A UNIFIED QUANTIZATION OF GRAVITY AND OTHER FUNDAMENTAL FORCES OF NATURE IMPLIES A BIG BANG ON THE QUANTUM LEVEL

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Abstract. In a recent paper we presented a model for a unified quantization of gravity with other fundamental sources of nature. After quantization we had to solve a Wheeler-DeWitt equation which was a hyperbolic equation in a fiber bundle which was equipped with a Lorentzian metric with a time function t ranging from 0 to infinity. The Lorentzian metric had a big bang singularity in $t = 0$ and the coefficients of the hyperbolic operator also inherited this singularity. The solutions of the Wheeler-DeWitt equation were products of spatial and temporal eigenfunctions and in order to prove that these solutions also experience a big bang singularity in $t = 0$ the temporal eigenfunctions $w(t)$ had to become unbounded if t tends to 0. In our former paper this was only proved in special cases where w could be expressed with the help of Bessel functions but not in general. In the present paper we prove that also in the general case the temporal solutions become unbounded near $t = 0$, or more precisely: $\lim_{t\to 0}(|w|^2 + t^2|\dot w|^2) = \infty$ and $\limsup_{t\to 0}|w|^2 = \infty$.

Contents

1. Introduction

In a recent paper [2] we presented a unified quantization of gravity and the fundamental forces of the Standard Model. We worked in a fiber bundle E with base space $\mathcal{S}_0 = \mathbb{R}^3$ where the fiber elements were Riemannian metrics.

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The gravitational Hamilton operator $\hat{H}_G$ works in the fibers and the (spatial) Hamilton operator of the Standard Model $\hat{H}_{SM}$ works in the base space $\mathcal{S}_0$. Let v resp. ψ be spatial eigendistributions of the respective Hamiltonians then the solutions u of the Wheeler-DeWitt equation can be written as a product

$$u = wv\psi, \tag{1.1}$$

where $w = w(t)$ satisfies an ODE

$$\ddot{w} + mt^{-1}w + \mu_0 t^{-2}w + m_2 t^{-\frac{2}{3}}w + m_3 t^2 w = 0 \qquad \forall t > 0. \tag{1.2}$$

Here

$$\mu_0 = \frac{32}{3}(|\lambda|^2 + 1). \tag{1.3}$$

$$m_2 = \frac{32}{3}\alpha_n^{-1}\lambda_1, \tag{1.4}$$

$$m_3 = \frac{64}{3}\alpha_N^{-2}\varLambda \tag{1.5}$$

and

$$m = 5, \tag{1.6}$$

where $(|\lambda|^2 + 1)$is the eigenvalue of v, $\lambda_1 \ge 0$ the eigenvalue of ψ, $\varLambda \in \mathbb{R}$ a cosmological constant and $\alpha_N > 0$ a coupling constant, cf. [2, equ. (41)].

The solution w of the ODE is referred to as a temporal eigenfunction defined in $\mathbb{R}_+$. The ODE has a singularity in $t = 0$ and the behaviour of $w(t)$ when t tends to 0 will decide if this singularity is a big bang singularity on a quantum level.

The solutions of the equation (1.2) with given initial values $w(t_0)$, $\dot{w}(t_0)$ at $t_0 \in \mathbb{R}_+$ exist for all time $t > 0$ and are smooth. In general they cannot be expressed by known functions. However, if m_2 or m_3 are equal to zero then the corresponding solutions w can be expressed with the help of Bessel functions and hence it could be proved that

$$\limsup_{t\to 0} |w| = \infty, \tag{1.7}$$

cf. [2, Section 5].

In the present paper we want to prove that the singularity in $t = 0$ is always a big bang singularity and not only in the special cases mentioned above. Let us rewrite the ODE (1.2) in the form

$$\ddot{w} + mt^{-1}w + t^{-2}\{\mu_0 + m_2 t^{2--\frac{2}{3}} + m_3 t^4\}w = 0 \qquad \forall\, t > 0, \tag{1.8}$$

then we look at the more general equation

$$\ddot{w} + mt^{-1}w + t^{-2}(\mu_0 + q_0(t))w = 0 \qquad \forall\, t > 0 \tag{1.9}$$

and we shall prove:

Theorem 1.1. *Let us assume that the constants m, μ_0 and the real function $q_0 \in C^1(\mathbb{R}_+)$ have the properties*

$$m > 1, \tag{1.10}$$

$$1 < \mu_0 - \frac{(m-1)^2}{4} \equiv 1 + \gamma, \qquad \gamma > 0, \tag{1.11}$$

and

$$\lim_{t \to 0} q_0(t) = 0. \tag{1.12}$$

Then any non-trivial solution w of (1.9) *satisfies*

$$\lim_{t \to 0} (|w|^2 + t^2 |\dot{w}|^2) = \infty \tag{1.13}$$

as well as

$$\limsup_{t \to 0} |w|^2 = \infty. \tag{1.14}$$

Note that the differential operator in (1.9) is symmetric with respect to the bilinear form

$$\langle u, v \rangle = \int_0^\infty uvt^m dt, \qquad u, v \in C_c^\infty(\mathbb{R}_+), \tag{1.15}$$

since

$$\ddot{w} + mt^{-1}\dot{w} = t^{-m} \frac{\partial}{dt}(t^m \dot{w}). \tag{1.16}$$

2. Proof of Theorem 1.1

We first simplify the ODE by using the ansatz

$$w(t) = t^{-\frac{m-1}{2}} \varphi(\log t). \tag{2.1}$$

Defining the variable

$$\tau = \log t, \qquad t > 0, \tag{2.2}$$

we denote the derivatives with respect to τ by primes. Thus, we obtain

$$\dot{w} = t^{-\frac{m-1}{2}} \varphi' t^{-1} - \tfrac{m-1}{2} t^{-\frac{m-1}{2}-1} \varphi \tag{2.3}$$

and

$$\ddot{w} = t^{-\frac{m-1}{2}-2} \varphi'' - mt^{-\frac{m-1}{2}-2} \varphi' + \tfrac{m-1}{2}(\tfrac{m-1}{2}+1) t^{-\frac{m-1}{2}-2} \varphi \tag{2.4}$$

from which we conclude, in view of (1.9) on page 2,

$$\varphi'' + \{\mu_0 - \tfrac{(m-1)^2}{2} + q_0\} \varphi = 0. \tag{2.5}$$

Let us define

$$q = \mu_0 - \tfrac{(m-1)^2}{2} + q_0 = 1 + \gamma + q_0, \tag{2.6}$$

cf. (1.11). Moreover, it will be convenient to switch from the variable τ to $-\tau$ and then rename $-\tau$ to τ such that τ is now defined by

$$\tau = -\log t, \tag{2.7}$$

thus,

$$t \to 0 \quad \Longleftrightarrow \quad \tau \to \infty. \tag{2.8}$$

We then consider the equation (2.5) for large τ

$$\tau \ge \tau_0 >> 1 \tag{2.9}$$

such that

$$q - 1 \ge \frac{\gamma}{2} > 0 \qquad \forall\, \tau \ge \tau_0, \tag{2.10}$$

where we used the assumption (1.12) on page 3.

In order to analyze the asymptotic behaviour of φ when τ tends to infinity we shall employ the so-called *Prüfer Substitution*, i.e., we rewrite the equation

$$\varphi'' + q\varphi = 0, \qquad \tau \ge \tau_0, \tag{2.11}$$

as a system of first order equations and switch to polar coordinates (r, θ) such that

$$\varphi'(\tau) = r(\tau)\cos\theta(\tau) \tag{2.12}$$

and

$$\varphi(\tau) = r(\tau)\sin\theta(\tau). \tag{2.13}$$

Then we conclude, in view of (2.11),

$$\begin{aligned} r'(\tau) &= -(q-1)r(\tau)\cos\theta(\tau)\sin\theta(\tau) \\ &= -(q-1)r(\tau)\tfrac{1}{2}\sin 2\theta(\tau) \qquad\qquad \forall\, \tau \ge \tau_0 \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \theta'(\tau) &= q(\tau)\sin^2\theta(\tau) + \cos^2\theta(\tau) \\ &= (q-1)\sin^2\theta(\tau) + 1 \qquad\qquad \forall\, \tau \ge \tau_0, \end{aligned} \tag{2.15}$$

for details we refer to [1, p. 91]. In view of (2.10) we obtain

$$\theta'(\tau) \ge 1 \qquad \forall\, \tau \ge \tau_0, \tag{2.16}$$

i.e., θ is a diffeomorphism of class C^2. Moreover, we can estimate

$$\theta'(\tau) \le (q-1) + 1 = 1 + \gamma + q_0 \le 1 + 2\gamma \qquad \forall\, \tau \ge \tau_0, \tag{2.17}$$

if τ_0 is large enough because of (1.12).

The flow (2.15) has solution of class C^2 for given initial values at τ_0. Since we want to prove Theorem 1.1 for any non-trivial solution w, we have to analyze the asymptotic behaviour of φ for two linearly independent real valued solutions φ and $\tilde\varphi$ of (2.11). Let (r, θ) resp. $(\tilde r, \tilde\theta)$ be the corresponding polar coordinates and let $k_0 \in \mathbb{N}$ be arbitrary then we set

$$\theta(\tau_0) = 2k_0\pi \tag{2.18}$$

and

$$\tilde{\theta}(\tau_0) = 2k_0\pi + \frac{\pi}{2} \tag{2.19}$$

yielding, in view of (2.12) and (2.13),

$$\varphi(\tau_0) = 0, \tag{2.20}$$

$$\varphi'(\tau_0) = r(\tau_0) \tag{2.21}$$

and

$$\tilde{\varphi}(\tau) = \tilde{r}(\tau_0), \tag{2.22}$$

$$\tilde{\varphi}'(\tau_0) = 0. \tag{2.23}$$

Choosing now

$$r(\tau_0) = \tilde{r}(\tau_0) = 1, \tag{2.24}$$

we conclude that φ, $\tilde{\varphi}$ are linearly independent.

In the following we shall only consider φ since the arguments for $\tilde{\varphi}$ are identical. The asymptotic behaviour of φ is then determined by r. From (2.14) and (2.18) we infer

$$\log r(\tau) = \log r(\tau) - \log r(\tau_0) = -\frac{1}{2}\int_{\tau_0}^{\tau}(q-1)\sin 2\theta(s)ds. \tag{2.25}$$

Introducing a new integration variable

$$x = 2\theta(s) \tag{2.26}$$

such that

$$dx = 2\theta' ds = 2\{(q-1)\sin^2\theta + 1\}, \tag{2.27}$$

because of (2.15), we observe that the right-hand side of (2.25) is equal to

$$-\frac{1}{4}\int_{2\theta(\tau_0)}^{2\theta(\tau)} \frac{\sin x}{\sin^2\frac{1}{2}x + \frac{1}{q-1}}dx. \tag{2.28}$$

We now employ

$$\frac{1}{q-1} - \frac{1}{\gamma} = \frac{\gamma - q + 1}{(q-1)\gamma} = -\frac{q_0}{(q-1)\gamma} \tag{2.29}$$

and infer

$$\begin{aligned} &\frac{\sin x}{\sin^2\frac{1}{2}x + \frac{1}{q-1}} - \frac{\sin x}{\sin^2\frac{1}{2}x + \frac{1}{\gamma}} \\ &= \sin x \frac{\frac{1}{q-1} - \frac{1}{\gamma}}{(\sin^2\frac{1}{2}x + \frac{1}{q-1})(\sin^2\frac{1}{2}x + \frac{1}{\gamma})} \\ &= -\frac{\sin x}{(\sin^2\frac{1}{2}x + \frac{1}{q-1})(\sin^2\frac{1}{2}x + \frac{1}{\gamma})}\frac{q_0}{(q-1)\gamma} \\ &= fq_0, \end{aligned} \tag{2.30}$$

where f is a uniformly bounded function in $\{\tau \ge \tau_0\}$. Hence we deduce

$$-\frac{1}{4}\int_{2\theta(\tau_0)}^{2\theta(\tau)} \frac{\sin x}{\sin^2 \frac{1}{2}x + \frac{1}{q-1}} dx = -\frac{1}{4}\int_{2\theta(\tau_0)}^{2\theta(\tau)} \frac{\sin x}{\sin^2 \frac{1}{2}x + \frac{1}{\gamma}} dx + R, \tag{2.31}$$

where R satisfies the estimate

$$|R| \le c \sup_{[\tau_0,\infty)} |q_0|(\tau - \tau_0) \qquad \forall\, \tau \ge \tau_0, \tag{2.32}$$

in view of (2.17). The constant c is independent of τ_0 if τ_0 is large enough.

The integral on the right-hand side of (2.31) is equal to

$$-\frac{1}{4}\big(\log\big(\sin^2\theta(\tau) + \tfrac{1}{\gamma}\big) - \log\big(\sin^2\theta(\tau_0) + \tfrac{1}{\gamma}\big)\big) \tag{2.33}$$

and hence uniformly bounded in $[\tau_0, \infty)$. Thus, we infer from (2.25), (2.28), (2.31) and (2.32)

$$r(\tau) \ge c t^{\epsilon} \qquad \forall\, \tau \ge \tau_0, \tag{2.34}$$

where the constant c depends on τ_0 and where $0 < \epsilon$ can be chosen arbitrarily small depending on τ_0, in view of the assumption (1.12) on page 3. Let us also recall that

$$\tau = -\log t. \tag{2.35}$$

We are now ready to prove the claim (1.13) on page 3. First we use

$$w(t) = t^{-\frac{(m-1)}{2}} \varphi(-\log t) \tag{2.36}$$

and (2.13) to express w^2 in the form

$$w^2 = t^{-(m-1)}\varphi^2 = t^{-(m-1)} r^2(\tau) \sin^2\theta(\tau) \tag{2.37}$$

as well as

$$\begin{aligned} t^2 \dot w^2 &= t^2\{-\tfrac{m-1}{2} t^{-\frac{m-1}{2}-1}\varphi - t^{-\frac{m-1}{2}-1}\varphi'\}^2 \\ &= t^{-(m-1)} r^2(\tau)\{\tfrac{m-1}{2}\sin\theta(\tau) + \cos\theta(\tau)\}^2. \end{aligned} \tag{2.38}$$

Here we also used (2.14). Combining these relations yields

$$w^2 + t^2\dot w^2 = t^{-(m-1)} r^2(\tau)\{\sin^2\theta(\tau) + \big(\tfrac{m-1}{2}\sin\theta(\tau) + \cos\theta(\tau)\big)^2\}. \tag{2.39}$$

The term in the braces is strictly positive

$$\sin^2\theta(\tau) + \big(\tfrac{m-1}{2}\sin\theta(\tau) + \cos\theta(\tau)\big)^2 \ge c_0 > 0 \qquad \forall\, \tau \ge \tau_0 \tag{2.40}$$

as can be immediately verified by setting $x = \theta(\tau)$ and considering the sum above in the interval $[0, 2\pi]$. The claim (1.13) then follows from the estimate (2.34) if τ_0 is large enough such that ϵ can be chosen

$$0 < \epsilon < m - 1. \tag{2.41}$$

Finally, let us prove the claim (2.14) by choosing in (2.13) a sequence

$$t_k = e^{-\tau_k}, \quad \tau_k \to \infty, \tag{2.42}$$

such that

$$\theta(\tau_k) = 2k\pi + \frac{\pi}{2}, \qquad k_0 \le k \in \mathbb{N}. \tag{2.43}$$

For these arguments we obtain

$$\varphi'(\tau_k) = 0 \tag{2.44}$$

and

$$r^2(\tau_k) = \varphi^2(\tau_k) \tag{2.45}$$

which, together with (2.37) and (2.39), implies the result and completes the proof of Theorem 1.1.

3. The oscillation behaviour of temporal eigenfunctions near the big bang

Writing a real valued temporal eigenfunction in the form (2.36) on page 6 it is evident that the corresponding function

$$\varphi = \varphi(\tau), \qquad \tau_0 \le \tau < \infty, \tag{3.1}$$

is responsible for the oscillation. Let us recall that φ satisfies the ODE

$$\varphi'' + q\varphi = 0 \qquad \forall\, \tau \in [\tau_0, \infty) \tag{3.2}$$

which is a special type of the so-called *Sturm-Liouville* equations. For solutions of these equations we can apply the Sturm comparison theorem which we shall cite for the special equations we have in mind:

Theorem 3.1 (Sturm Comparison Theorem)**.** *Suppose that*

$$q(\tau) \ge \tilde{q}(\tau) \qquad \forall\, \tau \in [a, b] \tag{3.3}$$

and let φ resp. $\tilde{\varphi}$ be non-trivial solutions of the equations

$$\varphi'' + q\varphi = 0 \tag{3.4}$$

resp.

$$\tilde{\varphi}'' + \tilde{q}\tilde{\varphi} = 0 \tag{3.5}$$

in the interval $[a, b]$. If $\tilde{\varphi}$ satisfies

$$\tilde{\varphi}(a) = \tilde{\varphi}(b) = 0, \tag{3.6}$$

then there exists $\tau \in (a, b]$ such that

$$\varphi(\tau) = 0, \tag{3.7}$$

i.e., between two zeros of $\tilde{\varphi}$ there exists at least one zero of φ.

A proof of the theorem can be found in [1, p. 95].

We shall compare φ with solutions $\tilde{\varphi}$ corresponding to constant $\tilde{q}$. The zeros of theses functions are well known. Indeed, let

$$\tilde{q} = \mu^2, \tag{3.8}$$

where $0 < \mu$ is a constant, then $\sin(\mu\tau)$ and $\cos(\mu\tau)$ are solutions of equation (3.5).

Let q be the function defined in (2.6) on page 3, namely,

$$q(\tau) = 1 + \gamma + q_0(\tau) \tag{3.9}$$

and let $0 < \epsilon < \gamma$ be arbitrary but small. Then there exists a constant $0 < c_1$ such that

$$\tilde{q} = \mu^2 = 1 + \gamma - \epsilon < q(\tau) \qquad \forall\, \tau > c_1. \tag{3.10}$$

Similarly there exists a constant $c_2 > 0$ such that

$$\tilde{q} = \mu^2 = 1 + \gamma + \epsilon > q(\tau) \qquad \forall\, \tau > c_2. \tag{3.11}$$

Therefore, in view of the Sturm comparison theorem, we can state

Theorem 3.2. *The oscillation behaviour near the big bang of the solutions $w(t)$ in Theorem 1.1 is asymptotically equal to the behaviour of the solutions of the ODE*

$$\ddot{w} + mt^{-1}w + \mu_0 t^{-2} w = 0 \qquad \forall\, t > 0. \tag{3.12}$$

References

[1] Simon Brendle, *Ordinary differential equations*, 2009, ODE pdf, Lecture notes.

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