

Cluster subalgebras and cotorsion pairs in Frobenius extriangulated categories

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Abstract

Nakaoka and Palu introduced the notion of extriangulated categories by extracting the similarities between exact categories and triangulated categories. In this paper, we study cotorsion pairs in a Frobenius extriangulated category \mathcal{C} . Especially, for a 2-Calabi-Yau extriangulated category \mathcal{C} with a cluster structure, we describe the cluster substructure in the cotorsion pairs. For rooted cluster algebras arising from \mathcal{C} with cluster tilting objects, we give a one-to-one correspondence between cotorsion pairs in \mathcal{C} and certain pairs of their rooted cluster subalgebras which we call complete pairs. Finally, we explain this correspondence by an example relating to a Grassmannian cluster algebra.

Key words: Frobenius extriangulated categories; 2-Calabi-Yau extriangulated (or triangulated) categories; Cotorsion pairs; Cluster algebras.

2010 Mathematics Subject Classification: 16S99; 16S70; 18E30.

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^{*}Supported by the Fundamental Research Funds for the Central Universities (Grant No. 1301030603) and the NSF of China (Grant No. 11601295)

[†]Supported by the NSF of China (Grants No. 11671221) and the China Postdoctoral Science Foundation (Grant No. 2016M591155)

[‡]Supported by the NSF of China (Grants No.11671221)

Introduction

Triangulated categories and exact categories are two fundamental structures in mathematics. They are also important tools in many mathematics branches. It is well known that these two kinds of categories have some similarities, there are even direct connections between them. For example, by a classical result of Happel [Ha], the stable category of a Frobenius category, which is a special exact category, is a triangulated category, where the triangulated structure inherits from the exact structure. By extracting the similarities between triangulated categories and exact categories, Nakaoka and Palu [NP] recently introduced the notion of extriangulated categories, whose extriangulated structures are given by \mathbb{E} -triangles with some axioms. Except triangulated categories and exact categories, there are many other examples for extriangulated categories [NP, ZhZ].

In recent years, the categorification is a topic of general interest. Roughly speaking, given a mathematical structure, a categorification is to find a category which has this structure. For example, some 2-Calabi-Yau triangulated categories or exact stably 2-Calabi-Yau categories [BMRRT, GLS2, GLS3, BIRS, FK] have cluster structures, which categorify the cluster algebras with or without coefficients introduced by Fomin and Zelevinsky [FZ]. Recall that an exact category \mathcal{B} is called *exact stably 2-Calabi-Yau* if it is Frobenius, that is, \mathcal{B} has enough projectives and enough injectives, which coincide, and the stable category $\overline{\mathcal{B}}$, which is triangulated [Ha], is 2-Calabi-Yau. Examples of exact stably 2-Calabi-Yau categories relating to categorification and cluster tilting theory, are the module category over the preprojective algebra of a Dynkin quiver, and the category of maximal Cohen-Macaulay modules over a 3-dimensional complete local commutative isolated Gorenstein singularity. In this paper we study extriangulated categories in the view point of the categorification of cluster algebras.

The aim of this paper is to give a classification of cotorsion pairs in a 2-Calabi-Yau extriangulated category, and study the cluster substructures in cotorsion pairs of an extriangulated category with a cluster structure, where a cotorsion pair consists of two subcategories which generate the whole extriangulated category by extensions given by \mathbb{E} -triangles. Here a k -linear extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is called 2-Calabi-Yau if there is a functorially isomorphism $\mathbb{E}(x, y) \simeq D\mathbb{E}(y, x)$, for any $x, y \in \mathcal{C}$, where D is k -duality. We will give a one-to-one correspondence between the cotorsion pairs of a 2-Calabi-Yau extriangulated category and the complete pairs of the rooted cluster subalgebras of the rooted cluster algebra categorified by the extriangulated category (see Section 5 for precise meaning). Here a rooted cluster algebra is just a cluster algebra associated with a fixed cluster. This correspondence is established recently in the case of triangulated category in [CZ].

The main examples of our 2-Calabi-Yau extriangulated categories are exact stably 2-Calabi-Yau categories and 2-Calabi-Yau triangulated categories. Applying the main results to exact stably 2-Calabi-Yau categories produces new results on classification of cotorsion pairs in an exact stably 2-Calabi-Yau category, cluster substructure on cotorsion pairs and a new connection between cotorsion pairs and pairs of cluster subalgebras of a rooted cluster algebras which are categorified by these exact stably 2-Calabi-Yau categories. In particular, one can get cor-

responding results on cluster subalgebras of cluster algebras categorified by module categories over preprojective algebras studied in [GLS2, GLS3]. Applications to 2-Calabi-Yau triangulated categories reproduce the results obtained recently by the first and the third author in [CZ]. The basic approach in the paper is to use the mutation pairs studied in [ZhZ] to establish a relation between the cotorsion pairs in extriangulated category and the cotorsion pairs in some triangulated quotient categories.

The paper is organized as follows: In Section 1, we review some elementary definitions and facts that we need to use. Moreover, we introduce the notion of 2-Calabi-Yau extriangulated category, and give some examples. In Section 2, we find a one-to-one correspondence between the cotorsion pair of 2-Calabi-Yau extriangulated category and certain quotient category. In Section 3, we introduce the notion of cluster tilting subcategory in a extriangulated category and discuss some properties. By using the notation of mutation introduced in [ZhZ], we study the mutation of cluster tilting subcategory in a 2-Calabi-Yau extriangulated category. In Section 4, we give a classification of cotorsion pairs in 2-Calabi-Yau extriangulated category with cluster tilting objects. Then we prove that the two subcategories in a cotorsion pair inherit the cluster structure in the extriangulated category. In Section 5, after we recall some definitions and properties on cluster algebras and rooted cluster algebras, we construct the main correspondence between the cotorsion pairs and the complete pairs of the rooted cluster algebra. In Section 6, to explain the correspondence, we give an example of Grassmannians cluster algebra of type E_6 which is categorified by an exact stably 2-Calabi-Yau category.

1 Preliminaries

In this section, we collect some definitions and results that will be used in this paper.

1.1 Functorially finite subcategories

Recall that a subcategories \mathcal{X} of an additive category \mathcal{C} is said to be *contravariantly finite* in \mathcal{C} if for every object M of \mathcal{C} , there exists some X in \mathcal{X} and a morphism $f: X \rightarrow M$ such that for every X' in \mathcal{X} the sequence

$$\mathrm{Hom}_{\mathcal{C}}(X', X) \xrightarrow{f \circ} \mathrm{Hom}_{\mathcal{C}}(X', M) \rightarrow 0$$

is exact. In this case f is called a right \mathcal{X} -approximation. Dually, we defined covariantly finite subcategories in \mathcal{C} and left \mathcal{X} -approximations. Furthermore, a subcategory of \mathcal{C} is said to be *functorially finite* in \mathcal{C} if it is both contravariantly finite and covariantly finite in \mathcal{C} . A morphism $f: A \rightarrow B$ in \mathcal{C} is *right minimal* if any endomorphism $g: A \rightarrow A$ such that $fg = f$ is an automorphism; and *left minimal* if any endomorphism $h: B \rightarrow B$ such that $hf = f$ is an automorphism. For more details, we refer to [AR].

1.2 Extriangulated categories

Let \mathcal{C} be an additive category. Suppose that \mathcal{C} is equipped with a biadditive functor

$$\mathbb{E}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow Ab.$$

For any pair of objects $A, C \in \mathcal{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an \mathbb{E} -*extension*. Thus formally, an \mathbb{E} -extension is a triplet (A, δ, C) . Let (A, δ, C) be an \mathbb{E} -extension. Since \mathbb{E} is a bifunctor, for any $a \in \mathcal{C}(A, A')$ and $c \in \mathcal{C}(C', C)$, we have \mathbb{E} -extensions

$$\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A') \quad \text{and} \quad \mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A).$$

We abbreviate them by $a_*\delta$ and $c^*\delta$. For any $A, C \in \mathcal{C}$, the zero element $0 \in \mathbb{E}(C, A)$ is called the split \mathbb{E} -extension.

Definition 1.1. [NP, Definition 2.3] Let $(A, \delta, C), (A', \delta', C')$ be any pair of \mathbb{E} -extensions. A *morphism*

$$(a, c): (A, \delta, C) \rightarrow (A', \delta', C')$$

of \mathbb{E} -extensions is a pair of morphisms $a \in \mathcal{C}(A, A')$ and $c \in \mathcal{C}(C, C')$ in \mathcal{C} , satisfying the equality

$$a_*\delta = c^*\delta'.$$

Simply we denote it as $(a, c): \delta \rightarrow \delta'$.

Let $A, C \in \mathcal{C}$ be any pair of objects. Sequences of morphisms in \mathcal{C}

$$A \xrightarrow{x} B \xrightarrow{y} C \quad \text{and} \quad A \xrightarrow{x'} B' \xrightarrow{y'} C$$

are said to be *equivalent* if there exists an isomorphism $b \in \mathcal{C}(B, B')$ which makes the following diagram commutative.

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \parallel & & \downarrow \simeq b & & \parallel \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C \end{array}$$

We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$. For any $A, C \in \mathcal{C}$, we denote as $0 = [A \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} A \oplus C \xrightarrow{(0, 1)} C]$.

Definition 1.2. [NP, Definition 2.9] Let \mathfrak{s} be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to any \mathbb{E} -extension $\delta \in \mathbb{E}(C, A)$. This \mathfrak{s} is called a *realization* of \mathbb{E} , if it satisfies the following condition:

- Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of \mathbb{E} -extensions, with

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C], \quad \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$

Then, for any morphism $(a, c): \delta \rightarrow \delta'$, there exists $b \in \mathcal{C}(B, B')$ which makes the following

diagram commutative.

$$\begin{array}{ccccc}
 A & \xrightarrow{x} & B & \xrightarrow{y} & C \\
 \downarrow a & & \downarrow b & & \downarrow c \\
 A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C'
 \end{array} \tag{1.1}$$

In this case, we say that sequence $A \xrightarrow{x} B \xrightarrow{y} C$ realizes δ , whenever it satisfies

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C].$$

Remark that this condition does not depend on the choices of the representatives of the equivalence classes. In the above situation, we say that (1.1) (or the triplet (a, b, c)) realizes (a, c) .

Now we recall the definition of extriangulated categories introduced recently by Nakaoka and Palu, which is main object to study in the paper.

Definition 1.3. [NP, Definition 2.12] We call the pair $(\mathbb{E}, \mathfrak{s})$ an *external triangulation* of \mathcal{C} if it satisfies the following conditions:

(ET1) $\mathbb{E}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Ab}$ is a biadditive functor.

(ET2) \mathfrak{s} is an additive realization of \mathbb{E} .

(ET3) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of \mathbb{E} -extensions, realized as

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C], \quad \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$

For any commutative square

$$\begin{array}{ccccc}
 A & \xrightarrow{x} & B & \xrightarrow{y} & C \\
 \downarrow a & & \downarrow b & & \\
 A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C'
 \end{array}$$

in \mathcal{C} , there exists a morphism $(a, c): \delta \rightarrow \delta'$ which is realized by (a, b, c) .

(ET3)^{op} Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of \mathbb{E} -extensions, realized by

$$A \xrightarrow{x} B \xrightarrow{y} C \quad \text{and} \quad A' \xrightarrow{x'} B' \xrightarrow{y'} C'$$

respectively. For any commutative square

$$\begin{array}{ccccc}
 A & \xrightarrow{x} & B & \xrightarrow{y} & C \\
 & & \downarrow b & & \downarrow c \\
 A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C'
 \end{array}$$

in \mathcal{C} , there exists a morphism $(a, c): \delta \rightarrow \delta'$ which is realized by (a, b, c) .

(ET4) Let (A, δ, D) and (B, δ', F) be \mathbb{E} -extensions realized by

$$A \xrightarrow{f} B \xrightarrow{f'} D \quad \text{and} \quad B \xrightarrow{g} C \xrightarrow{g'} F$$

respectively. Then there exist an object $E \in \mathcal{C}$, a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & D \\ \parallel & & \downarrow g & & \downarrow d \\ A & \xrightarrow{h} & C & \xrightarrow{h'} & E \\ & & \downarrow g' & & \downarrow e \\ & & F & \xlongequal{\quad} & F \end{array}$$

in \mathcal{C} , and an \mathbb{E} -extension $\delta'' \in \mathbb{E}(E, A)$ realized by $A \xrightarrow{h} C \xrightarrow{h'} E$, which satisfy the following compatibilities.

- (i) $D \xrightarrow{d} E \xrightarrow{e} F$ realizes $f'_*\delta'$,
- (ii) $d^*\delta'' = \delta$,
- (iii) $f_*\delta'' = e^*\delta'$.

(ET4)^{op} Let (D, δ, B) and (F, δ', C) be \mathbb{E} -extensions realized by

$$D \xrightarrow{f'} A \xrightarrow{f} B \quad \text{and} \quad F \xrightarrow{g'} B \xrightarrow{g} C$$

respectively. Then there exist an object $E \in \mathcal{C}$, a commutative diagram

$$\begin{array}{ccccc} D & \xrightarrow{f'} & E & \xrightarrow{f} & F \\ \parallel & & \downarrow h' & & \downarrow g' \\ D & \xrightarrow{f'} & A & \xrightarrow{f} & B \\ & & \downarrow h & & \downarrow g \\ & & C & \xlongequal{\quad} & C \end{array}$$

in \mathcal{C} , and an \mathbb{E} -extension $\delta'' \in \mathbb{E}(C, E)$ realized by $E \xrightarrow{h'} A \xrightarrow{h} C$, which satisfy the following compatibilities.

- (i) $D \xrightarrow{d} E \xrightarrow{e} F$ realizes $g'^*\delta$,
- (ii) $\delta' = e_*\delta''$,
- (iii) $d_*\delta = g^*\delta''$.

In this case, we call \mathfrak{s} an \mathbb{E} -triangulation of \mathcal{C} , and call the triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ an *externally triangulated category*, or for short, *extriangulated category* \mathcal{C} .

For an extriangulated category \mathcal{C} , we use the following notation:

- A sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called a *conflation* if it realizes some \mathbb{E} -extension $\delta \in \mathbb{E}(C, A)$.

- A morphism $f \in \mathcal{C}(A, B)$ is called an *inflation* if it admits some conflation $A \xrightarrow{f} B \longrightarrow C$.
- A morphism $f \in \mathcal{C}(A, B)$ is called a *deflation* if it admits some conflation $K \longrightarrow A \xrightarrow{f} B$.
- If a conflation $A \xrightarrow{x} B \xrightarrow{y} C$ realizes $\delta \in \mathbb{E}(C, A)$, we call the pair $(A \xrightarrow{x} B \xrightarrow{y} C, \delta)$ an *\mathbb{E} -triangle*, and write it in the following way.

$$A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \gg$$

- Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \gg$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{\delta'} \gg$ be any pair of \mathbb{E} -triangles. If a triplet (a, b, c) realizes $(a, c): \delta \rightarrow \delta'$ as in (1.1), then we write it as

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} \gg \\ \downarrow a & & \downarrow b & & \downarrow c \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \xrightarrow{\delta'} \gg \end{array}$$

and call (a, b, c) a *morphism of \mathbb{E} -triangles*.

Example 1.4. (1) Exact category \mathcal{B} can be viewed as an extriangulated category. For the definition and basic properties of an exact category, see [Bu]. In fact, a biadditive functor $\mathbb{E} := \text{Ext}_{\mathcal{B}}^1: \mathcal{B}^{\text{op}} \times \mathcal{B} \rightarrow \text{Ab}$. Let $A, C \in \mathcal{B}$ be any pair of objects. Define $\text{Ext}_{\mathcal{B}}^1(C, A)$ to be the collection of all equivalence classes of short exact sequences of the form $A \xrightarrow{x} B \xrightarrow{y} C$. We denote the equivalence class by $[A \xrightarrow{x} B \xrightarrow{y} C]$ as before. For any $\delta = [A \xrightarrow{x} B \xrightarrow{y} C] \in \text{Ext}_{\mathcal{B}}^1(C, A)$, define the realization $\mathfrak{s}(\delta)$ of $[A \xrightarrow{x} B \xrightarrow{y} C]$ to be δ itself. For more details, see [NP, Example 2.13].

(2) Let \mathcal{C} be an triangulated category with shift functor $[1]$. Put $\mathbb{E} := \mathcal{C}(-, -[1])$. For any $\delta \in \mathbb{E}(C, A) = \mathcal{C}(C, A[1])$, take a triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A[1]$$

and define as $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$. Then $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category. It is easy to see that extension closed subcategories of triangulated categories are also extriangulated categories. For more details, see [NP, Proposition 3.22].

(3) Let \mathcal{C} be an extriangulated category, and \mathcal{J} a subcategory of \mathcal{C} . If $\mathcal{J} \subseteq \mathcal{P} \cap \mathcal{I}$, where \mathcal{P} is the full category of projective objects in \mathcal{C} and \mathcal{I} is the full category of injective objects in \mathcal{C} , then \mathcal{C}/\mathcal{J} is an extriangulated category. This construction gives extriangulated categories which are not exact nor triangulated in general. For more details, see [NP, Proposition 3.30].

We recall some concepts from [NP]. Let \mathcal{C} be an extriangulated category.

- An object $P \in \mathcal{C}$ is called *projective* if for any \mathbb{E} -triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \gg$ and any morphism $c \in \mathcal{C}(P, C)$, there exists $b \in \mathcal{C}(P, B)$ satisfying $yb = c$. We denote the full subcategory of projective objects in \mathcal{C} by \mathcal{P} . Dually, the full subcategory of injective objects in \mathcal{C} is denoted by \mathcal{I} .

- We say \mathcal{C} has enough projectives, if for any object $C \in \mathcal{C}$, there exists an \mathbb{E} -triangle

$$A \xrightarrow{x} P \xrightarrow{y} C \xrightarrow{\delta} \gg$$

satisfying $P \in \mathcal{P}$. We can define the notion of having enough injectives dually.

- \mathcal{C} is said to be *Frobenius* if \mathcal{C} has enough projectives and enough injectives and if moreover the projectives coincide with the injectives. In this case one has the quotient category $\overline{\mathcal{C}}$ of \mathcal{C} by injectives, which is a triangulated category by [NP]. We refer to this category as the *stable category* of \mathcal{C} .

Remark 1.5. (1) If $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an exact category, then enough projectives and enough injectives agree with the usual definitions.

- (2) If $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is a triangulated category, then \mathcal{P} and \mathcal{I} consist of zero objects. Moreover it is Frobenius as an extriangulated category.

Example 1.6. [ZhZ, Corollary 4.10] Let \mathcal{C} be a triangulated category with Auslander-Reiten translation τ , and \mathcal{X} a functorially finite subcategory of \mathcal{C} , which satisfies $\tau\mathcal{X} = \mathcal{X}$. For any $A, C \in \mathcal{C}$, define $\mathbb{E}'(C, A) := \mathcal{C}(C, A[1])$ to be the collection of all equivalence classes of triangles of the form $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\delta} A[1]$, where f is \mathcal{X} -monic. $\mathfrak{s}'(\delta) = [A \xrightarrow{f} B \xrightarrow{g} C]$, for any $\delta \in \mathbb{E}'(C, A)$. Then $(\mathcal{C}, \mathbb{E}', \mathfrak{s}')$ is a Frobenius extriangulated category whose projective objects are precisely \mathcal{X} .

From now on to the end of the article, we always suppose that extriangulated category \mathcal{C} has enough projectives and enough injectives.

1.3 Mutations in extriangulated categories

We recall the notion of mutation pairs in extriangulated categories from [ZhZ, Definition 3.2].

Definition 1.7. Let \mathcal{X}, \mathcal{A} and \mathcal{B} be subcategories of an extriangulated category \mathcal{C} , and $\mathcal{X} \subseteq \mathcal{A}$ and $\mathcal{X} \subseteq \mathcal{B}$. The pair $(\mathcal{A}, \mathcal{B})$ is called an \mathcal{X} -mutation pair if it satisfies

- (1) For any $A \in \mathcal{A}$, there exists an \mathbb{E} -triangle

$$A \xrightarrow{\alpha} X \xrightarrow{\beta} B \xrightarrow{\delta} \gg$$

where $B \in \mathcal{B}$, α is a left \mathcal{X} -approximation of A and β is a right \mathcal{X} -approximation of B .

- (2) For any $B \in \mathcal{B}$, there exists an \mathbb{E} -triangle

$$A \xrightarrow{\alpha} X \xrightarrow{\beta} B \xrightarrow{\delta} \gg$$

where $A \in \mathcal{A}$, α is a left \mathcal{X} -approximation of A and β is a right \mathcal{X} -approximation of B .

Note that if \mathcal{C} is a triangulated category, \mathcal{X} -mutation pair is just the same as Liu-Zhu's definition [LZ, Definition 2.6]. In addition, if \mathcal{X} is a rigid subcategory of \mathcal{C} , i.e., $\text{Ext}_{\mathcal{C}}^1(\mathcal{X}, \mathcal{X}) = 0$, then \mathcal{X} -mutation pair is just the same as Iyama-Yoshino's definition [IY, Definition 2.5].

Let $\mathcal{X} \subseteq \mathcal{A}$ be subcategories of a category \mathcal{C} . We denote by \mathcal{A}/\mathcal{X} the category whose objects are objects of \mathcal{A} and whose morphisms are elements of $\text{Hom}_{\mathcal{A}}(A, B)/\mathcal{X}(A, B)$ for any $A, B \in \mathcal{A}$. Such category is called the quotient category of \mathcal{A} by \mathcal{X} . For any morphism $f: A \rightarrow B$ in \mathcal{A} , we denote by \bar{f} the image of f under the natural quotient functor $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{X}$.

Theorem 1.8. [ZhZ, Theorem 3.15] *Let \mathcal{C} be an extriangulated category and let $\mathcal{X} \subseteq \mathcal{A}$ be a subcategory of \mathcal{C} . We assume that the following two conditions concerning \mathcal{A} and \mathcal{X} :*

- (1) \mathcal{A} is extension closed,
- (2) $(\mathcal{A}, \mathcal{A})$ forms an \mathcal{X} -mutation pair.

We put $\mathcal{M} := \mathcal{A}/\mathcal{X}$. Then the category \mathcal{M} has the structure of a triangulated category with respect to the following shift functor and triangles:

- (I) For $A \in \mathcal{A}$, we take an \mathbb{E} -triangle

$$A \xrightarrow{a} X \xrightarrow{b} A\langle 1 \rangle \dashrightarrow,$$

where a is a left \mathcal{X} -approximation and b is a right \mathcal{X} -approximation. Then $\langle 1 \rangle$ gives a well-defined auto-equivalence of \mathcal{M} , which is the **shift functor** of \mathcal{M} .

- (II) For an \mathbb{E} -triangle $A \xrightarrow{f} B \xrightarrow{g} C \dashrightarrow$, with $A, B, C \in \mathcal{A}$ and f is \mathcal{X} -monic, there exists the following commutative diagram of \mathbb{E} -triangles:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \dashrightarrow \\ \parallel & & \downarrow & & \downarrow h \\ A & \xrightarrow{a} & X & \xrightarrow{b} & A\langle 1 \rangle \dashrightarrow \end{array}$$

Then we have a complex $A \xrightarrow{\bar{f}} B \xrightarrow{\bar{g}} C \xrightarrow{\bar{h}} A\langle 1 \rangle$. We define **triangles** in \mathcal{M} as the complexes which are isomorphic to a complex obtained in this way.

Lemma 1.9. *Assume that $(\mathcal{U}, \mathcal{V})$ is \mathcal{Y} -mutation pair in \mathcal{C} such that $\mathcal{U} \vee \mathcal{V} \subseteq \mathcal{A}$, where $\mathcal{U} \vee \mathcal{V}$ is the smallest subcategory of \mathcal{C} containing \mathcal{U} and \mathcal{V} , and $\mathcal{X} \subseteq \mathcal{Y}$. Then $(\bar{\mathcal{U}}, \bar{\mathcal{V}})$ is $\bar{\mathcal{Y}}$ -mutation pair in \mathcal{M} .*

Proof. For any $U \in \bar{\mathcal{U}}$, there exists an \mathbb{E} -triangles

$$U \xrightarrow{f} Y \xrightarrow{g} V \dashrightarrow,$$

where f is a left \mathcal{Y} -approximation, g is a right \mathcal{Y} -approximation and $V \in \mathcal{V}$. Thus we have following commutative diagram:

$$\begin{array}{ccccc} U & \xrightarrow{f} & Y & \xrightarrow{g} & V \dashrightarrow \\ \parallel & & \downarrow y & & \downarrow h \\ U & \xrightarrow{a} & X & \xrightarrow{b} & U\langle 1 \rangle \dashrightarrow. \end{array}$$

It follows that

$$U \xrightarrow{\bar{f}} Y \xrightarrow{\bar{g}} V \xrightarrow{\bar{h}} U\langle 1 \rangle$$

is a triangle in \mathcal{M} . It is easy to see that \bar{f} is a left $\overline{\mathcal{F}}$ -approximation of U , \bar{g} is a right $\overline{\mathcal{F}}$ -approximation of V and $V \in \overline{\mathcal{V}}$.

Dually, we can show that for any $V' \in \overline{\mathcal{V}}$, there exists a triangle

$$U' \xrightarrow{\bar{f}'} Y' \xrightarrow{\bar{g}'} V' \xrightarrow{\bar{h}'} U'\langle 1 \rangle$$

in \mathcal{M} , where \bar{f}' is a left $\overline{\mathcal{F}}$ -approximation of U' , \bar{g}' is a right $\overline{\mathcal{F}}$ -approximation of V' and $U' \in \overline{\mathcal{U}}$. This shows that $(\overline{\mathcal{U}}, \overline{\mathcal{V}})$ is $\overline{\mathcal{F}}$ -mutation pair in \mathcal{M} . \square

1.4 2-Calabi-Yau extriangulated categories

Motivated by the definition of 2-Calabi-Yau triangulated categories and exact stably 2-Calabi-Yau categories, we define 2-Calabi-Yau extriangulated categories.

Definition 1.10. Let \mathcal{C} be a k -linear Hom-finite extriangulated category over a field k . \mathcal{C} is called *2-Calabi-Yau* if there exists a bifunctorial isomorphism

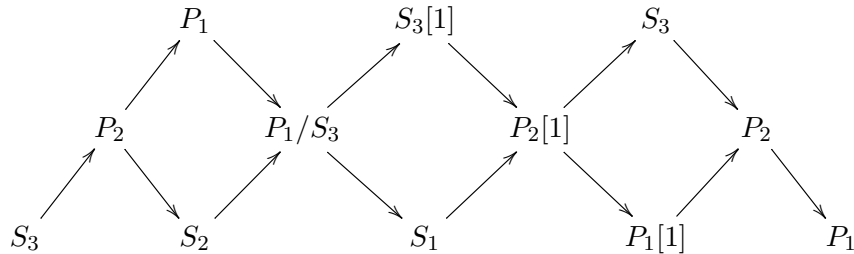
$$\mathbb{E}(A, B) \simeq \mathrm{DE}(B, A),$$

for any $A, B \in \mathcal{C}$, where $\mathrm{D} = \mathrm{Hom}_k(-, k)$ is the usual k -duality.

We give some examples for 2-Calabi-Yau extriangulated category.

Example 1.11.

- Any exact stably 2-Calabi-Yau category is a 2-Calabi-Yau extriangulated category.
- Any 2-Calabi-Yau triangulated category is a 2-Calabi-Yau extriangulated category.
- Let \mathcal{C}_Q be the cluster category of the path algebra kQ , where Q is the quiver $1 \rightarrow 2 \rightarrow 3$. We have the following AR-quiver for \mathcal{C}_Q , where S_i and P_i denote the simple and projective modules associated with vertex i respectively.



Then $\mathcal{B} = \mathrm{mod}kQ$ is an extension closed subcategory of \mathcal{C}_Q and it is easy to see that P_1 is the only indecomposable projective object in \mathcal{B} . Then $\mathcal{B}/\mathrm{add}P_1$ is clearly equivalent to the cluster category $\mathcal{C}_{Q'}$ where Q' is a quiver of type A_2 , which is a 2-Calabi-Yau triangulated category. Hence \mathcal{B} is a 2-Calabi-Yau extriangulated category.

2 Cotorsion pairs in extriangulated categories

We recall the definition of cotorsion pair in an extriangulated category from [NP].

Definition 2.1. [NP, Definition 4.1] Let $(\mathcal{X}, \mathcal{Y})$ be subcategories of an extriangulated categories \mathcal{C} . The pair $(\mathcal{X}, \mathcal{Y})$ is called a *cotorsion pair* on \mathcal{C} if it satisfies the following conditions.

- (1) $\mathbb{E}(\mathcal{X}, \mathcal{Y}) = 0$;
- (2) For any $C \in \mathcal{C}$, there exists an \mathbb{E} -triangle

$$Y^C \xrightarrow{f} X^C \xrightarrow{g} C \xrightarrow{\delta} \triangleright$$

satisfying $X^C \in \mathcal{X}, Y^C \in \mathcal{Y}$.

- (3) For any $C \in \mathcal{C}$, there exists an \mathbb{E} -triangle

$$C \xrightarrow{f} Y_C \xrightarrow{g} X_C \xrightarrow{\eta} \triangleright$$

satisfying $X_C \in \mathcal{X}, Y_C \in \mathcal{Y}$.

We call $I(\mathcal{X}) = \mathcal{X} \cap \mathcal{Y}$ the *core* of the cotorsion pair $(\mathcal{X}, \mathcal{Y})$, which is usually denoted by I . Note that any projective-injective object belongs to the core I , thus the subcategory Proj-Inj which consists of all projective-injective objects is contained in I . Moreover, if $I = \text{Proj-Inj}$, then we call $(\mathcal{X}, \mathcal{Y})$ a *t-structure*.

Note that if \mathcal{C} is a triangulated category, cotorsion pair is just the same as Iyama-Yoshino's definition [IY, Definition 2.2] or Nakaoka's definition [Na, Definition 2.1]. If \mathcal{C} is an exact category, cotorsion pair is just the same as Liu's definition [L, Definition 2.3].

Remark 2.2. Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair on an extriangulated category \mathcal{C} . Then

- (1) $C \in \mathcal{U}$ if and only if $\mathbb{E}(C, \mathcal{V}) = 0$;
- (2) $C \in \mathcal{V}$ if and only if $\mathbb{E}(\mathcal{U}, C) = 0$;
- (3) \mathcal{U} and \mathcal{V} are extension closed;
- (4) \mathcal{U} is a contravariantly finite in \mathcal{C} and \mathcal{V} is a covariantly finite in \mathcal{C} .
- (5) $\mathcal{P} \subseteq \mathcal{U}$ and $\mathcal{I} \subseteq \mathcal{V}$.

The following is an extriangulated version of Wakamatsu's Lemma.

Lemma 2.3. Let \mathcal{X} be an extension closed subcategory of an extriangulated category \mathcal{C} . Given an \mathbb{E} -triangle

$$K \xrightarrow{f} X_0 \xrightarrow{g} C \xrightarrow{\delta} \triangleright,$$

where g is a minimal right \mathcal{X} -approximation of C . Then $\mathbb{E}(X, K) = 0$, for any $X \in \mathcal{X}$.

Proof. There exists a long exact sequence

$$\mathcal{C}(X, X_0) \xrightarrow{\mathcal{C}(X, g)} \mathcal{C}(X, C) \rightarrow \mathbb{E}(X, K) \rightarrow \mathbb{E}(X, X_0) \xrightarrow{\mathbb{E}(X, g)} \mathbb{E}(X, C).$$

The first morphism in the sequence is epimorphism since $X_0 \xrightarrow{g} C$ is a right \mathcal{X} -approximation, so the second morphism is zero. We claim that the fourth map in the sequence is monomorphism whence the third map is zero. This forces $\mathbb{E}(X, K) = 0$ as desired.

To see that the fourth map is monomorphism, let $\eta \in \mathbb{E}(X, X_0)$ be any \mathbb{E} -extension, realized by an \mathbb{E} -triangle

$$X_0 \xrightarrow{u} A \xrightarrow{v} X \xrightarrow{\eta} \gg$$

such that $\mathbb{E}(X, g)(\eta) = 0$. Thus we obtain a morphism of \mathbb{E} -triangles

$$\begin{array}{ccccc} X_0 & \xrightarrow{u} & A & \xrightarrow{v} & X \xrightarrow{\eta} \gg \\ \downarrow g & & \downarrow h & & \parallel \\ C & \xrightarrow{x} & B & \xrightarrow{y} & X \xrightarrow{0} \gg \end{array}$$

By Corollary 3.5 in [NP], there exists a morphism $x': B \rightarrow C$ such that $x'x = 1$. Since \mathcal{X} is extension closed, we have $A \in \mathcal{X}$. Since g a right \mathcal{X} -approximation, there exists a morphism $t: A \rightarrow X_0$ such that $gt = x'h$. It follows that $g = x'xg = x'hu = gtu$. Since g is right minimal, we have that tu is an isomorphism. In particular, u is a section. By Corollary 3.5 in [NP], η splits and then $\eta = 0$, as desired. \square

We introduce two notations. Let \mathcal{X} be a subcategory of an extriangulated category \mathcal{C} . We set $\mathcal{X}^{\perp_{\mathbb{E}}} = \{M \in \mathcal{C} \mid \mathbb{E}(\mathcal{X}, M) = 0\}$ and ${}^{\perp_{\mathbb{E}}}\mathcal{X} = \{M \in \mathcal{C} \mid \mathbb{E}(M, \mathcal{X}) = 0\}$. Remark that $\mathcal{X}^{\perp_{\mathbb{E}}}$ and ${}^{\perp_{\mathbb{E}}}\mathcal{X}$ are extension closed.

Proposition 2.4. *Let \mathcal{X} be a contravariantly finite extension closed subcategory of an extriangulated category \mathcal{C} such that $\mathcal{P} \subseteq \mathcal{X}$. Then $(\mathcal{X}, \mathcal{X}^{\perp_{\mathbb{E}}})$ is a cotorsion pair in \mathcal{C} .*

Proof. For any object $C \in \mathcal{C}$, there exists an \mathbb{E} -triangle

$$C \xrightarrow{u} I_0 \xrightarrow{v} L \dashrightarrow,$$

where $I_0 \in \mathcal{I}$. Since \mathcal{X} is contravariantly finite and $\mathcal{P} \subseteq \mathcal{X}$, we can take two \mathbb{E} -triangles

$$M \xrightarrow{x} X_1 \xrightarrow{y} L \dashrightarrow \quad \text{and} \quad K \xrightarrow{f} X_2 \xrightarrow{g} C \dashrightarrow,$$

where v (resp. g) is a minimal right \mathcal{X} -approximation of L (resp. C). Since \mathcal{X} is extension closed, by Lemma 2.3, we obtain $M \in \mathcal{X}^{\perp_{\mathbb{E}}}$ (resp. $K \in \mathcal{X}^{\perp_{\mathbb{E}}}$). By Proposition 3.15 in [NP], we

obtain a commutative diagram

$$\begin{array}{ccccccc}
& & M & \xlongequal{\quad} & M & & \\
& & \downarrow & & \downarrow x & & \\
C & \longrightarrow & N & \longrightarrow & X_1 & \dashrightarrow & \\
\parallel & & \downarrow & & \downarrow y & & \\
C & \xrightarrow{u} & I_0 & \xrightarrow{v} & L & \dashrightarrow & \\
& & \downarrow & & \downarrow & & \\
& & \downarrow & & \downarrow & &
\end{array}$$

of \mathbb{E} -triangles. Since $I_0, M \in \mathcal{X}^{\perp_{\mathbb{E}}}$ and $\mathcal{X}^{\perp_{\mathbb{E}}}$ is extension closed, we have $N \in \mathcal{X}^{\perp_{\mathbb{E}}}$. Thus for any object $C \in \mathcal{C}$, there exists two \mathbb{E} -triangles

$$K \longrightarrow X_2 \longrightarrow C \dashrightarrow \quad \text{and} \quad C \longrightarrow N \longrightarrow X_1 \dashrightarrow$$

satisfying $K, N \in \mathcal{X}^{\perp_{\mathbb{E}}}$ and $X_2, X_1 \in \mathcal{X}$. Hence $(\mathcal{X}, \mathcal{X}^{\perp_{\mathbb{E}}})$ is a cotorsion pair. \square

We omit the dual statement.

Definition 2.5. Let \mathcal{X} be a subcategory of an extriangulated category \mathcal{C} . \mathcal{X} is called *rigid* if $\mathbb{E}(\mathcal{X}, \mathcal{X}) = 0$, i.e., $\mathbb{E}(A, B) = 0$, for any $A, B \in \mathcal{X}$. An object X is called *rigid* if $\text{add}X$ is rigid.

Lemma 2.6. Let \mathcal{X} be a functorially finite rigid subcategory of an extriangulated category \mathcal{C} such that $\mathcal{X}^{\perp_{\mathbb{E}}} = {}^{\perp_{\mathbb{E}}}\mathcal{X}$ and $\mathcal{P}, \mathcal{I} \subseteq \mathcal{X}$. Then $(\mathcal{X}^{\perp_{\mathbb{E}}}, \mathcal{X}^{\perp_{\mathbb{E}}})$ is an \mathcal{X} -mutation pair.

Proof. For any $M \in \mathcal{X}^{\perp_{\mathbb{E}}}$, there exists an \mathbb{E} -triangle

$$K \xrightarrow{f} X_0 \xrightarrow{g} M \dashrightarrow^{\delta},$$

where g is a minimal right \mathcal{X} -approximation of M . By Lemma 2.3, we have $\mathbb{E}(\mathcal{X}, K) = 0$. Namely $K \in \mathcal{X}^{\perp_{\mathbb{E}}} = {}^{\perp_{\mathbb{E}}}\mathcal{X}$ and then $\mathbb{E}(K, \mathcal{X}) = 0$. Applying the functor $\mathcal{C}(-, \mathcal{X})$ to the above \mathbb{E} -triangle, we obtain an exact sequence

$$\mathcal{C}(X_0, \mathcal{X}) \xrightarrow{\mathcal{C}(f, \mathcal{X})} \mathcal{C}(K, \mathcal{X}) \rightarrow \mathbb{E}(M, \mathcal{X}) = 0.$$

This shows that f is a left \mathcal{X} -approximation of K . Dually, we can show that $N \in \mathcal{X}^{\perp_{\mathbb{E}}}$, there exists an \mathbb{E} -triangle

$$N \xrightarrow{u} X_1 \xrightarrow{v} L \dashrightarrow^{\delta'},$$

where u is a left \mathcal{X} -approximation of M and v is a right \mathcal{X} -approximation of L .

Therefore, $(\mathcal{X}^{\perp_{\mathbb{E}}}, \mathcal{X}^{\perp_{\mathbb{E}}})$ is an \mathcal{X} -mutation pair. \square

Lemma 2.7. Let \mathcal{X} be a subcategory of an extriangulated category \mathcal{C} such that $\mathcal{X}^{\perp_{\mathbb{E}}} = {}^{\perp_{\mathbb{E}}}\mathcal{X}$. If $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair on an extriangulated category and $\mathcal{X} \subseteq \mathcal{U} \subseteq \mathcal{X}^{\perp_{\mathbb{E}}}$, then $\mathcal{X} \subseteq \mathcal{V} \subseteq \mathcal{X}^{\perp_{\mathbb{E}}}$.

Proof. Since $\mathcal{U} \subseteq \mathcal{X}^{\perp_{\mathbb{E}}} = {}^{\perp_{\mathbb{E}}}\mathcal{X}$, we have $\mathbb{E}(\mathcal{U}, \mathcal{X}) = 0$ and then $\mathcal{X} \subseteq \mathcal{V}$. Since $\mathcal{X} \subseteq \mathcal{U}$ and $\mathbb{E}(\mathcal{U}, \mathcal{V}) = 0$, we have $\mathbb{E}(\mathcal{X}, \mathcal{V}) = 0$ and then $\mathcal{V} \subseteq \mathcal{X}^{\perp_{\mathbb{E}}}$.

Lemma 2.8. *Let \mathcal{C} be an extriangulated category, $\mathcal{X} \subseteq \mathcal{A}$ subcategories of \mathcal{C} . Assume that $(\mathcal{A}, \mathcal{A})$ forms a \mathcal{X} -mutation pair and A, B are two objects in \mathcal{A} . If \mathcal{X} is a rigid subcategory of \mathcal{C} , then $\mathbb{E}(A, B) \simeq \mathcal{M}(A, B\langle 1 \rangle)$.*

Proof. Since $B \in \mathcal{A}$, there exists an \mathbb{E} -triangle in \mathcal{C}

$$B \xrightarrow{f} X_B \xrightarrow{g} B\langle 1 \rangle \xrightarrow{-\delta} \gg,$$

where $X_B \in \mathcal{X}$ and g is a right \mathcal{X} -approximation of $B\langle 1 \rangle$. Applying the functor $\mathcal{C}(A, -)$ to this \mathbb{E} -triangle, we have the following exact sequence

$$\mathcal{C}(A, X_B) \xrightarrow{\mathcal{C}(A, g)} \mathcal{C}(A, B\langle 1 \rangle) \rightarrow \mathbb{E}(A, B) \rightarrow 0$$

as $\mathbb{E}(A, X_B) = 0$ by [ZhZ, Lemma 3.5].

Note that $\text{Im } \mathcal{C}(A, g) = \mathcal{X}(A, B\langle 1 \rangle)$. It follows that

$$\mathcal{M}(A, B\langle 1 \rangle) = \mathcal{C}(A, B\langle 1 \rangle) / \mathcal{X}(A, B\langle 1 \rangle) = \mathcal{C}(A, B\langle 1 \rangle) / \text{Im } \mathcal{C}(A, g) \simeq \mathbb{E}(A, B).$$

□

The following theorem gives a one-to-one correspondence between cotorsion pairs whose core containing \mathcal{X} in \mathcal{C} and cotorsion pairs in \mathcal{M} , which generalizes Theorem 3.5 in [ZZ1]. In the following, $\overline{\mathcal{U}}$ denotes the subcategory of \mathcal{M} consisting of objects $U \in \mathcal{U}$.

Theorem 2.9. *Let \mathcal{X} be a functorially finite rigid subcategory of an extriangulated category \mathcal{C} such that $\mathcal{A} := \mathcal{X}^{\perp_{\mathbb{E}}} = {}^{\perp_{\mathbb{E}}}\mathcal{X}$ and $\mathcal{P}, \mathcal{I} \subseteq \mathcal{X}$. If $\mathcal{X} \subseteq \mathcal{U} \subseteq \mathcal{X}^{\perp_{\mathbb{E}}}$. Then $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair with core \mathcal{I} in \mathcal{C} if and only if $(\overline{\mathcal{U}}, \overline{\mathcal{V}})$ is a cotorsion pair with the core $\overline{\mathcal{I}}$ in $\mathcal{M} := \mathcal{A} / \mathcal{X}$.*

Proof. By Lemma 2.6, we know that $(\mathcal{A}, \mathcal{A})$ is an \mathcal{X} -mutation pair. Since \mathcal{A} is extension closed, by Theorem 1.8, we have that \mathcal{M} is a triangulated category.

We first show the “only if” part. Assume that $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in \mathcal{C} . Then $\mathbb{E}(\mathcal{U}, \mathcal{V}) = 0$. By Lemma 2.8, we have $\mathcal{M}(\mathcal{U}, \mathcal{V}\langle 1 \rangle) = 0$.

For any $A \in \mathcal{A}$, there exists an \mathbb{E} -triangle in \mathcal{C}

$$V \xrightarrow{f} U \xrightarrow{g} A \xrightarrow{-\delta} \gg,$$

where $V \in \mathcal{V}$ and $U \in \mathcal{U}$ as $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in \mathcal{C} . Applying the functor $\mathcal{C}(-, \mathcal{X})$ to this \mathbb{E} -triangle, we have the following exact sequence

$$\mathcal{C}(U, \mathcal{X}) \xrightarrow{\mathcal{C}(g, \mathcal{X})} \mathcal{C}(V, \mathcal{X}) \rightarrow \mathbb{E}(A, \mathcal{X}) = 0.$$

It follows that f is \mathcal{X} -monic. Thus we get the following commutative diagram:

$$\begin{array}{ccccc} V & \xrightarrow{f} & U & \xrightarrow{g} & A & \xrightarrow{\delta} & \rightarrow \\ \parallel & & \downarrow b & & \downarrow h & & \\ V & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & V\langle 1 \rangle & \xrightarrow{\eta} & \rightarrow \end{array}$$

Since V, U, A are in \mathcal{A} , there exists a standard triangle in \mathcal{M} :

$$V \xrightarrow{\bar{f}} U \xrightarrow{\bar{g}} A \xrightarrow{\bar{h}} V\langle 1 \rangle.$$

Therefore, $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in \mathcal{M} .

To prove the “if” part. Assume that $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in \mathcal{M} . Then $\mathcal{M}(\mathcal{U}, \mathcal{V}\langle 1 \rangle) = 0$. By Lemma 2.8, we have $\mathbb{E}(\mathcal{U}, \mathcal{V}) = 0$.

For any $A \in \mathcal{A}$, there exist two standard triangles in \mathcal{M}

$$\begin{array}{c} V_1 \xrightarrow{\bar{f}_1} U_1 \xrightarrow{\bar{g}_1} A \xrightarrow{\bar{h}_1} V_1\langle 1 \rangle \\ U_2 \xrightarrow{\bar{f}_2} A\langle 1 \rangle \xrightarrow{\bar{g}_2} V_2\langle 1 \rangle \xrightarrow{\bar{h}_2} U_2\langle 1 \rangle, \end{array}$$

where $U_1, U_2 \in \mathcal{U}$ and $V_1, V_2 \in \mathcal{V}$ as $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in \mathcal{M} and then

$$\begin{array}{c} V_1 \xrightarrow{\bar{f}_1} U_1 \xrightarrow{\bar{g}_1} A \xrightarrow{\bar{h}_1} V_1\langle 1 \rangle \\ A \xrightarrow{\bar{d}_2} V_2 \xrightarrow{\bar{e}_2} U_2 \xrightarrow{\bar{f}_2} A\langle 1 \rangle, \end{array}$$

are also two standard triangles in \mathcal{M} . We may assume that it is induced by the following commutative diagrams of \mathbb{E} -triangles in \mathcal{C} .

$$\begin{array}{ccc} V_1 \xrightarrow{f_1} U_1 \xrightarrow{g_1} A \xrightarrow{\delta_1} \rightarrow & A \xrightarrow{d_2} V_2 \xrightarrow{e_2} U_2 \xrightarrow{\delta_2} \rightarrow & \\ \parallel & \downarrow c_1 & \downarrow h_1 \\ V_1 \xrightarrow{\alpha_1} X_1 \xrightarrow{\beta_1} V_1\langle 1 \rangle \xrightarrow{\delta'_1} \rightarrow, & \parallel & \downarrow c_2 \\ & & \downarrow f_2 \\ & & A \xrightarrow{\alpha_2} X_2 \xrightarrow{\beta_2} A\langle 1 \rangle \xrightarrow{\delta'_2} \rightarrow. \end{array}$$

It follows that

$$V_1 \xrightarrow{f_1} U_1 \xrightarrow{g_1} A \xrightarrow{\delta_1} \rightarrow, \quad A \xrightarrow{d_2} V_2 \xrightarrow{e_2} U_2 \xrightarrow{\delta_2} \rightarrow,$$

are \mathbb{E} -triangles in \mathcal{A} , where $U_1, U_2 \in \mathcal{U}$ and $V_1, V_2 \in \mathcal{V}$.

For any $C \in \mathcal{C}$, there exists an \mathbb{E} -triangle

$$K \xrightarrow{f} X_0 \xrightarrow{g} C \xrightarrow{\delta} \rightarrow,$$

where g is a minimal right \mathcal{X} -approximation of C . By Lemma 2.3, we have that $\mathbb{E}(\mathcal{X}, K) = 0$ and then $K \in \mathcal{A}$. Thus there exists an \mathbb{E} -triangle in \mathcal{C} .

$$K \xrightarrow{d_2} V_2 \xrightarrow{e_2} U_2 \xrightarrow{\delta_2} \rightarrow,$$

where $V_2 \in \mathcal{V}$ and $U_2 \in \mathcal{U}$. By Proposition 3.15 in [NP], we obtain a commutative diagram

$$\begin{array}{ccccc}
K & \xrightarrow{f} & X_0 & \xrightarrow{f'} & C \xrightarrow{\delta} \gg \\
\downarrow d_2 & & \downarrow & & \parallel \\
V_2 & \xrightarrow{a} & U & \xrightarrow{b} & C \xrightarrow{\eta} \gg \\
\downarrow e_2 & & \downarrow & & \\
U_2 & \xlongequal{\quad} & U_2 & & \\
\downarrow \delta_g & & \downarrow \mathcal{E} & & \\
\Downarrow & & \Downarrow & &
\end{array}$$

of \mathbb{E} -triangles. Since \mathcal{U} is extension closed, we have $U \in \mathcal{U}$. Namely, for any $C \in \mathcal{C}$, there exists an \mathbb{E} -triangle

$$V_2 \xrightarrow{a} U \xrightarrow{b} C \xrightarrow{\eta} \gg,$$

where $V_2 \in \mathcal{V}$ and $U \in \mathcal{U}$.

Similarly, we can show that there exists an \mathbb{E} -triangle

$$C \xrightarrow{c} V \xrightarrow{d} U_1 \xrightarrow{\theta} \gg,$$

where $V_2 \in \mathcal{V}$ and $U \in \mathcal{U}$. Therefore, $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair in \mathcal{C} .

Finally, we have that $I(\overline{\mathcal{X}}) = \overline{\mathcal{X}} \cap \overline{\mathcal{Y}} = \overline{\mathcal{X} \cap \mathcal{Y}} = \overline{I(\mathcal{X})}$. □

This theorem immediately yields the following.

Corollary 2.10. *Let \mathcal{C} be a Frobenius extriangulated category. Then $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair with core I in \mathcal{C} if and only if $(\overline{\mathcal{U}}, \overline{\mathcal{V}})$ is a cotorsion pair with the core \overline{I} in $\overline{\mathcal{C}}$.*

Proof. It is easy to see that $(\mathcal{C}, \mathcal{C})$ forms a \mathcal{I} -mutation pair. This follows from Theorem 2.9 and Theorem 1.8. □

Apply to exact stably 2-Calabi-Yau categories (e.g. module categories over preprojective algebras of Dynkin quivers or the subcategories \mathcal{C}_M of module categories over preprojective algebras, where M is a terminal module, for details see [GLS1-2]), we have the following correspondence between cotorsion pairs in an exact stably 2-Calabi-Yau category and in its stable category.

Corollary 2.11. *Let \mathcal{B} be an exact stably 2-Calabi-Yau category. Then $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair with core I in \mathcal{B} if and only if $(\overline{\mathcal{U}}, \overline{\mathcal{V}})$ is a cotorsion pair with the core \overline{I} in $\overline{\mathcal{B}}$.*

3 Mutations of cluster tilting subcategories

Definition 3.1. Let \mathcal{C} be an extriangulated category, \mathcal{X} a subcategory of \mathcal{C} .

- \mathcal{X} is called *cluster tilting* if it satisfies the following conditions:

- (1) \mathcal{X} is a functorially finite in \mathcal{C} ;
- (2) $M \in \mathcal{X}$ if and only if $\mathbb{E}(M, \mathcal{X}) = 0$;

(3) $M \in \mathcal{X}$ if and only if $\mathbb{E}(\mathcal{X}, M) = 0$.

- An object X is called *cluster tilting* if $\text{add} X$ is cluster tilting.

This definition is a generalization of cluster tilting subcategories in triangulated categories [BMRRT, KR, KZ, IY, B] and in exact categories [GLS2-3, I].

By definition of a cluster tilting subcategory, we can immediately conclude:

Remark 3.2. Let \mathcal{C} be an extriangulated category.

- If \mathcal{X} is a cluster tilting subcategory of \mathcal{C} , then $\mathcal{P} \subseteq \mathcal{X}$ and $\mathcal{I} \subseteq \mathcal{X}$.
- \mathcal{X} is a cluster tilting subcategory of \mathcal{C} if and only if
 - (1) \mathcal{X} is rigid;
 - (2) For any $C \in \mathcal{C}$, there exists an \mathbb{E} -triangle $C \xrightarrow{a} X_1 \xrightarrow{b} X_2 \xrightarrow{\delta} C$, where $X_1, X_2 \in \mathcal{X}$;
 - (3) For any $C \in \mathcal{C}$, there exists an \mathbb{E} -triangle $X_3 \xrightarrow{c} X_4 \xrightarrow{d} C \xrightarrow{\eta} X_3$, where $X_3, X_4 \in \mathcal{X}$.

Proposition 3.3. *Let \mathcal{X} be a functorially finite rigid subcategory of a 2-Calabi-Yau extriangulated category \mathcal{C} such that $\mathcal{P} \subseteq \mathcal{X}$. Then $\mathcal{N} := \mathcal{X}^{\perp_{\mathbb{E}}}/\mathcal{X}$ is a 2-Calabi-Yau triangulated category.*

Proof. This follows from Lemma 2.6, Theorem 1.8 and Lemma 2.8. □

The following result is easy to verify and will be used in the sequel.

Lemma 3.4. *Let \mathcal{C} be an additive category and $\mathcal{X} \subseteq \mathcal{A}$ two subcategories of \mathcal{C} .*

- (1) *If \mathcal{X} is contravariantly finite in \mathcal{C} , then \mathcal{A} is contravariantly finite in \mathcal{C} if and only if \mathcal{A}/\mathcal{X} is contravariantly finite in \mathcal{C}/\mathcal{X} .*
- (2) *If \mathcal{X} is covariantly finite in \mathcal{C} , then \mathcal{A} is covariantly finite in \mathcal{C} if and only if \mathcal{A}/\mathcal{X} is covariantly finite in \mathcal{C}/\mathcal{X} .*
- (3) *If \mathcal{X} is functorially finite in \mathcal{C} , then \mathcal{A} is functorially finite in \mathcal{C} if and only if \mathcal{A}/\mathcal{X} is functorially finite in \mathcal{C}/\mathcal{X} .*

Proof. It is straightforward to check. □

Theorem 3.5. *Let \mathcal{C} be a 2-Calabi-Yau extriangulated category, and \mathcal{X} a functorially finite rigid subcategory of \mathcal{C} such that $\mathcal{P} \subseteq \mathcal{X}$. Denote $\mathcal{N} := \mathcal{X}^{\perp_{\mathbb{E}}}/\mathcal{X}$. The correspondence $\mathcal{R} \mapsto \overline{\mathcal{R}} := \mathcal{R}/\mathcal{X}$ gives*

- (1) *a one-one correspondence between rigid subcategories of \mathcal{C} containing \mathcal{X} and rigid subcategories of \mathcal{N} , and*
- (2) *a one-one correspondence between cluster tilting subcategories of \mathcal{C} containing \mathcal{X} and cluster tilting subcategories of \mathcal{N} .*

Proof. Obviously, any rigid subcategories \mathcal{R} of \mathcal{C} containing \mathcal{X} is contained in $\mathcal{X}^{\perp_{\mathbb{E}}} = {}^{\perp_{\mathbb{E}}}\mathcal{X}$.

(1). This follows from Lemma 2.8.

(2). By Lemma 2.8, it suffices to show that \mathcal{R} is a functorially finite subcategory of \mathcal{C} if and only if $\overline{\mathcal{R}}$ is a functorially finite subcategory of \mathcal{N} .

Since \mathcal{R} and \mathcal{X} are functorially finite subcategories of \mathcal{C} . By Lemma 3.4, we have that $\overline{\mathcal{R}}$ is a functorially finite subcategory of $\mathcal{N} := \mathcal{X}^{\perp_{\mathbb{E}}}/\mathcal{X}$.

Conversely, for any $C \in \mathcal{C}$, since \mathcal{X} are a functorially finite subcategories of \mathcal{C} , there exists an \mathbb{E} -triangle

$$K \xrightarrow{f} X_0 \xrightarrow{g} C \dashrightarrow,$$

where g is a right \mathcal{X} -approximation of C . Applying the functor $\text{Hom}_{\mathcal{C}}(\mathcal{X}, -)$ to the above \mathbb{E} -triangle, we have the following exact sequence

$$\text{Hom}_{\mathcal{C}}(\mathcal{X}, X_0) \xrightarrow{\text{Hom}_{\mathcal{C}}(\mathcal{X}, g)} \text{Hom}_{\mathcal{C}}(\mathcal{X}, C) \rightarrow \mathbb{E}(\mathcal{X}, K) \rightarrow \mathbb{E}(\mathcal{X}, X_0) = 0.$$

Since g is a right \mathcal{X} -approximation of C , we have that $\text{Hom}_{\mathcal{C}}(\mathcal{X}, g)$ is an epimorphism. It follows that $\mathbb{E}(\mathcal{X}, K) = 0$ and then $K \in \mathcal{X}^{\perp_{\mathbb{E}}}$. Since $\overline{\mathcal{R}}$ is a cluster tilting subcategory of \mathcal{N} , there exists a triangle

$$K \xrightarrow{u} R_0 \xrightarrow{v} R_1 \xrightarrow{w} K\langle 1 \rangle,$$

in \mathcal{N} , where $R_0, R_1 \in \mathcal{R}$. Without loss of generality, we can assume that the above triangle can be induced by this \mathbb{E} -triangle

$$K \xrightarrow{u} R_0 \xrightarrow{v} R_1 \dashrightarrow,$$

where $R_0, R_1 \in \mathcal{R}$. By Proposition 3.15 in [NP], we obtain a commutative diagram

$$\begin{array}{ccccc} K & \xrightarrow{f} & X_0 & \xrightarrow{g} & C \dashrightarrow \\ \downarrow u & & \downarrow & & \parallel \\ R_0 & \xrightarrow{a} & M & \xrightarrow{b} & C \longrightarrow \\ \downarrow v & & \downarrow & & \\ R_1 & \xlongequal{\quad} & R_1 & & \\ \downarrow & & \downarrow & & \\ \Upsilon & & \Upsilon & & \end{array}$$

of \mathbb{E} -triangles in \mathcal{C} . Since \mathcal{R} is rigid and $X_0, R_1 \in \mathcal{R}$, we have $M \in \mathcal{R}$. Applying the functor $\text{Hom}_{\mathcal{C}}(\mathcal{R}, -)$ to this \mathbb{E} -triangle

$$R_0 \xrightarrow{a} M \xrightarrow{b} C \dashrightarrow,$$

we have the following exact sequence

$$\text{Hom}_{\mathcal{C}}(\mathcal{R}, M) \xrightarrow{\text{Hom}_{\mathcal{C}}(\mathcal{R}, b)} \text{Hom}_{\mathcal{C}}(\mathcal{R}, C) \rightarrow \mathbb{E}(\mathcal{R}, R_0) = 0.$$

This shows that $\text{Hom}_{\mathcal{C}}(\mathcal{R}, b)$ is an epimorphism. Thus \mathcal{R} is a contravariantly finite subcategory of \mathcal{C} . Similarly, we can show that \mathcal{R} is a covariantly finite subcategory of \mathcal{C} . Therefore, \mathcal{R} is a functorially finite subcategory of \mathcal{C} . \square

Theorem 3.6. *Let \mathcal{C} be a 2-Calabi-Yau extriangulated category, and \mathcal{X} be a functorially finite rigid subcategory of \mathcal{C} such that $\mathcal{P} \subseteq \mathcal{X}$. If $(\mathcal{U}, \mathcal{V})$ is an \mathcal{X} -mutation pair in \mathcal{C} , then*

- (1) \mathcal{U} is a rigid subcategory of \mathcal{C} if and only if so is \mathcal{V} .
- (2) \mathcal{U} is a cluster subcategory of \mathcal{C} if and only if so is \mathcal{V} .

Proof. By Proposition 3.3, we have that $\mathcal{N} := \mathcal{X}^{\perp_{\mathbb{E}}}/\mathcal{X}$ is a 2-Calabi-Yau triangulated category. By Lemma 1.9, we obtain that $(\overline{\mathcal{U}}, \overline{\mathcal{V}}) := (\mathcal{U}/\mathcal{X}, \mathcal{V}/\mathcal{X})$ forms 0-mutation pair in \mathcal{N} . Thus we have $\overline{\mathcal{V}} = \overline{\mathcal{U}}\langle 1 \rangle$. In particular, $\overline{\mathcal{U}}$ is a rigid subcategory (resp. cluster tilting subcategory) of \mathcal{N} if and only if $\overline{\mathcal{V}}$ is a rigid subcategory (resp. cluster tilting subcategory) of \mathcal{N} . On the other hand, by Theorem 3.5, we have that $\overline{\mathcal{U}}$ (resp. $\overline{\mathcal{V}}$) is a rigid subcategory (resp. cluster tilting subcategory) of \mathcal{N} if and only if \mathcal{U} (resp. \mathcal{V}) is a rigid subcategory (resp. cluster tilting subcategory) of \mathcal{C} . Thus the assertions follow. \square

Definition 3.7. Let \mathcal{C} be a 2-Calabi-Yau extriangulated category. We call a functorially finite rigid subcategory \mathcal{X} of \mathcal{C} such that $\mathcal{P} \subseteq \mathcal{X}$ *almost complete cluster tilting* if there exists a cluster tilting subcategory \mathcal{R} of \mathcal{C} such that $\mathcal{X} \subseteq \mathcal{R}$ and $\mathcal{R} = \mathcal{X} \cup \{R_0\}$, where R_0 is an indecomposable object which is not isomorphic to any object in \mathcal{X} . Such R_0 is called a *complement* of a almost complete cluster tilting subcategory \mathcal{X} .

Theorem 3.8. *Let \mathcal{C} be a 2-Calabi-Yau extriangulated category. Then any almost complete cluster tilting subcategory \mathcal{X} of \mathcal{C} is contained in exactly two cluster tilting subcategory \mathcal{R} and \mathcal{Q} of \mathcal{C} . Both $(\mathcal{R}, \mathcal{Q})$ and $(\mathcal{Q}, \mathcal{R})$ form \mathcal{X} -mutation pair.*

Proof. We have that $\mathcal{N} := \mathcal{X}^{\perp_{\mathbb{E}}}/\mathcal{X}$ is a 2-Calabi-Yau triangulated category, and 0 is a almost complete cluster tilting subcategory of \mathcal{N} . Since any almost complete cluster tilting subcategory are exactly two cluster tilting subcategory in 2-Calabi-Yau triangulated category, see [IY, Theorem 5.3]. Then the object 0 in \mathcal{N} has two complements, say Q_0, R_0 , and both (Q_0, R_0) (R_0, Q_0) form 0-mutations in \mathcal{N} . By Theorem 3.5, we have that \mathcal{X} is contained in exactly two cluster tilting subcategories: $\mathcal{R} = \mathcal{X} \cup \{R_0\}$, $\mathcal{Q} = \mathcal{X} \cup \{Q_0\}$. It is easy to see that $(\mathcal{R}, \mathcal{Q})$ and $(\mathcal{Q}, \mathcal{R})$ form \mathcal{X} -mutation pairs. \square

For almost complete cluster tilting object, we have the following.

Corollary 3.9. *Let \mathcal{C} be a 2-Calabi-Yau extriangulated category. Then any basic almost complete cluster tilting object R of \mathcal{C} such that $\mathcal{P} \in \text{add}R$ is a direct summand of exactly two basic cluster tilting objects in \mathcal{C} .*

Now for an almost complete cluster tilting subcategory \mathcal{X} of \mathcal{C} , assume that Q_0, R_0 are two complements of \mathcal{X} in Theorem 3.8. Then there are two \mathbb{E} -triangles related to Q_0, R_0 :

$$Q_0 \xrightarrow{a} X \xrightarrow{b} R_0 \xrightarrow{\delta} \gg$$

$$R_0 \xrightarrow{a'} X' \xrightarrow{b'} Q_0 \xrightarrow{\delta'} \gg$$

where a, a' are the minimal left \mathcal{X} -approximations, and b, b' are the minimal right \mathcal{X} -approximations. These two \mathbb{E} -triangles are called *exchange \mathbb{E} -triangle*.

4 Cotorsion pairs in a 2-Calabi-Yau extriangulated category

Let \mathcal{C} be a 2-Calabi-Yau extriangulated category with a cluster tilting object. We will give a classification of cotorsion pairs in \mathcal{C} in the first subsection and study the cluster structures in a cotorsion pair inherited from \mathcal{C} in the second subsection.

4.1 Classification of cotorsion pairs

Let \mathcal{C} be a Frobenius extriangulated category. For any objects $A, B \in \mathcal{C}$, by Lemma 2.8, we have a functorially isomorphism

$$\mathbb{E}(A, B) \simeq \text{Hom}_{\overline{\mathcal{C}}}(A, B\langle 1 \rangle).$$

Thus \mathcal{C} is a 2-Calabi-Yau if and only if the stable category $\overline{\mathcal{C}}$ is a 2-Calabi-Yau.

Lemma 4.1. *Let \mathcal{C} be a 2-Calabi-Yau extriangulated category. Then*

- (1) \mathcal{X} is a rigid subcategory of \mathcal{C} if and only if $\overline{\mathcal{X}}$ is a rigid subcategory of $\overline{\mathcal{C}}$.
- (2) \mathcal{X} is a cluster tilting subcategory of \mathcal{C} if and only if $\overline{\mathcal{X}}$ is a cluster tilting subcategory of $\overline{\mathcal{C}}$.

Proof. It is straightforward to check. □

Proposition 4.2. *Let \mathcal{C} be a 2-Calabi-Yau extriangulated category with a cluster tilting object and let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair in \mathcal{C} . Then the core $\mathcal{I} = \text{add } I$, for some rigid object I which containing all indecomposable projective objects in \mathcal{C} and there exists a decomposition of triangulated category ${}^{\perp \mathbb{E}}\mathcal{I}/\mathcal{I} = \mathcal{X}/\mathcal{I} \oplus \mathcal{Y}/\mathcal{I}$.*

Proof. It is easy to see that the core $\mathcal{I} = \text{add } I$, for some rigid object I containing all indecomposable projective objects, $\mathcal{X} = {}^{\perp \mathbb{E}}\mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{\perp \mathbb{E}}$. It is straightforward to check that $(\mathcal{X}, \mathcal{X})$ forms a I -mutation pair. Since \mathcal{X} is extension closed, we have that \mathcal{X}/\mathcal{I} is a triangulated category.

By Proposition 3.3, we have that ${}^{\perp \mathbb{E}}\mathcal{I}/\mathcal{I}$ is a 2-Calabi-Yau triangulated category. By Lemma 4.1, ${}^{\perp \mathbb{E}}\mathcal{I}/\mathcal{I}$ has a cluster tilting object. Since $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair with core \mathcal{I} , by Corollary 2.10, we have that $(\overline{\mathcal{X}}, \overline{\mathcal{Y}})$ is a cotorsion pair in the stable category $\overline{\mathcal{C}}$ with core $\overline{\mathcal{I}}$. By Corollary 4.5 in [ZZ2], we obtain that $(\overline{\mathcal{Y}}, \overline{\mathcal{X}})$ is a cotorsion pair in $\overline{\mathcal{C}}$ with core $\overline{\mathcal{I}}$. By Corollary 2.10, we have that $(\mathcal{Y}, \mathcal{X})$ is a cotorsion pair in \mathcal{C} with core \mathcal{I} . Thus we obtain that \mathcal{X} is functorially finite subcategory in \mathcal{C} . By Lemma II.2.2 in [BIRS], we have a decomposition

$${}^{\perp \mathbb{E}}\mathcal{I}/\mathcal{I} = \mathcal{X}/\mathcal{I} \oplus \mathcal{Y}/\mathcal{I}.$$

□

Combining this result with the decomposition theorem of 2-Calabi-Yau triangulated categories with a cluster tilting object in [ZZ2], we have the classification of cotorsion pairs with given core \mathcal{I} in a 2-Calabi-Yau extriangulated category with a cluster tilting object.

Theorem 4.3. *Let \mathcal{C} be a 2-Calabi-Yau extriangulated category with a cluster tilting object and \mathcal{I} a rigid subcategory of \mathcal{C} such that $\mathcal{P} \subseteq \mathcal{I}$. Let ${}^{\perp\mathbb{E}}\mathcal{I}/\mathcal{I} = \bigoplus_{j \in J} A_j$ be the complete decomposition of ${}^{\perp\mathbb{E}}\mathcal{I}/\mathcal{I}$ (where all A_j are indecomposable triangulated categories). Then*

1. *all cotorsion pairs with core \mathcal{I} are obtained as preimages under $\pi : {}^{\perp\mathbb{E}}\mathcal{I} \rightarrow {}^{\perp\mathbb{E}}\mathcal{I}/\mathcal{I}$ of the pairs $(\bigoplus_{j \in L} A_j, \bigoplus_{j \in J-L} A_j)$ where L is a subset of J . There are $2^{ns({}^{\perp\mathbb{E}}\mathcal{I}/\mathcal{I})}$ cotorsion pairs with core \mathcal{I} , where $ns(\mathcal{C})$ is the number of indecomposable direct summands of such decomposition of \mathcal{C} .*
2. *$(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair with core \mathcal{I} if and only if so is $(\mathcal{Y}, \mathcal{X})$.*

Proof. The first assertion follows from Proposition 4.2 and [ZZ2, Theorem 4.4], the second is a consequence of the first one. □

As an application to exact stably 2-Calabi-Yau categories with cluster tilting objects, we get a classification of cotorsion pairs in these categories.

Corollary 4.4. *Let \mathcal{B} be an exact stably 2-Calabi-Yau category with a cluster tilting object and \mathcal{I} a rigid subcategory of \mathcal{B} such that $\mathcal{P} \subseteq \mathcal{I}$. Let ${}^{\perp 1}\mathcal{I}/\mathcal{I} = \bigoplus_{j \in J} A_j$ be the complete decomposition of the triangulated category ${}^{\perp 1}\mathcal{I}/\mathcal{I}$. Then all cotorsion pairs with core \mathcal{I} are obtained as preimages under $\pi : {}^{\perp 1}\mathcal{I} \rightarrow {}^{\perp 1}\mathcal{I}/\mathcal{I}$ of the pairs $(\bigoplus_{j \in L} A_j, \bigoplus_{j \in J-L} A_j)$ where L is a subset of J . There are $2^{ns({}^{\perp 1}\mathcal{I}/\mathcal{I})}$ cotorsion pairs with core \mathcal{I} , where $ns(\mathcal{B})$ is the number of indecomposable direct summands of such decomposition of \mathcal{B} and $\mathcal{I}^{\perp 1} = \{M \in \mathcal{B} \mid \text{Ext}_{\mathcal{B}}^1(\mathcal{I}, M) = 0\}$. Moreover if $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair in \mathcal{B} , then so is $(\mathcal{Y}, \mathcal{X})$.*

From this corollary, we can get a classification of cotorsion pairs in the module categories over preprojective algebras of Dynkin quivers or in the categories \mathcal{C}_M , where M is a terminal module over preprojective algebras; these categories are used to categorify some cyclic cluster algebras with coefficients by Geiß-Leclerc-Schröer in [GLS2-3]. This classification on cotorsion pairs may be used to study the cluster subalgebras of the cluster algebras categorified by these categories.

4.2 Cluster structures in cotorsion pairs

Any extension closed subcategory of an extriangulated category is an extriangulated category, we can talk about cluster tilting subcategories in it.

Definition 4.5. Let \mathcal{X} be a contravariantly finite (or covariantly finite) extension closed subcategory of an extriangulated category \mathcal{C} and let \mathcal{R} be a subcategory of \mathcal{X} . We call that \mathcal{R} is an \mathcal{X} -cluster tilting subcategory provided that \mathcal{R} is a cluster tilting subcategory in \mathcal{X} , i.e. \mathcal{X} is functorially finite and satisfies that for any object $X \in \mathcal{X}$, $R \in \mathcal{R}$ if and only if $\mathbb{E}(R, X) = 0$

if and only if $\mathbb{E}(X, R) = 0$. An object R in \mathcal{X} is called an \mathcal{X} -cluster tilting object if $\text{add } R$ is an \mathcal{X} -cluster tilting subcategory.

The relation on cluster tilting subcategory between \mathcal{C} and its cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is given by the following:

Proposition 4.6. *Let \mathcal{C} be a 2-Calabi-Yau extriangulated category with a cluster tilting object, and $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair in \mathcal{C} with core \mathcal{I} . Then*

1. *Any cluster tilting subcategory \mathcal{T} containing \mathcal{I} can be written uniquely as: $\mathcal{T} = \mathcal{T}_{\mathcal{X}} \oplus \mathcal{I} \oplus \mathcal{T}_{\mathcal{Y}}$, such that $\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I}$ is \mathcal{X} -cluster tilting, and $\mathcal{T}_{\mathcal{Y}} \oplus \mathcal{I}$ is \mathcal{Y} -cluster tilting.*
2. *Any \mathcal{X} -cluster tilting subcategory (or \mathcal{Y} -cluster tilting subcategory) contains \mathcal{I} , and can be written as $\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I}$ ($\mathcal{T}_{\mathcal{Y}} \oplus \mathcal{I}$ resp.). Furthermore $\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I} \oplus \mathcal{T}_{\mathcal{Y}}$ is a cluster tilting subcategory in \mathcal{C} .*
3. *There is a bijection between the set of cluster tilting subcategories containing \mathcal{I} in \mathcal{C} and the product of the set of \mathcal{X} -cluster tilting subcategories with the set of \mathcal{Y} -cluster tilting subcategories. The bijection is given by $\mathcal{T} \mapsto (\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I}, \mathcal{T}_{\mathcal{Y}} \oplus \mathcal{I})$, where $\mathcal{T} = \mathcal{T}_{\mathcal{X}} \oplus \mathcal{I} \oplus \mathcal{T}_{\mathcal{Y}}$.*

Proof. The proof of Proposition 5.5 in [ZZ2] works also in this setting with the help of Theorem 1.8. □

Definition 4.7. Let \mathcal{C} be a 2-Calabi-Yau extriangulated category with cluster tilting objects.

1. For a cluster tilting subcategory \mathcal{T} in a 2-Calabi-Yau extriangulated category \mathcal{C} , we define $Q(\mathcal{T})$ as an ice quiver whose exchangeable vertices are the isomorphism classes of indecomposable objects in \mathcal{T} which are not project objects and the frozen vertices are the isomorphism classes of indecomposable projective objects. For two vertices T_i and T_j (not both the frozen vertices), the number of arrows from T_i to T_j is the dimension of irreducible morphism space $\text{irr}(T_i, T_j) = \text{rad}(T_i, T_j) / \text{rad}^2(T_i, T_j)$ in \mathcal{T} .
2. For a \mathcal{X} -cluster tilting subcategory $\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I}$ in \mathcal{X} , we define $Q(\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I})$ as an ice quiver whose exchangeable vertices are the isomorphism classes of indecomposable objects in $\mathcal{T}_{\mathcal{X}}$ and the frozen vertices are the isomorphism classes of indecomposable objects in \mathcal{I} . For two vertices T_i and T_j (not both the frozen vertices), the number of arrows from T_i to T_j is the dimension of irreducible morphism space $\text{irr}(T_i, T_j)$ in $\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I}$. The quiver $Q(\mathcal{T}_{\mathcal{Y}} \oplus \mathcal{I})$ of \mathcal{Y} -cluster tilting subcategory is defined similarly.
3. For a cluster tilting subcategory $\overline{\mathcal{T}}$ in $\overline{\mathcal{C}}$, we define $Q(\overline{\mathcal{T}})$ as a quiver whose (exchangeable) vertices are the isomorphism classes of indecomposable objects of $\overline{\mathcal{T}}$. For two vertices \overline{T}_i and \overline{T}_j , the number of arrows from \overline{T}_i to \overline{T}_j is the dimension of irreducible morphism space $\text{irr}(\overline{T}_i, \overline{T}_j)$ in $\overline{\mathcal{T}}$.

The cluster structure in a 2-Calabi-Yau triangulated category or in an exact stably 2-Calabi-Yau category is defined in [BIRS] and [FK]. This structure is given by cluster tilting subcategories and ‘categorifies’ the cluster algebra associated to the quivers of the cluster tilting subcategories. Then one can use a cluster map [BIRS] to transform a cluster structure in these categories to the cluster algebra. The cluster map is also called the cluster character in [P]. In our setting, we similarly define a cluster structure in 2-Calabi-Yau extriangulated categories with cluster tilting subcategories and assume that there always be a cluster map from the cluster structure to a cluster algebra. We omit the related definitions and the detailed discussions, and refer to [BIRS] and [FK] for more details. However, we have the following equivalent description of the cluster structure, where the cases of the triangulated category and the exact category are proved in Theorem II.1.6 of [BIRS].

Theorem 4.8. *Let \mathcal{C} be a 2-Calabi-Yau extriangulated category with cluster tilting objects. If \mathcal{C} has no loops or 2-cycles (this means the quivers of any cluster tilting objects contains neither loops nor 2-cycles), then the cluster tilting subcategories determine a cluster structure for \mathcal{C} .*

Proof. Similar to the proof of Theorem II.1.6 in [BIRS]. □

From now to the end of the paper, we fix the following settings. Let \mathcal{C} be a 2-Calabi-Yau extriangulated category with cluster tilting objects, and cluster tiling objects form a cluster structure. Let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair of \mathcal{C} with core \mathcal{I} . Then by Proposition 4.6, we can write a cluster tilting object \mathcal{T} as $\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I} \oplus \mathcal{T}_{\mathcal{Y}}$ with $\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I}$ being \mathcal{X} -cluster tilting and $\mathcal{T}_{\mathcal{Y}} \oplus \mathcal{I}$ being \mathcal{Y} -cluster tilting.

Proposition 4.9. *Under above settings,*

1. *the quiver $Q(\overline{\mathcal{T}})$ is obtained from $Q(\mathcal{T})$ by deleting all the frozen vertices and arrows connected to these vertices.*
2. *the quiver $Q(\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I})$ is obtained from $Q(\mathcal{T})$ by deleting all the isomorphism classes of indecomposable direct summands of $\mathcal{T}_{\mathcal{Y}}$ and arrows connected to these vertices, and freezing the isomorphism classes of indecomposable direct summands of \mathcal{I} .*

Proof. 1. Since an object becomes zero object in $\overline{\mathcal{C}}$ if and only if it belongs to the projective subcategory *Proj*, the vertices of $Q(\overline{\mathcal{T}})$ are just the exchangeable vertices of $Q(\mathcal{T})$. It can be directly derived from the homomorphism theorem of groups that, for any two exchangeable vertices T_1 and T_2 in $Q(\mathcal{T})$, the dimension of $\text{irr}(T_1, T_2)$ in \mathcal{C} is equal to the dimension of $\text{irr}(\overline{T_1}, \overline{T_2})$ in $\overline{\mathcal{C}}$. Then by the Definition 4.7 of the quivers, we are done.

2. We only need to show that for any two vertices T_1 and T_2 of $Q(\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I})$ (not both frozen vertices), the dimension of $\text{irr}(T_1, T_2)$ in \mathcal{T} is the same as the dimension of $\text{irr}(T_1, T_2)$ in $\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I}$, or equivalently, a map f from T_1 to T_2 is irreducible in \mathcal{T} if and only if it is irreducible in $\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I}$. For the convenience, we assume that T_1 is an indecomposable direct summand of $\mathcal{T}_{\mathcal{X}}$. It is clear that if f is irreducible in \mathcal{T} , then it is irreducible in $\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I}$. Conversely, for an irreducible map f in $\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I}$, if it is not irreducible in \mathcal{T} , then it factor

through an object T_3 in $\mathcal{T}_{\mathcal{Y}}$. We write f as a composition of $f_1: T_1 \rightarrow T_3$ and $f_2: T_3 \rightarrow T_2$. By Proposition 4.2, f_1 factors through an object I_1 in \mathcal{I} , and we write it as a composition $g_1: T_1 \rightarrow I_1$ and $g_2: I_1 \rightarrow T_3$. Note that g_1 and f_2g_2 are both in the radical space of $\mathcal{X} \oplus \mathcal{I}$, this is contradict to the assumption that f is irreducible in $\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I}$. Therefore f is irreducible in \mathcal{T} . □

Lemma 4.10. *Under above settings, there are cluster structures in $\overline{\mathcal{C}}$, \mathcal{X} and \mathcal{Y} , which are induced from the cluster structure of \mathcal{C} . We call the cluster structures in \mathcal{X} and \mathcal{Y} the cluster substructure of the cluster structure in \mathcal{C} .*

Proof. It follows from Proposition 4.6 and the above proposition that the categories $\overline{\mathcal{C}}$, \mathcal{X} and \mathcal{Y} have no loops or 2-cycles. It follows from Theorem 4.8 that all of these categories, as extriangulated categories, have cluster structures, which are induced from the cluster structure of \mathcal{C} . □

5 Relation between cotorsion pairs and pairs of rooted cluster subalgebras

In this section, we study cotorsion pairs in a 2-Calabi-Yau extriangulated category with cluster tilting object in the point of view categorification. In the first subsection, we recall some basics on the cluster algebras introduced in [FZ] and the rooted cluster algebras introduced in [ADS]. In the second subsection, we give a correspondence between cotorsion pairs with certain pairs of rooted cluster subalgebras.

5.1 Rooted cluster subalgebras

Recall that a quiver is a quadruple (Q_0, Q_1, s, t) consisting of two sets: Q_0 (the set of vertices) and Q_1 (the set of arrows), and of two maps s, t which map each arrow $\alpha \in Q_1$ to its source $s(\alpha)$ and its target $t(\alpha)$, respectively. An ice quiver is a quiver Q associated a subset F (the set of frozen vertices) of its vertex set. The full subquiver of Q with vertex set $Q_0 \setminus F$ (the set of exchangeable vertices) is call the exchangeable part of Q . We assume in this section that there are no loops or 2-cycles, and no arrows between frozen vertices in the ice quivers. Let $m+n = |Q_0|$ the number of vertices in an ice quiver Q , and denote vertices by $Q_0 = \{1, 2, \dots, m+n\}$ and frozen vertices by $F = \{m+1, m+2, \dots, m+n\}$. By associating each vertex $1 \leq i \leq m+n$ an indeterminate element x_i , we have a set $\mathbf{x} = \{x_1, x_2, \dots, x_{m+n}\}$. Denote by $\mathbf{ex} = \{x_1, x_2, \dots, x_m\}$ and $\mathbf{fx} = \{x_{m+1}, x_{m+2}, \dots, x_{m+n}\}$. The cluster algebra \mathcal{A}_Q is a \mathbb{Z} -subalgebra of the rational function field $\mathcal{F} = \mathbb{Q}(x_1, \dots, x_{m+n})$ generated by generators obtained recursively from \mathbf{x} in the following manner. For an exchangeable vertex $1 \leq i \leq m$, we obtain a new triple $(\mu_i(\mathbf{ex}), \mathbf{fx}, \mu_i(Q))$ from $(\mathbf{ex}, \mathbf{fx}, Q)$ by a mutation μ_i at i defined as follows.

Firstly, $\mu_i(Q) = (Q')$ is obtained by:

- (a) reverse all arrows incident with i ;

- (b) inserting a new arrow $\gamma : j \rightarrow k$ for each path $j \xrightarrow{\alpha} i \xrightarrow{\beta} k$;
- (c) removing 2-cycles and arrows between frozen vertices.

Secondly, $\mu_i(\mathbf{ex}) = (\mathbf{ex} \setminus \{x_i\}) \cup \{x'_i\}$ where $x'_i \in \mathcal{F}$ is defined by the following exchange relation:

$$x_i x'_i = \prod_{\alpha:i \rightarrow j} x_j + \prod_{\alpha:j \rightarrow i} x_j.$$

Denote by \mathcal{X} the union of all possible sets of variables obtained from \mathbf{x} by successive mutations. Then the cluster algebra \mathcal{A}_Q is the \mathbb{Z} -subalgebra of \mathcal{F} generated by \mathcal{X} . We call each triple $\tilde{\Sigma} = (\tilde{\mathbf{ex}}, \tilde{\mathbf{fx}}, \tilde{Q},)$ obtained from $\Sigma = (\mathbf{ex}, \mathbf{fx}, Q)$ by successive mutations a seed, and $\tilde{\mathbf{x}} = \tilde{\mathbf{ex}} \sqcup \tilde{\mathbf{fx}}$ a cluster. The elements $\tilde{x}_1, \dots, \tilde{x}_{m+n}$ of a cluster $\tilde{\mathbf{x}}$ are cluster variables. The variables in the subset $\tilde{\mathbf{ex}}$ are exchangeable cluster variables and the variables in the subset $\tilde{\mathbf{fx}}$ are frozen cluster variables. A rooted cluster algebra associated to \mathcal{A}_Q is a pair (Σ, \mathcal{A}_Q) (or \mathcal{A}_Σ for brevity).

For the aim of constructing a category frame work of studying cluster algebras, rooted cluster morphisms are introduced in [ADS]. They are special ring homomorphisms between rooted cluster algebras which are compatible with cluster mutations. A sequence (x_1, \dots, x_l) is called $(\mathbf{ex}, \mathbf{fx}, Q)$ -admissible if $x_1 \in \mathbf{ex}$ and x_i is in $\mu_{x_{i-1}} \circ \dots \circ \mu_{x_1}(\mathbf{ex})$ for every $2 \leq i \leq l$. A rooted cluster morphism f from \mathcal{A}_Σ to $\mathcal{A}_{\Sigma'}$ is a ring homomorphism satisfies the following three conditions:

- (a) $f(\mathbf{ex}) \subset \mathbf{ex}' \sqcup \mathbb{Z}$;
- (b) $f(\mathbf{fx}) \subset \mathbf{x}' \sqcup \mathbb{Z}$;
- (c) For every $(f, \mathbf{x}, \mathbf{x}')$ -biadmissible sequence (x_1, \dots, x_l) , we have $f(\mu_{x_l} \circ \dots \circ \mu_{x_1, \mathbf{x}}(y)) = \mu_{f(x_l)} \circ \dots \circ \mu_{f(x_1), \mathbf{x}'}(f(y))$ for any y in \mathbf{x} . Here a $(f, \mathbf{x}, \mathbf{x}')$ -biadmissible sequence (x_1, \dots, x_l) is a Σ -admissible sequence such that $(f(x_1), \dots, f(x_l))$ is Σ' -admissible.

Definition 5.1. We call \mathcal{A}_Σ a rooted cluster subalgebra of $\mathcal{A}_{\Sigma'}$ if there is an injective rooted cluster morphism from \mathcal{A}_Σ to $\mathcal{A}_{\Sigma'}$.

We collect some definitions and results from [CZ] to introduce the complete pair of a rooted cluster algebra.

Definition-Proposition 5.2. Let Q be an ice quiver.

- (a) We call Q indecomposable if it is connected and its exchangeable part is also connected. We call a seed indecomposable if its quiver is indecomposable.
- (b) We call a full subquiver Q' of Q a connected component if the exchangeable part is connected and the frozen vertices are all the frozen vertices in Q which are connected directly to the exchangeable vertices of Q' .
- (c) Let $\Sigma = (\mathbf{ex}, \mathbf{fx}, Q)$ be a seed and \mathbf{ex}' be a subset of \mathbf{ex} . We call $\Sigma_f = (\mathbf{ex} \setminus \mathbf{ex}', \mathbf{fx} \sqcup \mathbf{ex}', Q_f)$ the freezing of Σ at \mathbf{ex}' , where Q_f is obtained from Q by freezing vertices corresponding to elements in \mathbf{ex}' and deleting the arrows between these vertices.

- (d) Let $\Sigma_1 = (\mathbf{ex}_1, \mathbf{fx}_1, Q_1)$ and $\Sigma_2 = (\mathbf{ex}_2, \mathbf{fx}_2, Q_2)$ be two seeds. Assume that there is a bijection between \mathbf{fx}_1 and \mathbf{fx}_2 , then we obtain a quiver Q by gluing Q_1 and Q_2 (as graphs) together at frozen vertices unified under this bijection. A gluing of Σ_1 and Σ_2 is a seed $\Sigma = (\mathbf{ex}, \mathbf{fx}, Q)$, where $\mathbf{ex} = \mathbf{ex}_1 \sqcup \mathbf{ex}_2$ and $\mathbf{fx} = \mathbf{fx}_1$ (or equivalently \mathbf{fx}_2).
- (e) Let Σ be a seed and Σ_f be a freezing of Σ . Then \mathcal{A}_{Σ_f} is a rooted cluster subalgebra of \mathcal{A}_{Σ} . Let Σ' be a gluing of some connected components of \mathcal{A}_{Σ_f} , then $\mathcal{A}_{\Sigma'}$ is a rooted cluster subalgebra of \mathcal{A}_{Σ_f} , and thus a rooted cluster subalgebra of \mathcal{A}_{Σ} . Conversely, all the rooted cluster subalgebras of \mathcal{A}_{Σ} is obtained from the above two ways. For more details, see [CZ].

Definition 5.3. Let $\Sigma = (\mathbf{ex}, \mathbf{fx}, Q)$ be a seed and Σ_f be the freezing of Σ at a subset \mathbf{ex}' of \mathbf{ex} . Denote by \mathbf{fx}_0 the set of isolated frozen variables of Σ_f . Let $\Sigma_1 = (\mathbf{ex}_1, \mathbf{fx}_1, Q_1)$ and $\Sigma_2 = (\mathbf{ex}_2, \mathbf{fx}_2, Q_2)$ be two seeds. We call the pair $(\mathcal{A}(\Sigma_1), \mathcal{A}(\Sigma_2))$ a complete pair of subalgebras of $\mathcal{A}(\Sigma)$ with coefficient set $\mathbf{fx} \sqcup \mathbf{ex}'$ if the following conditions are satisfied:

1. both Σ_1 and Σ_2 are gluings of some indecomposable components of Σ_f ;
2. $\mathbf{ex}_1 \cap \mathbf{ex}_2 = \emptyset$ and $\mathbf{ex} = \mathbf{ex}_1 \sqcup \mathbf{ex}_2 \sqcup \mathbf{ex}'$;
3. $\mathbf{fx} \cup \mathbf{fx}_0 \subseteq \mathbf{fx}_1 \cap \mathbf{fx}_2$.

5.2 Relations with cluster algebras

Let \mathcal{C} be a 2-Calabi-Yau extriangulated category with (non-zero) cluster tilting objects. Fix a cluster tilting object T , denote \mathcal{T} . Let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair with core \mathcal{I} . We denote the rooted cluster algebras corresponding to the quivers $Q(\mathcal{T})$, $Q(\overline{\mathcal{T}})$, $Q(\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I})$ and $Q(\mathcal{T}_{\mathcal{Y}} \oplus \mathcal{I})$ as $\mathcal{A}(\mathcal{T})$, $\mathcal{A}(\overline{\mathcal{T}})$, $\mathcal{A}(\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I})$ and $\mathcal{A}(\mathcal{T}_{\mathcal{Y}} \oplus \mathcal{I})$ respectively.

- $R(\mathcal{T})$: the subcategory of \mathcal{C} additive generated by rigid objects which are reachable from \mathcal{T} by a finite number of mutations, where the mutation is defined in Theorem 3.8.
- $R(\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I})$: the subcategory of \mathcal{C} additive generated by rigid objects which are reachable from $\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I}$ by finite number of mutations.
- $R(\mathcal{T}_{\mathcal{Y}} \oplus \mathcal{I})$: the subcategory of \mathcal{C} additive generated by rigid objects which are reachable from $\mathcal{T}_{\mathcal{Y}} \oplus \mathcal{I}$ by finite number of mutations.

Theorem 5.4. *Let \mathcal{C} be a 2-Calabi-Yau extriangulated category with cluster tilting objects, which form a cluster structure. The correspondence $(\mathcal{X}, \mathcal{Y}) \mapsto (\mathcal{A}(\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I}), \mathcal{A}(\mathcal{T}_{\mathcal{Y}} \oplus \mathcal{I}))$ given by the cluster map χ derives a bijection:*

$$\begin{aligned} & \{\text{cotorsion pairs in } \mathcal{C} \text{ with core } \mathcal{I}\} \\ & \quad \Downarrow \\ & \{\text{complete pairs of rooted cluster subalgebras of } \mathcal{A}(\mathcal{T}) \text{ with coefficient set } \chi(\mathcal{I})\}. \end{aligned}$$

This bijection induces the following bijection:

\{t-structures in \mathcal{C}\}

\Updownarrow

\{complete pairs of rooted cluster subalgebras of \mathcal{A}(\mathcal{T}) with coefficients set \chi(\text{Proj-Inj})\}.

Proof. For a cotorsion pair $(\mathcal{X}, \mathcal{Y})$, we have a decomposition of categories ${}^{\perp_{\mathbb{E}}}\mathcal{I}/\mathcal{I} = \mathcal{X}/\mathcal{I} \oplus \mathcal{Y}/\mathcal{I}$ from Proposition 4.2. Therefore morphisms between any two objects representing vertices $T_1 \in \mathcal{T}_{\mathcal{X}}$ and $T_2 \in \mathcal{T}_{\mathcal{Y}}$ factor through objects in \mathcal{I} . So there are no arrows between T_1 and T_2 in $Q(\mathcal{T})$, and thus $Q(\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I})$ and $Q(\mathcal{T}_{\mathcal{Y}} \oplus \mathcal{I})$ are both gluings of some indecomposable components of $Q(\mathcal{T})$. The rest two conditions of complete pairs in Definition 5.3 are clear. Therefore $(\mathcal{A}(\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I}), \mathcal{A}(\mathcal{T}_{\mathcal{Y}} \oplus \mathcal{I}))$ is a complete pair of rooted cluster subalgebras of $\mathcal{A}(\mathcal{T})$ with coefficient set $\chi(\mathcal{I})$. Conversely, if we have a complete pair $(\mathcal{A}', \mathcal{A}'')$ of rooted cluster subalgebras of $\mathcal{A}(\mathcal{T})$. From Proposition-Definition 5.2 (e) and the definition of complete pairs of rooted cluster subalgebras, the rooted cluster subalgebras \mathcal{A}' and \mathcal{A}'' are of the forms $\mathcal{A}(\mathcal{T}' \oplus \mathcal{I})$ and $\mathcal{A}(\mathcal{T}'' \oplus \mathcal{I})$ with $\mathcal{T} = \mathcal{T}' \oplus \mathcal{I} \oplus \mathcal{T}''$, and in the quiver $Q(\mathcal{T})$, there are no arrows between vertices in \mathcal{T}' and vertices in \mathcal{T}'' . Now we consider the pair $(\mathcal{T}', \mathcal{T}'')$ in the subfactor category ${}^{\perp_{\mathbb{E}}}\mathcal{I}/\mathcal{I}$, which is a 2-Calabi-Yau triangulated category by Proposition 3.3. Note that $\mathcal{T}' \oplus \mathcal{T}''$ is a cluster tilting subcategory in ${}^{\perp_{\mathbb{E}}}\mathcal{I}/\mathcal{I}$ by Proposition 4.6 and we have $\text{Hom}_{{}^{\perp_{\mathbb{E}}}\mathcal{I}/\mathcal{I}}(\mathcal{T}', \mathcal{T}'') = \text{Hom}_{{}^{\perp_{\mathbb{E}}}\mathcal{I}/\mathcal{I}}(\mathcal{T}'', \mathcal{T}') = 0$. Thus we have ${}^{\perp_{\mathbb{E}}}\mathcal{I}/\mathcal{I} = (\mathcal{T}' \oplus \mathcal{T}'') * (\mathcal{T}'\langle 1 \rangle \oplus \mathcal{T}''\langle 1 \rangle) = \mathcal{T}' * \mathcal{T}'\langle 1 \rangle \oplus \mathcal{T}'' * \mathcal{T}''\langle 1 \rangle = \mathcal{C}_1 \oplus \mathcal{C}_2$ as a decomposition of triangulated category by Proposition 3.5 in [ZZ]. Let $\pi: {}^{\perp_{\mathbb{E}}}\mathcal{I} \rightarrow {}^{\perp_{\mathbb{E}}}\mathcal{I}/\mathcal{I}$ be the natural projection. Then because $(\mathcal{C}_1, \mathcal{C}_2)$ is a cotorsion pair in ${}^{\perp_{\mathbb{E}}}\mathcal{I}/\mathcal{I}$ with core $\{0\}$, $(\mathcal{X}', \mathcal{Y}') = (\pi^{-1}(\mathcal{C}_1), \pi^{-1}(\mathcal{C}_2))$ is a cotorsion pair in \mathcal{C} with core \mathcal{I} by Corollary 2.10. It is clear that the above two kinds of processes are the inverse processes with each other. Since the t -structures are those cotorsion pairs with cores Proj-Inj , the second correspondence is clear from the first one. \square

Remark 5.5. Applying the theorem to 2-Calabi-Yau triangulated categories with cluster tilting objects, we recover the corresponding results in [CZ]. The application to exact stably 2-Calabi-Yau categories \mathcal{C} with cluster tilting objects yields a new correspondence between cotorsion pairs in the categories \mathcal{C} and the complete pairs of cluster subalgebras of the cluster algebras categorified by \mathcal{C} . We state it here in the following:

Corollary 5.6. *Let \mathcal{B} be an exact stably 2-Calabi-Yau category with cluster tilting objects, which form a cluster structure, \mathcal{I} a rigid subcategory of \mathcal{B} such that $\mathcal{P} \subseteq \mathcal{I}$. Then the correspondence $(\mathcal{X}, \mathcal{Y}) \mapsto (\mathcal{A}(\mathcal{T}_{\mathcal{X}} \oplus \mathcal{I}), \mathcal{A}(\mathcal{T}_{\mathcal{Y}} \oplus \mathcal{I}))$ given by the cluster map χ derives a bijection:*

\{cotorsion pairs in \mathcal{B} with core \mathcal{I}\}

\Updownarrow

\{complete pairs of rooted cluster subalgebras of \mathcal{A}(\mathcal{T}) with coefficient set \chi(\mathcal{I})\}.

6 An example

Here we explain the correspondence in Theorem 5.4 by an example relating to a Grassmannian cluster algebra. Recall that Fomin and Zelevinsky [FZ] proved that the homogenous coordinate

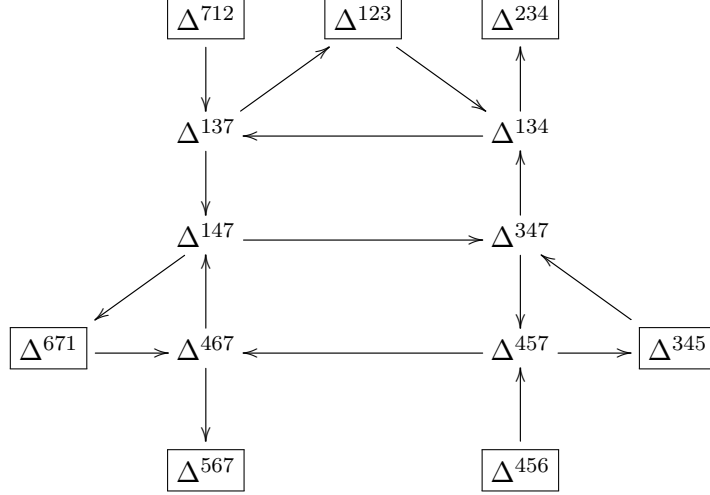


Figure 1: The initial quiver of the cluster algebra $\mathbb{C}(G_{3,7})$

ring of the Grassmannian $G_{2,n}$ of 2-planes in n -space has a cluster algebra structure. Scott [S] generalized this result to each Grassmannian $G_{r,n}$ by using Postnikov arrangements. Since such a cluster algebra has non-trivial coefficients, the Frobenius category is the candidate to categorify it [BIRS, FK]. In 2008, Geiß-Leclerc-Shröer [GLS1] described such a category in the representation category of a preprojective algebra, which provided a categorification the cluster algebra structure on the coordinate ring of the big cell in the Grassmannian. Recently, by relating to this result, Jensen-King-Su [JKS] found a ring $A_{r,n}$ whose category of maximal Cohen-Macaulay modules is a Frobenius category, which categorified the coordinate ring of Grassmannian $G_{r,n}$. Specially, we consider the Frobenius category relating to E_6 type cluster algebra of $G_{3,7}$.

Example 6.1. Recall that under the Plucker embedding $G_{3,7} \rightarrow \mathbb{P}(\wedge^3 \mathbb{C}^7)$, the coordinate ring $\mathbb{C}(G_{3,7})$ of $G_{3,7}$ is generated by the Plucker coordinates x_A of 3-subset A of $\{1,2,3,4,5,6,7\}$ which obey the Plucker relations. By viewing $\mathbb{C}(G_{3,7})$ as a cluster algebra, these coordinates are special cluster variables and these relations are special cluster exchange relations [S]. Following [S], we consider the initial quiver of $\mathbb{C}(G_{3,7})$ depicted in Figure 1, where we denote the vertex of cluster variable x_A as Δ^A and frame the frozen vertices.

After ordered mutations at points $\Delta^{147}, \Delta^{347}, \Delta^{137}, \Delta^{134}, \Delta^{467}, \Delta^{134}$ and Δ^{457} , we obtain a new quiver with cluster

$$\mathbf{x} = \mathbf{ex} \sqcup \mathbf{fx} = \{x_{126}, x_{135}, x_{235}, x_{367}, x', x''\} \sqcup \{x_{123}, x_{234}, x_{345}, x_{456}, x_{567}, x_{671}, x_{712}\},$$

which consists of ten Plucker coordinates and other two variables x' and x'' . The new quiver Q is in Figure 2, which is the standard bipartite quiver of E_6 type after deleting the frozen vertices. We still denote the point of Plucker coordinate x_A as Δ^A , and Δ' and Δ'' are the points of the two exceptional cluster variables x' and x'' respectively.

Then a new seed $\Sigma = (\mathbf{ex}, \mathbf{fx}, Q)$ of the cluster algebra $\mathbb{C}(G_{3,7})$ is obtained. Now we fix the

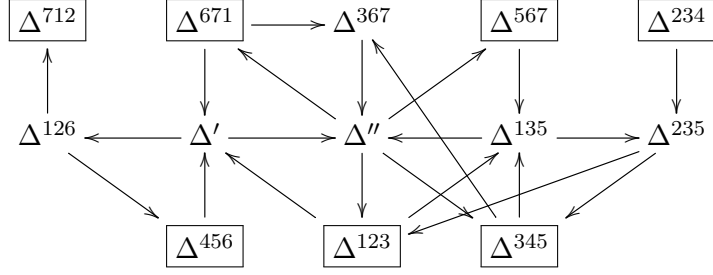


Figure 2: The standard quiver Q of the cluster algebra $\mathbb{C}(G_{3,7})$

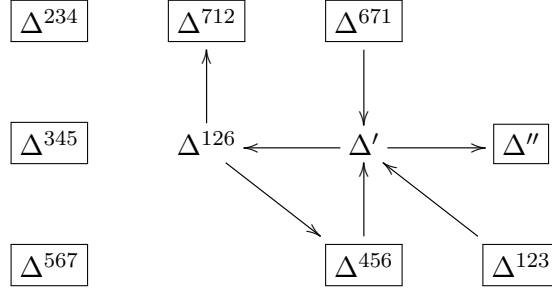


Figure 3: The quiver Q' of rooted cluster subalgebra \mathcal{A}' of $\mathbb{C}(G_{3,7})$

exchangable cluster variable x'' in \mathbf{ex} , while Δ'' becomes frozen in Q . Then note that Q can be separated as three indecomposable quivers at vertex Δ'' , correspondingly, the rooted cluster algebra $(\mathbb{C}(G_{3,7}), \Sigma)$ have three indecomposable rooted cluster subalgebra \mathcal{A}' , \mathcal{A}'' and \mathcal{A}''' with coefficients set $\{x''\} \sqcup \mathbf{fx}$. The quivers of these rooted cluster algebras are the Q' , Q'' and Q''' in Figure 3, 4 and 5 respectively.

So there are $C_3^0 + C_3^1 + C_3^2 + C_3^3 = 8$ complete pairs of rooted cluster subalgebra of $(\mathbb{C}(G_{3,7}), \Sigma)$ with coefficient set $\{x''\} \sqcup \mathbf{fx}$.

Now we recall the Frobenius category \mathcal{C} from [JKS] and correspond these complete subalgebra pairs to the cotorsion pairs in the category with the core corresponding to cluster variables in $\{x''\} \sqcup \mathbf{fx}$. In [JKS], Jensen-King-Su described a complete algebra $A_{3,7}$ as a quotient algebra of preprojective algebra of type \tilde{A}_7 up to some relations. The category $\mathbf{CM}(A_{3,7})$ of maximal Cohen-Macaulay $A_{3,7}$ -modules is a Hom-finite Krull-Schmidt stably 2-Calabi-Yau Frobenius category

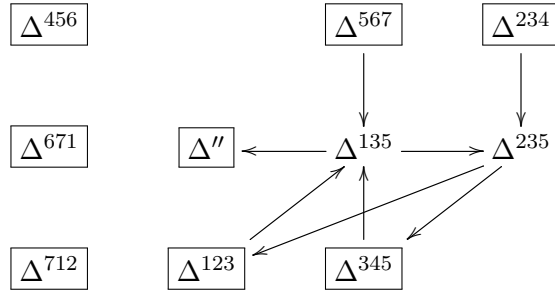


Figure 4: The quiver Q'' of rooted cluster subalgebra \mathcal{A}'' of $\mathbb{C}(G_{3,7})$

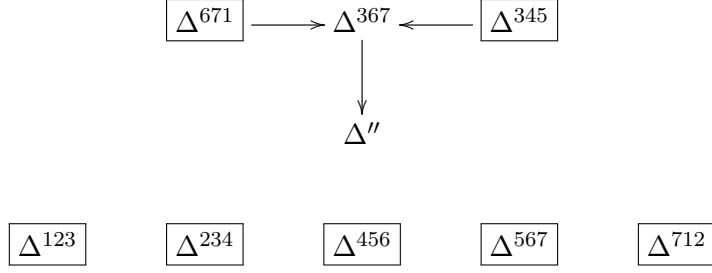


Figure 5: The quiver Q''' of rooted cluster subalgebra \mathcal{A}''' of $\mathbb{C}(G_{3,7})$

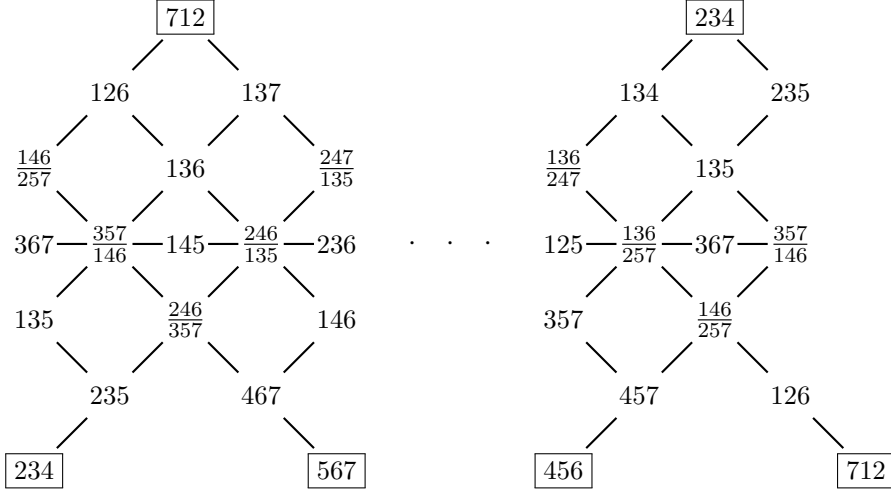


Figure 6: The Auslander-Reiten quiver of $\text{CM}(A_{3,7})$

with cluster tilting objects. The Auslander-Reiten quiver of $\text{CM}(A_{3,7})$ is depicted in Figure 6, where the objects are represented by ‘profiles’ of the modules, see Section 6 of [JKS] for details.

The quiver is periodical with two sides coincide under a Mobius reflection. The shape of omitted part is a two times repetition of the left side quiver. Here the framed objects in $T_{pi} = 123 \oplus 234 \oplus 345 \oplus 456 \oplus 567 \oplus 671 \oplus 712$ are projective-injective objects and $T = 126 \oplus \frac{146}{257} \oplus \frac{357}{146} \oplus 135 \oplus 235 \oplus 367 \oplus 123 \oplus 234 \oplus 345 \oplus 456 \oplus 567 \oplus 671 \oplus 712$ is a cluster tilting object. Under the cluster map $\chi : \text{CM}(A_{3,7}) \rightarrow \mathbb{C}(G_{3,7})$, T is mapped to \mathbf{x} , where the explicit correspondence is given in Table 1. With this correspondence, the quiver of $\text{End}_{\text{CM}(A_{3,7})}(T)$ is isomorphic to the quiver of the cluster \mathbf{x} , that is the Q in Figure 2. Moreover, the separation of Q at Δ'' corresponds to a separation of T as three indecomposable objects $T_1 = 126 \oplus \frac{146}{257} \oplus \frac{357}{146} \oplus 123 \oplus 234 \oplus 345 \oplus 456 \oplus 567 \oplus 671 \oplus 712$, $T_2 = \frac{357}{146} \oplus 135 \oplus 235 \oplus 123 \oplus 234 \oplus 345 \oplus 456 \oplus 567 \oplus 671 \oplus 712$ and $T_3 = \frac{357}{146} \oplus 367 \oplus 123 \oplus 234 \oplus 345 \oplus 456 \oplus 567 \oplus 671 \oplus 712$, whose quivers are isomorphic to Q_1 , Q_2 and Q_3 respectively. There are three indecomposable functionally finite extension-closed subcategories \mathcal{X}_1 , \mathcal{X}_2 and \mathcal{X}_3 in $\text{CM}(A_{3,7})$ with projective-injective object $I = \frac{357}{146} \oplus T_{pi}$. We draw up the Auslander-Reiten quivers of them in Figure 7, 8 and 9 respectively. The objects T_1 , T_2 and T_3 are cluster tilting objects of \mathcal{X}_1 , \mathcal{X}_2 and \mathcal{X}_3 respectively. These subcategories make

Rigid object in T	Cluster variable in \mathbf{x}
126	x_{126}
$\frac{146}{257}$	x'
$\frac{357}{146}$	x''
135	x_{135}
235	x_{235}
367	x_{367}
123	x_{123}
234	x_{234}
345	x_{345}
456	x_{456}
567	x_{567}
671	x_{671}
712	x_{712}

Table 1: Cluster map correspondence on the cluster tilting object T

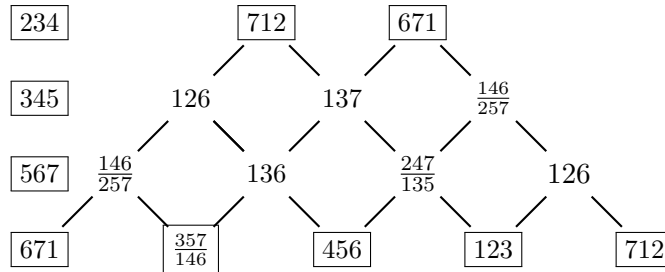


Figure 7: The Auslander-Reiten quiver of χ_1

up eight cotorsion pairs of $\text{CM}(A_{3,7})$ with core $I = \frac{357}{146} \oplus T_{pi}$. Finally, they correspond to eight complete subalgebras pairs of $(\mathbb{C}(G_{3,7}), \Sigma)$ consisted of rooted cluster subalgebras \mathcal{A}' , \mathcal{A}'' and \mathcal{A}''' described above.

Acknowledgement

The correspondence in Corollary 5.6 was asked by Osamu Iyama in a conference at Seoul Korea in 2014. We are grateful to him for this!

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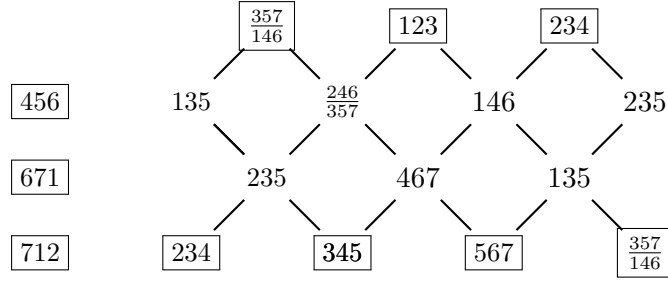


Figure 8: The Auslander-Reiten quiver of χ_2

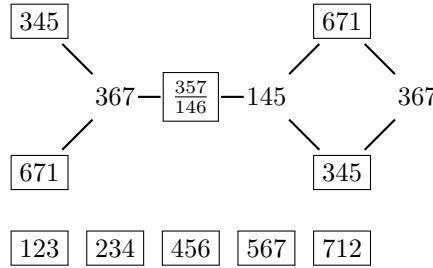


Figure 9: The Auslander-Reiten quiver of χ_3

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