phi} > \gamma$, where γ is the volume form on M. It is not hard to prove the properties that $** = (-1)^{p+q}, *\overline{\phi} = \overline{*\phi}$ and $\vartheta = - * \partial *$. There is a duality of the space $\mathfrak{C}^{p,q}$ and $\mathfrak{D}^{p,q}$, and the spaces $\mathfrak{C}^{p,q}$ form a complex under $\overline{\partial}$.

Proposition 2.5. $\mathfrak{C}^{p,q} = *\overline{\mathfrak{D}^{n-p,n-q}}$ *and* $\overline{\partial}\mathfrak{C}^{p,q} \subset \mathfrak{C}^{p,q+1}$.

We may, therefore, form the cohomology

$$H^{p,q}(\mathfrak{C}) = \{\phi \in \mathfrak{C}^{p,q} : \overline{\partial}\phi = 0\}/\overline{\partial}\mathfrak{C}^{p,q-1}.$$

In [**Ko-Ro 65**], Kohn–Rossi introduced the zero-boundary-value cohomology

$$H^{p,q}(0) = \frac{\{\phi \in \mathfrak{A}^{p,q}(\overline{M}) : \overline{\partial}\phi = 0, \phi \mid_{\partial M} = 0\}}{\overline{\partial}\{\phi \in \mathfrak{A}^{p,q-1}(\overline{M}) : \phi \mid_{\partial M} = 0, \overline{\partial}\phi \mid_{\partial M} = 0\}}.$$

Proposition 2.6. $H^{p,q}(\mathfrak{C}) = H^{p,q}(0)$.

Kohn–Rossi [**Ko-Ro 65**] also proved the following important Kohn–Rossi duality on pseudo-convex manifolds.

Proposition 2.7. *If M satisfies the condition $Z(q)$, $H^{p,q}(\overline{M})$ is naturally dual to $H^{n-p,n-q}(\mathfrak{C})$. In particular, $H^{n-p,n-q}(0) \cong (H^{p,q}(\overline{M}))^*$.*

Following [**Fo-Ko 72**], we now introduce space $\mathfrak{B}^{p,q}$ of forms on ∂M, according to the following equivalent definitions:

(1) $\mathfrak{B}^{p,q}$ is the space of (smooth) sections of the vector bundle $\wedge^{p,q}\mathbb{C}T^*M \cap \wedge^{p+q}\mathbb{C}T^*\partial M$ on ∂M.

(2) $\mathfrak{B}^{p,q}$ is the space of (p,q)-forms restricted to ∂M, which are pointwisely orthogonal to the ideal generated by $\overline{\partial} r$ (i.e., to all forms of the type $\overline{\partial} r \wedge \theta$).

(3) $\mathfrak{B}^{p,q}$ is the space of restrictions of elements $\mathfrak{D}^{p,q}$ to ∂M.

(4) Let $\tilde{\mathfrak{A}}^{p,q}$ and $\tilde{\mathfrak{C}}^{p,q}$ denote the sheaves of germs of $\mathfrak{A}^{p,q}$ and $\mathfrak{C}^{p,q}$ on $\overline{M}$, respectively. Then there is a natural injection:

$$0 \to \tilde{\mathfrak{C}}^{p,q} \to \tilde{\mathfrak{A}}^{p,q}.$$

The quotient sheaf $\tilde{\mathfrak{B}}^{p,q} = \tilde{\mathfrak{A}}^{p,q}/\tilde{\mathfrak{C}}^{p,q}$ is a locally free sheaf supported on ∂M, and $\mathfrak{B}^{p,q}$ is its space of sections.

In view of Proposition 2.5, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{\mathfrak{C}}^{p,q} & \longrightarrow & \tilde{\mathfrak{A}}^{p,q} & \longrightarrow & \tilde{\mathfrak{B}}^{p,q} & \longrightarrow & 0 \\ & & \downarrow \overline{\partial} & & \downarrow \overline{\partial} & & \downarrow \overline{\partial}_b & & \\ 0 & \longrightarrow & \tilde{\mathfrak{C}}^{p,q+1} & \longrightarrow & \tilde{\mathfrak{A}}^{p,q+1} & \longrightarrow & \tilde{\mathfrak{B}}^{p,q+1} & \longrightarrow & 0, \end{array}$$

where $\overline{\partial}_b$ is the quotient map induced by $\overline{\partial}$. Actually, $\overline{\partial}_b$ may be explicitly described on sections as follows: if $\phi \in \mathfrak{B}^{p,q}$, choosing $\phi' \in \mathfrak{A}^{p,q}$ such that $\phi' \mid_{\partial M} = \phi$, then $\overline{\partial}_b \phi$ is the projection of $\overline{\partial}\phi' \mid_{\partial M}$ onto $\mathfrak{B}^{p,q}$. Since $\overline{\partial}^2 = 0$, it follows that $\overline{\partial}_b^2 = 0$, so we have the boundary complex

$$0 \longrightarrow \mathfrak{B}^{p,0} \xrightarrow{\overline{\partial}_b} \mathfrak{B}^{p,1} \xrightarrow{\overline{\partial}_b} \cdots \xrightarrow{\overline{\partial}_b} \mathfrak{B}^{p,n-1} \longrightarrow 0.$$

Definition 2.4. The cohomology of the above boundary complex is called Kohn–Rossi cohomology and is denoted by $H^{p,q}_{KR}(\partial M)$.

Following [**Ta 75**], we now reformulate the definition in a way independent of the interior manifold.

Definition 2.5. Let X be a connected orientable manifold of real dimension $2n-1$. A CR-structure on X is an $(n-1)$-dimensional subbundle S of $\mathbb{C}T(X)$ (complexified tangent bundle) such that

1) $S \cap \bar{S} = \{0\}$,
2) If L, L' are local sections of S, then so is $[L, L']$.

A manifold with a CR structure is called a CR manifold.

There is a unique subbundle $\mathcal{H}$ of $T(X)$ such that

$$\mathbb{C}\mathcal{H} = S \oplus \bar{S}.$$

Furthermore, there is a unique homomorphism $J : \mathcal{H} \to \mathcal{H}$ such that

$$J^2 = -1.$$

The pair $(\mathcal{H}, J)$ is called the real expression of the CR structure.

Definition 2.6. With the notations in Definition 2.5, a smooth S^1-action on X is said to be holomorphic if it preserves the subbundle $\mathcal{H} \subset T(X)$ and commutes with J. It is said to be transversal if, in addition, the vector field $\mathcal{V}$ which generates the action, is transversal to $\mathcal{H}$ at all points of X.

Let $\{\mathcal{A}^k(X), d\}$ be the De Rham complex of X with complex coefficients, and let $H^k(X)$ be the De Rham cohomology groups. There is a natural filtration of the De Rham complex. In fact, for any integer p

and k, let $A^k(X) = \wedge^k(\mathbb{C}T(X)^*)$ and $F^p(A^k(X))$ be the subbundle of $A^k(X)$ consisting of all $\phi \in A^k(X)$ satisfying the equality

$$\phi(Y_1, \dots, Y_{p-1}, \bar{Z}_1, \dots, \bar{Z}_{k-p+1}) = 0,$$

for all $Y_1, \dots, Y_{p-1} \in \mathbb{C}T(X)_0$ and $Z_1, \dots, Z_{k-p+1} \in S_0$, and 0 being the origin of ϕ. Then

$$\begin{aligned} A^k(X) =& F^0(A^k(X)) \supset F^1(A^k(X)) \supset \cdots \supset F^k(A^k(X)) \\ & \supset F^{k+1}(A^k(X)) = 0. \end{aligned}$$

Setting $F^p(\mathcal{A}^k(X)) = \Gamma(F^p(A^k(X)))$, we have

$$\begin{aligned} \mathcal{A}^k(X) =& F^0(\mathcal{A}^k(X)) \supset F^1(\mathcal{A}^k(X)) \supset \cdots \supset F^k(\mathcal{A}^k(X)) \\ & \supset F^{k+1}(\mathcal{A}^k(X)) = 0. \end{aligned}$$

Since clearly $dF^p(\mathcal{A}^k(X)) \subseteq F^p(\mathcal{A}^{k+1}(X))$, the collection $\{F^p(\mathcal{A}^k(X))\}$ gives a filtration of the De Rham complex.

We denote $H^{p,q}_{KR}(X)$ be the groups $E^{p,q}_1(X)$ of the spectral sequence $\{E^{p,q}_r(X)\}$ associated with the filtration $\{F^p(\mathcal{A}^k(X))\}$. $H^{p,q}_{KR}(X)$ is the Kohn–Rossi cohomology group of type (p, q). More explicitly, let

$$A^{p,q}(X) = F^p(A^{p+q}(X)), \mathcal{A}^{p,q}(X) = \Gamma(A^{p,q}(X)),$$

and

$$C^{p,q}(X) = A^{p,q}(X)/A^{p+1,q-1}(X), \mathcal{C}^{p,q}(X) = \Gamma(C^{p,q}(X)).$$

Since $d : \mathcal{A}^{p,q}(X) \longrightarrow \mathcal{A}^{p,q+1}(X)$ maps $\mathcal{A}^{p+1,q-1}(X)$ into $\mathcal{A}^{p+1,q}(X)$, it induces an operator $d'' : \mathcal{C}^{p,q}(X) \longrightarrow \mathcal{C}^{p,q+1}(X)$. $H^{p,q}_{KR}(X)$ are then the cohomology groups of the complex $\{\mathcal{C}^{p,q}(X), d''\}$.

3. Proofs of the Main Theorems

Let us first recall the following fundamental theorem of Harvey–Lawson.

Theorem 3.1. *(*[**Ha-La 75**], [**Ha-La 00**]*) For any compact connected embeddable strongly pseudoconvex CR manifold X, there is a unique complex variety V in $\mathbb{C}^N$ for some N such that the boundary of V is X and V has only normal isolated singularities.*

The following proposition was the starting point of our investigation. It can be found in [**Ya 11**]. It was proved by using the results of Fornaess [**Fo 76**] and Prill [**Pr 67**].

Proposition 3.1. *Let X_1 and X_2 be two compact strongly pseudoconvex CR manifolds of dimension $2n-1 > 1$ which bound complex varieties V_1 and V_2 in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively. Suppose the singular set S_i of V_i, $i = 1, 2$, is either an empty set or a set consisting of only isolated normal singularities. If $\Phi\colon X_1 \longrightarrow X_2$ is a non-constant CR morphism,* *then Φ is surjective and it can be extended to a proper surjective holomorphic map from V_1 to V_2 such that $\Phi(S_1) \subseteq S_2$, $\Phi^{-1}(X_2) = X_1$ and $\Phi\colon V_1 - \Phi^{-1}(S_2) \longrightarrow V_2 - S_2$ is a covering map. Moreover, if S_2 does not have quotient singularity, then $\Phi^{-1}(S_2) = S_1$.*

Proof of the Main Theorem A. For $i = 1, 2$, let V_i be the complex variety in $\mathbb{C}^{N_i}$ as shown in Theorem 3.1 such that the boundary of V_i is X_i. Let S_i be the singular set of V_i. Then S_i consists of only finite number of isolated normal singularities. Let M_i be a resolution of singularities of V_i with exceptional set A_i.

Following [**La 72**], we consider the sheaf cohomology with support at infinity. Let us recall briefly the definition.

In general, let X be any compact connected strongly pseudoconvex CR manifold which bounds a complex variety V in $\mathbb{C}^N$ with only normal isolated singularities S. Let M be resolution of singularities of V with exceptional set A. Take a 1-convex exhaustion function ϕ on M such that $\phi \geq 0$ on M and $\phi(y) = 0$ if and only if $y \in A_i$. Let $M_r = \{y \in M : \phi(y) \leq r\}$. Then by Laufer [**La 72**],

$$\varinjlim_{r} H^q(M - M_r, \Omega^p) \cong H^q_\infty(M, \Omega^p).$$

On the other hand, by Andreoti and Grauert (Théorème 15, [**An-Gr 62**]), $H^q(M - A, \Omega^p)$ is isomorphic to $H^q(M - M_r, \Omega^p)$ for $q \leq n - 2$ and $H^{n-1}(M - A, \Omega^p) \to H^{n-1}(M - M_r, \Omega^p)$ is injective.

Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{A}_c^{p,*} & \longrightarrow & \mathfrak{A}^{p,*}(\overline{M}) & \longrightarrow & \overline{\mathfrak{A}}_\infty^{p,*} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{A}_c^{p,*} & \longrightarrow & \mathfrak{A}^{p,*}(M) & \longrightarrow & \mathfrak{A}_\infty^{p,*} & \longrightarrow & 0. \end{array}$$

It follows from Theorem 2.2, Theorem 2.3 and the five lemma that

$$H^q(\overline{\mathfrak{A}}_\infty^{p,*}) \cong H^q(\mathfrak{A}_\infty^{p,*}),\ q \geq 1. \tag{3.1}$$

We claim that the natural inclusion map ι from $\mathfrak{A}_c^{p,*}$ to $\mathfrak{C}^{p,*}$ induces isomorphisms from $H^q(\mathfrak{A}_c^{p,*})$ to $H^q(\mathfrak{C}^{p,*})$ for $1 \leq q \leq n-1$. To see this, recall that, on the other hand, $H^q(\mathfrak{A}_c^{p,*})$ is Serre dual to $H^{n-q}(M, \Omega^{n-p})$ by integration pairing. On the other hand, $H^q(\mathfrak{C}^{p,*})$ is Kohn–Rossi dual to $H^{n-q}(\overline{M}, \Omega^{n-p})$ and, hence, to $H^{n-q}(M, \Omega^{n-p})$ for $q \leq n-1$, again by integration pairing (cf. Propositions 2.6 and 2.7). Our claim follows easily, due to the fact that ι is compatible with these integration pairings. Now the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{A}_c^{p,*} & \longrightarrow & \mathfrak{A}^{p,*}(\overline{M}) & \longrightarrow & \overline{\mathfrak{A}}_\infty^{p,*} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{C}^{p,*} & \longrightarrow & \mathfrak{A}^{p,*}(\overline{M}) & \longrightarrow & \mathfrak{B}^{p,*} & \longrightarrow & 0 \end{array}$$

gives

$$H^q(\overline{\mathfrak{A}}_\infty^{p,*}) = H^q(\mathfrak{B}^{p,*}) := H_{KR}^{p,q}(X). \tag{3.2}$$

Combining (3.1) and (3.2), we have shown that for $1 \leq q \leq n-2$

$$H_{KR}^{p,q}(X) \cong H^q(M - A, \Omega^p) \cong H^q(V - S, \Omega^p).$$

Thus, in our case, we have for $1 \leq q \leq n-2$

$$H_{KR}^{p,q}(X_i) \cong H^q(M_i - A_i, \Omega^p) \cong H^q(V_i - S_i, \Omega^p). \tag{3.3}$$

Suppose on the contrary that there is a non-constant CR morphism Φ from X_1 to X_2. In view of Proposition 3.1, Φ can be extended to a proper surjective holomorphic map from V_1 to V_2 such that $\Phi(S_1) \subseteq S_2$, $\Phi^{-1}(X_2) = X_1$ and $\Phi : V_1 - \Phi^{-1}(S_2) \longrightarrow V_2 - S_2$ is a covering map. We claim that the induced map

$$\Phi^* : H^q(V_2 - S_2, \Omega^P) \longrightarrow H^q(V_1 - \Phi^{-1}(S_2), \Omega^P) \tag{3.4}$$

is injective. To see this, let $(\alpha^{p,q})$ be an element in $H^q(V_2 - S_2, \Omega^P)$, where $(\alpha^{p,q})$ is a $\overline{\partial}$-closed (p,q)-form. Suppose that $\Phi^*(\alpha^{p,q})$ is zero in $H^q(V_1 - \Phi^{-1}(S_2), \Omega^p)$. Then there exists $\beta^{p,q-1}$ which is a $(p, q-1)$ form on $V_1 - \Phi^{-1}(S_2)$ such that $\overline{\partial}\beta^{p,q-1} = \Phi^*(\alpha^{p,q})$. Since $\Phi\colon V_1 - \Phi^{-1}(S_2) \longrightarrow V_2 - S_2$ is a covering map, any form γ on $V_1 - \Phi^{-1}(S_2)$ can be pushed down via the trace map Φ_* to become a form $\Phi_*(\gamma)$ on $V_2 - S_2$ as follows. Take an open cover $\{U_j\}$ of $V_2 - S_2$ such that $\Phi^{-1}(U_j)$ is a disjoint union of $W_j^1, W_j^2, \cdots, W_j^d$, where each $W_j^i, i = 1, 2, \cdots, d$ is biholomorphic to U_j via Φ and d is the degree of the covering map Φ. Then $\Phi_*(\gamma)$ on U_j is defined as the sum of the pull back of γ restricted on $W_j^1, W_j^2, \cdots, W_j^d$ via the local inverse Φ^{-1}. For the form $\alpha^{p,q}$ defined on $V_2 - S_2$, it is clear that $\Phi_*\Phi^*(\alpha^{p,q}) = d\alpha^{p,q}$. It follows that

$$\overline{\partial}\Phi_*\beta^{p,q-1} = \Phi_*\overline{\partial}\beta^{p,q-1} = \Phi_*\Phi^*(\alpha^{p,q}) = d\alpha^{p,q}.$$

As a result, $\alpha^{p,q}$ is zero in $H^q(V_2 - S_2, \Omega^P)$. This completes the proof that the map Φ^* in (3.4) is injective.

We next prove that $H^q(V_1 - \Phi^{-1}(S_2), \Omega^p)$ is naturally isomorphic to $H^q(V_1 - S_1, \Omega^p)$. In view of Proposition 3.1, $\Phi^{-1}(S_2)$ is a finite set containing S_1. In particular, $\Phi^{-1}(S_2) - S_1$ is a finite set of smooth point in V_1. It follows that

$$H^q(V_1 - S_1, \Omega^p) \cong H^q(V_1 - \Phi^{-1}(S_2), \Omega^p).$$

This together with (3.4), we obtain a natural injective map

$$\Phi^* : H^q(V_2 - S_2, \Omega^p) \hookrightarrow H^q(V_1 - S_1, \Omega^p). \tag{3.5}$$

In view of (3.3), we have a natural injective map

$$\Phi^* : H^{p,q}_{KR}(X_2) \hookrightarrow H^{p,q}_{KR}(X_1).$$

In particular, $\dim H^{p,q}_{KR}(X_2) \leq \dim H^{p,q}_{KR}(X_1)$, which is a contradiction to our hypothesis. q.e.d.

In preparation of proving the Main Theorem B, we need to recall the theory developed by Lawson and Yau [**La-Ya 87**].

Theorem 3.2. *(*[**La-Ya 87**]*) Let $X \subseteq \mathbb{C}^N$ be a CR manifold of dimension $2n-1 > 1$, and suppose that X admits a transversal holomorphic S^1-action. Then there exists a holomorphic equivariant embedding $X \hookrightarrow V$ as a hypersurface in a n-dimensional algebraic variety $V \subseteq \mathbb{C}^N$ with a linear $\mathbb{C}^*$-action.*

Corollary 3.1. *(*[**La-Ya 87**]*) Let $X \subseteq \mathbb{C}^{n+1}$ be a CR manifold of dimension $2n-1 > 1$, and suppose X admits a transversal holomorphic S^1-action. Then after a holomorphic change of coordinates in $\mathbb{C}^{n+1}$, X is contained in an affine algebraic hypersurface $V \subseteq \mathbb{C}^{n+1}$. The hypersurface V has at most one singular point. It also has a $\mathbb{C}^*$-action and the embedding $X \hookrightarrow V$ is S^1-equivariant.*

Proof of the Main Theorem B. In view of Corollary 3.1, X_i is contained in an affine algebraic hypersurface $V_i \in \mathbb{C}^{n+1}$. The hypersurface V_i has precisely one singular point x_i. It also has a $\mathbb{C}^*$-action and the embedding $X_i \hookrightarrow V_i$ is S^1-equivariant. Recall that for $p + q = n-1$ and $1 \leq q \leq n-2$, there is a natural isomorphism $H^{p,q}_{KR}(X_i) \cong A_{x_i}(V_i)$, where $A_{x_i}(V_i)$ is the moduli algebra of V_i at x_i, i.e., $A_{x_i}(V_i) = \mathbb{C}\{z_0, \cdots, z_n\}/(f, \frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n})$, where $f \in \mathbb{C}\{z_0, \cdots, z_n\}$ is the function, which defines the hypersurface V_i in a neighborhood of x_i (see [**Ya 81**]). Our main hypothesis now implies that there is an algebra isomorphism:

$$A_{x_1}(V_1) \cong A_{x_2}(V_2).$$

It then follows from a result of Mather and Yau [**Ma-Ya 82**] that there is a holomorphic change of coordinates $H : U_1 \to U_2$, where U_i is a neighborhood of x_i in $\mathbb{C}^{n+1}$, so that

$$H(x_1) = x_2 \ \text{ and } \ H(U_1 \cap V_1) = U_2 \cap V_2.$$

As a consequence, we may assume that X_1 and X_2 are lying in the same variety as in the statement of Theorem 1.3. Therefore, any non-constant CR morphism from X_1 to X_2 is necessarily a CR-biholomorphism by Theorem 1.3. q.e.d.

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