# $L^p$ gradient estimate for elliptic equations with high-contrast conductivities in $\mathbb{R}^n$

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Uniform estimate for the solutions of elliptic equations with high-contrast conductivities in $\mathbb{R}^n$ is concerned. The space domain consists of a periodic connected sub-region and a periodic disconnected matrix block subset. The elliptic equations have fast diffusion in the connected sub-region and slow diffusion in the disconnected subset. Suppose $\epsilon \in (0,1]$ is the diameter of each matrix block and $\omega^2 \in (0,1]$ is the conductivity ratio of the disconnected matrix block subset to the connected sub-region. It is proved that the $W^{1,p}$ norm of the elliptic solutions in the connected sub-region are bounded uniformly in $\epsilon, \omega$; when $\epsilon \le \omega$, the $L^p$ norm of the elliptic solutions in the whole space are bounded uniformly in $\epsilon, \omega$; the $W^{1,p}$ norm of the elliptic solutions in perforated domains are bounded uniformly in $\epsilon$. However, the $L^p$ norm of the second order derivatives of the solutions in the connected sub-region may not be bounded uniformly in $\epsilon, \omega$.

*Keywords*: high-contrast conductivity, potentials, duality argument, embedding theory

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## 1. Introduction

Uniform estimate for the solutions of elliptic equations with high-contrast conductivities in $\mathbb{R}^n$ is concerned. The problem has applications in acoustic propagation in porous media, modeling of electromagnetic media, oil recovering process, the stress in composite materials [6, 8, 15, 16, 23]. Suppose $Y \equiv [0,1)^n$ is a cube in $\mathbb{R}^n$ for $n \ge 3$, $Y_m$ is a simply-connected sub-domain of $Y$ with $dist(Y_m, \partial Y) > 0$, $Y_f \equiv Y \setminus \overline{Y_m}$, $\tau \in (0,\infty)$, $\Omega_m^\tau \equiv \{x | x \in \tau(Y_m - j) \text{ for some } j \in \mathbb{Z}^n\}$ is a periodic disconnected subset of $\mathbb{R}^n$, $\Omega_f^\tau$ $(\equiv \mathbb{R}^n \setminus \overline{\Omega_m^\tau})$ denotes a periodic connected sub-region of $\mathbb{R}^n$, $\partial\Omega_m^\tau$ represents the boundary of $\Omega_m^\tau$, and $\mathbf{K}_{\alpha,\tau}(x) \equiv \begin{cases} 1 & \text{if } x \in \Omega_f^\tau \\ \alpha & \text{if } x \in \Omega_m^\tau \end{cases}$ for $\alpha, \tau > 0$. The equation that we consider is

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Psi + G) = V & \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} |\Psi|(x) = 0, \end{cases} \tag{1.1}$$

where $\omega, \epsilon \in (0,1]$ and $G, V$ are given functions. If $G, V$ are bounded with compact support, a solution of (1.1) in Hilbert space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ (see definition in section 2) exists uniquely for each $\omega, \epsilon$ by energy method. The $L^2$ norm of the gradient of the solution of (1.1) in the connected sub-region $\Omega_f^\epsilon$ is bounded uniformly in $\omega, \epsilon$ if $G, V$ are small in $\Omega_m^\epsilon$. However, the $L^2$ norm of the gradient of the solution of (1.1) in matrix blocks $\Omega_m^\epsilon$ can be very large when $\omega$ closes to 0. This is different from the case of uniform elliptic equations, in which uniform bound for $W^{1,p}$ or Lipschitz norm holds in the whole domain [4, 5, 14, 17, 22]. To consider nonlinear problems, it is necessary to know whether the uniform bound of the solution of (1.1) in $\omega, \epsilon$ can be extended to $L^p$ space for any $p \in (1, \infty)$. We note that the $L^p$ estimate of the second derivatives of the solution of (1.1) in the connected sub-region $\Omega_f^\epsilon$ may not be bounded uniformly in $\omega, \epsilon$ (see Remark 3.1).

There are some literatures related to this work. Lipschitz estimate and $W^{2,p}$ estimate for uniform elliptic equations with discontinuous coefficients had been proved in [17, 22]. Uniform Hölder, $W^{1,p}$, and Lipschitz estimates for uniform elliptic equations with Hölder periodic coefficients were shown in [4, 5]. Uniform $W^{1,p}$ estimate for uniform elliptic equations with continuous periodic coefficients was considered in [10] and the same problem with VMO periodic coefficients could be found in [26]. Uniform $W^{1,p}$ estimate for the Laplace equation in periodic perforated domains was considered in [21] and the same problem in Lipschitz estimate was studied in [25]. For non-uniform elliptic equations with smooth periodic coefficients, existence of $C^{2,\alpha}$ solution could be found in [14]. Uniform Hölder and Lipschitz estimates in $\omega, \epsilon$ for non-uniform elliptic equations with discontinuous periodic coefficients were shown in [28].

In this work, uniform $W^{1,p}$ estimate in $\omega, \epsilon$ for non-uniform elliptic equations with discontinuous coefficients in $\mathbb{R}^n$ is concerned. We remark that the uniform $W^{1,p}$ estimates for $(1.1)_1$ in bounded domains with Dirichlet boundary conditions are derived in [29], but the constants in the estimates are proportional to the size of the bounded domains. Indeed, the results and the estimate techniques in [29] can not be used in the problem here. To study the problem, we follow the approach developed by Avellaneda and Lin [5] for uniform (interior) $W^{1,p}$ estimates for elliptic equations with rapidly oscillating, periodic coefficients. Our main effort is the proof of the uniform $L^p$ gradient estimate for the solution of (1.1) for $V = 0$ case. First, we use the compactness argument to establish uniform Hölder and Lipschitz estimates. These estimates are then used to derive uniform bounds as well as error estimates for Green functions and their derivatives. Finally, the $L^p$ gradient estimate is obtained by representing solutions using Green functions and by applying the Potential theory. After the $L^p$ gradient estimate available, the other $L^p$ estimates for the solution of (1.1) can be obtained by using duality argument, Sobolev embedding theory, and extension theory. It is shown that the $W^{1,p}$ norm of the elliptic solutions in the connected sub-region $\Omega_f^\epsilon$ are bounded uniformly in $\epsilon, \omega$; when $\epsilon \leq \omega$, the $L^p$ norm of the solutions in the whole space $\mathbb{R}^n$ are bounded uniformly in $\epsilon, \omega$; the $W^{1,p}$ norm of the elliptic solutions in perforated domains $\Omega_f^\epsilon$ are bounded uniformly in $\epsilon$. In [28], Lipschitz estimate for the solution of (1.1) is derived if $G, V$ are piecewise smooth and if $V$ in $\Omega_m^\epsilon$ is small. Here no smoothness for $G, V$ and no smallness for $V$ in $\Omega_m^\epsilon$ are needed.

The rest of this work is organized as follows: Notation and main results are stated in section 2. In section 3, we list some auxiliary lemmas: a priori estimates for interface problems, uniform Hölder and Lipschitz estimates in $\omega, \epsilon$ for non-uniform elliptic equations, and a convergence result for non-uniform elliptic equations. To derive uniform $W^{1,p}$ estimate for the solution of (1.1), we need uniform bounds as well as error estimates for Green functions and their derivatives. The derivation of these results for Green functions is in section 4. Proofs of main results are given in section 5. In section 6, we present the proofs of some lemmas claimed in section 3.

## 2. Notation and main result

$W^{s,p}(D)$ is a Sobolev space with norm $\|\cdot\|_{W^{s,p}(D)}$, $C^{k,\alpha}(D)$ is a Hölder space with norm $\|\cdot\|_{C^{k,\alpha}(D)}$, $W^{s,p}_{loc}(D) \equiv \{\zeta |\ \zeta \in W^{s,p}(\mathbb{D})$ for any compact subset $\mathbb{D}$ of $D\}$, $C^{0,\alpha}_{loc}(D) \equiv \{\zeta |\ \zeta \in C^{0,\alpha}(\mathbb{D})$ for any compact subset $\mathbb{D}$ of $D\}$, and $[\zeta]_{C^{0,\alpha}}$ is the Hölder semi-norm of $\zeta$, where $s \geq -1, p \in [1,\infty], k \geq 0,\ \alpha \in [0,1)$ (see [2]). $H^1_{loc}(D) \equiv W^{1,2}_{loc}(D)$, $L^p(D) \equiv W^{0,p}(D)$, $H^s(D) \equiv W^{s,2}(D)$, and $C^k(D) \equiv C^{k,0}(D)$. $C(\mathbb{R}^n)$ is a space of continuous functions in $\mathbb{R}^n$, $C^\infty(\mathbb{R}^n)$ is a space of infinitely differentiable functions in $\mathbb{R}^n$, $C^\infty_0(D)$ is a space of infinitely differentiable functions with compact support in $D$, and $C^\infty_{per}(\mathbb{R}^n)$ is the subset of $C^\infty(\mathbb{R}^n)$ of $(0,1)^n$-periodic functions. $H^s_{per}(D)$ is the closure of $C^\infty_{per}(\mathbb{R}^n)$ under the $H^s$ norm and $\|\zeta\|_{H^s_{per}(D)} \equiv \|\zeta\|_{H^s(D\cap(0,1)^n)}$ for $s \geq 1$. $\mathcal{D}^{1,2}(\mathbb{R}^n) \equiv \{\zeta \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) | \nabla\zeta \in L^2(\mathbb{R}^n)\}$ under the norm $\|\zeta\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)} \equiv \|\nabla\zeta\|_{L^2(\mathbb{R}^n)}$ is a Hilbert space (see page 168 [19]). $supp(\zeta)$ denotes the support of $\zeta$. Define $\|\zeta_1, \cdots, \zeta_i\|_{\mathbf{B}_1} \equiv \|\zeta_1\|_{\mathbf{B}_1} + \cdots + \|\zeta_i\|_{\mathbf{B}_1}$ and $\|\zeta\|_{\mathbf{B}_1\cap\mathbf{B}_2} \equiv \|\zeta\|_{\mathbf{B}_1} + \|\zeta\|_{\mathbf{B}_2}$ for Banach spaces $\mathbf{B}_1, \mathbf{B}_2$. $B(x)$ is a ball centered at $x$ and $B_r(x)$ is a ball centered at $x$ with radius $r > 0$. For any set $D$, $|D|$ is the volume of $D$, $\overline{D}$ is the closure of $D$, $\mathcal{X}_D$ is the characteristic function on $D$, $dist(x, D)$ is the distance from $x$ to $D$, $\partial D$ is the boundary of $D$, $D/r \equiv \{x|\ rx \in D\}$ for $r > 0$, and

$$⨍_D \zeta(x)dx \equiv \frac{1}{|D|}\int_D \zeta(x)dx \qquad \text{if } \zeta \in L^1(D).$$

Define $|\!|\!|\zeta|\!|\!|_{C^{0,\alpha}(D\cap\overline{\Omega^\epsilon_f})} \equiv \|\eta\|_{C^{0,\alpha}(D/\epsilon\cap\overline{\Omega^\epsilon_f}/\epsilon)}$ and $|\!|\!|\zeta|\!|\!|_{C^{0,\alpha}(D\cap\overline{\Omega^\epsilon_m})} \equiv \|\eta\|_{C^{0,\alpha}(D/\epsilon\cap\overline{\Omega^\epsilon_m}/\epsilon)}$ where $\eta(x) = \zeta(\epsilon x)$, $\epsilon, \alpha \in (0,1)$. If $\vec{\mathbf{n}}^\tau$ is a outward normal vector on $\tau(\partial Y_m - j)$ for $\tau > 0$ and $j \in \mathbb{Z}^n$, we define, for any function $\zeta$ and $x \in \tau(\partial Y_m - j)$,

$$\zeta_{,\pm}(x) \equiv \lim_{t\to 0^+} \zeta(x \pm t\vec{\mathbf{n}}^\tau), \qquad \lfloor\zeta\rfloor_{\tau(\partial Y_m - j)}(x) = \zeta_{,+}(x) - \zeta_{,-}(x). \tag{2.1}$$

Next we give two statements:

A1. $\omega, \epsilon \in (0,1]$,

A2. $Y_m$ with diameter less than 1 is a smooth simply-connected sub-domain of $Y \subset \mathbb{R}^n, n \geq 3$.

The assumption on $Y_m$ (that is, A2) is only for convenience of presentation. More general case is possible. Our main result is Theorem 2.1. Others are the consequences of Theorem 2.1.

**Theorem 2.1.** *Under A1–A2, $p \in (1, \infty)$, and $G \in [L^p(\mathbb{R}^n)]^n$ with compact support, a $W^{1,p}_{loc}(\mathbb{R}^n)$ solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Psi + G) = 0 \quad \text{in } \mathbb{R}^n \\ \lim_{|x| \to \infty} |\Psi|(x) = 0 \end{cases} \tag{2.2}$$

*exists uniquely and satisfies*

$$\|\mathbf{K}_{\omega,\epsilon} \nabla \Psi\|_{L^p(\mathbb{R}^n)} \leq c \|\mathbf{K}_{1/\omega,\epsilon} G\|_{L^p(\mathbb{R}^n)}, \tag{2.3}$$

*where $c$ is independent of $\omega, \epsilon$.*

By Theorem 2.1, Potential theory, duality argument, and Sobolev embedding theory, we have the following two results.

**Corollary 2.1.** *Under A1–A2, $p \in (1, \infty)$, and $G \in [L^p(\mathbb{R}^n)]^n$ with support in $B_r(0)$ for $r > 0$, then the solution of (2.2) satisfies*

$$\begin{cases} \|\mathbf{K}_{\omega,\epsilon} \Psi\|_{L^p(B_t(0))} \leq c_{t,r} \|\mathbf{K}_{1/\omega,\epsilon} G\|_{L^p(\mathbb{R}^n)} & \text{for } p \in (1, \infty), \\ \|\mathbf{K}_{\omega,\epsilon} \Psi\|_{L^p(\mathbb{R}^n)} \leq c \|\mathbf{K}_{1/\omega,\epsilon} G\|_{L^{\frac{np}{n+p}}(\mathbb{R}^n)} & \text{for } p \in (\frac{n}{n-1}, \infty). \end{cases} \tag{2.4}$$

*Here $t > 0$; $c_{t,r}$ is independent of $\omega, \epsilon$ but dependent on $t, r$; $c$ is independent of $\omega, \epsilon, r$.*

**Corollary 2.2.** *Under A1–A2, $p \in (1, \infty)$, and $V \in W^{-1,p}(\mathbb{R}^n)$ with support in $B_r(0)$ for $r > 0$, a $W^{1,p}_{loc}(\mathbb{R}^n)$ solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Psi) = V \quad \text{in } \mathbb{R}^n \\ \lim_{|x| \to \infty} |\Psi|(x) = 0 \end{cases} \tag{2.5}$$

*exists uniquely and satisfies*

$$\|\mathbf{K}_{\omega,\epsilon} \Psi, \mathbf{K}_{\omega,\epsilon} \nabla \Psi\|_{L^p(B_t(0))} \leq c_{t,r} \|\mathbf{K}_{1/\omega,\epsilon} V\|_{W^{-1,p}(\mathbb{R}^n)}, \tag{2.6}$$

*where $t > 0$ and $c_{t,r}$ is independent of $\omega, \epsilon$ but dependent on $t, r$. In addition to $V \in L^p(\mathbb{R}^n)$ with compact support, then*

$$\begin{cases} \|\mathbf{K}_{\omega,\epsilon} \Psi\|_{L^p(\mathbb{R}^n)} \leq c \|\mathbf{K}_{1/\omega,\epsilon} V\|_{L^{\frac{np}{n+2p}}(\mathbb{R}^n)} & \text{for } p \in (\frac{n}{n-2}, \infty), \\ \|\mathbf{K}_{\omega,\epsilon} \nabla \Psi\|_{L^p(\mathbb{R}^n)} \leq c \|\mathbf{K}_{1/\omega,\epsilon} V\|_{L^{\frac{np}{n+p}}(\mathbb{R}^n)} & \text{for } p \in (\frac{n}{n-1}, \infty), \end{cases} \tag{2.7}$$

*where $c$ is independent of $\omega, \epsilon$.*

Theorem 2.1 and Corollaries 2.1, 2.2 are proved in section 5. For $\epsilon \leq \omega$ case, Theorem 2.1 and Corollaries 2.1, 2.2 imply the following estimate.

**Corollary 2.3.** *Under A1–A2 and $\epsilon \leq \omega$ and if both $G, V$ are smooth enough with support in $B_r(0)$ for $r > 0$, then a $W^{1,p}(\mathbb{R}^n)$ solution of (1.1) exists uniquely and* *satisfies*

$$\begin{cases} \|\Psi, \mathbf{K}_{\omega,\epsilon}\nabla\Psi\|_{L^p(B_t(0))} \le c_{t,r}\big(\|\mathbf{K}_{1/\omega,\epsilon}G\|_{L^p(\mathbb{R}^n)} \\ \qquad\qquad +\|\mathbf{K}_{1/\omega,\epsilon}V\|_{W^{-1,p}(\mathbb{R}^n)}\big) & \text{if } p\in(1,\infty), \\ \|\Psi, \mathbf{K}_{\omega,\epsilon}\nabla\Psi\|_{L^p(\mathbb{R}^n)} \le c\big(\|\mathbf{K}_{1/\omega,\epsilon}G\|_{L^p(\mathbb{R}^n)\cap L^{\frac{np}{n+p}}(\mathbb{R}^n)} \\ \qquad\qquad +\|\mathbf{K}_{1/\omega,\epsilon}V\|_{L^{\frac{np}{n+p}}(\mathbb{R}^n)\cap L^{\frac{np}{n+2p}}(\mathbb{R}^n)}\big) & \text{if } p\in(\frac{n}{n-2},\infty). \end{cases}$$

*Here* $t>0$*;* $c_{t,r}$ *is independent of* $\omega,\epsilon$ *but dependent on* $t,r$*;* $c$ *is independent of* $\omega,\epsilon$*.*

**Proof.** Let $\Pi_\epsilon\Psi|_{\Omega_f^\epsilon}$ denote the extension function of $\Psi|_{\Omega_f^\epsilon}$ on $\mathbb{R}^n$ (see Theorem 2.1 [1]). If $\epsilon\le\omega$, by Theorem 2.1 [1] and Poincaré inequality, the solution of (1.1) satisfies

$$\begin{aligned} \|\Psi\|_{L^p(B_t(0)\cap\Omega_m^\epsilon)} &\le \|\Psi-\Pi_\epsilon\Psi|_{\Omega_f^\epsilon}\|_{L^p(B_t(0)\cap\Omega_m^\epsilon)} + \|\Pi_\epsilon\Psi|_{\Omega_f^\epsilon}\|_{L^p(B_t(0)\cap\Omega_m^\epsilon)} \\ &\le c\|\epsilon(\nabla\Psi-\nabla\Pi_\epsilon\Psi|_{\Omega_f^\epsilon})\|_{L^p(B_t(0)\cap\Omega_m^\epsilon)} + \|\Pi_\epsilon\Psi|_{\Omega_f^\epsilon}\|_{L^p(B_t(0)\cap\Omega_m^\epsilon)} \\ &\le c(\|\mathbf{K}_{\omega,\epsilon}\nabla\Psi\|_{L^p(B_t(0))} + \|\Psi\|_{L^p(B_t(0)\cap\Omega_f^\epsilon)}), \end{aligned} \tag{2.8}$$

where $c$ is independent of $\omega,\epsilon$. (2.8), Theorem 2.1, and Corollaries 2.1, 2.2 imply the result. □

For $\omega\le\epsilon$ case, Theorem 2.1, Corollaries 2.1, 2.2, and compactness argument imply the following estimate in a perforated domain $\Omega_f^\epsilon$.

**Corollary 2.4.** *Under A2,* $\epsilon\in(0,1]$*, and* $p\in(\frac{n}{n-2},\infty)$ *and if both* $G\in L^p(\Omega_f^\epsilon)$*,* $V\in L^{\frac{np}{n+p}}(\Omega_f^\epsilon)$ *have compact support, then a* $W^{1,p}(\Omega_f^\epsilon)$ *solution of*

$$\begin{cases} -\nabla\cdot(\nabla\Psi+G)=V & \text{in } \Omega_f^\epsilon \\ (\nabla\Psi+G)\cdot\vec{\mathbf{n}}^\epsilon=0 & \text{on } \partial\Omega_m^\epsilon \\ \lim_{|x|\to\infty}|\Psi|(x)=0 \end{cases} \tag{2.9}$$

*exists uniquely and satisfies*

$$\|\Psi\|_{W^{1,p}(\Omega_f^\epsilon)} \le c\big(\|G\|_{L^p(\Omega_f^\epsilon)\cap L^{\frac{np}{n+p}}(\Omega_f^\epsilon)} + \|V\|_{L^{\frac{np}{n+p}}(\Omega_f^\epsilon)\cap L^{\frac{np}{n+2p}}(\Omega_f^\epsilon)}\big), \tag{2.10}$$

*where* $c$ *is a positive constant independent of* $\epsilon$ *and* $\vec{\mathbf{n}}^\epsilon$ *is a normal vector on* $\partial\Omega_m^\epsilon$*,*

**Proof.** Consider the following equation

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}\nabla\Psi_\omega+G\mathcal{X}_{\Omega_f^\epsilon})=V\mathcal{X}_{\Omega_f^\epsilon} & \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty}|\Psi_\omega|(x)=0. \end{cases}$$

By Theorem 2.1 and Corollaries 2.1, 2.2,

$$\|\mathbf{K}_{\omega,\epsilon}\Psi_\omega, \mathbf{K}_{\omega,\epsilon}\nabla\Psi_\omega\|_{L^p(\mathbb{R}^n)} \le c\big(\|G\|_{L^p(\Omega_f^\epsilon)\cap L^{\frac{np}{n+p}}(\Omega_f^\epsilon)} + \|V\|_{L^{\frac{np}{n+p}}(\Omega_f^\epsilon)\cap L^{\frac{np}{n+2p}}(\Omega_f^\epsilon)}\big),$$

where $c$ is a positive constant independent of $\omega, \epsilon$. Fixing $\epsilon$ and letting $\omega \to 0$, by compactness principle, there is a sequence $\{\Psi_\omega\}$ converging weakly to $\Psi \in W^{1,p}(\Omega_f^\epsilon)$ which satisfies (2.9), (2.10). $\square$

**Remark 2.1.** Let us mention that there is an interesting problem related to (1.1). If $Y_m$ is replaced by, say, two disjoint balls with a distance $\delta > 0$ between them, how would the constant $c$ in (2.3) depend on $\delta$? For what value of $p$, is the constant $c$ in (2.3) independent of $\delta$? In this setting, Theorem 1.9 in [18] gives an $L^\infty$ estimate (independent of $\epsilon, \delta$) for the gradient provided $\omega^2$ stays away from 0. Also Theorem 1.2 in [7] gives an estimate on the gradient in terms of $\delta$ for $\omega^2 = 0$ and $\epsilon = 1$. We shall pursuit this problem in the later work.

## 3. Some auxiliary lemmas

Here we list some lemmas. Basically they can be derived by modifying the arguments in [28]. If one result is not obvious, we give its proof in section 6. Otherwise we refer the reader to [28]. The listed lemmas are an imbedding result (Lemma 3.1), uniform estimates for interface problems (Lemmas 3.2–3.5), and a convergence result (Lemma 3.6).

**Lemma 3.1.** *There is a constant $c$ independent of $\epsilon \in (0,1]$ such that*

$$\begin{cases} \|\zeta\|_{L^{\frac{2n}{n-2}}(\Omega_f^\epsilon)} \le \|\Pi_\epsilon \zeta\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \le c\|\nabla\zeta\|_{L^2(\Omega_f^\epsilon)} \\ \|\zeta\|_{L^{\frac{2n}{n-2}}(\Omega_m^\epsilon)} \le c\|\nabla\zeta\|_{L^2(\mathbb{R}^n)} \end{cases} \quad \text{if } \zeta \in \mathcal{D}^{1,2}(\mathbb{R}^n).$$

*Here $\Pi_\epsilon$ is the extension operator for functions defined in $\Omega_f^\epsilon$ (see Theorem 2.1 [1]). If $\zeta$ is defined in $\mathbb{R}^n$, then $\Pi_\epsilon\zeta$ means $\Pi_\epsilon\zeta \equiv \Pi_\epsilon(\zeta|_{\Omega_f^\epsilon})$.*

**Proof.** This lemma follows from Theorem 4.31 [2] and Theorem 2.1 [1]. $\square$

By A2, one can find a positive constant $\mathbf{d}_0$ and smooth simply-connected domains $\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}, \mathcal{A}, \mathbf{S}_2$ satisfying

$$\begin{cases} Y_m \subset \mathbf{S}_0 \subset \mathbf{S}_1 \subset \mathbf{S} \subset \mathcal{A} \subset Y \subset \mathbf{S}_2, \\ dist(Y_m, \partial\mathbf{S}_0), dist(\mathbf{S}_0, \partial\mathbf{S}_1), dist(\mathbf{S}_1, \partial\mathbf{S}), dist(\mathbf{S}, \partial\mathcal{A}) \ge \mathbf{d}_0 > 0, \\ dist(\mathcal{A}, \partial Y), dist(Y, \partial\mathbf{S}_2), dist(\mathbf{S}_2, \Omega_m^1 \setminus Y_m) \ge \mathbf{d}_0 > 0, \\ \text{diameter of } Y_m \text{ is less than } 1 - 10\mathbf{d}_0. \end{cases} \tag{3.1}$$

**Lemma 3.2.** *Let $p \in (1, \infty)$ and $\omega \in (0,1]$. Any solution of*

$$-\nabla \cdot (\mathbf{K}_{\omega^2,1}\nabla U + Q) = F \qquad in\ \mathbf{S}_2 \tag{3.2}$$

*satisfies*

$$\|\mathbf{K}_{\omega,1}U, \mathbf{K}_{\omega,1}\nabla U\|_{L^p(Y)} \le cJ_\omega,$$

*where c is a constant independent of $\omega$ and*

$$J_\omega \equiv \|U\|_{L^p(\mathbf{S}_2\setminus\overline{Y})} + \|\mathbf{K}_{1/\omega,1}Q\|_{L^p(\mathbf{S}_2)} + \|\mathbf{K}_{1/\omega,1}F\|_{W^{-1,p}(\mathbf{S}_2)}. \tag{3.3}$$

Proof of Lemma 3.2 is given in section 6.

**Lemma 3.3.** *Let $p \in (n,\infty)$ and $\omega \in (0,1]$. Any solution of*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,1}\nabla U + Q) = F & \text{in } Y\setminus\partial Y_m \\ \lfloor U\rfloor_{\partial Y_m} = 0 \\ \lfloor \mathbf{K}_{\omega^2,1}\nabla U + Q\rfloor_{\partial Y_m}\cdot\vec{\mathbf{n}}_y = \zeta \end{cases} \tag{3.4}$$

*satisfies*

$$\begin{aligned} &\|\mathbf{K}_{\omega^i,1}U\|_{C^{k+1,1-n/p}(\overline{\mathbf{S}\setminus Y_m})\cap C^{k+1,1-n/p}(\overline{Y_m})} \le c\big(\|U\|_{L^2(Y_f)} + \|\zeta\|_{C^{k,1-n/p}(\partial Y_m)} \\ &\quad + \|\mathbf{K}_{\omega^{i-2},1}Q\|_{C^{k,1-n/p}(\overline{Y_f})\cap C^{k,1-n/p}(\overline{Y_m})} + \|\mathbf{K}_{\omega^{i-2},1}F\|_{W^{k,p}(Y_f)\cap W^{k,p}(Y_m)}\big), \end{aligned}$$

*where $i,k \in \{0,1\}$, c is independent of $\omega$, and $\vec{\mathbf{n}}_y$ is the unit outward normal vector on $\partial Y_m$. See (2.1) for $(3.4)_{2,3}$ and (3.1) for $\mathbf{S}$.*

Lemma 3.3 can be proved by a modification of the argument of Lemma 3.2, so we skip its proof. The following result is a local estimate around the interface.

**Lemma 3.4.** *Let $\omega \in (0,1]$, $\tau \in (1,\infty)$, $x_0 \in \tau\partial Y_m$, and $B_{1/2}(x_0) \subset \tau Y$. Any solution of*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,\tau}\nabla U) = F\mathcal{X}_{\tau Y_f} & \text{in } B_{1/2}(x_0)\setminus\tau\partial Y_m \\ \lfloor U\rfloor_{B_{1/2}(x_0)\cap\tau\partial Y_m} = 0 \\ \lfloor \mathbf{K}_{\omega^2,\tau}\nabla U\rfloor_{B_{1/2}(x_0)\cap\tau\partial Y_m}\cdot\vec{\mathbf{n}}_y = \zeta \end{cases} \tag{3.5}$$

*satisfies*

$$\begin{aligned} &\|\mathbf{K}_{\omega,\tau}U\|_{C^{k+1,\alpha}(\overline{B_{1/8}(x_0)\cap\tau Y_f})\cap C^{k+1,\alpha}(\overline{B_{1/8}(x_0)\cap\tau Y_m})} \le c(\|\mathbf{K}_{\omega,\tau}U\|_{L^2(B_{1/2}(x_0))} \\ &\quad + \|\zeta\|_{C^{k,\alpha}(\overline{B_{1/2}(x_0)\cap\tau\partial Y_m})} + \|F\|_{C^k(\overline{B_{1/2}(x_0)\cap\tau Y_f})}), \end{aligned}$$

*where $k \in \{0,1\}$, $\alpha \in (0,1)$, $\vec{\mathbf{n}}_y$ is the unit outward normal vector on $\tau\partial Y_m$, and c is a constant independent of $\omega,\tau$. See (2.1) for $(3.5)_{2,3}$.*

Proof of Lemma 3.4 is given in section 6. We find $\mathbb{X}_\omega^{(i)} \in H^1_{per}(\mathbb{R}^n)$ satisfying, for $\omega \in (0,1]$ and $i \in \{1,\cdots,n\}$,

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,1}(\nabla\mathbb{X}_\omega^{(i)} + \vec{e}_i)) = 0 \quad \text{in } Y, \\ \displaystyle\int_{Y_f}\mathbb{X}_\omega^{(i)}(y)dy = 0, \end{cases} \tag{3.6}$$

and find $\mathbb{X}_0^{(i)} \in H^1_{per}(\Omega^1_f) \cap H^1_{per}(\Omega^1_m)$ satisfying, for $i \in \{1, \cdots, n\}$,

$$\begin{cases} \mathbb{X}_0^{(i)} = 0 & \text{in } Y_m, \\ -\nabla \cdot (\nabla \mathbb{X}_0^{(i)} + \vec{e}_i) = 0 & \text{in } Y_f, \\ (\nabla \mathbb{X}_0^{(i)} + \vec{e}_i)_{,+} \cdot \vec{\mathbf{n}}_y = 0 & \text{on } \partial Y_m, \\ \displaystyle\int_{Y_f} \mathbb{X}_0^{(i)}(y)dy = 0, \end{cases} \tag{3.7}$$

where $\vec{e}_i$ is the unit vector in the $i$-th coordinate direction and $\vec{\mathbf{n}}_y$ is a unit outward normal vector on $\partial Y_m$. See (2.1) for $(3.7)_3$. By Lax-Milgram Theorem and Poincaré inequality [13], $\mathbb{X}_\omega^{(i)}$ exists uniquely. By energy method, Theorem 6.30 [13], and Lemma 3.3, the solutions of (3.6)–(3.7) satisfy

$$\|\mathbb{X}_\omega^{(i)}\|_{W^{2,\infty}(Y_f) \cap W^{2,\infty}(Y_m)} \leq c, \tag{3.8}$$

where $c$ is a constant independent of $\omega$. Define $\mathbb{X}_{\omega,\tau}^{(i)}(x) \equiv \tau \mathbb{X}_\omega^{(i)}(\frac{x}{\tau})$ and $\mathbb{X} \equiv (\mathbb{X}_\omega^{(1)}, \mathbb{X}_\omega^{(2)}, \cdots, \mathbb{X}_\omega^{(n)})$, $\mathbb{X}_{\omega,\tau} \equiv (\mathbb{X}_{\omega,\tau}^{(1)}, \mathbb{X}_{\omega,\tau}^{(2)}, \cdots, \mathbb{X}_{\omega,\tau}^{(n)})$ for $\omega \in [0,1]$ and $\tau \in (0, \infty)$. Denote by $\Xi_\omega$ for $\omega \in [0,1]$ a $n \times n$ matrix function whose $(i,j)$-component is $\partial_i \mathbb{X}_\omega^{(j)}$. By remark in pages 17-19, 94-95 [15],

$$\mathcal{K}_\omega \equiv \int_{Y_f \cup Y_m} \mathbf{K}_{\omega^2,1}(I + \Xi_\omega(y))dy \qquad \text{for } \omega \in [0,1] \tag{3.9}$$

is a symmetric positive definite matrix dependent only on $\omega$. Here $I$ is the identity matrix. By (3.8), it is not difficult to see that there are positive constants $d_1, d_2$ independent of $\omega$ such that

$$d_1 I \leq \mathcal{K}_\omega \leq d_2 I. \tag{3.10}$$

One example below shows that the $L^p$ norm of the second order derivatives of the solution of (1.1) may not be bounded uniformly in $\omega$.

**Remark 3.1.** Suppose $\eta$ is a bell-shaped smooth function satisfying $\eta \in C_0^\infty(B_1(0))$, $\eta \in [0,1]$, and $\eta(x) = 1$ in $B_{1/2}(0)$. Employ $\eta$, $\mathbb{X}_{\epsilon,\epsilon}^{(1)}$ for $\epsilon \in (0,1]$, and (3.6) to obtain

$$\begin{cases} -\nabla \cdot \left(\mathbf{K}_{\epsilon^2,\epsilon} \nabla(\eta \mathbb{X}_{\epsilon,\epsilon}^{(1)}) - \mathbf{K}_{\epsilon^2,\epsilon} \mathbb{X}_{\epsilon,\epsilon}^{(1)} \nabla \eta + \mathbf{K}_{\epsilon^2,\epsilon} \eta \vec{e}_1\right) & \\ \qquad = -\mathbf{K}_{\epsilon^2,\epsilon}(\nabla \mathbb{X}_{\epsilon,\epsilon}^{(1)} + \vec{e}_1)\nabla \eta & \text{in } B_1(0), \\ |\eta \mathbb{X}_{\epsilon,\epsilon}^{(1)}|(x) = 0 & \text{for } x \notin B_1(0). \end{cases}$$

By (3.8), we see

$$\|\mathbb{X}_{\epsilon,\epsilon}^{(1)} \nabla \eta - \eta \vec{e}_1\|_{W^{1,\infty}(B_1(0))} + \|(\nabla \mathbb{X}_{\epsilon,\epsilon}^{(1)} + \vec{e}_1)\nabla \eta\|_{L^\infty(B_1(0))}$$

is bounded uniformly in $\epsilon$, but $\|\eta \mathbb{X}_{\epsilon,\epsilon}^{(1)}\|_{W^{2,p}(B_1(0) \cap \Omega_f^\epsilon)}$ for $p \in (1, \infty)$ is not bounded uniformly in $\epsilon$.

Next we state one result but omit its proof because the proof is similar to the arguments of Lemmas 4.1–4.3, 5.1–5.3 [28].

**Lemma 3.5.** *Let $\ell > 0$ and $\omega, \epsilon \in (0,1]$. Any solution of*

$$-\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}\nabla U + Q) = F \qquad \text{in } \mathbb{R}^n \tag{3.11}$$

*satisfies*

$$\begin{aligned}&[U]_{C^{0,\mu}(\overline{B_{1/2}(x^*)\cap\Omega_f^\epsilon})}\\ &\qquad\le c\big(\|\mathbf{K}_{\omega,\epsilon}U\|_{L^2(B_1(x^*))} + \|\mathbf{K}_{1/\omega,\epsilon}Q, \mathbf{K}_{1/\omega,\epsilon}F\|_{L^{n+\ell}(B_1(x^*))}\big),\end{aligned} \tag{3.12}$$

$$\begin{aligned}\|\mathbf{K}_{\omega,\epsilon}\nabla U\|_{L^\infty(B_{1/2}(x^*))} &\le c\big(\|U\|_{L^\infty(B_1(x^*)\cap\Omega_f^\epsilon)} + \omega\|U\|_{L^2(B_1(x^*)\cap\Omega_m^\epsilon)}\\ &+|\!|\!|Q|\!|\!|_{C^{0,\mu}(\overline{B_1(x^*)\cap\Omega_f^\epsilon})} + \omega^{-1}|\!|\!|Q|\!|\!|_{C^{0,\mu}(\overline{B_1(x^*)\cap\Omega_m^\epsilon})}\\ &+\|\epsilon^{\mu/2-1}\mathbf{K}_{1/\omega,\epsilon}Q, \mathbf{K}_{1/\omega,\epsilon}F\|_{L^{n+\ell}(B_1(x^*))}\big),\end{aligned} \tag{3.13}$$

*where $\mu \equiv \frac{\ell}{n+\ell}$, $x^* \in \mathbb{R}^n$, and $c$ is a constant independent of $\omega, \epsilon, x^*$.*

*In addition to $Q = 0$ in $\mathbb{R}^n$ and $F = 0$ in $\Omega_m^\epsilon$, any solution of (3.11) satisfies*

$$\begin{aligned}\|\nabla U\|_{L^\infty(B_{1/2}(x^*))} &\le c\big(\|U\|_{L^\infty(B_1(x^*)\cap\Omega_f^\epsilon)}\\ &+\omega\|U\|_{L^2(B_1(x^*)\cap\Omega_m^\epsilon)} + \|F\|_{L^{n+\ell}(B_1(x^*))}\big),\end{aligned} \tag{3.14}$$

*where $x^* \in \mathbb{R}^n$ and $c$ is a constant independent of $\omega, \epsilon, x^*$.*

**Corollary 3.1.** *Let $x^* \in \mathbb{R}^n$, $\tau, r \in (0,\infty)$, and $\omega \in (0,1]$. Any solution of*

$$-\nabla\cdot(\mathbf{K}_{\omega^2,\tau}\nabla U) = 0 \qquad \text{in } B_r(x^*) \tag{3.15}$$

*satisfies*

$$|\mathbf{K}_{\omega,\tau}U|(x^*) \le c\left|\fint_{B_r(x^*)} |\mathbf{K}_{\omega,\tau}U(y)|^2 dy\right|^{1/2}, \tag{3.16}$$

*for some constant $c$ independent of $\omega, \tau, x^*, r$.*

**Proof.** First we assume $x^* = 0$ and define $\Phi(y) = U(ry)$. Then (3.15) implies

$$-\nabla\cdot(\mathbf{K}_{\omega^2,\tau/r}\nabla\Phi) = 0 \qquad \text{in } B_1(0).$$

Note $\tau/r \le 1$ or $1 < \tau/r$. If $\tau/r \le 1$ (resp. $1 < \tau/r$), inequality (3.12) (resp. Theorem 7.26 [13] and Lemma 3.4) implies

$$[\Phi]_{C^{0,\alpha}(\overline{B_{1/4}(0)\cap\Omega_f^\tau/r})} \le c\|\mathbf{K}_{\omega,\tau/r}\Phi\|_{L^2(B_1(0))}, \tag{3.17}$$

where $\alpha, c$ are positive constants independent of $\omega, \tau, r$.

Suppose $0 \in \Omega_f^\tau$, by (3.17),

$$\begin{aligned}|U(0)| = |\Phi(0)| &\le \left|\Phi(0) - \fint_{B_{1/4}(0)\cap\Omega_f^\tau/r}\Phi(y)dy\right| + \left|\fint_{B_{1/4}(0)\cap\Omega_f^\tau/r}\Phi(y)dy\right|\\ &\le c([\Phi]_{C^{0,\alpha}(\overline{B_{1/4}(0)\cap\Omega_f^\tau/r})} + \|\Phi\|_{L^2(B_1(0)\cap\Omega_f^\tau/r)}) \le c\left|\fint_{B_r(0)}|\mathbf{K}_{\omega,\tau}U(y)|^2dy\right|^{\frac{1}{2}}.\end{aligned} \tag{3.18}$$

So (3.16) holds for $x^* = 0 \in \Omega_f^\tau$ case. If $0 \ne x^* \in \Omega_f^\tau$, by translation, we see that (3.16) is true for $x^* \in \Omega_f^\tau$.

Suppose $0 \in \Omega_m^\tau$. If $\tau/r \leq 1$, by maximal principle [13] and $0 \in \frac{\tau}{r}Y_m$, maximal value of $|\Phi|$ in the region $\frac{\tau}{r}Y_m$ is bounded by the maximal value of $|\Phi|$ on the boundary of $\frac{\tau}{r}Y_m$. Since (3.16) holds in $\Omega_f^\tau$ for any $\tau \in (0,\infty)$,

$$|U(0)| = |\Phi(0)| \leq \max_{z\in\frac{\tau}{r}\partial Y_m} |\Phi(z)| \leq \max_{z\in\frac{\tau}{r}\partial Y_m} c\left| \fint_{B_{\mathbf{d}_0}(z)} |\mathbf{K}_{\omega,\tau/r}\Phi(y)|^2 dy\right|^{1/2}$$
$$\leq c\left| \fint_{B_1(0)} |\mathbf{K}_{\omega,\tau/r}\Phi(y)|^2 dy\right|^{1/2} = c\left| \fint_{B_r(0)} |\mathbf{K}_{\omega,\tau}U(y)|^2 dy\right|^{1/2},$$

where $\mathbf{d}_0$ is defined in (3.1) and $c$ is independent of $\omega, \tau/r$. If $\tau/r > 1$, we argue as (3.18) and we conclude that (3.16) holds for $0 \in \Omega_m^\tau$ case. If $0 \neq x^* \in \Omega_m^\tau$, by translation, we see that (3.16) is true for $x^* \in \Omega_m^\tau$. □

By Lax-Milgram Theorem [13], Lemma 3.1, and Theorem 2.1 [1], we see that if $F \in C_0^\infty(\mathbb{R}^n)$, a $\mathcal{D}^{1,2}(\mathbb{R}^n)$ solution of

$$-\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}\nabla U) = F\mathcal{X}_{\Omega_f^\epsilon} \qquad \text{in } \mathbb{R}^n \tag{3.19}$$

exists uniquely for all $\omega, \epsilon \in (0,1]$. Moreover, the solution of (3.19) satisfies, for any $x \in \mathbb{R}^n$,

$$\|\mathbf{K}_{\omega,\epsilon}U\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} + \|\mathbf{K}_{\omega,\epsilon}\nabla U\|_{L^2(\mathbb{R}^n)} \leq c\|F\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}, \tag{3.20}$$

where $c$ is independent of $x, \omega, \epsilon$. By (3.12), (3.14), and (3.20), we also have

$$\|U\|_{W^{1,\infty}(\mathbb{R}^n)} \leq c\|F\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)\cap L^{n+\ell}(\mathbb{R}^n)}, \tag{3.21}$$

where $\ell > 0$ and $c$ is a constant independent of $\omega, \epsilon$. By [3] and remark in pages 17-19, 94-95 [15], we see that there is a sequence of the solution $\{U_\epsilon\}$ of (3.19) satisfying, for each fixed $\omega \in (0,1]$ and $r > 0$,

$$\begin{cases} U_\epsilon \to U & \text{in } L^2(B_r(0)) \text{ strongly} \\ \mathbf{K}_{\omega^2,\epsilon}\nabla U_\epsilon \to \mathcal{K}_\omega\nabla U & \text{in } L^2(B_r(0)) \text{ weakly} \\ F\mathcal{X}_{\Omega_f^\epsilon} \to |Y_f|F & \text{in } L^2(B_r(0)) \text{ weakly} \end{cases} \quad \text{as } \epsilon \to 0, \tag{3.22}$$

where $|Y_f|$ is the volume of $Y_f$ and $\mathcal{K}_\omega$ is the one in (3.9). The $U$ in (3.22) satisfies

$$-\nabla\cdot(\mathcal{K}_\omega\nabla U) = |Y_f|F \qquad \text{in } \mathbb{R}^n. \tag{3.23}$$

Moreover, we have

**Lemma 3.6.** *Let $\ell > 0$, $\omega, \epsilon \in (0,1]$, and $F \in C_0^\infty(\mathbb{R}^n)$. The difference between the solution of (3.19) and the solution of (3.23) satisfies*

$$\|U_\epsilon - U\|_{L^\infty(\mathbb{R}^n)} \leq c\epsilon\|F\|_{W^{1,\frac{2n}{n+2}}(\mathbb{R}^n)\cap W^{2,n+\ell}(\mathbb{R}^n)},$$

*where $c$ is a constant independent of $\omega, \epsilon$.*

Proof of Lemma 3.6 is given in section 6.

## 4. Approximation of the Green functions

This section containing three subsections presents uniform bounds and approximations for Green functions. The first subsection is the uniform bounds of the zero order and the first order derivatives of Green functions $\Gamma_\tau^\omega(x,y)$ of the Poisson equations. The second subsection is an approximation of the second derivatives of $\Gamma_\tau^\omega(x,y)$ for $\tau \in (0,1]$. The third subsection is an approximation of the second derivatives of $\Gamma_\tau^\omega(x,y)$ for $\tau \in [\frac{1}{n+1},\infty)$.

### 4.1. *Green functions of the Poisson equations*

For any $\tau \in (0,\infty)$ and $\omega \in (0,1]$, let $\Gamma_\tau^\omega$ denote the Green function of

$$\begin{cases} -\nabla_y \cdot (\mathbf{K}_{\omega^2,\tau}(y)\nabla_y \Gamma_\tau^\omega(x,y)) = \delta(x-y) \quad \text{in } \mathbb{R}^n, \\ \Gamma_\tau^\omega(x,y) \to 0 \quad \text{as } |x-y| \to \infty. \end{cases} \tag{4.1}$$

By Theorem 5.4 [20], remark in pages $62, 67$ in [20], and Lemma 3.3, function $\Gamma_\tau^\omega$ exists uniquely in $H^1_{loc}(\mathbb{R}^n \setminus \{x\}) \cap W^{1,1}_{loc}(\mathbb{R}^n)$ and

$$\begin{cases} \Gamma_\tau^\omega(x,\cdot) \in C(\mathbb{R}^n \setminus \{x\}), \\ \Gamma_\tau^\omega(x,y) = \Gamma_\tau^\omega(y,x) \qquad \text{for } x \neq y, \\ \Gamma_\tau^\omega(x,y) = \tau^{2-n}\Gamma_1^\omega(\frac{x}{\tau},\frac{y}{\tau}). \end{cases} \tag{4.2}$$

**Lemma 4.1.** *For any $\tau \in (0,\infty)$ and $\omega \in (0,1]$, there is a constant $c$ independent of $\tau,\omega,x,y$ such that*

$$\begin{cases} \big|\Gamma_\tau^\omega(x,y)\big| \leq c|x-y|^{2-n}\mathbf{K}_{1/\omega,\tau}(x)\mathbf{K}_{1/\omega,\tau}(y), \\ |\Gamma_\tau^\omega(x,y)| \leq c|x-y|^{2-n} \qquad \textit{if } |x-y| \geq \tau(1-6\mathbf{d}_0), \\ |\nabla_x\Gamma_\tau^\omega(x,y)| \leq c|x-y|^{1-n}\mathbf{K}_{1/\omega,\tau}(x)\mathbf{K}_{1/\omega,\tau}(y), \\ |\nabla_y\Gamma_\tau^\omega(x,y)| \leq c|x-y|^{1-n} \qquad \textit{if } |x-y| \geq \tau(1-5\mathbf{d}_0). \end{cases} \tag{4.3}$$

*For any $\omega \in (0,1]$ and $x \in \Omega_f^1 \cup \Omega_m^1$, there is a constant $d > 0$ so that*

$$|\nabla_x\nabla_y\Gamma_1^\omega(x,y)| \leq c|x-y|^{-n}\mathbf{K}_{1/\omega^2,1}(x) \qquad \textit{if } 0 < |x-y| < d, \tag{4.4}$$

*where $c$ is a constant independent of $x,y,\omega$. See (3.1) for $\mathbf{d}_0$; (4.4) holds only in a small neighborhood of $x$.*

**Proof.** *Proof of $(4.3)_1$.* First we assume $\omega,\tau \in (0,1]$ and $x,y \in \Omega_f^\tau$. Set $r \equiv |x-y|$ for $x,y \in \Omega_f^\tau$. Let $F \in C_0^\infty(B_{r/3}(y))$ and find $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ satisfying

$$-\nabla \cdot (\mathbf{K}_{\omega^2,\tau}\nabla\Phi) = \mathbf{K}_{\omega,\tau}F.$$

By Lax-Milgram Theorem [13], Lemma 3.1, and Theorem 2.1 [1], $\Phi$ is solvable uniquely in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. By Definition 5.1 and remark in pages $59, 62, 67$ [20] as well as Corollary 3.1, we see, for $x \in \Omega_f^\tau$,

$$\begin{cases} \Phi(x) = \int_{B_{r/3}(y)} \Gamma_\tau^\omega(x,z)\mathbf{K}_{\omega,\tau}(z)F(z)dz, \\ |\Phi(x)| \le c\left| \fint_{B_{r/3}(x)} |\mathbf{K}_{\omega,\tau}\Phi(z)|^2 dz \right|^{1/2} \le c\left| \fint_{B_{r/3}(x)} \left|\mathbf{K}_{\omega,\tau}\Phi(z)\right|^{\frac{2n}{n-2}} dz \right|^{\frac{n-2}{2n}}, \end{cases} \tag{4.5}$$

where $c$ is independent of $\tau, \omega, r, x, y$. Lemma 3.1, (4.5), Theorem 2.1 [1], and Hölder inequality [13] imply

$$\begin{aligned} &\left| \int_{B_{r/3}(y)} \Gamma_\tau^\omega(x,z)\mathbf{K}_{\omega,\tau}(z)F(z)dz \right| \le c\left| \fint_{B_{r/3}(x)} \left|\mathbf{K}_{\omega,\tau}\Phi(z)\right|^{\frac{2n}{n-2}} dz \right|^{\frac{n-2}{2n}} \\ &\qquad \le cr^{\frac{2-n}{2}} \|\mathbf{K}_{\omega,\tau}\nabla\Phi\|_{L^2(\mathbb{R}^n)} \le cr^{\frac{4-n}{2}} \|F\|_{L^2(B_{r/3}(y))}, \end{aligned} \tag{4.6}$$

where $c$ is independent of $\tau, \omega, r, x, y$. Since $y \in \Omega_f^\tau$, equations (4.1), (4.6) and Corollary 3.1 imply

$$\left|\Gamma_\tau^\omega(x,y)\right| \le c\left| \fint_{B_{r/3}(y)} |\mathbf{K}_{\omega,\tau}(z)\Gamma_\tau^\omega(x,z)|^2 dz \right|^{\frac{1}{2}} \le \frac{c}{r^{n-2}},$$

where $c$ is independent of $\tau, \omega, r, x, y$. So $(4.3)_1$ holds for $x, y \in \Omega_f^\tau$, $\omega, \tau \in (0,1]$. Together with $(4.2)_3$, we see that $(4.3)_1$ also holds for $x, y \in \Omega_f^\tau, \tau \in [1,\infty)$ and $\omega \in (0,1]$. By a similar argument, the other cases of $(4.3)_1$ also hold.

Inequality $(4.3)_2$ follows from Theorem 3.1 [13], $(3.1)_4$, and $(4.3)_1$.

*Proof of $(4.3)_3$.* If $y \ne 0$, we define $r \equiv |y|$. For $\tau \in (0,\infty)$ and $\omega \in (0,1]$, both (4.1) and $(4.2)_2$ imply

$$-\nabla_z \cdot (\mathbf{K}_{\omega^2,\tau}(z)\nabla_z\Gamma_\tau^\omega(z,y)) = 0 \qquad \text{for } z \in B_{r/2}(0).$$

If $\Phi(z) \equiv \Gamma_\tau^\omega(\frac{r}{2}z, y)$, then

$$-\nabla \cdot (\mathbf{K}_{\omega^2,2\tau/r}(z)\nabla\Phi(z)) = 0 \qquad \text{for } z \in B_1(0).$$

Suppose $2\tau/r > 1$, by Lemma 3.4,

$$|\nabla\Phi(0)| \le c\mathbf{K}_{1/\omega,2\tau/r}(0)\|\mathbf{K}_{\omega,2\tau/r}\Phi\|_{L^2(B_1(0))},$$

where $c$ is a constant independent of $\omega, \tau/r$. By $(4.3)_1$,

$$|\nabla_x\Gamma_\tau^\omega(0,y)| = \frac{2}{r}|\nabla\Phi(0)| \le c|y|^{1-n}\mathbf{K}_{1/\omega,\tau}(0)\mathbf{K}_{1/\omega,\tau}(y),$$

where $c$ is independent of $y, \omega, \tau/r$. Suppose $2\tau/r \le 1$, by (3.14) and $(4.3)_2$,

$$|\nabla_x\Gamma_\tau^\omega(0,y)| = \frac{2}{r}|\nabla\Phi(0)| \le \frac{c}{r}\|\Phi\|_{L^\infty(B_1(0))} \le c|y|^{1-n},$$

where $c$ is independent of $y, \omega, \tau/r$. If $x \ne 0$ and $x \ne y$, then we shift $x$ to 0 and repeat the above argument to obtain the estimate for $|\nabla_x\Gamma_\tau^\omega(x,y)|$ in $(4.3)_3$ for $\tau \in (0,\infty)$ and $\omega \in (0,1]$.

Inequality $(4.3)_4$ follows from (4.1), $(4.3)_{2,3}$, $(3.1)_4$, maximal principle [13], and Lemma 3.3.

*Proof of (4.4).* If $x \in \Omega_f^1$, set $\tilde{d} \equiv dist(x, \Omega_m^1) > 0$. If $y \in B_{\tilde{d}/4}(x) \setminus \{x\}$ and $r \equiv |x-y|$, then, by (4.1) and $(4.2)_2$,

$$-\Delta_z \partial_{y_i} \Gamma_1^\omega(z,y) = 0 \qquad \text{in } B_{r/2}(x),$$

where $\partial_{y_i}$ is the partial derivative with respect to $y_i$ for $i \in \{1, \cdots, n\}$. By Theorem 2.10 [13] and $(4.3)_3$,

$$|\nabla_x \partial_{y_i} \Gamma_1^\omega(x,y)| \le \frac{c}{r} \|\partial_{y_i} \Gamma_1^\omega(\cdot, y)\|_{L^\infty(B_{r/2}(x))} \le c|x-y|^{-n},$$

where $c$ is a constant independent of $x, y, \omega$. So we prove (4.4) for $x \in \Omega_f^1$. (4.4) for $x \in \Omega_m^1$ is proved in a similar way. □

**Lemma 4.2.** *The Green function $\Gamma_1^\omega$ for $\omega \in (0,1]$ in (4.1) satisfies*

$$\begin{cases} -\nabla_x \cdot (\mathbf{K}_{\omega^2,1}(x) \nabla_x \partial_{y_k} \Gamma_1^\omega(x,y)) = 0 & \text{in } \mathbb{R}^n \setminus \{y\}, y \notin \partial\Omega_m^1, \\ \sup\limits_{|x-y| \in [r_1, r_2]} |\nabla_x \nabla_y \Gamma_1^\omega(x,y)| < \infty & \text{for any } r_1, r_2 > 0, \end{cases} \tag{4.7}$$

*where $y_k$ is the $k$-th component of $y = (y_1, \cdots, y_n)$, $\partial_{y_k}$ is the partial derivative with respect to $y_k$, and $k \in \{1, \cdots, n\}$.*

**Proof.** By (4.1), (4.2), $(4.3)_3$, Corollary 6.3 [13], and Lemma 3.4, we obtain $(4.7)_1$. By $(4.3)_3$, $(4.7)_1$, Corollary 6.3 [13], and Lemma 3.4, there is a constant $c$ such that, if $|y-z| > t$ for any $t > 0$,

$$\|\partial_{y_k} \Gamma_1^\omega(\cdot, y)\|_{C^1(\overline{B_{t/2}(z) \cap \Omega_f^1}) \cap C^1(\overline{B_{t/2}(z) \cap \Omega_m^1})} \le c. \tag{4.8}$$

(4.8) implies $(4.7)_2$. □

**Remark 4.1.** (1) By Lax-Milgram Theorem [13], Theorem 2.1 [1], and Lemma 3.1, we see that for any $\omega, \epsilon \in (0,1]$ and $F, Q \in L^\infty(\mathbb{R}^n)$ with compact support, a $\mathcal{D}^{1,2}(\mathbb{R}^n)$ solution of

$$-\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Phi + Q) = F \qquad \text{in } \mathbb{R}^n$$

exists uniquely. By Definition 5.1 and remark in pages $59, 62, 67$ [20], Green Theorem, and $(4.3)_3$ of Lemma 4.1,

$$\Phi(x) = \int_{\mathbb{R}^n} \Gamma_\epsilon^\omega(x,y) F(y) dy - \int_{\mathbb{R}^n} \nabla_y \Gamma_\epsilon^\omega(x,y) Q(y) dy \qquad \text{in } \mathbb{R}^n. \tag{4.9}$$

(2) Tracing the proof of Lemma 4.1 [13] as well as employing $(4.3)_3$ of Lemma 4.1, we see that if $\omega \in (0,1]$ and $F \in L^\infty(\mathbb{R}^n)$ with compact support, then the $\mathcal{D}^{1,2}(\mathbb{R}^n)$ solution of

$$-\nabla \cdot (\mathbf{K}_{\omega^2,1} \nabla \Phi) = F \qquad \text{in } \mathbb{R}^n$$

satisfies $\Phi \in C^1(\Omega_f^1) \cap C^1(\Omega_m^1)$ and

$$\nabla \Phi(x) = \int_{\mathbb{R}^n} \nabla_x \Gamma_1^\omega(x,y) F(y) dy \qquad \text{for any } x \in \Omega_f^1 \cup \Omega_m^1.$$

Tracing the proof of Lemma 4.2 [13] and employing Lemma 4.1, we obtain

**Lemma 4.3.** *If* $\omega \in (0,1]$, $Q \in L^\infty(\mathbb{R}^n) \cap C^{0,\alpha}_{loc}(\Omega^1_f) \cap C^{0,\alpha}_{loc}(\Omega^1_m)$ *with support in* $B_r(0)$ *for some* $r > 0$, *and* $\alpha \in (0,1)$, *then the* $\mathcal{D}^{1,2}(\mathbb{R}^n)$ *solution of*

$$-\nabla \cdot (\mathbf{K}_{\omega^2,1}\nabla\Phi + Q) = 0 \qquad \text{in } \mathbb{R}^n$$

*satisfies, for any* $x \in \Omega^1_f \cup \Omega^1_m$, $t > r$, *and* $j \in \{1, \cdots, n\}$,

$$\partial_j\Phi(x) = -\int_{B_t(0)} \partial_{x_j}\nabla_y\Gamma^\omega_1(x,y)Q(y)dy + \mathbf{K}_{\frac{1}{\omega^2},1}(x)Q(x)\int_{\partial B_t(0)} \nabla_y\Gamma(x,y)\mathbf{n}_j \, d\sigma_y$$
$$+\mathbf{K}_{\frac{1}{\omega^2},1}(x)Q(x)\int_{B_t(0)} \partial_{x_j}\nabla_y\Gamma(x,y)dy. \tag{4.10}$$

*Here* $\Gamma$ *is the fundamental solution of the Laplace equation in* $\mathbb{R}^n$, $x_j$ *is the* $j$*-th component of* $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, *and* $\mathbf{n}_j$ *is the* $j$*-th component of* $\vec{\mathbf{n}} = (\mathbf{n}_1, \mathbf{n}_2, \cdots, \mathbf{n}_n)$ *denoting a unit outward normal vector on* $\partial B_t(0)$. *Moreover, there is a constant* $c$ *independent of* $\omega, r$ *such that*

$$\left|\nabla\Phi(x) + \int_{\mathbb{R}^n} \nabla_x\nabla_y\Gamma^\omega_1(x,y)Q(y)dy\right| \le c\mathbf{K}_{\frac{1}{\omega^2},1}(x)|Q(x)| \quad \text{for } x \in \Omega^1_f \cup \Omega^1_m. \tag{4.11}$$

**Proof.** Define, for any $t > r > \delta > 0$ and $x \in \mathbb{R}^n$,

$$\Phi_\delta(x) \equiv -\int_{B_t(0)} \nabla_y\Gamma^\omega_1(x,y)\eta_\delta(|x-y|)Q(y)dy,$$

where $\eta_\delta(x) = \eta(x/\delta)$ and $\eta \in C^\infty_0(\mathbb{R})$ is an even function satisfying $\eta \in [0,1]$, $\eta(x) = 0$ in $|x| \le 1/2$, $\eta(x) = 1$ in $|x| \ge 1$, and $\eta'(x) \ge 0$ for $x \ge 0$. By $(4.3)_3$ and $(4.7)_2$, we see $\Phi_\delta \in C^1(\mathbb{R}^n)$. By $(4.3)_3$, $\Phi_\delta$ converges to

$$\Phi(x) \equiv -\int_{B_t(0)} \nabla_y\Gamma^\omega_1(x,y)Q(y)dy \qquad \text{in } L^\infty(\mathbb{R}^n) \text{ as } \delta \to 0.$$

Define $\varphi_\omega(x,\cdot)$ and $x \in \Omega^1_f \cup \Omega^1_m$ as

$$\varphi_\omega(x,y) \equiv \Gamma^\omega_1(x,y) - \mathbf{K}_{1/\omega^2,1}(x)\Gamma(x,y) \quad \text{in } \mathbb{R}^n.$$

Note for any $x \in \Omega^1_f \cup \Omega^1_m$, there is a small neighborhood $B_d(x)$ of $x$ such that

$$\Delta_y\varphi_\omega(x,y) = 0 \qquad \text{for } y \in B_d(x).$$

So $\varphi_\omega(x,\cdot)$ is smooth in a neighborhood $B_d(x)$ of $x \in \Omega^1_f \cup \Omega^1_m$ and is piecewise smooth in $\mathbb{R}^n$.

If $x \in \Omega_f^1 \cup \Omega_m^1$, $t > r > \delta$, and $j \in \{1, \cdots, n\}$, by Green's formula,

$$\begin{aligned}\partial_j \Phi_\delta(x) &= -\int_{B_t(0)} \partial_{x_j}\big(\nabla_y \Gamma_1^\omega(x,y)\eta_\delta(|x-y|)\big)Q(y)dy \\ &= -\int_{B_t(0)} \partial_{x_j}\big(\nabla_y \Gamma_1^\omega(x,y)\eta_\delta(|x-y|)\big)(Q(y)-Q(x))dy \\ &\quad + \mathbf{K}_{\frac{1}{\omega^2},1}(x)Q(x)\int_{\partial B_t(0)} \nabla_y \Gamma(x,y)\mathbf{n}_j \; d\sigma_y \\ &\quad - Q(x)\int_{B_t(0)} \partial_{x_j}\big(\nabla_y \varphi_\omega(x,y)\eta_\delta(|x-y|)\big)dy.\end{aligned}$$

By $(4.3)_{1,3}$, (4.4), and following the proof of Lemma 4.2 [13], if $\delta$ closes to 0, then $\partial_j \Phi_\delta(x)$ converges to $\partial_j \Phi(x)$ in (4.10) for any $x \in \Omega_f^1 \cup \Omega_m^1$, $t > r$, and $j \in \{1, \cdots, n\}$. So we prove (4.10).

If $x \notin B_r(0)$, (4.11) is from (4.10) because of $Q(x) = 0$. If $x \in B_r(0)$ and $r < t$, by (2.13) in [13] and $i, j \in \{1, \cdots, n\}$,

$$\begin{aligned}\left|\int_{B_t(0)} \partial_{x_j}\partial_{y_i}\Gamma(x,y)dy\right| &= \left|\int_{B_t(0)\setminus B_{t-|x|}(x)} \partial_{x_j}\partial_{y_i}\Gamma(x,y)dy\right| \\ &\le c\int_{B_{t+|x|}(x)\setminus B_{t-|x|}(x)} |x-y|^{-n}dy \le c\ln\frac{t+|x|}{t-|x|} \le c\ln\frac{t+r}{t-r}, \end{aligned} \tag{4.12}$$

$$\left|\int_{\partial B_t(0)} \partial_{y_i}\Gamma(x,y)\mathbf{n}_j \; d\sigma_y\right| \le c\left|\frac{t}{t-r}\right|^{n-1}, \tag{4.13}$$

where $c$ is a constant. If $t$ is much larger than $r$, then the right hand sides of (4.12)–(4.13) are bounded by a constant independent of $\omega, r$. Together with (4.10), we obtain (4.11). So we prove the lemma. □

Next we derive an approximation of the second derivatives of $\Gamma_\tau^\omega$ for $\tau \in (0, \infty)$. The case of $\tau \in (0, 1]$ is considered in subsection 4.2 and the case of $\tau \in [\frac{1}{n+1}, \infty)$ is in subsection 4.3.

## 4.2. *For* $\tau \in (0,1]$ *case*

Let $\Gamma_0^\omega$ for $\omega \in (0, 1]$ denote the Green function of

$$\begin{cases} -\nabla_y \cdot (\mathcal{K}_\omega \nabla_y \Gamma_0^\omega(x,y)) = \delta(x-y) & \text{in } \mathbb{R}^n, \\ \Gamma_0^\omega(x,y) \to 0 \quad \text{as } |x-y| \to \infty, \end{cases} \tag{4.14}$$

where $\mathcal{K}_\omega$ is the symmetric positive definite matrix in (3.9). By (3.10) and change of variables, the $\Gamma_0^\omega$ in (4.14) can be transformed to the fundamental solution of the Laplace equation in a new coordinate system. By the results in page 17 [13], we see $\Gamma_0^\omega(x, \cdot) \in H_{loc}^1(\mathbb{R}^n \setminus \{x\}) \cap C(\mathbb{R}^n \setminus \{x\})$ and there is a constant $c$ independent of $\omega$

such that

$$\begin{cases} \Gamma_0^\omega(x,y) = \Gamma_0^\omega(y,x) & \text{for } x \neq y, \\ \Gamma_0^\omega(x,y) = \nu^{2-n}\ \Gamma_0^\omega(\frac{x}{\nu},\frac{y}{\nu}) & \text{for any } \nu > 0, \\ |\Gamma_0^\omega(x,y)| \leq c|x-y|^{2-n} & \text{for } x \neq y, \\ |\nabla_y \Gamma_0^\omega(x,y)| \leq c|x-y|^{1-n} & \text{for } x \neq y. \end{cases} \tag{4.15}$$

**Lemma 4.4.** *For any $\omega, \tau \in (0,1]$,*

$$\sup_{|x-y|\geq 1-4\mathbf{d}_0} |\Gamma_\tau^\omega(x,y) - \Gamma_0^\omega(x,y)| \leq c\tau^\lambda, \tag{4.16}$$

*where $c, \lambda$ are positive constants independent of $\omega, \tau$. See (3.1) for $\mathbf{d}_0$.*

**Proof.** Fix $x \in \mathbb{R}^n$ and let $c_1 = \sup\limits_{|x-y|\geq 1-6\mathbf{d}_0} |\Gamma_\tau^\omega(x,y) - \Gamma_0^\omega(x,y)|$. By (3.1), $(4.3)_2$, and $(4.15)_3$, the $c_1$ is independent of $\omega, \tau, x$. By $(4.3)_4$ and $(4.15)_4$, $\Gamma_\tau^\omega(x,\cdot)$ and $\Gamma_0^\omega(x,\cdot)$ are Lipschitz continuous functions in $\mathbb{R}^n \setminus B_{1-5\mathbf{d}_0}(x)$. In other words,

$$|\nabla_y \Gamma_\tau^\omega(x,y)| + |\nabla_y \Gamma_0^\omega(x,y)| \leq c_2 \quad \text{for } y \in \mathbb{R}^n \setminus B_{1-5\mathbf{d}_0}(x), \tag{4.17}$$

where $c_2$ is independent of $\omega, \tau, x$. Define $\theta_{\omega,\tau,x} \equiv \sup\limits_{|x-y|\geq 1-4\mathbf{d}_0} |\Gamma_\tau^\omega(x,y) - \Gamma_0^\omega(x,y)|$ for any fixed $\omega, \tau, x$. By $(4.3)_2$ and $(4.15)_3$, there is a $y_*$ satisfying

$$|x - y_*| \geq 1 - 4\mathbf{d}_0 \qquad \text{and} \qquad \theta_{\omega,\tau,x} = |\Gamma_\tau^\omega(x,y_*) - \Gamma_0^\omega(x,y_*)| \leq c_1.$$

Pick a number $\beta > 9$ so that $\rho \equiv \frac{\theta_{\omega,\tau,x}}{\beta c_2} \leq \mathbf{d}_0$, where $c_2$ is from (4.17). By (4.17), we see, for any $y \in B_\rho(y_*)$,

$$\begin{aligned} |\Gamma_\tau^\omega(x,y) - \Gamma_0^\omega(x,y)| &\geq |\Gamma_\tau^\omega(x,y_*) - \Gamma_0^\omega(x,y_*)| \\ &\quad -|\Gamma_\tau^\omega(x,y) - \Gamma_\tau^\omega(x,y_*)| - |\Gamma_0^\omega(x,y) - \Gamma_0^\omega(x,y_*)| \geq \frac{\theta_{\omega,\tau,x}}{5}. \end{aligned} \tag{4.18}$$

Let $F \in C_0^\infty(B_\rho(y_*))$, $F \in [0,1]$, $F = 1$ on $B_{\rho/2}(y_*)$, $\|\nabla F\|_{L^\infty(B_\rho(y_*))} \leq \frac{4}{\rho}$, $\|\nabla^2 F\|_{L^\infty(B_\rho(y_*))} \leq \frac{16}{\rho^2}$. Put $F$ in the right hand side of (3.19) to get $U_\tau$ and of (3.23) to get $U$. By Lemma 3.6, for any $\ell > 0$ and $\omega, \tau \in (0,1]$,

$$\|U_\tau - U\|_{L^\infty(\mathbb{R}^n)} \leq c_3 \tau |\rho|^{\frac{-(n+2\ell)}{n+\ell}}, \tag{4.19}$$

where $c_3$ is a constant independent of $\omega, \tau, x$. By (3.19), (3.23), (4.17)–(4.18), and Remark 4.1,

$$\begin{aligned} |U_\tau(x) - U(x)| &= \left| \int_{\Omega_f^\tau} \Gamma_\tau^\omega(x,y) F(y) dy - \int_{\mathbb{R}^n} \Gamma_0^\omega(x,y) |Y_f| F(y) dy \right| \\ &\geq \int_{\Omega_f^\tau} |\Gamma_\tau^\omega - \Gamma_0^\omega|(x,y) F(y) dy - \left| \int_{\Omega_f^\tau} \Gamma_0^\omega(x,y) F(y) dy - \int_{\mathbb{R}^n} \Gamma_0^\omega(x,y)|Y_f| F(y) dy \right| \\ &\geq c_4 \rho^{n+1} - \sum_{j\in\mathbb{Z}^n} \tau^n |Y_f| \big| \Gamma_0^\omega(x,y_j) F(y_j) - \Gamma_0^\omega(x,z_j) F(z_j) \big| \quad \text{for } y_j, z_j \in \tau(Y-j) \\ &\geq c_4 \rho^{n+1} \left( 1 - \frac{c_5 \tau}{\rho^2} \right), \end{aligned} \tag{4.20}$$

where $c_4, c_5$ are independent of $\omega, \tau, x$. If $1 - \frac{c_5\tau}{\rho^2} > 1/2$, (4.19) and (4.20) imply $\rho^{n+1} \le c\tau|\rho|^{\frac{-(n+2\ell)}{n+\ell}}$, which implies (4.16). If $1 - \frac{c_5\tau}{\rho^2} \le 1/2$, then $\rho \le c\tau^{1/2}$. That again implies (4.16). So this lemma is proved. □

**Lemma 4.5.** *For any $\omega, \tau \in (0,1]$ and $|x-y| > 1 - 3\mathbf{d}_0$,*

$$\left|\nabla_y \Gamma_\tau^\omega(x,y) - \big(I + \nabla\mathbb{X}(y/\tau)\big)\nabla_y \Gamma_0^\omega(x,y)\right| \le c\tau^\lambda \mathbf{K}_{1/\omega,\tau}(y), \tag{4.21}$$

*where $I$ is the identity matrix and $c, \lambda$ are positive constants independent of $\omega, \tau, x, y$. See (3.1) for $\mathbf{d}_0$ and remark after (3.8) for $\mathbb{X}$.*

**Proof.** Define $\mathbb{T}_\omega^{(i_1,i_2)}$ as follows. For $i_1, i_2 \in \{1, \cdots, n\}$, we find $\mathbb{T}_\omega^{(i_1,i_2)} \in H^1_{per}(\mathbb{R}^n)$ satisfying

$$\begin{cases} \nabla\cdot\left(\mathbf{K}_{\omega^2,1}\left(\nabla\mathbb{T}_\omega^{(i_1,i_2)} + \mathbb{X}_\omega^{(i_2)}\vec{e}_{i_1}\right)\right) = -\mathbf{K}_{\omega^2,1}\left(\delta_{i_1,i_2} + \partial_{i_1}\mathbb{X}_\omega^{(i_2)}\right) \quad \text{in } Y, \\ \displaystyle\int_{Y_f} \mathbb{T}_\omega^{(i_1,i_2)}(y)dy = 0. \end{cases}$$

Here $\mathbb{X}_\omega^{(i_2)}$ is defined in (3.6), $\vec{e}_{i_1}$ denotes a unit vector in the $i_1$ direction, $\delta_{i_1,i_2} = \begin{cases} 1 & \text{if } i_1 = i_2 \\ 0 & \text{if } i_1 \neq i_2 \end{cases}$. By Lax-Milgram Theorem [13], (3.8), and Lemma 3.3, function $\mathbb{T}_\omega^{(i_1,i_2)}$ is uniquely solvable and satisfies

$$\|\mathbb{T}_\omega^{(i_1,i_2)}\|_{C^{1,\alpha}(\overline{Y_f})\cap C^{1,\alpha}(\overline{Y_m})} \le c \qquad \text{for some } \alpha \in (0,1), \tag{4.22}$$

where $c$ is independent of $\omega$. For fixed $x \in \mathbb{R}^n$, we define, in $|x-y| > 1 - 4\mathbf{d}_0$,

$$\Phi_{\omega,\tau}(y) \equiv \Gamma_\tau^\omega(x,y) - \Gamma_0^\omega(x,y) - \mathbb{X}_{\omega,\tau}(y)\nabla_y\Gamma_0^\omega(x,y) - \mathbb{T}_{\omega,\tau}(y)\nabla_y^2\Gamma_0^\omega(x,y),$$

where $\mathbb{T}_{\omega,\tau}(y) \equiv \tau^2\mathbb{T}_\omega(\frac{y}{\tau})$ and $\mathbb{T}_\omega \equiv (\mathbb{T}_\omega^{(i_1,i_2)})$ are $n\times n$ periodic matrix functions with period $Y$. Then $\Phi_{\omega,\tau}$ satisfies, in $\mathbb{R}^n \setminus B_{1-4\mathbf{d}_0}(x)$,

$$-\nabla\cdot(\mathbf{K}_{\omega^2,\tau}(\nabla\Phi_{\omega,\tau} + \mathbb{T}_{\omega,\tau}\nabla_y^3\Gamma_0^\omega)) = \mathbf{K}_{\omega^2,\tau}(\mathbb{X}_{\omega,\tau}\nabla_y\Delta_y\Gamma_0^\omega + \nabla\mathbb{T}_{\omega,\tau}\nabla_y^3\Gamma_0^\omega).$$

By (3.13), Lemma 4.4, (3.8), and (4.22), we know that if $\omega, \tau \in (0,1]$,

$$\|\mathbf{K}_{\omega,\tau}\nabla\Phi_{\omega,\tau}\|_{L^\infty(\mathbb{R}^n\setminus B_{1-3\mathbf{d}_0}(x))} \le c\tau^\lambda,$$

where $c, \lambda$ are positive constants independent of $\omega, \tau, x$. Which implies (4.21). □

Let us fix $\omega \in (0,1]$, $y \in \mathbb{R}^n \setminus \partial\Omega_m^\tau$ and define

$$\begin{cases} \varphi_\tau(x) \equiv \nabla_y\Gamma_\tau^\omega(x,y) \\ \varphi_0(x) \equiv (I + \nabla\mathbb{X}(\frac{y}{\tau}))\nabla_y\Gamma_0^\omega(x,y) \end{cases} \qquad \text{for } x \in \mathbb{R}^n \setminus B_{1-3\mathbf{d}_0}(y). \tag{4.23}$$

By Lemma 4.5, if $\omega, \tau \in (0,1]$, then

$$\|\varphi_\tau - \varphi_0\|_{L^\infty(\mathbb{R}^n\setminus B_{1-3\mathbf{d}_0}(y))} \le c\tau^\lambda\mathbf{K}_{1/\omega,\tau}(y), \tag{4.24}$$

where $c, \lambda$ are positive constants independent of $\omega, \tau, y$. By $(4.7)_1$,

$$-\nabla \cdot (\mathbf{K}_{\omega^2,\tau} \nabla \varphi_\tau) = 0 \qquad \text{in } \mathbb{R}^n \setminus B_{1-3\mathbf{d}_0}(y).$$

As $\tau \to 0$, by $(4.3)_4$, (4.24), and an analogous argument as (3.19)–(3.23), we obtain

$$-\nabla \cdot (\mathcal{K}_\omega \nabla \varphi_0) = 0 \qquad \text{in } \mathbb{R}^n \setminus B_{1-2\mathbf{d}_0}(y). \tag{4.25}$$

Tracing the proof of Lemma 4.5, we see that (4.23)–(4.25) imply

**Lemma 4.6.** *For any $\omega, \tau \in (0,1]$ and $|x-y| \geq 1 - \mathbf{d}_0$,*

$$\Big|\nabla_x \nabla_y \Gamma_\tau^\omega(x,y) - \big(I + \nabla \mathbb{X}(\frac{x}{\tau})\big)\big(I + \nabla \mathbb{X}(\frac{y}{\tau})\big)\nabla_x \nabla_y \Gamma_0^\omega(x,y)\Big| \leq c\tau^\lambda \mathbf{K}_{\frac{1}{\omega},\tau}(x)\mathbf{K}_{\frac{1}{\omega},\tau}(y),$$

*where $c, \lambda$ are positive constants independent of $\omega, \tau, x, y$.*

**Proof.** Define, for fixed $\omega \in (0,1]$, $y \in \mathbb{R}^n$ and $|x-y| > 1 - 3\mathbf{d}_0$,

$$\Phi_{\omega,\tau} \equiv \varphi_\tau - \varphi_0 - \mathbb{X}_{\omega,\tau}\nabla\varphi_0 - \mathbb{T}_{\omega,\tau}\nabla^2\varphi_0,$$

where $\varphi_\tau, \varphi_0$ are defined in (4.23) and $\mathbb{X}_{\omega,\tau}, \mathbb{T}_{\omega,\tau}$ are same as those in Lemma 4.5. Following the proof of Lemma 4.5, we see that Lemma 4.6 holds. □

Lemma 4.6 then implies

**Corollary 4.1.** *For any $\omega \in (0,1]$ and $|x-y| \geq 1$,*

$$\begin{aligned} &\big|\nabla_x \nabla_y \Gamma_1^\omega(x,y) - (I + \nabla\mathbb{X}(x))(I + \nabla\mathbb{X}(y))\nabla_x\nabla_y\Gamma_0^\omega(x,y)\big| \\ &\quad \leq c|x-y|^{-n-\lambda}\mathbf{K}_{1/\omega,1}(x)\mathbf{K}_{1/\omega,1}(y), \end{aligned} \tag{4.26}$$

*where $\lambda, c$ are positive constants independent of $\omega, x, y$.*

**Proof.** For $x, y \in \Omega_f^1$ case. By $(4.2)_3$ and (4.15), we see $\Gamma_\tau^\omega(x,y) = \tau^{2-n}\Gamma_1^\omega(\frac{x}{\tau}, \frac{y}{\tau})$, $\Gamma_0^\omega(x,y) = \tau^{2-n}\Gamma_0^\omega(\frac{x}{\tau}, \frac{y}{\tau})$ for $\tau > 0$. Define $x = \frac{\xi}{\tau}$, $y = \frac{\eta}{\tau}$, then $\xi, \eta \in \Omega_f^\tau$. Assume $|\xi - \eta| = 1$. By Lemma 4.6, if $\omega, \tau \in (0,1]$,

$$\begin{aligned} &\big|\nabla_x\nabla_y\Gamma_1^\omega(x,y) - (I + \nabla\mathbb{X}(x))(I + \nabla\mathbb{X}(y))\nabla_x\nabla_y\Gamma_0^\omega(x,y)\big| \\ &= \Big|\nabla_x\nabla_y\Gamma_1^\omega(\frac{\xi}{\tau}, \frac{\eta}{\tau}) - \big(I + \nabla\mathbb{X}(\frac{\xi}{\tau})\big)\big(I + \nabla\mathbb{X}(\frac{\eta}{\tau})\big)\nabla_x\nabla_y\Gamma_0^\omega(\frac{\xi}{\tau}, \frac{\eta}{\tau})\Big| \\ &= \Big|\nabla_\xi\nabla_\eta\Gamma_\tau^\omega(\xi,\eta) - \big(I + \nabla\mathbb{X}(\frac{\xi}{\tau})\big)\big(I + \nabla\mathbb{X}(\frac{\eta}{\tau})\big)\nabla_\xi\nabla_\eta\Gamma_0^\omega(\xi,\eta)\Big|\tau^n \leq c\tau^{n+\lambda}, \end{aligned}$$

where $c, \lambda$ are positive constants independent of $\omega, \tau$.

If we take $\tau = \frac{1}{|x-y|}$, then $|\xi - \eta| = 1$. Note $|x-y| \geq 1$ because of $\tau \leq 1$. So we prove (4.26) for $x, y \in \Omega_f^1$ case. The other cases of (4.26) are proved in a similar way. □

### 4.3. *For* $\tau \in [\frac{1}{n+1}, \infty)$ *case*

In this subsection, $\tau \in [\frac{1}{n+1}, \infty)$ and notation in (3.1) is used. For any $\nu \in (0, \infty)$, $\omega \in (0, 1]$, and $j \in \mathbb{Z}^n$, let $\mathcal{M}^{(j)}_{\omega,\nu}$ denote the Green function of

$$\begin{cases} -\nabla_y \cdot (\mathbf{K}^{(j)}_{\omega^2,\nu}(y) \nabla_y \mathcal{M}^{(j)}_{\omega,\nu}(x, y)) = \delta(x - y) \quad \text{in } \mathbb{R}^n, \\ \mathcal{M}^{(j)}_{\omega,\nu}(x, y) \to 0 \quad \text{as } |x - y| \to \infty, \end{cases} \tag{4.27}$$

where

$$\mathbf{K}^{(j)}_{\omega^2,\nu}(y) \equiv \begin{cases} 1 & \text{if } y \in \mathbb{R}^n \setminus \nu(\overline{Y_m} - j), \\ \omega^2 & \text{if } y \in \nu(Y_m - j). \end{cases}$$

Similar to $\Gamma^\omega_\tau$ in (4.1), by Theorem 5.4 [20], remark in pages $62, 67$ in [20], and Lemma 3.3, $\mathcal{M}^{(j)}_{\omega,\nu}$ exists uniquely in $H^1_{loc}(\mathbb{R}^n \setminus \{x\}) \cap W^{1,1}_{loc}(\mathbb{R}^n)$ and satisfies

$$\begin{cases} \mathcal{M}^{(j)}_{\omega,\nu}(x, \cdot) \in C(\mathbb{R}^n \setminus \{x\}), \\ \mathcal{M}^{(j)}_{\omega,\nu}(x, y) = \mathcal{M}^{(j)}_{\omega,\nu}(y, x) \qquad \text{for } x \neq y, \\ \mathcal{M}^{(j)}_{\omega,\nu}(x, y) = \nu^{2-n} \mathcal{M}^{(0)}_{\omega,1}(\frac{x}{\nu} + j, \frac{y}{\nu} + j). \end{cases} \tag{4.28}$$

If $(x, y) \notin \overline{Y_m} \times \overline{Y_m}$ and if $x \neq y$, we define

$$\mathcal{E}_\omega(x, y) \equiv \mathcal{M}^{(0)}_{\omega,1}(x, y) - \Gamma(x, y), \tag{4.29}$$

where $\Gamma$ is the fundamental solution of the Laplace operator in $\mathbb{R}^n$. By (4.28),

$$\mathcal{E}_\omega(x, y) = \mathcal{E}_\omega(y, x) \qquad \text{for } (x, y) \notin \overline{Y_m} \times \overline{Y_m} \text{ and } x \neq y. \tag{4.30}$$

If $x \in \mathbb{R}^n \setminus \overline{Y_m}$ is fixed, $\mathcal{E}_\omega(x, \cdot)$ is a piecewise smooth function in $\mathbb{R}^n$ satisfying

$$\begin{cases} -\Delta_y \mathcal{E}_\omega(x, y) = 0 & \text{in } \mathbb{R}^n \setminus \partial Y_m, \\ \lfloor \mathcal{E}_\omega \rfloor_{\partial Y_m}(x, y) = 0, & \\ \lfloor \mathbf{K}^{(0)}_{\omega^2,1} \nabla_y \mathcal{E}_\omega \rfloor_{\partial Y_m} \cdot \vec{\mathbf{n}}_y(x, y) = (\omega^2 - 1) \nabla_y \Gamma(x, y) \cdot \vec{\mathbf{n}}_y & \text{on } \partial Y_m, \\ \mathcal{E}_\omega(x, y) \to 0 \text{ as } |x - y| \to \infty, & \end{cases} \tag{4.31}$$

where $\vec{\mathbf{n}}_y$ is a unit normal vector on $\partial Y_m$. See (2.1) for $(4.31)_{2,3}$.

$(\overline{\mathcal{A}} - i) \cap (\overline{\mathcal{A}} - j) = \emptyset$ for $i \neq j$ and $i, j \in \mathbb{Z}^n$, where $\overline{\mathcal{A}}$ is the closure of $\mathcal{A}$ (see (3.1) for $\mathcal{A}$). If $x \in \Omega^\tau_f \setminus \bigcup_{j \in \mathbb{Z}^n} \tau(\mathbf{S}_1 - j)$, then the distance from $x$ to $\partial\Omega^\tau_m$ is of order $\tau$. Let $\phi \in C^\infty_0(\mathbb{R}^n)$ be a bell-shaped function satisfying $\phi \in [0, 1]$, $\phi(x) = 1$ in $x \in \mathbf{S}$, $supp(\phi) \subset \mathcal{A}$. If $\mathcal{X}^\dagger_{\mathcal{A}}(x, y) \equiv \phi(x)\phi(y) \in [0, 1]$, then $\mathcal{X}^\dagger_{\mathcal{A}}$ satisfies

$$\begin{cases} \mathcal{X}^\dagger_{\mathcal{A}}(x, y) = \mathcal{X}^\dagger_{\mathcal{A}}(y, x) \qquad \text{for } x, y \in \mathbb{R}^n, \\ \mathcal{X}^\dagger_{\mathcal{A}}(x, y) = \begin{cases} 1 & \text{if } x, y \in \mathbf{S}, \\ 0 & \text{if } x \notin \mathcal{A} \text{ or } y \notin \mathcal{A}. \end{cases} \end{cases} \tag{4.32}$$

See (3.1) for $\mathbf{S}_1, \mathbf{S}, \mathcal{A}$. Define, for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n \setminus \{x\}$,

$$\begin{aligned}\mathcal{G}_1^\omega(x,y) \equiv \Gamma(x,y)\Big(1 - \sum_{j\in\mathbb{Z}^n} \mathcal{X}_{\mathcal{A}}^\dagger(x+j, y+j)\Big) \\ + \sum_{j\in\mathbb{Z}^n} \mathcal{M}_{\omega,1}^{(0)}(x+j,y+j)\mathcal{X}_{\mathcal{A}}^\dagger(x+j,y+j),\end{aligned} \tag{4.33}$$

and define

$$\mathcal{G}_\tau^\omega(x,y) \equiv \tau^{2-n}\mathcal{G}_1^\omega(\frac{x}{\tau}, \frac{y}{\tau}) \qquad \text{for } \tau \in [\tfrac{1}{n+1}, \infty). \tag{4.34}$$

By symmetry property of $\Gamma, \mathcal{M}_{\omega,1}^{(0)}, \mathcal{X}_{\mathcal{A}}^\dagger$ (see (4.28) and (4.32)), we know

$$\mathcal{G}_1^\omega(x,y) = \mathcal{G}_1^\omega(y,x) \qquad \text{for } x \neq y. \tag{4.35}$$

If $(x,y) \notin \bigcup_{j\in\mathbb{Z}^n}(\overline{Y_m} - j) \times (\overline{Y_m} - j)$ and $x \neq y$, equation (4.33) can written as, by (4.29),

$$\mathcal{G}_1^\omega(x,y) \equiv \Gamma(x,y) + \sum_{j\in\mathbb{Z}^n} \mathcal{E}_\omega(x+j,y+j)\mathcal{X}_{\mathcal{A}}^\dagger(x+j,y+j). \tag{4.36}$$

**Remark 4.2.** (1) For $\tau \in [\frac{1}{n+1}, \infty)$, $\omega \in (0,1]$, and $j \in \mathbb{Z}^n$, we consider

$$-\nabla \cdot (\mathbf{K}_{\omega^2,\tau}\nabla\Phi) = 0 \quad \text{in } \tau(\mathbf{S} - j).$$

By maximal principle (see Theorem 3.1 and Lemma 3.4 in [13]) and Lemma 3.3,

$$\begin{cases}\|\Phi\|_{L^\infty(\tau(\mathbf{S}-j))} \leq \|\Phi\|_{L^\infty(\tau(\partial\mathbf{S}-j))}, \\ \tau\|\nabla\Phi\|_{L^\infty(\tau(\mathbf{S}_1-j))} \leq c\|\Phi\|_{L^\infty(\tau(\partial\mathbf{S}-j))},\end{cases} \tag{4.37}$$

where $c$ is independent of $\omega, \tau$. See (3.1) for $\mathbf{S}_1, \mathbf{S}$.

(2) Tracing the proof of Lemma 4.2 and employing (4.27)–(4.28), we see *The Green function $\mathcal{M}_{\omega,1}^{(0)}$ in (4.27) satisfies*

$$-\nabla_x \cdot (\mathbf{K}_{\omega^2,1}^{(0)}(x)\nabla_x\partial_{y_k}\mathcal{M}_{\omega,1}^{(0)}(x,y)) = 0 \qquad \text{in } \mathbb{R}^n \setminus \{y\}, y \notin \partial Y_m, \tag{4.38}$$

*where $y_k$ is the $k$-th component of $y = (y_1, \cdots, y_n)$ and $\partial_{y_k}$ is the partial derivative with respect to $y_k$ for $k \in \{1, \cdots, n\}$.*

**Lemma 4.7.** *Let $\omega \in (0,1]$. (1) There is a constant $c$ independent of $\omega, x, y$ such that*

$$\begin{cases}\big|\mathcal{M}_{\omega,1}^{(0)}(x,y)\big| \leq c|x-y|^{2-n}\mathbf{K}_{1/\omega,1}^{(0)}(x)\mathbf{K}_{1/\omega,1}^{(0)}(y), & \\ |\mathcal{M}_{\omega,1}^{(0)}(x,y)| \leq c|x-y|^{2-n} & \text{if } |x-y| \geq 1 - 6\mathbf{d}_0, \\ |\mathcal{E}_\omega(x,y)| \leq c & \text{if } (x,y) \notin \mathbf{S}_0 \times \mathbf{S}_0, \\ \|\mathcal{E}_\omega(x,\cdot)\|_{C^2(\mathbb{R}^n\setminus\mathbf{S}_1)} \leq c & \text{for } x \in \mathbb{R}^n, \\ \|\nabla_y\mathcal{E}_\omega(\cdot,y)\|_{C^1(\mathbb{R}^n\setminus\mathbf{S})} \leq c & \text{for } y \notin \partial Y_m, \\ \Big|\big(1 - \mathcal{X}_{\mathcal{A}}^\dagger(x,y)\big)\big(\Gamma(x,y) - \mathcal{M}_{\omega,1}^{(0)}(x,y)\big)\Big| \leq c & \text{for } x \neq y.\end{cases} \tag{4.39}$$

*(2) For any $\tau \in [\frac{1}{n+1}, \infty)$, there is a constant $c$ independent of $\omega, \tau, x, y$ such that if $x, y \in \Omega_f^\tau$ or if $|x-y| \ge \tau(1-6\mathbf{d}_0)$, then*

$$|\mathcal{G}_\tau^\omega(x,y)| \le c|x-y|^{2-n}. \tag{4.40}$$

*(3) If $x \in \mathbf{S}_2 \setminus Y$ and $y \in Y$, there is a constant $c$ independent of $\omega, x, y$ such that*

$$\begin{cases} \left|\mathcal{M}_{\omega,1}^{(0)}(x,y)\right| \le c|x-y|^{2-n}, \\ \left|\nabla_y \mathcal{M}_{\omega,1}^{(0)}(x,y)\right| \le c|x-y|^{1-n}. \end{cases} \tag{4.41}$$

*See (3.1) for $\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}, \mathcal{A}, \mathbf{S}_2, \mathbf{d}_0$.*

**Proof.** Following the arguments of Lemmas 4.1–4.3, 5.1–5.3 [28], we have

*Let $\ell > 0$ and $\omega, \nu \in (0,1]$. Any solution of*

$$-\nabla \cdot (\mathbf{K}_{\omega^2,\nu}^{(0)} \nabla \Phi) = F \qquad \text{in } B_1(x^*)$$

*satisfies*

$$[\Phi]_{C^{0,\mu}(\overline{B_{1/2}(x^*) \cap (\mathbb{R}^n \setminus \nu Y_m)})} \le c\big(\|\mathbf{K}_{\omega,\nu}^{(0)}\Phi\|_{L^2(B_1(x^*))} + \|\mathbf{K}_{1/\omega,\nu}^{(0)} F\|_{L^{n+\ell}(B_1(x^*))}\big),$$

*where $\mu \equiv \frac{\ell}{n+\ell}$, $x^* \in \mathbb{R}^n$, and $c$ is independent of $\omega, \nu, x^*$.*

Then we follow the proofs of Corollary 3.1 and Lemma 4.1 to obtain $(4.39)_{1,2}$.

Note

$$\|\Gamma(x,\cdot)\|_{C^{2,\alpha}(\partial Y_m)} \le c \quad \text{if } x \in \mathbb{R}^n \setminus \mathbf{S}_0, \tag{4.42}$$

where $\alpha \in (0,1)$ and $c$ is independent of $x$. $(4.39)_1$ and (4.42) imply

$$\|\mathcal{E}_\omega(x,\cdot)\|_{L^\infty(\partial Y_m)} \le c \quad \text{if } x \in \mathbb{R}^n \setminus \mathbf{S}_0, \tag{4.43}$$

where $c$ is independent of $\omega, x$. (4.30)–(4.31), (4.43), and Theorem 3.1 [13] imply $(4.39)_3$.

By (4.27), $\Delta_y \mathcal{E}_\omega(x,y) = 0$ for $x \in \mathbb{R}^n, y \in \mathbb{R}^n \setminus \mathbf{S}_0$. Corollary 6.3 [13] and $(4.39)_3$ imply $(4.39)_4$.

By (4.31), $(4.39)_3$, (4.42), and Lemma 3.3,

$$\|\mathcal{E}_\omega(x,\cdot)\|_{C^2(\overline{\mathbf{S}_2 \setminus Y_m}) \cap C^2(\overline{Y_m})} \le c \quad \text{if } x \in \mathbb{R}^n \setminus \mathbf{S}_0,$$

where $c$ is independent of $\omega, x$. Together with $(4.39)_4$, we have

$$\|\mathcal{E}_\omega(x,\cdot)\|_{C^2(\mathbb{R}^n \setminus Y_m) \cap C^2(\overline{Y_m})} \le c \quad \text{if } x \in \mathbb{R}^n \setminus \mathbf{S}_0, \tag{4.44}$$

where $c$ is independent of $\omega, x$. By Corollary 6.3 [13], (4.38) of Remark 4.2, and (4.44), we have $(4.39)_5$.

If $x \notin \mathbf{S}$ or $y \notin \mathbf{S}$, $(4.39)_3$ implies $|\Gamma(x,y) - \mathcal{M}_{\omega,1}^{(0)}(x,y)| = |\mathcal{E}_\omega(x,y)| \le c$, where $c$ is a constant independent of $\omega$. If $x, y \in \mathbf{S}$, (4.32) implies $1 - \mathcal{X}_{\mathcal{A}}^\dagger(x,y) = 0$. So we see that $(4.39)_6$ holds.

If $|x-y|>n\tau$, then $\mathcal{G}^\omega_\tau(x,y)=\Gamma(x,y)$ by (4.32)–(4.34). So (4.40) holds clearly. If $|x-y|\le n\tau$ and $x,y\in\Omega^\tau_f$, equations (4.34) and (4.36) imply

$$\begin{aligned}\mathcal{G}^\omega_\tau(x,y)&=\Gamma(x,y)+\tau^{2-n}\mathcal{E}_\omega(\frac{x}{\tau}+j,\frac{y}{\tau}+j)\mathcal{X}^\dagger_{\mathcal{A}}(\frac{x}{\tau}+j,\frac{y}{\tau}+j)\\&=\tau^{2-n}\mathcal{M}^{(0)}_{\omega,1}(\frac{x}{\tau}+j,\frac{y}{\tau}+j)+\tau^{2-n}(\mathcal{X}^\dagger_{\mathcal{A}}-1)\mathcal{E}_\omega(\frac{x}{\tau}+j,\frac{y}{\tau}+j)\end{aligned}\qquad(4.45)$$

for some $j\in\mathbb{Z}^n$ satisfying $\frac{x}{\tau}+j,\frac{y}{\tau}+j\in Y$. $(4.39)_{1,6}$ and (4.45) imply that (4.40) holds for $x,y\in\Omega^\tau_f$. If $|x-y|\in[(1-6\mathbf{d}_0)\tau,n\tau]$, then $x\in\Omega^\tau_f$ or $y\in\Omega^\tau_f$ by (3.1). By $(4.39)_{2,6}$ and (4.45), we know (4.40) holds for $|x-y|\in[(1-6\mathbf{d}_0)\tau,n\tau]$. So (4.40) is proved.

By (3.1), (4.27), $(4.39)_1$, and maximal principle (see Theorem 3.1 [13]), we see that $(4.41)_1$ holds. Let $x\in\mathbf{S}_2\setminus Y$ be fixed and let $c$ be a constant independent of $\omega$. If $y\in Y\cap B_{\mathbf{d}_0}(x)$, we define $r=|x-y|$. See (3.1) for $\mathbf{d}_0$. Then, by (4.27),

$$-\Delta_z\mathcal{M}^{(0)}_{\omega,1}(x,z)=0\qquad\text{in }B_{r/2}(y).$$

Theorem 2.10 [13] and $(4.39)_1$ imply, for $y\in Y\cap B_{\mathbf{d}_0}(x)$,

$$|\nabla_y\mathcal{M}^{(0)}_{\omega,1}(x,y)|\le\frac{c}{r}\|\mathcal{M}^{(0)}_{\omega,1}(x,\cdot)\|_{L^\infty(B_{r/2}(y))}\le\frac{c}{|x-y|^{n-1}}.\qquad(4.46)$$

By (4.27),

$$-\nabla_z\cdot(\mathbf{K}^{(0)}_{\omega^2,1}\nabla_z\mathcal{M}^{(0)}_{\omega,1}(x,z))=0\qquad\text{in }Y\setminus B_{\mathbf{d}_0/2}(x).$$

Corollary 6.3 [13], Lemma 3.3, (3.1), and $(4.39)_1$ imply, for $y\in Y\setminus B_{\mathbf{d}_0}(x)$,

$$|\nabla_y\mathcal{M}^{(0)}_{\omega,1}(x,y)|\le c\sup_{z\in\mathbf{S}_2\setminus(B_{\mathbf{d}_0/2}(x)\cup Y_m)}|\mathcal{M}^{(0)}_{\omega,1}(x,z)|\le\frac{c}{|x-y|^{n-1}}.\qquad(4.47)$$

Equations (4.46)–(4.47) imply $(4.41)_2$. □

Modifying the argument of Lemma 4.3, we have

**Lemma 4.8.** *For any $\omega\in(0,1]$, $Q\in L^\infty(\mathbb{R}^n)\cap C^{0,\alpha}_{loc}(\mathbb{R}^n\setminus\overline{Y_m})\cap C^{0,\alpha}_{loc}(Y_m)$ with support in $\mathcal{A}$, and $\alpha\in(0,1)$, the $\mathcal{D}^{1,2}(\mathbb{R}^n)$ solution of*

$$-\nabla\cdot(\mathbf{K}^{(0)}_{\omega^2,1}\nabla\Phi+Q)=0\qquad\text{in }\mathbb{R}^n$$

*satisfies, for $x\in Y_f\cup Y_m$ and $j\in\{1,\cdots,n\}$,*

$$\begin{aligned}\partial_j\Phi(x)=&-\int_{\mathcal{A}}\partial_{x_j}\nabla_y\mathcal{M}^{(0)}_{\omega,1}(x,y)Q(y)dy+\mathbf{K}^{(0)}_{1/\omega^2,1}(x)Q(x)\int_{\partial\mathbf{S}_2}\nabla_y\Gamma(x,y)\mathbf{n}_j\,d\sigma_y\\&+\mathbf{K}^{(0)}_{1/\omega^2,1}(x)Q(x)\int_{\mathbf{S}_2}\partial_{x_j}\nabla_y\Gamma(x,y)dy.\end{aligned}$$

*Here $x_j$ is the $j$-th component of $x=(x_1,\cdots,x_n)$, $\mathbf{n}_j$ is the $j$-th component of $\vec{\mathbf{n}}=(\mathbf{n}_1,\cdots,\mathbf{n}_n)$ denoting a unit outward normal vector on $\partial\mathbf{S}_2$, and $\Gamma$ is the fundamental solution of the Laplace operator in $\mathbb{R}^n$. Moreover, there is a constant $c$ independent of $\omega$ such that*

$$\left|\nabla\Phi(x)+\int_{\mathcal{A}}\nabla_x\nabla_y\mathcal{M}^{(0)}_{\omega,1}(x,y)Q(y)dy\right|\le c\mathbf{K}^{(0)}_{1/\omega^2,1}(x)|Q(x)|\quad\text{for }x\in Y_f\cup Y_m.$$

**Lemma 4.9.** *For any* $\tau \in [\frac{1}{n+1}, \infty)$ *and* $\omega \in (0,1]$,

$$\sup_{|x-y|\geq 1-4\mathbf{d}_0} |\Gamma^\omega_\tau(x,y) - \mathcal{G}^\omega_\tau(x,y)| \leq c\tau^{2-n}, \tag{4.48}$$

*where* $c$ *is a positive constant independent of* $\omega, \tau$. *See (3.1) for* $\mathbf{d}_0$.

**Proof.** Let $x \in \mathbb{R}^n$ be fixed and define

$$c_1 \equiv \sup_{|x-y|\geq 1-6\mathbf{d}_0} |\Gamma^\omega_\tau(x,y) - \mathcal{G}^\omega_\tau(x,y)|.$$

Claim I: $c_1$ is a constant independent of $\omega, \tau, x$.
Proof of Claim I: From (3.1), the distance between $\mathbb{R}^n \setminus \bigcup_{j\in\mathbb{Z}^n} \tau(\mathbf{S}_1 - j)$ and $\Omega^\tau_m$ as well as the distance between $\tau(\mathbf{S}_1 - j)$ and $\tau(\partial\mathbf{S} - j)$ for any $j \in \mathbb{Z}^n$ both are greater than $\mathbf{d}_0\tau$. So

- (S1) if $x \in \tau(\mathbf{S}_1 - j)$ for some $j \in \mathbb{Z}^n$, then $\|\Gamma^\omega_\tau(x,\cdot) - \mathcal{G}^\omega_\tau(x,\cdot)\|_{L^\infty(\tau(\mathbf{S}-j))}$ is bounded independent of $\omega, \tau, x$ by $(4.37)_1$ with $\Phi = \Gamma^\omega_\tau(x,\cdot) - \mathcal{G}^\omega_\tau(x,\cdot)$, Theorem 3.1 [13], $(4.3)_1$, (4.36), and $(4.39)_3$.
- (S2) if $x \in \tau(\mathbf{S}_1 - j)$ for some $j \in \mathbb{Z}^n$, then

$$\sup_{\substack{|x-y|\geq 1-6\mathbf{d}_0 \\ y\notin\tau(\mathbf{S}-j)}} |\Gamma^\omega_\tau(x,y) - \mathcal{G}^\omega_\tau(x,y)| \leq \sup_{\substack{|x-y|\geq 1-6\mathbf{d}_0 \\ y\notin\tau(\mathbf{S}-j)}} |\Gamma^\omega_\tau(x,y)| + |\mathcal{G}^\omega_\tau(x,y)| \tag{4.49}$$

  is bounded independent of $\omega, \tau, x$ by Theorem 3.1 [13], $(4.3)_1$, (4.36), $(4.39)_3$.
- (S3) if $x \in \mathbb{R}^n \setminus \bigcup_{j\in\mathbb{Z}^n} \tau(\mathbf{S}_1 - j)$, then $c_1$ is independent of $\omega, \tau, x$ by $(4.37)_1$, (4.49), Theorem 3.1 [13], $(4.3)_1$, and $(4.39)_1$.

(S1)–(S3) imply Claim I.

Claim II: There is a constant $c_2$ independent of $\omega, \tau, x$ such that

$$\sup_{|x-y|\geq 1-5\mathbf{d}_0} |\nabla_y\Gamma^\omega_\tau(x,y) - \nabla_y\mathcal{G}^\omega_\tau(x,y)| \leq c_2. \tag{4.50}$$

Proof of Claim II: We note

- (S4) if $x \in \tau(\mathbf{S}_1 - j)$ for $j \in \mathbb{Z}^n$, then $\|\nabla_y\Gamma^\omega_\tau(x,\cdot) - \nabla_y\mathcal{G}^\omega_\tau(x,\cdot)\|_{L^\infty(\tau(\mathbf{S}_1-j))}$ is bounded independent of $\omega, \tau, x$ by $(4.37)_2$ with $\Phi = \Gamma^\omega_\tau(x,\cdot) - \mathcal{G}^\omega_\tau(x,\cdot)$ and (S1).
- (S5) if $x \in \tau(\mathbf{S}_1 - j)$ for some $j \in \mathbb{Z}^n$, then

$$\sup_{\substack{|x-y|>1-5\mathbf{d}_0 \\ y\notin\tau(\mathbf{S}_1-j)}} |\nabla_y\Gamma^\omega_\tau(x,y)| + |\nabla_y\mathcal{G}^\omega_\tau(x,y)|$$

  is bounded independent of $\omega, \tau, x$ by $(4.3)_{3,4}$, Theorem 2.10 [13], (4.36), and $(4.39)_4$.
- (S6) if $x \in \mathbb{R}^n \setminus \bigcup_{j\in\mathbb{Z}^n} \tau(\mathbf{S}_1 - j)$, then (4.50) holds by $(4.37)_2$, Theorem 2.10 [13], (4.3), (4.36), and (4.39).

(S4)–(S6) imply Claim II.

Define $\theta_{\omega,\tau,x} \equiv \sup_{|x-y|\geq 1-4\mathbf{d}_0} |\Gamma^\omega_\tau(x,y) - \mathcal{G}^\omega_\tau(x,y)|$ for fixed $\omega, \tau, x$. Clearly, $\theta_{\omega,\tau,x} \leq c_1$ by Claim I. By $(4.3)_2$ and (4.40), there is a $y_*$ satisfying

$$1 - 4\mathbf{d}_0 \leq |x - y_*| \qquad \text{and} \qquad \theta_{\omega,\tau,x} = |\Gamma^\omega_\tau(x,y_*) - \mathcal{G}^\omega_\tau(x,y_*)| \leq c_1. \tag{4.51}$$

Take a number $\beta > 9$ so that

$$\rho \equiv \frac{\theta_{\omega,\tau,x}}{\beta c_2} \leq \mathbf{d}_0, \tag{4.52}$$

where $c_2$ is from (4.50). From (4.50), (4.51), and mean value theorem,

$$|\Gamma^\omega_\tau(x,y) - \mathcal{G}^\omega_\tau(x,y)| \geq \theta_{\omega,\tau,x}/5 \qquad \text{for } y \in B_\rho(y_*). \tag{4.53}$$

Take $F \in C^\infty_0(B_\rho(y_*))$ so that

$$0 \leq F \leq 1 \quad \text{and} \quad F = 1 \text{ on } B_{\rho/\beta}(y_*). \tag{4.54}$$

Because of (4.52), at most one $j \in \mathbb{Z}^n$ satisfies $supp(F) \cap \tau(\mathcal{A} - j) \neq \emptyset$.

Suppose $y_* \in \tau(\overline{Y} - j)$ for some $j \in \mathbb{Z}^n$. We find $\Phi, \Phi^* \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ such that

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,\tau}\nabla\Phi) = F & \text{in } \mathbb{R}^n, \\ -\nabla\cdot(\mathbf{K}^{(j)}_{\omega^2,\tau}\nabla\Phi^*) = F & \text{in } \mathbb{R}^n. \end{cases} \tag{4.55}$$

By Lax-Milgram Theorem [13], Lemma 3.1, and Theorem 2.1 [1], $\Phi$ and $\Phi^*$ are uniquely solvable and are piecewise smooth by Lemma 3.3. Moreover,

$$\begin{cases} -\Delta(\Phi - \Phi^*) = 0 & \text{in } \mathbb{R}^n \setminus \partial\Omega^\tau_m, \\ \lfloor \Phi - \Phi^* \rfloor_{\partial\Omega^\tau_m} = 0, & \\ \lfloor \mathbf{K}_{\omega^2,\tau}\nabla(\Phi - \Phi^*)\rfloor_{\partial\Omega^\tau_m}\cdot\vec{\mathbf{n}}^\tau = \begin{cases} 0 & \text{on } \tau(\partial Y_m - j), \\ (\omega^2 - 1)\nabla\Phi^*\cdot\vec{\mathbf{n}}^\tau & \text{on } \partial\Omega^\tau_m \setminus \tau(\partial Y_m - j). \end{cases} \end{cases} \tag{4.56}$$

See (2.1) for $(4.56)_{2,3}$.

Claim III: There is a constant $c$ independent of $\omega, \tau, x$ such that

$$\|\Phi - \Phi^*\|_{L^\infty(\mathbb{R}^n)} \leq c\tau^{2-n}\rho^n. \tag{4.57}$$

Proof of Claim III: Because of $y_* \in \tau(\overline{Y} - j)$, there is a smooth domain $\mathbf{D}_\tau$ such that

(S7) $\mathbf{D}_\tau \supset \overline{\tau(Y_m - j) \cup supp(F)}$, $\mathbf{D}_\tau \cap \overline{\tau(Y_m - i)} = \emptyset$ for $i \neq j$,
(S8) $dist(\partial\mathbf{D}_\tau, supp(F))$ is greater than $\tau(1 - 4\mathbf{d}_0)$.

By (S7) and maximal principle (see Theorem 3.1 and Lemma 3.4 [13]), (4.56) implies

$$\|\Phi - \Phi^*\|_{L^\infty(\mathbf{D}_\tau)} \leq \|\Phi - \Phi^*\|_{L^\infty(\partial\mathbf{D}_\tau)}. \tag{4.58}$$

By (4.9), (S8), $(4.3)_2$, $(4.39)_2$, and $(4.28)_3$, equation (4.55) implies, for $z \in \mathbb{R}^n \setminus \mathbf{D}_\tau$,

$$\begin{cases} |\Phi(z)| = \left|\int_{\mathbb{R}^n} \Gamma^\omega_\tau(z,y)F(y)dy\right| \leq \dfrac{c\|F\|_{L^1(\mathbb{R}^n)}}{dist(z, supp(F))^{n-2}}, \\ |\Phi^*(z)| = \left|\int_{\mathbb{R}^n} \mathcal{M}^{(j)}_{\omega,\tau}(z,y)F(y)dy\right| \leq \dfrac{c\|F\|_{L^1(\mathbb{R}^n)}}{dist(z, supp(F))^{n-2}}, \end{cases}$$

where $c$ is independent of $z, \omega, \tau, x$, and $F$. By (S8),

$$\max_{z\in\mathbb{R}^n\setminus\mathbf{D}_\tau}\{|\Phi(z)|,|\Phi^*(z)|\}\le c\tau^{2-n}\|F\|_{L^1(B_\rho(y_*))},$$

where $c$ is independent of $\omega, \tau, x$, and $F$. So we have

$$\|\Phi-\Phi^*\|_{L^\infty(\mathbb{R}^n\setminus\mathbf{D}_\tau)}\le c\tau^{2-n}\rho^n. \tag{4.59}$$

Equations (4.58)–(4.59) then imply Claim III.

By Remark 4.1, (4.33), $(4.39)_6$, (4.53)–(4.54), and (4.57),

$$\begin{aligned}\rho^{n+1}\le c&\left|\int\left(\Gamma^\omega_\tau(x,y)-\mathcal{G}^\omega_\tau(x,y)\right)F(y)dy\right|\le c|\Phi(x)-\Phi^*(x)|\\&+c\left|\int\left(1-\mathcal{X}^\dagger_{\mathcal{A}}(\frac{x}{\tau}+j,\frac{y}{\tau}+j)\right)\left(\Gamma(x,y)-\frac{\mathcal{M}^{(0)}_{\omega,1}(\frac{x}{\tau}+j,\frac{y}{\tau}+j)}{\tau^{n-2}}\right)F(y)dy\right|\\&\le c\tau^{2-n}\rho^n,\end{aligned}\tag{4.60}$$

where $c$ is independent of $\omega, \tau, x$. In (4.60), $\Gamma(x,y)=\tau^{2-n}\Gamma(\frac{x}{\tau}+j,\frac{y}{\tau}+j)$ is used. (4.60) implies $\rho\le c\tau^{2-n}$ where $c$ is independent of $\omega, \tau, x$. So (4.48) holds. The proof is complete. □

**Lemma 4.10.** *For any* $\tau\in[\frac{1}{n+1},\infty)$, $\omega\in(0,1]$, *and* $|x-y|\ge 1-3\mathbf{d}_0$,

$$\left|\nabla_y\Gamma^\omega_\tau(x,y)-\nabla_y\mathcal{G}^\omega_\tau(x,y)\right|\le c\tau^{2-n}\mathbf{K}_{1/\omega,\tau}(y),\tag{4.61}$$

*where $c$ is a positive constant independent of $\omega, \tau$. See (3.1) for $\mathbf{d}_0$.*

**Proof.** If $x\in\tau(\overline{Y}-j)$ for some $j\in\mathbb{Z}^n$, then (4.1), (4.27), and (4.32)–(4.34) imply

$$\begin{cases}-\Delta_y(\Gamma^\omega_\tau-\mathcal{G}^\omega_\tau)=\begin{cases}0 & \text{in } \mathbb{R}^n\setminus(\tau(\mathcal{A}\setminus\mathbf{S}-j)\cup\partial\Omega^\tau_m),\\ 2\tau^{-n}(\nabla_y\mathcal{X}^\dagger_{\mathcal{A}}\nabla_y\mathcal{E}_\omega)(\frac{x}{\tau}+j,\frac{\cdot}{\tau}+j) & \\ \quad+\tau^{-n}(\mathcal{E}_\omega\Delta_y\mathcal{X}^\dagger_{\mathcal{A}})(\frac{x}{\tau}+j,\frac{\cdot}{\tau}+j) & \text{in } \tau(\mathcal{A}\setminus\mathbf{S}-j),\end{cases}\\ \lfloor\Gamma^\omega_\tau-\mathcal{G}^\omega_\tau\rfloor_{\partial\Omega^\tau_m}=0,\\ \lfloor\mathbf{K}_{\omega^2,\tau}\nabla_y(\Gamma^\omega_\tau-\mathcal{G}^\omega_\tau)\rfloor_{\partial\Omega^\tau_m}\cdot\vec{\mathbf{n}}^\tau\\ \quad=-(1-\omega^2)\left(1-\mathcal{X}^\dagger_{\mathcal{A}}(\frac{x}{\tau}+j,\frac{\cdot}{\tau}+j)\right)\nabla_y\Gamma(x,\cdot)\cdot\vec{\mathbf{n}}^\tau \quad \text{on } \partial\Omega^\tau_m.\end{cases}$$

(3.1), $(4.39)_4$, Corollary 6.3 [13], Lemma 3.4, and Lemma 4.9 imply (4.61). □

**Lemma 4.11.** *For any* $\tau\in[\frac{1}{n+1},\infty)$, $\omega\in(0,1]$, *and* $|x-y|>1-2\mathbf{d}_0$,

$$\left|\nabla_x\nabla_y\Gamma^\omega_\tau(x,y)-\nabla_x\nabla_y\mathcal{G}^\omega_\tau(x,y)\right|\le c\tau^{2-n}\mathbf{K}_{1/\omega,\tau}(x)\mathbf{K}_{1/\omega,\tau}(y),\tag{4.62}$$

*where $c$ is a positive constant independent of $\omega, \tau$. See (3.1) for $\mathbf{d}_0$.*

**Proof.** This lemma is proved by following the argument for Lemma 4.10 and using (4.2) and (4.34)–(4.35). Let $y=(y_1,\cdots,y_n)\in\Omega^\tau_f\cup\Omega^\tau_m$ be fixed and define $\Phi(x)\equiv\partial_{y_k}\Gamma^\omega_\tau(x,y)$, $\Phi^*(x)\equiv\partial_{y_k}\mathcal{G}^\omega_\tau(x,y)$ for $k\in\{1,\cdots,n\}$.

If $y \in \tau(\overline{Y} - j)$ for some $j \in \mathbb{Z}^n$, then $(4.7)_1$ and (4.38) of Remark 4.2 imply

$$\begin{cases} -\Delta(\Phi - \Phi^*) = \begin{cases} 0 & \text{in } \mathbb{R}^n \setminus (\tau(\mathcal{A} \setminus \mathbf{S} - j) \cup \partial\Omega_m^\tau \cup B_{1-3\mathbf{d}_0}(y)), \\ \frac{2}{\tau^{n+1}}\partial_{y_k}\left(\nabla_x \mathcal{X}_{\mathcal{A}}^\dagger \nabla_x \mathcal{E}_\omega\right)(\frac{\cdot}{\tau} + j, \frac{y}{\tau} + j) \\ \quad + \frac{1}{\tau^{n+1}}\partial_{y_k}\left(\mathcal{E}_\omega \Delta_x \mathcal{X}_{\mathcal{A}}^\dagger\right)(\frac{\cdot}{\tau} + j, \frac{y}{\tau} + j) \\ \qquad \text{in } \tau(\mathcal{A} \setminus \mathbf{S} - j) \setminus B_{1-3\mathbf{d}_0}(y), \end{cases} \\ \lfloor \Phi - \Phi^* \rfloor_{\partial\Omega_m^\tau \setminus B_{1-3\mathbf{d}_0}(y)} = 0, \\ \lfloor \mathbf{K}_{\omega^2,\tau} \nabla(\Phi - \Phi^*) \rfloor_{\partial\Omega_m^\tau \setminus B_{1-3\mathbf{d}_0}(y)} \cdot \vec{\mathbf{n}}^\tau = \frac{1-\omega^2}{\tau}\partial_{y_k}\mathcal{X}_{\mathcal{A}}^\dagger(\frac{\cdot}{\tau} + j, \frac{y}{\tau} + j)\nabla_x \Gamma(\cdot, y) \cdot \vec{\mathbf{n}}^\tau \\ \quad -(1-\omega^2)\left(1 - \mathcal{X}_{\mathcal{A}}^\dagger(\frac{\cdot}{\tau} + j, \frac{y}{\tau} + j)\right)\nabla_x \partial_{y_k}\Gamma(\cdot, y) \cdot \vec{\mathbf{n}}^\tau \qquad \text{on } \partial\Omega_m^\tau \setminus B_{1-3\mathbf{d}_0}(y). \end{cases}$$

We get (4.62) by (3.1), $(4.39)_5$, Corollary 6.3 [13], Lemma 3.4, and Lemma 4.10. □

By Lemma 4.11 and tracing the argument of Corollary 4.1, we have

**Corollary 4.2.** *For any $\omega \in (0, 1]$ and $|x - y| \le n + 1$,*

$$\left|\nabla_x \nabla_y \Gamma_1^\omega(x, y) - \nabla_x \nabla_y \mathcal{G}_1^\omega(x, y)\right| \le c|x - y|^{-2}\mathbf{K}_{1/\omega,1}(x)\mathbf{K}_{1/\omega,1}(y),$$

*where $c$ is a constant independent of $\omega$.*

## 5. Proofs of Theorem 2.1 and its related results

This section gives the proofs of Theorem 2.1, Corollary 2.1, Corollary 2.2.

### 5.1. *Proof of Theorem 2.1*

Notations in (3.1) and (4.32) are used. We first consider (2.2) for $\epsilon = 1$ case and assume $G \in L^\infty(\mathbb{R}^n) \cap C^{0,\alpha}_{loc}(\Omega^1_f) \cap C^{0,\alpha}_{loc}(\Omega^1_m)$ with compact support and $\alpha \in (0, 1)$. Let $\eta \in C_0^\infty(\mathbb{R})$ be a bell-shaped function satisfying $\eta \in [0, 1]$, $\eta(z) = 1$ for $|z| < n$, and $\eta(z) = 0$ for $|z| \ge n + 1$, and let $\nabla_x \nabla_y \Gamma_1^\omega(x, y) = \mathcal{S}^0(x, y) + \mathcal{S}^c(x, y)$, where

$$\mathcal{S}^0(x, y) \equiv \eta(|x - y|)\nabla_x \nabla_y \mathcal{G}_1^\omega + (1 - \eta(|x - y|))(I + \nabla\mathbb{X}(x))(I + \nabla\mathbb{X}(y))\nabla_x \nabla_y \Gamma_0^\omega.$$

See (4.14), (4.33) for $\Gamma_0^\omega, \mathcal{G}_1^\omega$. Define

$$\int_{\mathbb{R}^n} \nabla_x \nabla_y \Gamma_1^\omega(x, y)G(y)dy = \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_4, \tag{5.1}$$

where

$$\mathcal{F}_1(x) \equiv \int_{\mathbb{R}^n} \eta(|x - y|)G(y)\nabla_x \nabla_y \left(\sum_{k \in \mathbb{Z}^n} \mathcal{M}^{(0)}_{\omega,1}(x + k, y + k)\mathcal{X}_{\mathcal{A}}^\dagger(x + k, y + k)\right)dy,$$

$$\mathcal{F}_2(x) \equiv \int_{\mathbb{R}^n} \eta(|x - y|)G(y)\nabla_x \nabla_y \left(\Gamma(x, y)\left(1 - \sum_{k \in \mathbb{Z}^n} \mathcal{X}_{\mathcal{A}}^\dagger(x + k, y + k)\right)\right)dy,$$

$$\mathcal{F}_3(x) \equiv \int_{\mathbb{R}^n} (1 - \eta(|x - y|))G(y)(I + \nabla\mathbb{X}(x))(I + \nabla\mathbb{X}(y))\nabla_x \nabla_y \Gamma_0^\omega(x, y)dy,$$

$$\mathcal{F}_4(x) \equiv \int_{\mathbb{R}^n} \mathcal{S}^c(x, y)G(y)dy.$$

**Step 1:** From (3.1) and (4.32), we know $\mathcal{X}^{\dagger}_{\mathcal{A}}(x+k, y+k) = \phi(x+k)\phi(y+k) \neq 0$ only if $|x-y| < \sqrt{n}$ and $x+k, y+k \in Y$. If $x \in Y - j$ for some $j \in \mathbb{Z}^n$, we see that, by change of variable and the definition of the bell-shaped function $\eta$,

$$\begin{aligned}
\mathcal{F}_1(x) &= \int_{Y-j} G(y)\nabla_x\nabla_y \Big(\mathcal{M}^{(0)}_{\omega,1}(x+j, y+j)\phi(x+j)\phi(y+j)\Big)\, dy \\
&= \nabla\phi(x+j)\int_Y G(y-j)\Big(\phi(y)\nabla_y\mathcal{M}^{(0)}_{\omega,1}(x+j,y) + \mathcal{M}^{(0)}_{\omega,1}(x+j,y)\nabla\phi(y)\Big)\, dy \\
&\quad +\phi(x+j)\int_Y G(y-j)\phi(y)\nabla_x\nabla_y\mathcal{M}^{(0)}_{\omega,1}(x+j,y)dy \\
&\quad +\phi(x+j)\int_Y G(y-j)\nabla\phi(y)\nabla_x\mathcal{M}^{(0)}_{\omega,1}(x+j,y)dy. \qquad (5.2)
\end{aligned}$$

Define $\mathcal{V}^{(1)}_j(z) \equiv \int_Y \mathcal{M}^{(0)}_{\omega,1}(z,y)G(y-j)\nabla\phi(y)dy$ for $z \in \mathbb{R}^n$. By the same reasoning as mentioned in Remark 4.1, $\mathcal{V}^{(1)}_j$ is the unique $\mathcal{D}^{1,2}(\mathbb{R}^n)$ solution of

$$-\nabla\cdot\Big(\mathbf{K}^{(0)}_{\omega^2,1}(z)\nabla\mathcal{V}^{(1)}_j(z)\Big) = G(z-j)\nabla\phi(z) \qquad \text{in } \mathbb{R}^n.$$

Similarly, define $\mathcal{V}^{(2)}_j(z) \equiv \int_Y G(y-j)\phi(y)\nabla_y\mathcal{M}^{(0)}_{\omega,1}(z,y)dy$ for $z \in \mathbb{R}^n$. $\mathcal{V}^{(2)}_j$ is the unique $\mathcal{D}^{1,2}(\mathbb{R}^n)$ solution of

$$-\nabla\cdot\Big(\mathbf{K}^{(0)}_{\omega^2,1}(z)\nabla\mathcal{V}^{(2)}_j(z) - \phi(z)G(z-j)\Big) = 0 \qquad \text{in } \mathbb{R}^n.$$

By Lemma 3.2,

$$\begin{aligned}
&\|\mathbf{K}_{\omega,1}\mathcal{V}^{(1)}_j, \mathbf{K}_{\omega,1}\nabla\mathcal{V}^{(1)}_j, \mathbf{K}_{\omega,1}\mathcal{V}^{(2)}_j, \mathbf{K}_{\omega,1}\nabla\mathcal{V}^{(2)}_j\|_{L^p(Y)} \\
&\quad \leq c\big(\|\mathcal{V}^{(1)}_j, \mathcal{V}^{(2)}_j\|_{L^p(\mathbf{S}_2\setminus\overline{Y})} + \|\mathbf{K}_{1/\omega,1}G(\cdot - j)\|_{L^p(Y)}\big), \qquad (5.3)
\end{aligned}$$

where $p \in (1,\infty)$ and $c$ is independent of $\omega, j$. If $z \in \mathbf{S}_2 \setminus \overline{Y}$, by (4.41),

$$|\mathcal{V}^{(1)}_j(z)| + |\mathcal{V}^{(2)}_j(z)| \leq c\int_Y |G(y-j)|(|z-y|^{2-n} + |z-y|^{1-n})dy,$$

where $c$ is independent of $\omega, j$. Tracing the proof of Lemma 7.12 [13], we see

$$\|\mathcal{V}^{(1)}_j, \mathcal{V}^{(2)}_j\|_{L^p(\mathbf{S}_2\setminus\overline{Y})} \leq c\|G(\cdot - j)\|_{L^p(Y)} \qquad \text{for } p \in (1,\infty), \qquad (5.4)$$

where $c$ is independent of $\omega, j$. By Lemma 4.8 and $supp(G(y-j)\phi(y)) \subset \mathcal{A}$,

$$\left|\int_Y \nabla_z\nabla_y\mathcal{M}^{(0)}_{\omega,1}(z,y)G(y-j)\phi(y)dy\right| \leq \big|\nabla\mathcal{V}^{(2)}_j(z)\big| + c|G(z-j)\phi(z)|\mathbf{K}_{\frac{1}{\omega^2},1}(z), \quad (5.5)$$

where $z \in Y_f \cup Y_m$ and $c$ is independent of $\omega, j$.

Since $supp(\phi(x+k_1)) \cap supp(\phi(x+k_2)) = \emptyset$ if $k_1 \neq k_2$, Remark 4.1 and (5.2)–(5.5) imply

$$\begin{aligned}
\|\mathbf{K}_{\omega,1}\mathcal{F}_1\|^p_{L^p(\mathbb{R}^n)} &\leq c\sum_{j\in\mathbb{Z}^n}\|\mathbf{K}_{\omega,1}\mathcal{V}^{(1)}_j, \mathbf{K}_{\omega,1}\nabla\mathcal{V}^{(1)}_j, \mathbf{K}_{\omega,1}\mathcal{V}^{(2)}_j, \mathbf{K}_{\omega,1}\nabla\mathcal{V}^{(2)}_j\|^p_{L^p(Y)} \\
&\quad +c\sum_{j\in\mathbb{Z}^n}\|\mathbf{K}_{1/\omega,1}G(\cdot - j)\|^p_{L^p(Y)} \leq c\|\mathbf{K}_{1/\omega,1}G\|^p_{L^p(\mathbb{R}^n)}, \qquad (5.6)
\end{aligned}$$

where $c$ is independent of $\omega$.

**Step 2:** $\mathcal{F}_2$ can be written as $\mathcal{F}_2 = \mathcal{F}_{21} - \mathcal{F}_{22}$, where

$$\begin{cases} \mathcal{F}_{21}(x) = \int_{\mathbb{R}^n} \eta(|x-y|) G(y) \nabla_x \nabla_y \Gamma(x,y) dy, \\ \mathcal{F}_{22}(x) = \int_{\mathbb{R}^n} \eta(|x-y|) G(y) \nabla_x \nabla_y \Big( \Gamma(x,y) \sum_{k\in\mathbb{Z}^n} \mathcal{X}_{\mathcal{A}}^{\dagger}(x+k, y+k) \Big) dy. \end{cases}$$

It is not difficult to see, by Theorem 3 in page 39 [27] and (2.13) in [13],

$$\|\mathcal{F}_{21}\|_{L^p(\mathbb{R}^n)} \le c \|G\|_{L^p(\mathbb{R}^n)} \qquad \text{for } p \in (1,\infty),$$

where $c$ is independent of $\omega$. Repeating the argument from (5.2) to (5.6), we obtain

$$\|\mathcal{F}_{22}\|_{L^p(\mathbb{R}^n)} \le c \|G\|_{L^p(\mathbb{R}^n)} \qquad \text{for } p \in (1,\infty),$$

where $c$ is independent of $\omega$. Therefore,

$$\|\mathcal{F}_{2}\|_{L^p(\mathbb{R}^n)} \le c \|G\|_{L^p(\mathbb{R}^n)} \qquad \text{for } p \in (1,\infty), \tag{5.7}$$

where $c$ is independent of $\omega$.

**Step 3:** Since $\mathcal{K}_\omega$ is a symmetric positive definite matrix (see (3.9)–(3.10)), we do change of variables (see the remark before Lemma 4.4) so that the $\Gamma_0^\omega$ in (4.14) can be transformed to the fundamental solution of the Laplace equation in a new coordinate system. By Theorem 3 in page 39 [27] and (2.13) in [13], we obtain

$$\|\mathcal{F}_{3}\|_{L^p(\mathbb{R}^n)} \le c \|G\|_{L^p(\mathbb{R}^n)} \qquad \text{for } p \in (1,\infty), \tag{5.8}$$

where $c$ is independent of $\omega$.

**Step 4:** We note

$$\mathcal{F}_4(x) \equiv \int_{\mathbb{R}^n} \eta(|x-y|) \mathcal{S}_1^c(x,y) G(y) + (1-\eta(|x-y|)) \mathcal{S}_2^c(x,y) G(y) dy,$$

where

$$\begin{cases} \mathcal{S}_1^c(x,y) = \nabla_x \nabla_y \Gamma_1^\omega(x,y) - \nabla_x \nabla_y \mathcal{G}_1^\omega(x,y), \\ \mathcal{S}_2^c(x,y) = \nabla_x \nabla_y \Gamma_1^\omega(x,y) - (I + \nabla \mathbb{X}(x))(I + \nabla \mathbb{X}(y)) \nabla_x \nabla_y \Gamma_0^\omega(x,y). \end{cases}$$

Define, for any $\zeta \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$,

$$\mathcal{P}_1(\zeta)(x) \equiv \begin{cases} \int_{\mathbb{R}^n} \eta(|x-y|) \mathcal{S}_1^c(x,y) \zeta(y) dy & \text{if } x \in \Omega_f^1, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega_f^1. \end{cases} \tag{5.9}$$

Clearly, $\mathcal{P}_1$ is a linear map. For any $\delta > 0$, by Fubini Theorem [24], Corollary 4.2, and change of variable,

$$\begin{aligned} \delta |\{x \in \mathbb{R}^n |\ |\mathcal{P}_1(\zeta)(x)| > \delta\}| &\le \int_{\Omega_f^1} |\mathcal{P}_1(\zeta)| dx \le \int_{\Omega_f^1} \int_{\mathbb{R}^n} \eta(|x-y|) |\mathcal{S}_1^c(x,y) \zeta(y)| dy dx \\ &\le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\eta(|x|)}{|x|^2} \mathcal{X}_{\Omega_f^1}(x+y) |\mathbf{K}_{1/\omega,1}(y) \zeta(y)| dx dy \le c \|\mathbf{K}_{1/\omega,1} \zeta\|_{L^1(\mathbb{R}^n)}, \end{aligned} \tag{5.10}$$

where $c$ is independent of $\omega$. By (5.9) and Corollary 4.2, if $x \in \Omega_f^1$,

$$|\mathcal{P}_1(\zeta)(x)| \le \int_{\mathbb{R}^n} \eta(|x-y|)|\mathcal{S}_1^c(x,y)\zeta(y)|dy \le c\|\mathbf{K}_{1/\omega,1}\zeta\|_{L^\infty(\mathbb{R}^n)}, \tag{5.11}$$

where $c$ is independent of $\omega$. By (5.10)–(5.11) and Theorem 5 in page 21 [27], we see

$$\|\mathcal{P}_1(\zeta)\|_{L^p(\mathbb{R}^n)} \le c\|\mathbf{K}_{1/\omega,1}\zeta\|_{L^p(\mathbb{R}^n)} \qquad \text{for } p \in (1,\infty), \tag{5.12}$$

where $c$ is independent of $\omega$. Define, for any $\zeta \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$,

$$\mathcal{P}_2(\zeta)(x) \equiv \begin{cases} 0 & \text{if } x \in \Omega_f^1, \\ \displaystyle\int_{\mathbb{R}^n} \eta(|x-y|)\mathcal{S}_1^c(x,y)\zeta(y)dy & \text{if } x \in \mathbb{R}^n \setminus \Omega_f^1. \end{cases}$$

Repeat the above argument for $\mathcal{P}_1(\zeta)$ to see, for any $\delta > 0$,

$$\begin{cases} \mathcal{P}_2 \text{ is a linear map,} \\ \delta|\{x \in \mathbb{R}^n |\ |\mathcal{P}_2(\zeta)(x)| > \delta\}| \le c\omega^{-1}\|\mathbf{K}_{1/\omega,1}\zeta\|_{L^1(\mathbb{R}^n)}, \\ \|\mathcal{P}_2(\zeta)\|_{L^\infty(\mathbb{R}^n)} \le c\omega^{-1}\|\mathbf{K}_{1/\omega,1}\zeta\|_{L^\infty(\mathbb{R}^n)}. \end{cases}$$

By Theorem 5 in page 21 [27], we obtain

$$\|\mathcal{P}_2(\zeta)\|_{L^p(\mathbb{R}^n)} \le c\omega^{-1}\|\mathbf{K}_{1/\omega,1}\zeta\|_{L^p(\mathbb{R}^n)} \qquad \text{for } p \in (1,\infty), \tag{5.13}$$

where $c$ is independent of $\omega$. Similarly, if we define, for any $\zeta \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$,

$$\mathcal{P}_3(\zeta)(x) \equiv \begin{cases} \displaystyle\int_{\mathbb{R}^n} (1-\eta(|x-y|))\mathcal{S}_2^c(x,y)\zeta(y)dy & \text{if } x \in \Omega_f^1, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega_f^1, \end{cases}$$

$$\mathcal{P}_4(\zeta)(x) \equiv \begin{cases} 0 & \text{if } x \in \Omega_f^1, \\ \displaystyle\int_{\mathbb{R}^n} (1-\eta(|x-y|))\mathcal{S}_2^c(x,y)\zeta(y)dy & \text{if } x \in \mathbb{R}^n \setminus \Omega_f^1, \end{cases}$$

as well as employ Theorem 5 in page 21 [27] and Corollary 4.1, then

$$\begin{cases} \|\mathcal{P}_3(\zeta)\|_{L^p(\mathbb{R}^n)} \le c\|\mathbf{K}_{1/\omega,1}\zeta\|_{L^p(\mathbb{R}^n)}, \\ \|\mathcal{P}_4(\zeta)\|_{L^p(\mathbb{R}^n)} \le c\omega^{-1}\|\mathbf{K}_{1/\omega,1}\zeta\|_{L^p(\mathbb{R}^n)}, \end{cases}$$

for any $p \in (1,\infty)$. Therefore we have

$$\begin{aligned} \|\mathbf{K}_{\omega,1}\mathcal{F}_4\|_{L^p(\mathbb{R}^n)} &\le \|\mathcal{P}_1(G), \mathcal{P}_3(G), \omega\mathcal{P}_2(G), \omega\mathcal{P}_4(G)\|_{L^p(\mathbb{R}^n)} \\ &\le c\|\mathbf{K}_{1/\omega,1}G\|_{L^p(\mathbb{R}^n)}, \end{aligned} \tag{5.14}$$

where $p \in (1,\infty)$ and $c$ is a constant independent of $\omega$.

If $G \in L^\infty(\mathbb{R}^n) \cap C^{0,\alpha}_{loc}(\Omega_f^1) \cap C^{0,\alpha}_{loc}(\Omega_m^1)$ with compact support, a $\mathcal{D}^{1,2}(\mathbb{R}^n)$ solution of (2.2) for $\epsilon = 1$ case exists uniquely by Remark 4.1. Equations (4.11), (5.1), (5.6), (5.7), (5.8), and (5.14) imply that the solution satisfies (2.3) for $\epsilon = 1$ case. For $G \in L^p(\mathbb{R}^n)$ with compact support, the unique existence and estimate (2.3) of the solution of (2.2) for $\epsilon = 1$ case can be proved by a limiting argument. If $\epsilon \ne 1$, Theorem 2.1 is proved by a scaling argument.

### 5.2. *Proof of Corollary 2.1*

First consider (2.2) for $\epsilon = 1$ case and assume $G \in L^\infty(\mathbb{R}^n) \cap C^{0,\alpha}_{loc}(\Omega^1_f) \cap C^{0,\alpha}_{loc}(\Omega^1_m)$ with support in $B_r(0)$ for some $r > 0$ and $\alpha \in (0,1)$. By Remark 4.1, $(4.2)_2$, and $(4.3)_3$, the solution of (2.2) satisfies

$$|\mathbf{K}_{\omega,1}\Psi|(x) \leq c\int_{B_r(0)} |x-y|^{1-n}|\mathbf{K}_{1/\omega,1}G|(y)dy \qquad \text{for } x \in \mathbb{R}^n, \tag{5.15}$$

where $c$ is a constant independent of $\omega, r$. By Lemma 7.12 [13] and (5.15), we obtain $(2.4)_1$. By Theorem 1 in page 119 [27] and (5.15), we obtain $(2.4)_2$. For $G \in L^p(\mathbb{R}^n)$ with compact support, the estimate (2.4) for equation (2.2) with $\epsilon = 1$ case can be proved by a limiting argument. If $\epsilon \neq 1$, (2.4) is proved by a scaling argument.

### 5.3. *Proof of Corollary 2.2*

Corollary 2.2 is proved by using Theorem 2.1, Corollary 2.1, Potential theory, duality argument, and Sobolev embedding theory. Details are as follows. Assume $V \in L^\infty(\mathbb{R}^n)$ with support in $B_r(0)$ for some $r > 0$. By Remark 4.1 and Lemma 3.1, a $\mathcal{D}^{1,2}(\mathbb{R}^n)$ solution of (2.5) exists uniquely and

$$\|\mathbf{K}_{\omega,\epsilon}\Psi\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq c_r(\|V\|_{H^{-1}(\mathbb{R}^n)} + \omega^{-1}\|V\|_{H^{-1}(\Omega^\epsilon_m)}), \tag{5.16}$$

where $c_r$ is independent of $\omega, \epsilon$ but dependent on $r$. For any $\zeta \in C^\infty_0(B_t(0))$ with $t > 0$, by Fubini Theorem [24] and Remark 4.1,

$$\int_{B_t(0)} \mathbf{K}_{\omega,\epsilon}\nabla\Psi\ \zeta dx = \int_{B_r(0)}\left(\int_{B_t(0)} \nabla_x\Gamma^\omega_\epsilon(x,y)\mathbf{K}_{\omega,\epsilon}(x)\zeta(x)dx\right)V(y)dy. \tag{5.17}$$

If we define

$$\varphi(y) \equiv \int_{B_t(0)} \nabla_x\Gamma^\omega_\epsilon(x,y)\mathbf{K}_{\omega,\epsilon}(x)\zeta(x)dx \qquad \text{in } \mathbb{R}^n,$$

then $(4.3)_3$ and Remark 4.1 imply that $\varphi$ is the $\mathcal{D}^{1,2}(\mathbb{R}^n)$ solution of

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}\nabla\varphi - \mathbf{K}_{\omega,\epsilon}\zeta) = 0 \quad \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} |\varphi|(x) = 0. \end{cases}$$

By Theorem 2.1 and Corollary 2.1, we see, for any $q \in (1,\infty)$,

$$\|\mathbf{K}_{\omega,\epsilon}\varphi, \mathbf{K}_{\omega,\epsilon}\nabla\varphi\|_{L^q(B_\ell(0))} \leq c_{\ell,t}\|\zeta\|_{L^q(\mathbb{R}^n)}, \tag{5.18}$$

where $\ell > 0$ and $c_{\ell,t}$ is independent of $\omega, \epsilon$ but dependent on $\ell, t$. By (5.17)–(5.18), Theorem 2.1 [1], and Hölder inequality,

$$\begin{aligned} \int_{B_t(0)} \mathbf{K}_{\omega,\epsilon}\nabla\Psi\ \zeta dx &= \int_{B_r(0)} \varphi V(y)dy \leq \|\Pi_\epsilon\varphi|_{\Omega^\epsilon_f}\|_{W^{1,q}(B_{1+r}(0))}\|V\|_{W^{-1,p}(\mathbb{R}^n)} \\ &\quad + \|\varphi - \Pi_\epsilon\varphi|_{\Omega^\epsilon_f}\|_{W^{1,q}(B_{1+r}(0)\cap\Omega^\epsilon_m)}\|V\|_{W^{-1,p}(\Omega^\epsilon_m)} \\ &\leq c_{t,r}\|\zeta\|_{L^q(\mathbb{R}^n)}(\|V\|_{W^{-1,p}(\mathbb{R}^n)} + \omega^{-1}\|V\|_{W^{-1,p}(\Omega^\epsilon_m)}), \end{aligned} \tag{5.19}$$

where $\frac{1}{q}+\frac{1}{p}=1$ and $c_{t,r}$ is independent of $\omega,\epsilon$ but dependent on $t,r$. Here $\Pi_\epsilon\varphi|_{\Omega_f^\epsilon}$ denotes the extension function of $\varphi|_{\Omega_f^\epsilon}$ on $\mathbb{R}^n$ (see Theorem 2.1 [1]). Since $C_0^\infty(B_t(0))$ is dense in $L^q(B_t(0))$ for any $q\in(1,\infty)$ and $t>0$, we obtain

$$\|\mathbf{K}_{\omega,\epsilon}\nabla\Psi\|_{L^p(B_t(0))}\le c_{t,r}(\|V\|_{W^{-1,p}(\mathbb{R}^n)}+\omega^{-1}\|V\|_{W^{-1,p}(\Omega_m^\epsilon)}),\tag{5.20}$$

where $p\in(1,\infty)$ and $c_{t,r}$ is independent of $\omega,\epsilon$ but dependent on $t,r$.

If $p\in[2,\infty)$, then

$$\begin{cases}\|V\|_{H^{-1}(\mathbb{R}^n)}\le c_r\|V\|_{W^{-1,p}(\mathbb{R}^n)},\\ \|V\|_{H^{-1}(\Omega_m^\epsilon)}\le c_r\|V\|_{W^{-1,p}(\Omega_m^\epsilon)},\end{cases}\tag{5.21}$$

where $c_r$ depends on $r$. (5.16) and (5.20)–(5.21) imply that a $W^{1,p}_{loc}(\mathbb{R}^n)$ solution of (2.5) for $p\in[2,\frac{2n}{n-2}]$ exists and satisfies (2.6). That is,

$$\|\mathbf{K}_{\omega,\epsilon}\Psi,\mathbf{K}_{\omega,\epsilon}\nabla\Psi\|_{L^p(B_t(0))}\le c_{t,r}(\|V\|_{W^{-1,p}(\mathbb{R}^n)}+\omega^{-1}\|V\|_{W^{-1,p}(\Omega_m^\epsilon)}),\tag{5.22}$$

where $c_{t,r}$ is independent of $\omega,\epsilon$ but dependent on $t,r$. If $p\in(\frac{2n}{n-2},\frac{2n}{n-4}]$, we know

$$\begin{cases}\|V\|_{W^{-1,\frac{2n}{n-2}}(\mathbb{R}^n)}\le c_r\|V\|_{W^{-1,p}(\mathbb{R}^n)},\\ \|V\|_{W^{-1,\frac{2n}{n-2}}(\Omega_m^\epsilon)}\le c_r\|V\|_{W^{-1,p}(\Omega_m^\epsilon)},\end{cases}\tag{5.23}$$

where $c_r$ depends on $r$. By Theorem 7.26 [13], (5.22)–(5.23), Lemma 3.1, and Theorem 2.1 [1], if $\frac{2n}{n-2}<n$ and $p\in(\frac{2n}{n-2},\frac{2n}{n-4}]$, then

$$\begin{aligned}\|\mathbf{K}_{\omega,\epsilon}\Psi\|_{L^{\frac{2n}{n-4}}(B_t(0))}&\le c_{t,r}\|\mathbf{K}_{\omega,\epsilon}\Psi,\mathbf{K}_{\omega,\epsilon}\nabla\Psi\|_{L^{\frac{2n}{n-2}}(B_t(0))}\\ &\le c_{t,r}(\|V\|_{W^{-1,p}(\mathbb{R}^n)}+\omega^{-1}\|V\|_{W^{-1,p}(\Omega_m^\epsilon)}),\end{aligned}\tag{5.24}$$

where $c_{t,r}$ is independent of $\omega,\epsilon$ but dependent on $t,r$. Theorem 7.26 [13], (5.20), and (5.24) imply that, if $\frac{2n}{n-2}\ge n$ and $p\in(\frac{2n}{n-2},\infty)$ or if $\frac{2n}{n-2}<n$ and $p\in(\frac{2n}{n-2},\frac{2n}{n-4}]$, then a $W^{1,p}_{loc}(\mathbb{R}^n)$ solution of (2.5) exists and satisfies (2.6). Repeating the above process, we see that a $W^{1,p}_{loc}(\mathbb{R}^n)$ solution of (2.5) for $p\in[2,\infty)$ exists and satisfies (2.6). Since (2.6) holds for $p\in[2,\infty)$, a modification of the argument from (5.17) to (5.19) shows, for $p\in(1,2]$,

$$\|\mathbf{K}_{\omega,\epsilon}\Psi\|_{L^p(B_t(0))}\le c_{t,r}(\|V\|_{W^{-1,p}(\mathbb{R}^n)}+\omega^{-1}\|V\|_{W^{-1,p}(\Omega_m^\epsilon)}),\tag{5.25}$$

where $c_{t,r}$ is a positive constant independent of $\omega,\epsilon$ but dependent on $t,r$. (5.20) and (5.25) imply that a $W^{1,p}_{loc}(\mathbb{R}^n)$ solution of (2.5) exists and (2.6) holds for $p\in(1,2]$. So a $W^{1,p}_{loc}(\mathbb{R}^n)$ solution of (2.5) exists and (2.6) holds for $p\in(1,\infty)$.

By Remark 4.1 and Lemma 4.1, we know, for any $x\in\mathbb{R}^n$,

$$\begin{cases}|\mathbf{K}_{\omega,\epsilon}\Psi|(x)\le c\displaystyle\int_{\mathbb{R}^n}|x-y|^{2-n}|\mathbf{K}_{1/\omega,\epsilon}V|(y)dy,\\ |\mathbf{K}_{\omega,\epsilon}\nabla\Psi|(x)\le c\displaystyle\int_{\mathbb{R}^n}|x-y|^{1-n}|\mathbf{K}_{1/\omega,\epsilon}V|(y)dy,\end{cases}$$

where $c$ is a constant independent of $\omega,\epsilon,r$. By Theorem 1 in page 119 [27], we obtain (2.7).

Uniqueness of the $W^{1,p}_{loc}(\mathbb{R}^n)$ solution of (2.5) for $p \in (1,\infty)$ is due to Theorem 3.1 and Lemma 3.4 [13]. So Corollary 2.2 holds for $V \in L^\infty(\mathbb{R}^n)$ with compact support case. If $V \in W^{-1,p}(\mathbb{R}^n)$ with compact support, Corollary 2.2 can be proved by a limiting argument.

## 6. Appendix

In this section we give the proofs of Lemma 3.2, Lemma 3.4, and Lemma 3.6. Let $\Gamma(x-y)$ denote the fundamental solution of the Laplace equation in $\mathbb{R}^n$ (see section 6.2 [11]). Define a single-layer and a double-layer potentials as, for any smooth function $\zeta$ on the boundary $\partial\mathbb{D}$ of a smooth, bounded, and simply-connected domain $\mathbb{D}$,

$$\begin{cases} \mathbf{E}_{\partial\mathbb{D}}(\zeta)(x) \equiv \displaystyle\int_{\partial\mathbb{D}} \Gamma(x-y)\zeta(y)d\sigma_y \\ \mathcal{L}_{\partial\mathbb{D}}(\zeta)(x) \equiv \displaystyle\int_{\partial\mathbb{D}} \nabla_y\Gamma(x-y)\cdot \vec{\mathbf{n}}_y\ \zeta(y)d\sigma_y \end{cases} \qquad \text{for } x \in \partial\mathbb{D},$$

where $\vec{\mathbf{n}}_y$ is the unit vector outward normal to $\partial\mathbb{D}$ at $y \in \partial\mathbb{D}$. A modification of the argument of Lemma 3.2 [28], we have

**Lemma 6.1.** *Let $I$ denote the identity operator. For any $p \in (1,\infty)$, $k \in \{1,2\}$, and $\alpha \in (0,1)$, the linear operators*

$$\begin{cases} \mathbf{E}_{\partial\mathbb{D}} : W^{-1/p,p}(\partial\mathbb{D}) \to W^{1-1/p,p}(\partial\mathbb{D}) \\ \mathcal{L}_{\partial\mathbb{D}} : W^{1-1/p,p}(\partial\mathbb{D}) \to W^{2-1/p,p}(\partial\mathbb{D}) \\ \mathbf{E}_{\partial\mathbb{D}} : C^{k-1,\alpha}(\partial\mathbb{D}) \to C^{k,\alpha}(\partial\mathbb{D}) \\ \mathcal{L}_{\partial\mathbb{D}} : C^{k-1,\alpha}(\partial\mathbb{D}) \to C^{k,\alpha}(\partial\mathbb{D}) \end{cases}$$

*are bounded; the operator $I-\gamma\mathcal{L}_{\partial\mathbb{D}}$ with $\gamma \in [-2,2]$ is continuously invertible in space $W^{1-1/p,p}(\partial\mathbb{D})$ and in space $C^{k,\alpha}(\partial\mathbb{D})$; and there is a constant $c$ independent of $\gamma \in [-2,2]$ so that*

$$\begin{cases} \|\zeta\|_{W^{1-1/p,p}(\partial\mathbb{D})} \le c\|(I-\gamma\mathcal{L}_{\partial\mathbb{D}})(\zeta)\|_{W^{1-1/p,p}(\partial\mathbb{D})} & \text{for } \zeta \in W^{1-1/p,p}(\partial\mathbb{D}), \\ \|\zeta\|_{C^{k,\alpha}(\partial\mathbb{D})} \le c\|(I-\gamma\mathcal{L}_{\partial\mathbb{D}})(\zeta)\|_{C^{k,\alpha}(\partial\mathbb{D})} & \text{for } \zeta \in C^{k,\alpha}(\partial\mathbb{D}). \end{cases}$$

### 6.1. *Proof of Lemma 3.2*

We shall use the notation (3.1). Denote by $c$ a constant independent of $\omega$.

**Step 1:** We claim that if $Q \in C_0^\infty(\mathbf{S}_2 \setminus \overline{Y_m}) + C_0^\infty(Y_m)$ and $F \in L^p(\mathbf{S}_2)$, then any solution of

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,1}\nabla U + Q) = F & \text{in } \mathbf{S}_2 \\ U = 0 & \text{on } \partial\mathbf{S}_2 \end{cases} \tag{6.1}$$

satisfies

$$\|\mathbf{K}_{\omega,1}U, \mathbf{K}_{\omega,1}\nabla U\|_{L^p(\mathbf{S}_2)} \le cJ_\omega \qquad \text{for any } p \in (1,\infty). \tag{6.2}$$

See (3.3) for $J_\omega$.

Proof of the claim: Let us find a function $\eta \in C^2(\overline{\mathbf{S}_2})$ satisfying

$$\begin{cases} \eta \in [0,1], \quad \eta(x) = 1 \text{ if } dist(x, \partial \mathbf{S}_2) \le \mathbf{d}_0/2, \\ \eta(x) = 0 \text{ if } dist(x, \partial \mathbf{S}_2) > \mathbf{d}_0, \quad \|\nabla \eta\|_{L^\infty(\mathbf{S}_2)} \le c. \end{cases}$$

By $(3.1)_3$, $\nabla \eta = 0$ in $Y$. See (3.1) for $\mathbf{d}_0$. If we define $\Phi \equiv \eta U$, the support of $\Phi$ is in a neighborhood of $\partial \mathbf{S}_2$ of thickness $\mathbf{d}_0$ and $\Phi$ satisfies, by (6.1),

$$\begin{cases} -\nabla \cdot (\nabla \Phi - U \nabla \eta + \eta Q) = \eta F - \nabla \eta (\nabla U + Q) & \text{in } \mathbf{S}_2 \setminus \overline{Y_m}, \\ \Phi = 0 & \text{on } \partial \mathbf{S}_2 \cup \partial Y_m. \end{cases} \tag{6.3}$$

By [9],

$$\|\Phi\|_{W^{1,p}(\mathbf{S}_2 \setminus \overline{Y_m})} \le c J_\omega \qquad \text{for } p \in (1, \infty). \tag{6.4}$$

Multiply $(6.3)_1$ by $\zeta \in W^{1,q}(\mathbf{S}_2 \setminus \overline{Y_m})$ for $q \in (1,\infty)$ and integrate it over $\mathbf{S}_2 \setminus \overline{Y_m}$ to see

$$\int_{\partial \mathbf{S}_2} \frac{\partial U}{\partial \mathbf{n}} \zeta d\sigma = \int_{\mathbf{S}_2 \setminus \overline{Y_m}} \big(\nabla \Phi - U \nabla \eta + \eta Q\big) \nabla \zeta - \zeta \left(\eta F - \nabla \eta \big(\nabla U + Q\big)\right) dx,$$

where $\frac{\partial U}{\partial \mathbf{n}}|_{\partial \mathbf{S}_2} = \nabla U \cdot \vec{\mathbf{n}}|_{\partial \mathbf{S}_2}$ is the normal derivative of $U$ on $\partial \mathbf{S}_2$ and $\vec{\mathbf{n}}$ is a unit normal vector on $\partial \mathbf{S}_2$. Which implies the $U$ of (6.1) satisfies, by (6.4),

$$\left\| \frac{\partial U}{\partial \mathbf{n}} \right\|_{W^{-1/p,p}(\partial \mathbf{S}_2)} \le c J_\omega \qquad \text{for } p \in (1, \infty). \tag{6.5}$$

Let $\widehat{U}$ in $Y_m$ be a solution of

$$\begin{cases} -\nabla \cdot (\omega^2 \nabla \widehat{U} + Q) = F & \text{in } Y_m, \\ \widehat{U} = 0 & \text{on } \partial Y_m, \end{cases} \tag{6.6}$$

and $\widehat{U}$ in $\mathbf{S}_2 \setminus \overline{Y_m}$ be a solution of

$$\begin{cases} -\nabla \cdot (\nabla \widehat{U} + Q) = F & \text{in } \mathbf{S}_2 \setminus \overline{Y_m}, \\ \widehat{U} = 0 & \text{on } \partial Y_m \cup \partial \mathbf{S}_2. \end{cases} \tag{6.7}$$

By [9], there is a $\widehat{U} \in W^{1,p}(\mathbf{S}_2)$ for $p \in (1,\infty)$ satisfying (6.6)–(6.7) and

$$\|\mathbf{K}_{\omega,1} \widehat{U}, \mathbf{K}_{\omega,1} \nabla \widehat{U}\|_{L^p(\mathbf{S}_2)} \le c J_\omega. \tag{6.8}$$

Define $\check{U} \equiv U - \widehat{U}$ in $\mathbf{S}_2$. Equations (6.1) and (6.6)–(6.7) imply

$$\begin{cases} -\Delta \check{U} = 0 & \text{in } \mathbf{S}_2 \setminus \partial Y_m, \\ \lfloor \check{U} \rfloor_{\partial Y_m} = 0, & \\ \lfloor \mathbf{K}_{\omega^2,1} \nabla \check{U} \rfloor_{\partial Y_m} \cdot \vec{\mathbf{n}}_y = \zeta_\omega, & \\ \check{U} = 0 & \text{on } \partial \mathbf{S}_2, \end{cases} \tag{6.9}$$

where $\zeta_\omega \equiv -\lfloor \mathbf{K}_{\omega^2,1}\nabla\widehat{U}\rfloor_{\partial Y_m}\cdot\vec{\mathbf{n}}_y$ because of $Q\in C_0^\infty(\mathbf{S}_2\setminus\overline{Y_m})+C_0^\infty(Y_m)$. See (2.1) for $(6.9)_{2,3}$. By (6.8),

$$\|\zeta_\omega\|_{W^{-1/p,p}(\partial Y_m)} \le cJ_\omega \qquad \text{for } p\in(1,\infty). \tag{6.10}$$

By Green's formula, (2.1), (6.9), and Theorem 6.5.1 [11], we see that

$$\begin{cases} \check{U}/2+\mathcal{L}_{\partial Y_m}(\check{U}) = \mathbf{E}_{\partial Y_m}(\nabla\check{U}_{,-}\cdot\vec{\mathbf{n}}_y|_{\partial Y_m}) \\ \check{U}/2-\mathcal{L}_{\partial Y_m}(\check{U}) = -\mathbf{E}_{\partial Y_m}(\nabla\check{U}_{,+}\cdot\vec{\mathbf{n}}_y|_{\partial Y_m})+\mathbf{E}_{\partial\mathbf{S}_2}(\frac{\partial\check{U}}{\partial\mathbf{n}}|_{\partial\mathbf{S}_2}) \end{cases} \quad \text{on } \partial Y_m,$$

where $\frac{\partial\check{U}}{\partial\mathbf{n}}|_{\partial\mathbf{S}_2}$ is the normal derivative of $\check{U}$ on $\partial\mathbf{S}_2$. Therefore $(6.9)_{2,3}$ imply

$$\left(I-\frac{2(1-\omega^2)}{\omega^2+1}\mathcal{L}_{\partial Y_m}\right)\check{U} = \frac{2\mathbf{E}_{\partial\mathbf{S}_2}(\frac{\partial\check{U}}{\partial\mathbf{n}}|_{\partial\mathbf{S}_2})}{1+\omega^2}-\frac{2\mathbf{E}_{\partial Y_m}(\zeta_\omega)}{1+\omega^2} \quad \text{on } \partial Y_m. \tag{6.11}$$

By Lemma 6.1, (6.5), and (6.10)–(6.11),

$$\|\check{U}\|_{W^{1-1/p,p}(\partial Y_m)} \le c\left(\|\zeta_\omega\|_{W^{-1/p,p}(\partial Y_m)}+\|\frac{\partial\check{U}}{\partial\mathbf{n}}\|_{W^{-1/p,p}(\partial\mathbf{S}_2)}\right) \le cJ_\omega. \tag{6.12}$$

By (6.8), (6.9), and (6.12), we obtain (6.2).

**Step 2:** Claim if $Q\in C_0^\infty(\mathbf{S}_2\setminus\overline{Y_m})+C_0^\infty(Y_m)$, and $F\in L^p(\mathbf{S}_2)$, any solution of (6.1) satisfies

$$\|\mathbf{K}_{\omega,1}U,\mathbf{K}_{\omega,1}\nabla U\|_{L^p(\mathbf{S}_2)} \le c\hat{J}_\omega \qquad \text{for any } p\in(1,\infty), \tag{6.13}$$

where $\hat{J}_\omega \equiv \|\mathbf{K}_{1/\omega,1}Q\|_{L^p(\mathbf{S}_2)}+\|\mathbf{K}_{1/\omega,1}F\|_{W^{-1,p}(\mathbf{S}_2)}$.

For any $\zeta\in C_0^\infty(\mathbf{S}_2\setminus\overline{Y_m})$ and $q\in[2,\infty)$, let $\varphi$ be the solution of

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,1}\nabla\varphi)=\zeta & \text{in } \mathbf{S}_2, \\ \varphi=0 & \text{on } \partial\mathbf{S}_2. \end{cases} \tag{6.14}$$

By Lax-Milgram Theorem and Theorem 7.26 [13], the $H^1$ solution of (6.14) exists uniquely and

$$\|\varphi\|_{L^{\frac{2n}{n-2}}(\mathbf{S}_2\setminus\overline{Y_m})} \le c\|\varphi\|_{H^1(\mathbf{S}_2\setminus\overline{Y_m})} \le c\|\zeta\|_{L^2(\mathbf{S}_2\setminus\overline{Y_m})} \le c\|\zeta\|_{L^q(\mathbf{S}_2\setminus\overline{Y_m})}.$$

So, if $q\in[2,\frac{2n}{n-2}]$, then

$$\|\varphi\|_{L^q(\mathbf{S}_2\setminus\overline{Y_m})} \le c\|\zeta\|_{L^q(\mathbf{S}_2\setminus\overline{Y_m})}.$$

Together with (6.2), we have

$$\|\mathbf{K}_{\omega,1}\varphi,\mathbf{K}_{\omega,1}\nabla\varphi\|_{L^q(\mathbf{S}_2)} \le c\|\zeta\|_{L^q(\mathbf{S}_2\setminus\overline{Y_m})}. \tag{6.15}$$

If $q>\frac{2n}{n-2}$, by (6.15),

$$\|\varphi\|_{W^{1,\frac{2n}{n-2}}(\mathbf{S}_2\setminus\overline{Y_m})} \le c\|\zeta\|_{L^{\frac{2n}{n-2}}(\mathbf{S}_2\setminus\overline{Y_m})} \le c\|\zeta\|_{L^q(\mathbf{S}_2\setminus\overline{Y_m})}. \tag{6.16}$$

By Theorem 7.26 [13],

$$\begin{cases} \|\varphi\|_{L^{\frac{2n}{n-4}}(\mathbf{S}_2\setminus\overline{Y_m})} \le c\|\varphi\|_{W^{1,\frac{2n}{n-2}}(\mathbf{S}_2\setminus\overline{Y_m})} & \text{if } n>4, q>\frac{2n}{n-4}, \\ \|\varphi\|_{L^q(\mathbf{S}_2\setminus\overline{Y_m})} \le c\|\varphi\|_{W^{1,\frac{2n}{n-2}}(\mathbf{S}_2\setminus\overline{Y_m})} & \text{if } \begin{cases} n\le 4, q>\frac{2n}{n-2} \text{ or} \\ n>4, q\le\frac{2n}{n-4}. \end{cases} \end{cases} \tag{6.17}$$

So if $n \le 4$ and $q > \frac{2n}{n-2}$ or if $n > 4$ and $q \in [2, \frac{2n}{n-4}]$, the solution of (6.14) satisfies, by (6.2) and (6.16)–(6.17),

$$\|\mathbf{K}_{\omega,1}\varphi, \mathbf{K}_{\omega,1}\nabla\varphi\|_{L^q(\mathbf{S}_2)} \le c\|\zeta\|_{L^q(\mathbf{S}_2\setminus\overline{Y_m})}. \tag{6.18}$$

Repeating above argument, we see that the solution of (6.14) satisfies (6.18) under $\zeta \in C_0^\infty(\mathbf{S}_2 \setminus \overline{Y_m})$ and $q \in [2,\infty)$ case. Since $C_0^\infty(\mathbf{S}_2 \setminus \overline{Y_m})$ is dense in $L^q(\mathbf{S}_2 \setminus \overline{Y_m})$ for $q \in [2,\infty)$, (6.18) also holds if $\zeta \in L^q(\mathbf{S}_2 \setminus \overline{Y_m})$ and $q \in [2,\infty)$.

For any $\zeta \in L^q(\mathbf{S}_2 \setminus \overline{Y_m})$ with $q \in [2,\infty)$, let $\varphi$ be its corresponding solution obtained from (6.14). Multiply (6.1) by $\varphi$ and employ Theorem 2.1 [1] to obtain

$$\begin{aligned} -\int_{\mathbf{S}_2\setminus\overline{Y_m}} U\zeta dx &= \int_{\mathbf{S}_2} U\nabla\cdot(\mathbf{K}_{\omega^2,1}\nabla\varphi)dx = \int_{\mathbf{S}_2} Q\nabla\varphi\, dy - \int_{\mathbf{S}_2} F\ \Pi_1\varphi|_{\mathbf{S}_2\setminus\overline{Y_m}} dx \\ &\quad - \int_{Y_m} F(\varphi - \Pi_1\varphi|_{\mathbf{S}_2\setminus\overline{Y_m}})dx \le c\|\zeta\|_{L^q(\mathbf{S}_2\setminus\overline{Y_m})}\hat{J}_\omega, \end{aligned} \tag{6.19}$$

where the $p$ in $\hat{J}_\omega$ satisfies $p \in (1,2]$ and $\frac{1}{p}+\frac{1}{q} = 1$. Here $\Pi_1\varphi|_{\mathbf{S}_2\setminus\overline{Y_m}}$ is the extension function of $\varphi|_{\mathbf{S}_2\setminus\overline{Y_m}}$ on $\mathbf{S}_2$ (see Theorem 2.1 [1] or Lemma 3.1). So

$$\|U\|_{L^p(\mathbf{S}_2\setminus\overline{Y_m})} \le c\hat{J}_\omega \quad \text{for } 1 < p \le 2. \tag{6.20}$$

Equations (6.2) and (6.20) imply (6.13) for $1 < p \le 2$.

Similarly take $\varphi$ to be the solution of (6.14) with $\zeta \in L^q(\mathbf{S}_2 \setminus \overline{Y_m})$ for $q \in (1,2]$. Since (6.13) holds for $1 < p \le 2$, the solution of (6.14) satisfies

$$\|\mathbf{K}_{\omega,1}\varphi, \mathbf{K}_{\omega,1}\nabla\varphi\|_{L^q(\mathbf{S}_2)} \le c\|\zeta\|_{L^q(\mathbf{S}_2\setminus\overline{Y_m})} \quad \text{for } q \in (1,2].$$

Again we multiply (6.1) by $\varphi$ from (6.14) with $\zeta \in L^q(\mathbf{S}_2 \setminus \overline{Y_m})$ and $q \in (1,2]$ as well as argue as (6.19) to see that (6.13) holds for $2 \le p < \infty$. Therefore we prove (6.13) for $1 < p < \infty$.

**Step 3:** Since $C_0^\infty(\mathbf{S}_2 \setminus \overline{Y_m})$ (resp. $C_0^\infty(Y_m)$, $L^p(\mathbf{S}_2)$, and $L^p(Y_m)$) is dense in $L^p(\mathbf{S}_2 \setminus \overline{Y_m})$ (resp. $L^p(Y_m)$, $W^{-1,p}(\mathbf{S}_2)$, and $W^{-1,p}(Y_m)$) for $p \in (1,\infty)$, we see the solution of (6.1) satisfies (6.13) for $Q \in L^p(\mathbf{S}_2), F \in W^{-1,p}(\mathbf{S}_2)$ by using Lax-Milgram Theorem [13] and a limiting argument.

**Step 4:** Let $\hat{\eta}$ be a smooth function satisfying $\hat{\eta} \in C_0^\infty(\mathbf{S}_2)$, $\hat{\eta} \in [0,1]$, $\hat{\eta} = 1$ in $Y$, $\|\nabla\hat{\eta}\|_{W^{1,\infty}(\mathbf{S}_2)} \le c$. Multiply (3.2) by $\hat{\eta}$ to see

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,1}\nabla(\hat{\eta}U) - U\nabla\hat{\eta} + \hat{\eta}Q) = \hat{\eta}F - \nabla\hat{\eta}(\nabla U + Q) & \text{in } \mathbf{S}_2, \\ \hat{\eta}U = 0 & \text{on } \partial\mathbf{S}_2. \end{cases} \tag{6.21}$$

Employing the result of Step 3 to (6.21), we obtain Lemma 3.2.

### 6.2. *Proof of Lemma 3.4*

After translation, we assume $x_0 = 0 \in \tau\partial Y_m$. For each $\tau > 1$, we find a $C^2$ domain $\mathbf{D}_\tau$ such that

$$B_{1/4}(x_0) \cap \tau Y_m \subset \mathbf{D}_\tau \subset B_{1/2}(x_0) \cap \tau Y_m \quad \text{and} \quad \overline{B_{1/4}(x_0) \cap \tau\partial Y_m} \subset \partial\mathbf{D}_\tau.$$

For any $z \in \partial\mathbf{D}_\tau$, there exists a ball $B(z)$ and a $C^2$ one-to-one mapping $\xi_{z,\tau}$ of $\overline{B(z)}$ onto $\overline{\xi_{z,\tau}(B(z))} \subset \mathbb{R}^n$ satisfying

$$\xi_{z,\tau}(B(z)\cap \mathbf{D}_\tau) \subset \mathbb{R}^n_+,\ \xi_{z,\tau}(B(z)\cap \partial\mathbf{D}_\tau) \subset \partial\mathbb{R}^n_+,\ \xi_{z,\tau}(B(z)\setminus \overline{\mathbf{D}_\tau}) \subset \mathbb{R}^n_-. \quad (6.22)$$

Here $\mathbb{R}^n_+ \equiv \{x = (x_1, \cdots, x_n) | x_n > 0\}, \partial\mathbb{R}^n_+ \equiv \{x | x_n = 0\}, \mathbb{R}^n_- \equiv \{x | x_n < 0\}$. Since $\partial\mathbf{D}_\tau$ is compact, there exist a finite number $\ell_\tau$ of open balls $\{B(z_i)\}_{i=1}^{\ell_\tau}$ and one-to-one mappings $\{\xi_{z_i,\tau}\}_{i=1}^{\ell_\tau}$ such that

$$\begin{cases} z_i \in \partial\mathbf{D}_\tau \text{ for } i \in \{1, \cdots, \ell_\tau\}, \\ \text{(6.22) holds for each ball } B(z_i) \text{ and } i \in \{1, \cdots, \ell_\tau\}, \\ \partial\mathbf{D}_\tau \subset \bigcup_{i=1}^{\ell_\tau} B(z_i). \end{cases}$$

Since $Y_m$ is smooth, it is possible to choose domains $\mathbf{D}_\tau$ for all $\tau > 1$ such that

$$\begin{cases} \text{the number } \ell_\tau \text{ is bounded above by a constant independent of } \tau, \\ \|\xi_{z,\tau}\|_{C^2(\overline{B(z)})}, \|\xi_{z,\tau}^{-1}\|_{C^2(\overline{\Phi_{z,\tau}(B(z))})} \le c_*, \text{where } c_* \text{ is independent of } \tau, z. \end{cases}$$

Define $\mathbb{U}$ in $\mathbb{R}^n$ by

$$\mathbb{U} \equiv \begin{cases} U & \text{in } B_{1/4}(x_0), \\ 0 & \text{elsewhere.} \end{cases}$$

Let $\eta$ be a bell-shaped function satisfying $\eta \in C_0^\infty(B_{1/4}(x_0))$, $\eta \in [0,1]$, $\eta = 1$ in $B_{1/5}(x_0)$, $\|\nabla\eta\|_{W^{1,\infty}(B_{1/4}(x_0))} \le c$. Multiply (3.5) by $\eta$ to get

$$\begin{cases} -\nabla\cdot(\mathbb{K}_{\omega^2,\tau}\nabla(\eta\mathbb{U})) = -\mathbb{K}_{\omega^2,\tau}(\mathbb{U}\Delta\eta + 2\nabla\mathbb{U}\nabla\eta) + \eta F\mathcal{X}_{\tau Y_f} & \text{in } B_{\frac{1}{2}}(x_0)\setminus\partial\mathbf{D}_\tau, \\ \lfloor \eta\mathbb{U}\rfloor_{\partial\mathbf{D}_\tau} = 0, & \\ \lfloor \mathbb{K}_{\omega^2,\tau}\nabla(\eta\mathbb{U})\rfloor_{\partial\mathbf{D}_\tau}\cdot\vec{\mathbf{n}}_y = (1-\omega^2)\mathbb{U}\nabla\eta\cdot\vec{\mathbf{n}}_y + \eta\zeta & \text{on } \partial\mathbf{D}_\tau, \\ \eta\mathbb{U} = 0 & \text{on } \partial B_{\frac{1}{2}}(x_0). \end{cases}$$

Here $\vec{\mathbf{n}}_y$ also denotes the unit outward normal vector on $\partial\mathbf{D}_\tau$ and $\mathbb{K}_{\omega^2,\tau}$ is defined to be $\omega^2$ in $\mathbf{D}_\tau$ and 1 elsewhere. Then we follow the argument of Lemma 3.2 and employ Lemma 6.1 to obtain Lemma 3.4.

### 6.3. *Proof of Lemma 3.6*

For $i_1, i_2 \in \{1, \cdots, n\}$, we find $\mathbb{Y}_\omega^{(i_1,i_2)} \in H^1_{per}(\mathbb{R}^n)$ satisfying, in the cell $Y$,

$$\begin{cases} \nabla\cdot\left(\mathbf{K}_{\omega^2,1}\left(\nabla\mathbb{Y}_\omega^{(i_1,i_2)} + \mathbb{X}_\omega^{(i_2)}\vec{e}_{i_1}\right)\right) \\ \qquad = \frac{\mathcal{K}_\omega^{(i_1,i_2)}}{|Y_f|}\mathcal{X}_{Y_f} - \mathbf{K}_{\omega^2,1}\left(\delta_{i_1,i_2} + \partial_{i_1}\mathbb{X}_\omega^{(i_2)}\right) \quad \text{in } Y, \\ \int_{Y_f} \mathbb{Y}_\omega^{(i_1,i_2)}(y)dy = 0. \end{cases} \quad (6.23)$$

Here $\mathbb{X}_\omega^{(i)}$ is defined in (3.6), $\mathcal{K}_\omega^{(i_1,i_2)}$ is the $(i_1,i_2)$-component of $\mathcal{K}_\omega$ (see (3.22)), $\delta_{i_1,i_2} = \begin{cases} 1 & \text{if } i_1 = i_2, \\ 0 & \text{if } i_1 \ne i_2, \end{cases}$ and $\vec{e}_{i_1}$ denotes a unit vector in the $i_1$ direction. By Lax-Milgram Theorem [13], (3.8), and Lemma 3.3, function $\mathbb{Y}_\omega^{(i_1,i_2)}$ is solvable uniquely and

$$\|\mathbb{Y}_\omega^{(i_1,i_2)}\|_{C^{1,\alpha}(\overline{Y_f})\cap C^{1,\alpha}(\overline{Y_m})} \le c, \tag{6.24}$$

where $\alpha \in (0,1)$ and $c$ is a constant independent of $\omega$. Define $n \times n$ periodic matrix functions $\mathbb{Y}_\omega \equiv (\mathbb{Y}_\omega^{(i_1,i_2)})$ and $\mathbb{Y}_{\omega,\epsilon}(x) \equiv \epsilon^2 \mathbb{Y}_\omega(\frac{x}{\epsilon})$.

By Theorem 3 in page 39 [27], (3.10), $F \in C_0^\infty(\mathbb{R}^n)$, and the proof of Lemma 4.4 [13], the $\mathcal{D}^{1,2}(\mathbb{R}^n)$ solution of (3.23) satisfies

$$\|\nabla^2 U\|_{W^{1,\frac{2n}{n+2}}(\mathbb{R}^n)\cap W^{2,n+\ell}(\mathbb{R}^n)} \le c\|F\|_{W^{1,\frac{2n}{n+2}}(\mathbb{R}^n)\cap W^{2,n+\ell}(\mathbb{R}^n)}, \tag{6.25}$$

where $\ell > 0$ and $c$ is a constant. Define

$$\Phi_\epsilon \equiv U_\epsilon - U - \mathbb{X}_{\omega,\epsilon}\nabla U - \mathbb{Y}_{\omega,\epsilon}\nabla^2 U \qquad \text{in } \mathbb{R}^n.$$

See remark after (3.8) for $\mathbb{X}_{\omega,\epsilon}$. By (3.6), (3.19), (3.23), and (6.23), we see $\Phi_\epsilon \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ satisfies

$$-\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}(\nabla\Phi_\epsilon + \mathbb{Y}_{\omega,\epsilon}\nabla^3 U)) = \mathbf{K}_{\omega^2,\epsilon}(\mathbb{X}_{\omega,\epsilon}\nabla\Delta U + \nabla\mathbb{Y}_{\omega,\epsilon}\nabla^3 U) \qquad \text{in } \mathbb{R}^n.$$

By energy method, Theorem 2.1 [1], Lemma 3.1, (3.8), and (6.24)–(6.25),

$$\|\mathbf{K}_{\omega,\epsilon}\Phi_\epsilon\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} + \|\mathbf{K}_{\omega,\epsilon}\nabla\Phi_\epsilon\|_{L^2(\mathbb{R}^n)} \le c\epsilon\|F\|_{W^{1,\frac{2n}{n+2}}(\mathbb{R}^n)\cap W^{2,n+\ell}(\mathbb{R}^n)}, \tag{6.26}$$

where $\ell > 0$ and $c$ is independent of $\omega, \epsilon$. By (3.12) of Lemma 3.5 and (6.24)–(6.26), for any $\epsilon, \omega \in (0,1]$ and $x \in \mathbb{R}^n$,

$$[\Phi_\epsilon]_{C^{0,\mu}(\overline{B_1(x)\cap\Omega_f^\epsilon})} \le c\epsilon\|F\|_{W^{1,\frac{2n}{n+2}}(\mathbb{R}^n)\cap W^{2,n+\ell}(\mathbb{R}^n)}, \tag{6.27}$$

where $\mu \equiv \frac{\ell}{n+\ell}$, $\ell > 0$, and $c$ is independent of $x, \omega, \epsilon$. (6.26)–(6.27) imply, for any $x \in \Omega_f^\epsilon$,

$$\begin{aligned}
|\Phi_\epsilon(x)| &\le \left|\Phi_\epsilon(x) - \fint_{B_1(x)\cap\Omega_f^\epsilon} \Phi_\epsilon(y)dy\right| + \left|\fint_{B_1(x)\cap\Omega_f^\epsilon} \Phi_\epsilon(y)dy\right| \\
&\le c([\Phi_\epsilon]_{C^{0,\mu}(\overline{B_1(x)\cap\Omega_f^\epsilon})} + \|\Phi_\epsilon\|_{L^{\frac{2n}{n-2}}(B_1(x)\cap\Omega_f^\epsilon)}) \\
&\le c\epsilon\|F\|_{W^{1,\frac{2n}{n+2}}(\mathbb{R}^n)\cap W^{2,n+\ell}(\mathbb{R}^n)},
\end{aligned}$$

which implies

$$\|U_\epsilon - U\|_{L^\infty(\Omega_f^\epsilon)} \le c\epsilon\|F\|_{W^{1,\frac{2n}{n+2}}(\mathbb{R}^n)\cap W^{2,n+\ell}(\mathbb{R}^n)}, \tag{6.28}$$

where $c$ is independent of $\omega, \epsilon$. (3.21), (6.25), (6.28), and triangle inequality imply Lemma 3.6.

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