Elliptic and parabolic equations in fractured media

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The elliptic and the parabolic equations with Dirichlet boundary conditions in fractured media are considered. The fractured media consist of a periodic connected high permeability sub-region and a periodic disconnected matrix block subset with low permeability. Let $\epsilon \in (0,1]$ denote the size ratio of the matrix blocks to the whole domain and let $\omega^2 \in (0,1]$ denote the permeability ratio of the disconnected subset to the connected sub-region. It is proved that the $W^{1,p}$ norm of the elliptic and the parabolic solutions in the high permeability sub-region are bounded uniformly in ω, ϵ. However, the $W^{1,p}$ norm of the solutions in the low permeability subset may not be bounded uniformly in ω, ϵ. For the elliptic and the parabolic equations in periodic perforated domains, it is also shown that the $W^{1,p}$ norm of their solutions are bounded uniformly in ϵ.

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1. Introduction

The $W^{1,p}$ estimates for the solutions of the elliptic and the parabolic equations with Dirichlet boundary conditions in fractured media are concerned. The problem arises from two-phase problems, flows in fractured media, and the stress in composite materials (see [3, 9, 15]). Let Ω be a smooth simply-connected domain in $\mathbb{R}^n$ for $n \geq 3$, $\partial\Omega$ be the boundary of Ω, $Y \equiv (0,1)^n$ consist of a smooth sub-domain Y_m completely surrounded by another connected sub-domain Y_f $(\equiv Y \setminus \overline{Y_m})$, $\epsilon \in (0,1]$, $\Omega(2\epsilon) \equiv \{x \in \Omega : dist(x, \partial\Omega) \geq 2\epsilon\}$, $\Omega_m^\epsilon \equiv \{x : x \in \epsilon(Y_m + j) \subset \Omega(2\epsilon) \text{ for some } j \in \mathbb{Z}^n\}$ be a disconnected subset of Ω, Ω_f^ϵ $(\equiv \Omega \setminus \overline{\Omega_m^\epsilon})$ denote a connected sub-region of Ω, and $\mathbf{K}_{\nu,\epsilon}(x) \equiv \begin{cases} 1 & \text{if } x \in \Omega_f^\epsilon \\ \nu & \text{if } x \in \Omega_m^\epsilon \end{cases}$ for any $\nu, \epsilon > 0$.

The elliptic equation that we consider is

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla U + G) = F & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\omega, \epsilon \in (0,1]$ and G, F are given functions. If G, F are bounded, a solution of (1.1) in Hilbert space $H^1(\Omega)$ exists uniquely for each ω, ϵ by Lax-Milgram Theorem [12]. The L^2 norm of the gradient of the solution of (1.1) in the connected sub-region Ω_f^ϵ is bounded uniformly in ω, ϵ if G, F are small in Ω_m^ϵ. However, the L^2 norm of the gradient of the solution of (1.1) in matrix blocks Ω_m^ϵ can be very large when ω closes to 0. The parabolic equation that we consider is, for any $\omega, \epsilon \in (0, 1]$,

$$\begin{cases} \partial_t U - \nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla U) = F & \text{in } \Omega \times (0, T), \\ U = 0 & \text{on } \partial\Omega \times (0, T), \\ U(x, 0) = U_0(x) & \text{in } \Omega. \end{cases} \tag{1.2}$$

If F, U_0 are smooth, a solution of (1.2) in Hilbert space $L^2([0, T]; H^1(\Omega))$ exists uniquely for each ω, ϵ. The L^2 norm of the gradient of the solution of (1.2) in the connected sub-region $\Omega_f^\epsilon \times (0, T)$ is bounded uniformly in ω, ϵ if F is small in $\Omega_m^\epsilon \times (0, T)$. However, the L^2 norm of the gradient of the solution of (1.2) in matrix blocks $\Omega_m^\epsilon \times (0, T)$ can be very large when ω closes to 0. One also notes that for the elliptic and the parabolic equations in periodic perforated domains, the H^1 norm of their solutions are bounded uniformly in ϵ.

There are some literatures related to this work. Lipschitz estimate and $W^{2,p}$ estimate for uniform elliptic equations with discontinuous coefficients had been proved in [15, 18]. Uniform Hölder, $W^{1,p}$, and Lipschitz estimates for uniform elliptic equations with Hölder periodic coefficients were shown in [4, 5]. Uniform $W^{1,p}$ estimate for uniform elliptic equations with continuous periodic coefficients was considered in [6] and the same problem with VMO periodic coefficients could be found in [22]. Uniform $W^{1,p}$ estimate for the Laplace equation in periodic perforated domains was considered in [19] and the same problem in Lipschitz estimate was studied in [21]. Uniform Hölder, $W^{1,p}$, and Lipschitz estimates in ϵ for uniform parabolic equations with oscillating periodic coefficients were obtained in [10]. For non-uniform elliptic equations with smooth periodic coefficients, existence of $C^{2,\alpha}$ solution could be found in [13]. Uniform Hölder estimate in ϵ for non-uniform parabolic equations with discontinuous periodic coefficients was shown in [23].

Here we present uniform $W^{1,p}$ estimate for the solutions of the non-uniform elliptic and the non-uniform parabolic equations with Dirichlet boundary conditions in fractured media. It is proved that the $W^{1,p}$ norm of the elliptic and the parabolic solutions in the high permeability sub-region Ω_f^ϵ are bounded uniformly in ω, ϵ. However, the solutions in the low permeability subset may not be bounded uniformly in ω, ϵ. For the elliptic and the parabolic equations in perforated domains, it is also shown that the $W^{1,p}$ norm of their solutions are bounded uniformly in ϵ. A three-step compactness argument introduced in [4, 5] will be employed to obtain the uniform estimate for non-uniform elliptic equations. Different from the approach in [10], we apply semigroup theory and the uniform estimate results for non-uniform elliptic equations to prove the uniform estimate for non-uniform parabolic equations.

The rest of this work is organized as follows: Notation and main results are stated in section 2. In section 3, we present a priori estimates for some interface problems and present some local uniform Lipschitz and local uniform $W^{1,p}$ estimates in ω, ϵ for the solutions of elliptic equations in fracture media. Proofs of the main results are given in section 4. The proof of local uniform Lipschitz estimate for the solutions of elliptic equations in fracture media (claimed in section 3) is given in section 5.

2. Notation and main result

Let $C^{k,\alpha}$ denote the Hölder space with norm $\|\cdot\|_{C^{k,\alpha}}$, $W^{s,p}$ the Sobolev space with norm $\|\cdot\|_{W^{s,p}}$, and $[\varphi]_{C^{0,\alpha}}$ the Hölder semi-norm of φ for $k \geq 0$, $\alpha \in [0,1]$, $s \geq -1, p \in [1,\infty]$ (see [2, 12]). $L^p = W^{0,p}$ and $H^1 = W^{1,2}$. $C_0^\infty(D)$ is the space of infinitely differentiable functions with support in D and $C_{per}^\infty(\mathbb{R}^n)$ is the space of infinitely differentiable Y-periodic functions in $\mathbb{R}^n$. $W_0^{s,p}(D)$ is the closure of $C_0^\infty(D)$ under the $W^{s,p}$ norm and $W_{per}^{s,p}(\mathbb{R}^n)$ is the closure of $C_{per}^\infty(\mathbb{R}^n)$ under $W^{s,p}$ norm and $\|\varphi\|_{W_{per}^{s,p}(\mathbb{R}^n)} \equiv \|\varphi\|_{W^{s,p}(Y)}$ for $s \geq 1$, $p \in [1,\infty]$. $\mathcal{A}_m \equiv \{x : x \in Y_m + j \text{ for some } j \in \mathbb{Z}^n\}$ and $\mathcal{A}_f \equiv \mathbb{R}^n \setminus \overline{\mathcal{A}_m}$. $\mathcal{H}_{per}^1(\mathbb{R}^n) \equiv \{\varphi \in W_{per}^{1,2}(\mathbb{R}^n) : \int_{Y_f} \varphi(y)dy = 0\}$ and $\mathcal{H}_{per}^1(\mathcal{A}_f) \equiv \{\varphi|_{\mathcal{A}_f} : \varphi \in \mathcal{H}_{per}^1(\mathbb{R}^n)\}$. Let $\|\varphi_1, \cdots, \varphi_m\|_{\mathbf{B}_1} \equiv \|\varphi_1\|_{\mathbf{B}_1} + \cdots + \|\varphi_m\|_{\mathbf{B}_1}$ and $\|\varphi\|_{\mathbf{B}_1 \cap \mathbf{B}_2} \equiv \|\varphi\|_{\mathbf{B}_1} + \|\varphi\|_{\mathbf{B}_2}$. Set $rD = D/r^{-1} \equiv \{x : x/r \in D\}$, $\overline{D}$ be the closure of D, ∂D be the boundary of D, $\mathcal{X}_D$ is the characteristic function on D, and let $B_r(x)$ denote a ball centered at x with radius r. For any $\varphi \in L^1(D)$,

$$(\varphi)_D \equiv \fint_D \varphi(y)dy \equiv \frac{1}{|D|}\int_D \varphi(y)dy.$$

$\mathbb{K}_{\omega,\nu}(x) \equiv \begin{cases} 1 & \text{if } x \in \nu\mathcal{A}_f \\ \omega & \text{if } x \in \nu\mathcal{A}_m \end{cases}$ for $\omega \in [0,1], \nu \in (0,\infty)$. If $\vec{\mathbf{n}}$ is an outward normal vector on ∂Y_m, we define, for any function φ in Y and $x \in \partial Y_m$,

$$\lfloor \varphi \rfloor_{\partial Y_m}(x) = \varphi_{,+}(x) - \varphi_{,-}(x) \quad \text{where} \quad \varphi_{,\pm}(x) \equiv \lim_{t \to 0^+} \varphi(x \pm t\vec{\mathbf{n}}). \tag{2.1}$$

Similarly, if $\vec{\mathbf{n}}_\epsilon$ is an outward normal vector on $\partial\Omega_m^\epsilon$, we define, for any function φ in Ω and $x \in \partial\Omega_m^\epsilon$,

$$\lfloor \varphi \rfloor_{\partial\Omega_m^\epsilon}(x) = \varphi_{,+}(x) - \varphi_{,-}(x) \quad \text{where} \quad \varphi_{,\pm}(x) \equiv \lim_{t \to 0^+} \varphi(x \pm t\vec{\mathbf{n}}_\epsilon).$$

Next we give two statements:

A1. Ω is a bounded smooth simply-connected domain in $\mathbb{R}^n$ for $n \geq 3$,
A2. Y_m is a smooth simply-connected sub-domain of Y.

A1–A2 will be assumed throughout this paper except in subsection 5.1. Our main results are the following:

Theorem 2.1. *Suppose A1–A2 and*

A3. $\omega, \epsilon \in (0,1]$, $p \in (1,\infty)$, $G \in L^p(\Omega)$, $F \in W^{-1,p}(\Omega)$,

then a $W^{1,p}(\Omega)$ *solution of (1.1) exists uniquely and satisfies*

$$\begin{cases} \|\mathbf{K}_{\omega/\epsilon,\epsilon}U, \mathbf{K}_{\omega,\epsilon}\nabla U\|_{L^p(\Omega)} \\ \quad \leq c\big(\|\mathbf{K}_{1/\omega,\epsilon}G\|_{L^p(\Omega)} + \|F\|_{W^{-1,p}(\Omega)} + \omega^{-1}\|F\|_{W^{-1,p}(\Omega_m^\epsilon)}\big) \quad \text{for } \frac{\omega}{\epsilon} \leq 1, \\ \|U, \mathbf{K}_{\omega,\epsilon}\nabla U\|_{L^p(\Omega)} \\ \quad \leq c\big(\|\mathbf{K}_{1/\omega,\epsilon}G\|_{L^p(\Omega)} + \|F\|_{W^{-1,p}(\Omega)} + \omega^{-1}\|F\|_{W^{-1,p}(\Omega_m^\epsilon)}\big) \quad \text{for } \frac{\omega}{\epsilon} \geq 1, \end{cases}$$

where c is a constant independent of ω, ϵ.

Theorem 2.1 implies an analogous result for perforated domains.

Theorem 2.2. *Suppose A1–A2 and*

A4. $\epsilon \in (0,1]$, $p \in (1,\infty)$, $G \in L^p(\Omega_f^\epsilon)$, $F \in W^{-1,p}(\Omega)$, $\|F\|_{W^{-1,p}(\Omega_m^\epsilon)} = 0$,

then a $W^{1,p}(\Omega_f^\epsilon)$ solution of

$$\begin{cases} -\nabla \cdot (\nabla U + G) = F & \text{in } \Omega_f^\epsilon \\ (\nabla U + G) \cdot \vec{\mathbf{n}}_\epsilon = 0 & \text{on } \partial\Omega_m^\epsilon \\ U = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

exists uniquely and satisfies

$$\|U\|_{W^{1,p}(\Omega_f^\epsilon)} \leq c(\|G\|_{L^p(\Omega_f^\epsilon)} + \|F\|_{W^{-1,p}(\Omega)}), \tag{2.3}$$

where $\vec{\mathbf{n}}_\epsilon$ is a unit normal vector on $\partial\Omega_m^\epsilon$ and c is a constant independent of ϵ.

For any $\omega, \epsilon \in (0,1]$ and $p \in (1,\infty)$, let us define

$$\begin{cases} \mathbb{B}_p \equiv \{\varphi : \ \varphi \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega_f^\epsilon) \cap W^{2,p}(\Omega_m^\epsilon), \lfloor \mathbf{K}_{\omega^2,\epsilon} \nabla\varphi \cdot \vec{\mathbf{n}}_\epsilon \rfloor_{\partial\Omega_m^\epsilon} = 0\}, \\ \mathbb{D}_p \equiv \{\varphi : \ \varphi \in W^{2,p}(\Omega_f^\epsilon), \varphi|_{\partial\Omega} = 0, \nabla\varphi \cdot \vec{\mathbf{n}}_\epsilon|_{\partial\Omega_m^\epsilon} = 0\}, \end{cases}$$

where $\vec{\mathbf{n}}_\epsilon$ is a normal vector on $\partial\Omega_m^\epsilon$. By Lemma 3.4 [23], $\mathbb{B}_p$ with norm $\|\varphi\|_{\mathbb{B}_p} \equiv \|\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla\varphi)\|_{L^p(\Omega)}$ is a Banach space. If $\overline{\mathbb{B}_p}$ denotes the closure of $\mathbb{B}_p$ in L^p space, then $\overline{\mathbb{B}_p} = L^p(\Omega)$. Also note $\mathbb{D}_p$ with norm $\|\varphi\|_{\mathbb{D}_p} \equiv \|\Delta\varphi\|_{L^p(\Omega_f^\epsilon)}$ is a Banach space. The function spaces $C([0,T];\mathbf{B}), C^\sigma([0,T];\mathbf{B})$ for $\sigma \in (0,1]$ are defined as those in pages $1, 3$ [17].

Theorem 2.3. *Suppose A1–A2 and*

A5. $\omega, \epsilon, \sigma \in (0,1]$, $p \in (n,\infty)$, $\epsilon \leq \omega$, $F \in C^\sigma([0,T];L^p(\Omega))$, $U_0 \in \mathbb{B}_p$,

then a $C([0,T];W^{1,p}(\Omega))$ solution of (1.2) exists uniquely and satisfies

$$\|U\|_{C^1([0,T];L^p(\Omega))} + \|\mathbf{K}_{\omega,\epsilon} \nabla U\|_{C([0,T];L^p(\Omega))} \leq c\left(\|U_0\|_{\mathbb{B}_p} + \|F\|_{C^\sigma([0,T];L^p(\Omega))}\right),$$

where c is a constant independent of ω, ϵ.

Theorem 2.4. *Suppose A1–A2 and*

A6. $\epsilon, \sigma \in (0,1]$, $p \in (n,\infty)$, $F \in C^\sigma([0,T];L^p(\Omega_f^\epsilon))$, $U_0 \in \mathbb{D}_p$,

then a $C([0,T];W^{1,p}(\Omega_f^\epsilon))$ solution of

$$\begin{cases} \partial_t U - \Delta U = F & \text{in } \Omega_f^\epsilon \times (0,T) \\ \nabla U \cdot \vec{\mathbf{n}}_\epsilon = 0 & \text{on } \partial\Omega_m^\epsilon \times (0,T) \\ U = 0 & \text{on } \partial\Omega \times (0,T) \\ U(x,0) = U_0(x) & \text{in } \Omega_f^\epsilon \end{cases} \tag{2.4}$$

exists uniquely and satisfies

$$\|U\|_{C^1([0,T];L^p(\Omega_f^\epsilon))} + \|\nabla U\|_{C([0,T];L^p(\Omega_f^\epsilon))} \leq c \left(\|U_0\|_{\mathbb{D}_p} + \|F\|_{C^\sigma([0,T];L^p(\Omega_f^\epsilon))} \right),$$

where $\vec{\mathbf{n}}_\epsilon$ *is a normal vector on* $\partial\Omega_m^\epsilon$ *and* c *is a constant independent of* ϵ.

3. Preliminaries

From Theorem 2.1 [1], we know

Lemma 3.1. *For* $p \in [1,\infty)$ *and* $\epsilon \in (0,1]$, *there is a constant* $c(Y_f, p)$ *and a linear continuous extension operator* $\mathcal{P}_\epsilon : W^{1,p}(\Omega_f^\epsilon) \to W^{1,p}(\Omega)$ *such that if* $\varphi \in W^{1,p}(\Omega_f^\epsilon)$, *then*

$$\begin{cases} \mathcal{P}_\epsilon\varphi = \varphi \quad \text{in } \Omega_f^\epsilon, \\ \|\mathcal{P}_\epsilon\varphi\|_{L^p(\Omega)} \leq c\|\varphi\|_{L^p(\Omega_f^\epsilon)}, \\ \|\nabla\mathcal{P}_\epsilon\varphi\|_{L^p(\Omega)} \leq c\|\nabla\varphi\|_{L^p(\Omega_f^\epsilon)}, \\ 0 < d_1 \leq \mathcal{P}_\epsilon\varphi \leq d_2 \quad \text{if } 0 < d_1 \leq \varphi \leq d_2 \text{ for some constants } d_1, d_2, \\ \mathcal{P}_\epsilon\varphi = \zeta \quad \text{in } \Omega \text{ if } \varphi = \zeta|_{\Omega_f^\epsilon} \text{ for some linear function } \zeta \text{ in } \Omega. \end{cases}$$

Moreover, if $\zeta(x) \equiv \varphi(rx)$ *in* $B_1(0) \cap \Omega_f^\epsilon/r$ *for any* $r > \epsilon$, *then* $\mathcal{P}_{\epsilon/r}\zeta(x) = \mathcal{P}_\epsilon\varphi(rx)$ *in* $B_{1/2}(0)$.

Remark 3.1. Tracing the proof of Theorem 7.25 [12], we know that if $0 \in \partial Y_m$ and $p, \nu \in [1,\infty)$, there exist a constant $c(Y_f)$ and a linear continuous extension operator $\mathcal{P}_\nu : W^{1,p}(B_1(0) \cap \nu Y_f) \to W^{1,p}(B_1(0))$ such that, for any $\varphi \in W^{1,p}(B_1(0) \cap \nu Y_f)$,

$$\begin{cases} \mathcal{P}_\nu\varphi = \varphi \quad \text{in } B_1(0) \cap \nu Y_f, \\ \|\mathcal{P}_\nu\varphi\|_{L^p(B_1(0))} \leq c\|\varphi\|_{L^p(B_1(0)\cap\nu Y_f)}, \\ \|\nabla\mathcal{P}_\nu\varphi\|_{L^p(B_1(0))} \leq c\|\nabla\varphi\|_{L^p(B_1(0)\cap\nu Y_f)}. \end{cases}$$

Lemma 3.2. *Let* $\omega \in (0,1], \nu \in (0,\infty)$, $\varphi \in H^1(B_1(0))$, *and* $\mathcal{P}_\nu\varphi|_{\nu\mathcal{A}_f}$ *denote the extension of* $\varphi|_{\nu\mathcal{A}_f}$ *on* $B_1(0)$. *There is a constant* c *independent of* ω, ν *such that*

$$\left\|\mathbb{K}_{\omega,\nu}\big(\varphi - (\mathcal{P}_\nu\varphi|_{\nu\mathcal{A}_f})_{B_1(0)}\big)\right\|_{L^2(B_1(0))} \leq c\|\mathbb{K}_{\omega,\nu}\nabla\varphi\|_{L^2(B_1(0))}.$$

See section 2 for $\mathbb{K}_{\omega,\nu}$.

Proof. By Poincaré inequality [12], Lemma 3.1, and Remark 3.1, the extension function $\mathcal{P}_\nu\varphi|_{\nu\mathcal{A}_f} \in H^1(B_1(0))$ satisfies

$$\begin{aligned} &\left\|\mathcal{P}_\nu\varphi|_{\nu\mathcal{A}_f} - (\mathcal{P}_\nu\varphi|_{\nu\mathcal{A}_f})_{B_1(0)}\right\|_{L^2(B_1(0))} \\ &\quad \leq c\left\|\nabla\mathcal{P}_\nu\varphi|_{\nu\mathcal{A}_f}\right\|_{L^2(B_1(0))} \leq c\|\nabla\varphi\|_{L^2(B_1(0)\cap\nu\mathcal{A}_f)}, \end{aligned} \tag{3.1}$$

where c is independent of ω, ν. (3.1), Lemma 3.1, Remark 3.1, and Poincaré inequality imply

$$\begin{aligned}
&\big\|\mathbb{K}_{\omega,\nu}(\varphi - (\mathcal{P}_\nu\varphi|_{\nu\mathcal{A}_f})_{B_1(0)})\big\|_{L^2(B_1(0))} \\
&\quad \le \big\|\mathbb{K}_{\omega,\nu}(\mathcal{P}_\nu\varphi|_{\nu\mathcal{A}_f} - (\mathcal{P}_\nu\varphi|_{\nu\mathcal{A}_f})_{B_1(0)})\big\|_{L^2(B_1(0))} \\
&\qquad +\omega\big\|\varphi - \mathcal{P}_\nu\varphi|_{\nu\mathcal{A}_f}\big\|_{L^2(B_1(0)\cap\nu\mathcal{A}_m)} \\
&\quad \le c\|\nabla\varphi\|_{L^2(B_1(0)\cap\nu\mathcal{A}_f)} + c\omega\big\|\nabla\varphi - \nabla\mathcal{P}_\nu\varphi|_{\nu\mathcal{A}_f}\big\|_{L^2(B_1(0)\cap\nu\mathcal{A}_m)} \\
&\quad \le c\|\mathbb{K}_{\omega,\nu}\nabla\varphi\|_{L^2(B_1(0))}. \qquad \square
\end{aligned}$$

If $0 \in \partial\Omega$, by A1 and rotation, there is a smooth function $\Psi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\begin{cases} \Psi(0) = |\nabla\Psi(0)| = 0, \\ B_1(0)\cap\Omega/r = B_1(0)\cap\{(x', x_n)\in\mathbb{R}^n :\ rx_n > \Psi(rx')\} \quad \text{if } r\in(0,1]. \end{cases} \tag{3.2}$$

If $r = 0$, we define $B_1(0)\cap\Omega/r \equiv B_1(0)\cap\{(x', x_n)\in\mathbb{R}^n :\ x_n > 0\}$. Set

$$\breve{\mathbf{K}}_{\nu,\epsilon,r} \equiv \begin{cases} 1 & \text{in } \Omega_f^\epsilon/r \\ \nu & \text{in } \Omega_m^\epsilon/r \end{cases} \qquad \text{for } \nu, \epsilon, r \in (0,1]. \tag{3.3}$$

Similar to Lemma 3.2, we also have, by Poincaré inequality [12], Lemma 3.1, and Remark 3.1,

Lemma 3.3. *If $\omega, \epsilon, r \in (0,1]$, $0 \in \partial\Omega$ and $\varphi \in H^1(B_2(0)\cap\Omega/r)$ with $\varphi|_{\partial\Omega/r} = 0$, there is a constant c independent of ω, ϵ, r such that*

$$\|\breve{\mathbf{K}}_{\omega,\epsilon,r}\varphi\|_{L^2(B_1(0)\cap\Omega/r)} \le c\|\breve{\mathbf{K}}_{\omega,\epsilon,r}\nabla\varphi\|_{L^2(B_1(0)\cap\Omega/r)}.$$

3.1. *Interface problem*

Let $\Gamma(x-y)$ denote the fundamental solution of the Laplace equation in $\mathbb{R}^n$, see §6.2 [7]. Define a single-layer and a double-layer potentials as, for any smooth function φ on the boundary ∂Y_m of Y_m,

$$\begin{cases} \mathcal{S}_{\partial Y_m}(\varphi)(x) \equiv \displaystyle\int_{\partial Y_m} \Gamma(x-y)\varphi(y)dy \\ \mathcal{L}_{\partial Y_m}(\varphi)(x) \equiv \displaystyle\int_{\partial Y_m} \nabla_y\Gamma(x-y)\vec{\mathbf{n}}_y\ \varphi(y)dy \end{cases} \quad \text{for } x\in\partial Y_m,$$

where $\vec{\mathbf{n}}_y$ is the unit vector outward normal to ∂Y_m. By tracing the argument of Lemma 3.2 [23], we know

Lemma 3.4. *For any $p \in (1,\infty)$ and $s \in \{0,1\}$, the linear operators*

$$\begin{cases} \mathcal{S}_{\partial Y_m} : W^{s-\frac{1}{p},p}(\partial Y_m) \to W^{s+1-\frac{1}{p},p}(\partial Y_m) \\ \mathcal{L}_{\partial Y_m} : W^{s+1-\frac{1}{p},p}(\partial Y_m) \to W^{s+2-\frac{1}{p},p}(\partial Y_m) \end{cases}$$

are bounded; the operator $I - \ell\mathcal{L}_{\partial Y_m}$ is continuously invertible in $W^{s+1-\frac{1}{p},p}(\partial Y_m)$ for any $\ell \in [-2,2]$; and there is a constant c independent of ℓ so that

$$\|\varphi\|_{W^{s+1-\frac{1}{p},p}(\partial Y_m)} \le c\|(I - \ell\mathcal{L}_{\partial Y_m})(\varphi)\|_{W^{s+1-\frac{1}{p},p}(\partial Y_m)} \quad \text{for } \varphi \in W^{s+1-\frac{1}{p},p}(\partial Y_m),$$

where I is the identity operator.

Let us introduce some notations:

$$\begin{cases} \widetilde{\partial Y} \text{ is an open portion of the boundary } \partial Y, \\ \mathbf{D}_1, \mathbf{D}_2 \text{ are smooth domains satisfying } Y_m \subset \mathbf{D}_1 \subset \mathbf{D}_2 \subset Y, \\ dist(Y_m, \partial\mathbf{D}_1), dist(\mathbf{D}_1, \partial\mathbf{D}_2), dist(\mathbf{D}_2, \partial Y \setminus \widetilde{\partial Y}) > d_0 > 0. \end{cases}$$

Lemma 3.5. *Suppose $\omega \in (0,1]$, any solution Φ of*

$$\begin{cases} -\nabla\cdot(\mathbb{K}_{\omega^2,1}\nabla\Phi + V) = \zeta & \text{in } Y \\ \Phi = 0 & \text{on } \widetilde{\partial Y} \end{cases} \tag{3.4}$$

satisfies

$$\begin{cases} \|\mathbb{K}_{\omega^\sigma,1}\Phi\|_{W^{1,p}(\mathbf{D}_1\setminus\overline{Y_m})\cap W^{1,p}(Y_m)} \le c\big(\|\Phi\|_{L^2(Y_f)} \\ \qquad +\|\mathbb{K}_{\omega^{\sigma-2},1}V\|_{L^p(Y)} + \|\mathbb{K}_{\omega^{\sigma-2},1}\zeta\|_{W^{-1,p}(Y_f)\cap W^{-1,p}(Y_m)}\big), \\ \|\Phi\|_{W^{2,p}(\mathbf{D}_1\setminus\overline{Y_m})\cap W^{2,p}(Y_m)} \le c\big(\|\Phi\|_{L^2(Y_f)} \\ \qquad +\|\mathbb{K}_{\omega^{-2},1}V\|_{W^{1,p}(Y_f)\cap W^{1,p}(Y_m)} + \|\mathbb{K}_{\omega^{-2},1}\zeta\|_{L^p(Y)}\big), \end{cases} \tag{3.5}$$

where $p \in [2,\infty)$, $\sigma \in \{0,1\}$, and c is a constant independent of ω.

Proof. Define $\mathcal{I}_{\omega,\sigma} \equiv \|\mathbb{K}_{\omega^{\sigma-2},1}V\|_{L^p(Y)} + \|\mathbb{K}_{\omega^{\sigma-2},1}\zeta\|_{W^{-1,p}(Y_f)\cap W^{-1,p}(Y_m)}$ and let c denote a constant independent of ω. First we prove $(3.5)_1$.

Step 1: Assume $V \in W_0^{1,p}(Y_f) \cap W_0^{1,p}(Y_m)$ and $\zeta \in L^p(Y)$. Consider the following problem

$$\begin{cases} -\nabla\cdot(\mathbb{K}_{\omega^2,1}\nabla\phi + V) = \zeta & \text{in } \mathbf{D}_2, \\ \phi = 0 & \text{on } \partial\mathbf{D}_2. \end{cases} \tag{3.6}$$

The unique existence of a H^1 solution of (3.6) is from Lax-Milgram Theorem [12]. By energy method, the solution satisfies

$$\|\phi\|_{H^1(\mathbf{D}_2\setminus Y_m)} \le c\mathcal{I}_{\omega,1}. \tag{3.7}$$

Let $\eta \in C^\infty(\mathbf{D}_2\setminus Y_m)$, $\eta \in [0,1]$, $\eta = 1$ in $\mathbf{D}_2\setminus\mathbf{D}_1$, $\eta = 0$ on ∂Y_m, $\|\eta\|_{W^{1,\infty}(\mathbf{D}_2\setminus Y_m)} \le c$. Multiply $(3.6)_1$ by η to get

$$\begin{cases} -\nabla\cdot(\nabla(\eta\phi) - \phi\nabla\eta + \eta V) = \eta\zeta - (\nabla\phi + V)\nabla\eta & \text{in } \mathbf{D}_2\setminus Y_m, \\ \eta\phi = 0 & \text{on } \partial\mathbf{D}_2 \cup \partial Y_m. \end{cases} \tag{3.8}$$

By (3.7)–(3.8), [8], Theorem 7.26 [12], and an iterative argument, we have

$$\|\phi\|_{W^{1,p}(\mathbf{D}_2\setminus\mathbf{D}_1)} \le c\mathcal{I}_{\omega,\sigma}. \tag{3.9}$$

Let φ in Y_m satisfy

$$\begin{cases} -\nabla\cdot(\omega^2\nabla\varphi+V)=\zeta & \text{in } Y_m,\\ \varphi=0 & \text{on } \partial Y_m, \end{cases} \tag{3.10}$$

and φ in $\mathbf{D}_2\setminus\overline{Y_m}$ satisfy

$$\begin{cases} -\nabla\cdot(\nabla\varphi+V)=\zeta & \text{in } \mathbf{D}_2\setminus\overline{Y_m},\\ \varphi=0 & \text{on } \partial(\mathbf{D}_2\setminus\overline{Y_m}). \end{cases} \tag{3.11}$$

By [8] again,

$$\|\varphi\|_{W^{1,p}(\mathbf{D}_2\setminus\overline{Y_m})}+\omega^\sigma\|\varphi\|_{W^{1,p}(Y_m)}\le c\mathcal{I}_{\omega,\sigma}. \tag{3.12}$$

If we define $\psi\equiv\phi-\varphi$ in $\mathbf{D}_2$, then (3.6) and (3.10)–(3.11) imply

$$\begin{cases} \Delta\psi=0 & \text{in } \mathbf{D}_2\setminus\partial Y_m,\\ \lfloor\psi\rfloor_{\partial Y_m}=0 & \text{on } \partial Y_m,\\ \lfloor\mathbb{K}_{\omega^2,1}\nabla\psi\rfloor_{\partial Y_m}\cdot\vec{\mathbf{n}}_y=\mathcal{F} & \text{on } \partial Y_m,\\ \psi=0 & \text{on } \partial\mathbf{D}_2, \end{cases} \tag{3.13}$$

where $\vec{\mathbf{n}}_y$ is the unit vector outward normal to ∂Y_m. See (2.1) for $(3.13)_{2,3}$. Since $V\in W_0^{1,p}(Y_f)\cap W_0^{1,p}(Y_m)$,

$$\mathcal{F}\equiv\left(\omega^2\nabla\varphi_{,-}-\nabla\varphi_{,+}\right)\cdot\vec{\mathbf{n}}_y|_{\partial Y_m}.$$

By (3.12),

$$\|\mathcal{F}\|_{W^{-\frac{1}{p},p}(\partial Y_m)}\le c\mathcal{I}_{\omega,\sigma}. \tag{3.14}$$

By Green's formula, (3.13), and Theorem 6.5.1 [7],

$$\begin{cases} \psi/2+\mathcal{L}_{\partial Y_m}(\psi)=\mathcal{S}_{\partial Y_m}(\nabla\psi_{,-}\cdot\vec{\mathbf{n}}_y|_{\partial Y_m})\\ \psi/2-\mathcal{L}_{\partial Y_m}(\psi)=-\mathcal{S}_{\partial Y_m}(\nabla\psi_{,+}\cdot\vec{\mathbf{n}}_y|_{\partial Y_m})+\mathcal{S}_{\partial\mathbf{D}_2}(\partial_{\mathbf{n}_y}\psi|_{\partial\mathbf{D}_2}) \end{cases} \quad \text{on } \partial Y_m,$$

where $\partial_{\mathbf{n}_y}\psi|_{\partial\mathbf{D}_2}$ is the normal derivative of ψ on $\partial\mathbf{D}_2$. So

$$\left(I-\frac{2(1-\omega^2)}{\omega^2+1}\mathcal{L}_{\partial Y_m}\right)\psi=\frac{2}{\omega^2+1}\left(\mathcal{S}_{\partial\mathbf{D}_2}(\partial_{\mathbf{n}_y}\psi|_{\partial\mathbf{D}_2})-\mathcal{S}_{\partial Y_m}(\mathcal{F})\right)\text{ on } \partial Y_m. \tag{3.15}$$

Then (3.9), (3.12)–(3.15), and Lemma 3.4 imply

$$\|\psi\|_{W^{1-\frac{1}{p},p}(\partial Y_m)}\le c\left(\|\mathcal{F}\|_{W^{-\frac{1}{p},p}(\partial Y_m)}+\|\partial_{\mathbf{n}_y}\psi\|_{W^{-\frac{1}{p},p}(\partial\mathbf{D}_2)}\right)\le c\mathcal{I}_{\omega,\sigma}. \tag{3.16}$$

(3.13) and (3.16) imply

$$\|\psi\|_{W^{1,p}(\mathbf{D}_2)}\le c\mathcal{I}_{\omega,\sigma}.$$

Together with (3.12), we obtain

$$\|\mathbb{K}_{\omega^\sigma,1}\phi\|_{W^{1,p}(\mathbf{D}_2\setminus\overline{Y_m})\cap W^{1,p}(Y_m)}\le c\mathcal{I}_{\omega,\sigma}. \tag{3.17}$$

Note $W_0^{1,p}(Y_f)$ (resp. $W_0^{1,p}(Y_m)$) is dense in $L^p(Y_f)$ (resp. $L^p(Y_m)$) and $L^p(Y)$ is dense in $W^{-1,p}(Y)$. By a limiting argument, we see that if $V \in L^p(Y)$ and $\zeta \in W^{-1,p}(Y)$, any solution of (3.6) satisfies (3.17).

Step 2: Let η be a smooth function satisfying $\eta \in C_0^\infty(\mathbf{D}_2)$, $\eta \in [0,1]$, $\eta = 1$ in $\mathbf{D}_1$, $\|\eta\|_{W^{1,\infty}(\mathbf{D}_2)} \le c$. Multiply (3.4) by η to obtain

$$\begin{cases} -\nabla \cdot (\mathbb{K}_{\omega^2,1}\nabla(\Phi\eta) - \Phi\nabla\eta + V\eta) = \zeta\eta - (\nabla\Phi + V)\nabla\eta & \text{in } \mathbf{D}_2, \\ \Phi\eta = 0 & \text{on } \partial\mathbf{D}_2. \end{cases}$$

By the result of Step 1, we have

$$\|\mathbb{K}_{\omega^\sigma,1}\Phi\|_{W^{1,p}(\mathbf{D}_1\setminus\overline{Y_m})\cap W^{1,p}(Y_m)} \le c(\|\Phi\|_{L^p(\mathbf{D}_2\setminus\mathbf{D}_1)} + \mathcal{I}_{\omega,\sigma}). \tag{3.18}$$

Let $\tilde\eta$ be another smooth function satisfying $\tilde\eta \in C_0^\infty(Y_f)$, $\tilde\eta \in [0,1]$, $\tilde\eta = 1$ in $\mathbf{D}_2\setminus\mathbf{D}_1$, $\|\tilde\eta\|_{W^{1,\infty}(Y)} \le c$. Multiply (3.4) by $\tilde\eta\Phi$ and use energy method and Theorem 7.26 [12] to get

$$\|\Phi\|_{L^p(\mathbf{D}_2\setminus\mathbf{D}_1)} \le c(\|\Phi\|_{L^2(Y_f)} + \mathcal{I}_{\omega,\sigma}).$$

Together with (3.18), we obtain $(3.5)_1$. $(3.5)_2$ are proved in a similar way as $(3.5)_1$, so we skip it. □

By a similar argument as Lemma 3.5, we also have the following local estimate:

Lemma 3.6. *If $\omega \in (0,1]$, $\nu \in (1,\infty)$, $x_0 \in \nu\partial Y_m$, and $B_1(x_0) \subset \nu Y$, then any solution Φ of*

$$-\nabla \cdot (\mathbb{K}_{\omega^2,\nu}\nabla\Phi) = 0 \qquad \text{in } \nu Y \tag{3.19}$$

satisfies

$$\|\mathbb{K}_{\omega^\sigma,\nu}\Phi\|_{W^{2,p}(B_{1/3}(x_0)\cap\nu Y_f)\cap W^{2,p}(B_{1/3}(x_0)\cap\nu Y_m)} \le c\|\mathbb{K}_{\omega^\sigma,\nu}\Phi\|_{L^2(B_1(x_0))}, \tag{3.20}$$

where $p \in [2,\infty)$, $\sigma \in \{0,1\}$, and c is a constant independent of ω, ν.

Proof. After translation, we assume x_0 is the origin. For each $\nu > 1$, we find a smooth domain $\mathbf{D}_\nu$ such that

$$B_{1/2}(x_0) \cap \nu Y_m \subset \mathbf{D}_\nu \subset B_1(x_0) \cap \nu Y_m \quad \text{and} \quad \overline{B_{1/2}(x_0) \cap \nu\partial Y_m} \subset \partial\mathbf{D}_\nu.$$

Since $\mathbf{D}_\nu$ is smooth, for any $z \in \partial\mathbf{D}_\nu$ there is a ball $B(z)$ centered at z and there is a smooth one-to-one mapping $\varphi_{z,\nu}$ of $\overline{B(z)}$ onto $\overline{\varphi_{z,\nu}(B(z))} \subset \mathbb{R}^n$ satisfying

$$\varphi_{z,\nu}(B(z)\cap\mathbf{D}_\nu) \subset \mathbb{R}^n_+,\ \varphi_{z,\nu}(B(z)\cap\partial\mathbf{D}_\nu) \subset \partial\mathbb{R}^n_+,\ \varphi_{z,\nu}(B(z)\setminus\overline{\mathbf{D}_\nu}) \subset \mathbb{R}^n_-. \tag{3.21}$$

Here $\mathbb{R}^n_+ \equiv \{x = (x_1,\cdots,x_n) : x_n > 0\}, \partial\mathbb{R}^n_+ \equiv \{x : x_n = 0\}, \mathbb{R}^n_- \equiv \{x : x_n < 0\}$. Since $\partial\mathbf{D}_\nu$ is compact for each $\nu > 1$, there exist a finite number ℓ_ν of open balls $\{B(z_i)\}_{i=1}^{\ell_\nu}$ and one-to-one mappings $\{\varphi_{z_i,\nu}\}_{i=1}^{\ell_\nu}$ such that

$$\begin{cases} z_i \in \partial\mathbf{D}_\nu \text{ for } i \in \{1,\cdots,\ell_\nu\}, \\ (3.21) \text{ holds for each ball } B(z_i) \text{ and } i \in \{1,\cdots,\ell_\nu\}, \\ \partial\mathbf{D}_\nu \subset \bigcup_{i=1}^{\ell_\nu} B(z_i). \end{cases}$$

Since Y_m is smooth, it is possible to choose domains $\mathbf{D}_\nu$ for all $\nu > 1$ such that

$$\begin{cases} \text{the number } \ell_\nu \text{ is bounded above by a constant independent of } \nu, \\ \|\varphi_{z,\nu}\|_{C^{3,0}(\overline{B(z)})}, \|\varphi_{z,\nu}^{-1}\|_{C^{3,0}(\overline{\varphi_{z,\nu}(B(z))})} \le c_*, \text{where } c_* \text{ is independent of } \nu, z. \end{cases}$$

By assumption $x_0 = 0 \in \nu\partial Y_m$, we define $\widehat{\mathbb{K}}_{\omega^2,\nu}$ and ϕ in $\mathbb{R}^n$ by

$$\widehat{\mathbb{K}}_{\omega^2,\nu} \equiv \begin{cases} \omega^2 & \text{in } \mathbf{D}_\nu, \\ 1 & \text{elsewhere,} \end{cases} \qquad \phi \equiv \begin{cases} \Phi & \text{in } B_{1/2}(x_0), \\ 0 & \text{elsewhere.} \end{cases}$$

Let $\eta \in C_0^\infty(B_{1/2}(x_0))$ be a bell-shaped function satisfying $\eta \in [0,1]$, $\eta = 1$ in $B_{1/3}(x_0)$, $\|\nabla\eta\|_{W^{1,\infty}(B_{1/2}(x_0))} \le c$. Multiply (3.19) by η to get

$$\begin{cases} -\nabla\cdot\left(\widehat{\mathbb{K}}_{\omega^2,\nu}\nabla(\eta\phi) - \widehat{\mathbb{K}}_{\omega^2,\nu}\phi\nabla\eta\right) = -\widehat{\mathbb{K}}_{\omega^2,\nu}\nabla\phi\nabla\eta & \text{in } B_1(x_0), \\ \eta\phi = 0 & \text{on } \partial B_1(x_0). \end{cases}$$

Then we follow the argument of Step 1 of Lemma 3.5 to see that (3.20) holds. □

Let $\mathbb{X}_{\omega,1}^{(j)} \in \mathcal{H}_{per}^1(\mathbb{R}^n)$ for $\omega \in (0,1]$ be a function satisfying

$$\nabla\cdot(\mathbb{K}_{\omega^2,1}(\nabla\mathbb{X}_{\omega,1}^{(j)} + \vec{e}_j)) = 0 \quad \text{in } Y, \tag{3.22}$$

and let $\mathbb{X}_{0,1}^{(j)} \in \mathcal{H}_{per}^1(\mathcal{A}_f) \cap H^1(\mathcal{A}_m)$ be a function satisfying $\mathbb{X}_{0,1}^{(j)}(x) = 0$ in $\mathcal{A}_m$ and

$$\begin{cases} \nabla\cdot(\nabla\mathbb{X}_{0,1}^{(j)} + \vec{e}_j) = 0 & \text{in } Y_f, \\ (\nabla\mathbb{X}_{0,1}^{(j)} + \vec{e}_j)\cdot\vec{\mathbf{n}}_y = 0 & \text{on } \partial Y_m, \end{cases}$$

where $\vec{e}_j, j = 1, \cdots, n$ is the unit vector in the j-th coordinate direction, and $\vec{\mathbf{n}}_y$ is a unit normal vector on ∂Y_m. By Lax-Milgram Theorem [12], $\mathbb{X}_{\omega,1}^{(j)}$ for $\omega \in [0,1]$ and $j = 1, \cdots, n$ is uniquely solvable. By Theorem 6.30 [12] and $(3.5)_2$ of Lemma 3.5,

$$\|\mathbb{X}_{\omega,1}^{(j)}\|_{W^{2,p}(Y_f)\cap W^{2,p}(Y_m)} \le c(n, Y_m) \qquad \text{for } \omega \in [0,1],\ p \in [2,\infty). \tag{3.23}$$

Define $\mathbb{X}_{\omega,1} \equiv (\mathbb{X}_{\omega,1}^{(1)}, \cdots, \mathbb{X}_{\omega,1}^{(n)})$ and $\mathbb{X}_{\omega,\epsilon}(x) \equiv \epsilon\mathbb{X}_{\omega,1}(\frac{x}{\epsilon})$ for $\omega \in [0,1], \epsilon \in (0,1]$. Denote by Ξ_ω for $\omega \in [0,1]$ a $n \times n$ matrix function whose (i,j)-component is $\partial_i\mathbb{X}_{\omega,1}^{(j)}$. By remark in pages 17-19, 94-95 [14],

$$\mathcal{K}_\omega \equiv \int_{Y_f\cup Y_m} \mathbb{K}_{\omega^2,1}(I + \Xi_\omega)dy \qquad \text{for } \omega \in [0,1] \tag{3.24}$$

is a symmetric positive definite matrix dependent only on ω. Here I is the identity matrix. By (3.23), it is not difficult to see, for $\omega \in [0,1]$,

$$\begin{cases} d_3 I \le \mathcal{K}_\omega \le d_4 I \ \text{ where } d_3, d_4 \text{ are positive constants,} \\ \mathcal{K}_\omega \text{ is a continuous function of } \omega. \end{cases} \tag{3.25}$$

3.2. *Local Lipschitz and local L^p gradient estimates*

We have the following Lipschitz estimate:

Lemma 3.7. *Suppose $\omega, \epsilon \in (0,1]$, any solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Phi) = 0 & \text{in } B_1(0) \cap \Omega \\ \Phi = 0 & \text{on } B_1(0) \cap \partial\Omega \end{cases}$$

satisfies

$$\|\nabla \Phi\|_{L^\infty(B_{1/2}(0)\cap\Omega)} \le c \|\mathbf{K}_{\omega,\epsilon} \Phi\|_{L^2(B_1(0)\cap\Omega)}, \tag{3.26}$$

where c is a constant independent of ω, ϵ.

Proof of Lemma 3.7 is given in section 5. Next is the local L^p gradient estimate:

Lemma 3.8. *Let $\omega, \epsilon, r \in (0,1]$, $p \in (2,\infty)$, and either $B_{2r}(x_0) \subset \Omega$ or $x_0 \in \partial\Omega$. Any solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Phi) = 0 & \text{in } B_{2r}(x_0) \cap \Omega \\ \Phi = 0 & \text{on } B_{2r}(x_0) \cap \partial\Omega \end{cases} \tag{3.27}$$

satisfies

$$\left(\fint_{B_{r/2}(x_0)} |\mathbf{K}_{\omega,\epsilon} \nabla \Phi|^p \mathcal{X}_\Omega \; dx \right)^{1/p} \le c \left(\fint_{B_r(x_0)} |\mathbf{K}_{\omega,\epsilon} \nabla \Phi|^2 \mathcal{X}_\Omega \; dx \right)^{1/2}, \tag{3.28}$$

where c is a constant independent of ω, ϵ, r, x_0.

Proof. Let c denote a constant independent of ω, ϵ, r, x_0.

Case I: For $B_{2r}(x_0) \subset \Omega$ case. By translation, we move x_0 to the origin (that is, $x_0 = 0 \in \Omega$). Let $d \in \mathbb{R}$ and $\varphi(y) = \Phi(ry)$. By (3.27), we know

$$-\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon/r} \nabla(\varphi + d)) = 0 \quad \text{in } B_2(0).$$

If $\epsilon/r \le 1$ (resp. $\epsilon/r > 1$), Lemma 3.7 (resp. Theorem 9.11 [12] and Lemma 3.6) implies

$$\|\mathbf{K}_{\omega,\epsilon/r} \nabla \varphi\|_{L^p(B_{1/2}(0))} \le c \|\mathbf{K}_{\omega,\epsilon/r}(\varphi + d)\|_{L^2(B_1(0))},$$

where c is also independent of d. By Lemma 3.2,

$$\|\mathbf{K}_{\omega,\epsilon/r} \nabla \varphi\|_{L^p(B_{1/2}(0))} \le c \|\mathbf{K}_{\omega,\epsilon/r} \nabla \varphi\|_{L^2(B_1(0))}.$$

Which implies (3.28).

Case II: For $x_0 \in \partial\Omega$ case. Set $x_0 = 0 \in \partial\Omega$ by translation and set $\varphi(y) = \Phi(ry)$. By (3.3) and (3.27),

$$\begin{cases} -\nabla \cdot (\breve{\mathbf{K}}_{\omega^2,\epsilon,r} \nabla \varphi) = 0 & \text{in } B_2(0) \cap \partial\Omega/r, \\ \varphi = 0 & \text{on } B_2(0) \cap \partial\Omega/r. \end{cases} \tag{3.29}$$

If $\epsilon/r \le 1$ (resp. $\epsilon/r > 1$), Lemma 3.7 (resp. Theorem 9.13 [12]) implies that the φ in (3.29) satisfies

$$\|\breve{\mathbf{K}}_{\omega,\epsilon,r}\nabla\varphi\|_{L^p(B_{1/2}(0)\cap\Omega/r)} \le c\|\breve{\mathbf{K}}_{\omega,\epsilon,r}\varphi\|_{L^2(B_1(0)\cap\Omega/r)}.$$

By Lemma 3.3, we obtain

$$\|\breve{\mathbf{K}}_{\omega,\epsilon,r}\nabla\varphi\|_{L^p(B_{1/2}(0)\cap\Omega/r)} \le c\|\breve{\mathbf{K}}_{\omega,\epsilon,r}\nabla\varphi\|_{L^2(B_1(0)\cap\Omega/r)}.$$

Which implies (3.28). □

4. Proof of main results

Proof of Theorem 2.1: Suppose $\omega, \epsilon \in (0,1]$, let us find $U \in H^1(\Omega)$ satisfying

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}\nabla U + \mathbf{K}_{\omega,\epsilon}G) = 0 & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

By Lax-Milgram Theorem [12], U exists uniquely if $G \in L^2(\Omega)$. If we define $\mathbf{T} : L^2(\Omega) \to L^2(\Omega)$ by $\mathbf{T}G = \mathbf{K}_{\omega,\epsilon}\nabla U$, then $\mathbf{T}$ is a linear and bounded operator on $L^2(\Omega)$ by energy method. Lemma 3.8 implies that the operator $\mathbf{T}$ satisfies (1.9) of Theorem 1.3 [22] for any $G \in L^p(\Omega), p \in (2,\infty)$. So $\mathbf{T}$ is a bounded and linear operator in $L^p(\Omega)$ for $p \in (2,\infty)$ by Theorem 1.3 [22]. By Poincaré inequality [12] and Lemma 3.1, the solution of (4.1) satisfies

$$\begin{aligned} \|U\|_{L^p(\Omega_f^\epsilon)} &\le \|\mathcal{P}_\epsilon U|_{\Omega_f^\epsilon}\|_{L^p(\Omega)} \le c\|\nabla\mathcal{P}_\epsilon U|_{\Omega_f^\epsilon}\|_{L^p(\Omega)} \le c\|\nabla U\|_{L^p(\Omega_f^\epsilon)}, \\ \|U\|_{L^p(\Omega_m^\epsilon)} &\le \|U - \mathcal{P}_\epsilon U|_{\Omega_f^\epsilon}\|_{L^p(\Omega_m^\epsilon)} + \|\mathcal{P}_\epsilon U|_{\Omega_f^\epsilon}\|_{L^p(\Omega_m^\epsilon)} \\ &\le c\epsilon\|\nabla U - \nabla\mathcal{P}_\epsilon U|_{\Omega_f^\epsilon}\|_{L^p(\Omega_m^\epsilon)} + c\|U\|_{L^p(\Omega_f^\epsilon)}, \end{aligned}$$

where c is independent of ω, ϵ. Function $\mathcal{P}_\epsilon U|_{\Omega_f^\epsilon}$ above denotes the extension of $U|_{\Omega_f^\epsilon}$ on Ω. So we have

Lemma 4.1. *Under A1–A2, if $\omega, \epsilon \in (0,1]$, $p \in [2,\infty)$, and $G \in L^p(\Omega)$, then a $W^{1,p}(\Omega)$ solution U of (4.1) exists uniquely and*

$$\begin{cases} \|\mathbf{K}_{\omega/\epsilon,\epsilon}U, \mathbf{K}_{\omega,\epsilon}\nabla U\|_{L^p(\Omega)} \le c\|G\|_{L^p(\Omega)} & \text{for } \frac{\omega}{\epsilon} \le 1, \\ \|U, \mathbf{K}_{\omega,\epsilon}\nabla U\|_{L^p(\Omega)} \le c\|G\|_{L^p(\Omega)} & \text{for } \frac{\omega}{\epsilon} \ge 1, \end{cases}$$

where c is a constant independent of ω, ϵ.

By a duality argument, Poincaré inequality [12], and Lemmas 3.1, 4.1, we have

Lemma 4.2. *Under A1–A2, if $\omega, \epsilon \in (0,1]$, $p \in (1,2]$, and $G \in L^p(\Omega)$, then a $W^{1,p}(\Omega)$ solution U of (4.1) exists uniquely and*

$$\begin{cases} \|\mathbf{K}_{\omega/\epsilon,\epsilon}U, \mathbf{K}_{\omega,\epsilon}\nabla U\|_{L^p(\Omega)} \le c\|G\|_{L^p(\Omega)} & \text{for } \frac{\omega}{\epsilon} \le 1, \\ \|U, \mathbf{K}_{\omega,\epsilon}\nabla U\|_{L^p(\Omega)} \le c\|G\|_{L^p(\Omega)} & \text{for } \frac{\omega}{\epsilon} \ge 1, \end{cases}$$

where c is a constant independent of ω, ϵ.

If $\omega, \epsilon \in (0,1]$ and $G, F \in L^\infty(\Omega)$, then the $H^1(\Omega)$ solution of

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla U) = F & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

and the $H^1(\Omega)$ solution of

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \varphi - \mathbf{K}_{\omega,\epsilon} G) = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases} \tag{4.3}$$

exist uniquely by Lax-Milgram Theorem [12]. Lemma 4.1 and Lemma 4.2 imply that the solution of (4.3) satisfies

$$\begin{cases} \|\mathbf{K}_{\omega/\epsilon,\epsilon}\varphi, \mathbf{K}_{\omega,\epsilon}\nabla\varphi\|_{L^r(\Omega)} \le c\|G\|_{L^r(\Omega)} & \text{for } \frac{\omega}{\epsilon} \le 1, \\ \|\varphi, \mathbf{K}_{\omega,\epsilon}\nabla\varphi\|_{L^r(\Omega)} \le c\|G\|_{L^r(\Omega)} & \text{for } \frac{\omega}{\epsilon} \ge 1, \end{cases} \tag{4.4}$$

where $r \in (1,\infty)$ and c is a constant independent of ω, ϵ. Multiply (4.2) by the solution of (4.3), multiply (4.3) by the solution of (4.2), integrate by part, as well as employ (4.4), Lemma 3.1, and Hölder inequality to get

$$\begin{aligned} \int_\Omega \mathbf{K}_{\omega,\epsilon}\nabla U \; G dx &= \int_\Omega \varphi \; F dy = \int_\Omega \mathcal{P}_\epsilon \varphi|_{\Omega_f^\epsilon} F dy + \int_{\Omega_m^\epsilon} (\varphi - \mathcal{P}_\epsilon\varphi|_{\Omega_f^\epsilon}) F dy \\ &\le c\|G\|_{L^r(\Omega)}(\|F\|_{W^{-1,p}(\Omega)} + \omega^{-1}\|F\|_{W^{-1,p}(\Omega_m^\epsilon)}), \end{aligned}$$

where $\frac{1}{r} + \frac{1}{p} = 1$ and c is independent of ω, ϵ. Since $L^\infty(\Omega)$ is dense in $L^r(\Omega)$ for any $r \in (1,\infty)$, we obtain

$$\|\mathbf{K}_{\omega,\epsilon}\nabla U\|_{L^p(\Omega)} \le c(\|F\|_{W^{-1,p}(\Omega)} + \omega^{-1}\|F\|_{W^{-1,p}(\Omega_m^\epsilon)}),$$

where $\frac{1}{r} + \frac{1}{p} = 1$ and c is a constant independent of ω, ϵ. By Poincaré inequality [12] and Lemma 3.1, it is easy to see that

$$\begin{cases} \|\mathbf{K}_{\omega/\epsilon,\epsilon} U\|_{L^p(\Omega)} \le c(\|F\|_{W^{-1,p}(\Omega)} + \omega^{-1}\|F\|_{W^{-1,p}(\Omega_m^\epsilon)}) & \text{for } \frac{\omega}{\epsilon} \le 1, \\ \|U\|_{L^p(\Omega)} \le c(\|F\|_{W^{-1,p}(\Omega)} + \omega^{-1}\|F\|_{W^{-1,p}(\Omega_m^\epsilon)}) & \text{for } \frac{\omega}{\epsilon} \ge 1, \end{cases}$$

where $p \in (1,\infty)$ and c is a constant independent of ω, ϵ. Together with Lemma 4.1 and Lemma 4.2, we see that Theorem 2.1 holds for $G \in L^p(\Omega), F \in L^\infty(\Omega)$. If $G \in L^p(\Omega), F \in W^{-1,p}(\Omega)$, Theorem 2.1 can be proved by a limiting argument.

Proof of Theorem 2.2: Suppose $G, F \in L^p(\Omega_f^\epsilon)$ for $p \in (1,\infty)$, let us do zero extension for G, F. That is, set $\widetilde{G} \equiv \begin{cases} G & \text{on } \Omega_f^\epsilon \\ 0 & \text{on } \Omega_m^\epsilon \end{cases}$ and $\widetilde{F} \equiv \begin{cases} F & \text{on } \Omega_f^\epsilon \\ 0 & \text{on } \Omega_m^\epsilon \end{cases}$. Then $\widetilde{G}, \widetilde{F} \in L^p(\Omega)$. Let $U_{\omega,\epsilon}$ denote the solution of (1.1) with G, F replaced by $\widetilde{G}, \widetilde{F}$ above. By Theorem 2.1,

$$\begin{cases} \|\mathbf{K}_{\omega/\epsilon,\epsilon} U_{\omega,\epsilon}, \mathbf{K}_{\omega,\epsilon}\nabla U_{\omega,\epsilon}\|_{L^p(\Omega)} \le c(\|\widetilde{G}\|_{L^p(\Omega)} + \|\widetilde{F}\|_{W^{-1,p}(\Omega)}) & \text{for } \frac{\omega}{\epsilon} \le 1, \\ \|U_{\omega,\epsilon}, \mathbf{K}_{\omega,\epsilon}\nabla U_{\omega,\epsilon}\|_{L^p(\Omega)} \le c(\|\widetilde{G}\|_{L^p(\Omega)} + \|\widetilde{F}\|_{W^{-1,p}(\Omega)}) & \text{for } \frac{\omega}{\epsilon} \ge 1, \end{cases} \tag{4.5}$$

where c is independent of ω, ϵ. If we fix ϵ, then we see, by (4.5) and Lemma 3.1,

- There is a subsequence of $\{U_{\omega,\epsilon}\}$ (same notation for subsequence) such that $U_{\omega,\epsilon}|_{\Omega_f^\epsilon}$ converges weakly to U in $W^{1,p}(\Omega_f^\epsilon)$ as $\omega \to 0$.
- The limit function U satisfies (2.2) and (2.3).

So Theorem 2.2 is proved for $G, F \in L^p(\Omega_f^\epsilon), p \in (1, \infty)$. For general case, Theorem 2.2 can be proved by a limiting argument.

Based on the above uniform results (that is, Theorem 2.1, Theorem 2.2), we then apply semigroup theory to obtain the uniform estimates for parabolic equations.

Proof of Theorem 2.3: By A1–A2, A5 as well as by tracing the proof of Theorem 2.1 [23], we know the solution of (1.2) exists uniquely and satisfies, for $p \in (n, \infty)$,

$$\|U\|_{C^1([0,T];L^p(\Omega))} + \|U\|_{C([0,T];\mathbb{B}_p)} \le c\big(\|U_0\|_{\mathbb{B}_p} + \|F\|_{C^\sigma([0,T];L^p(\Omega))}\big),$$

where c is a constant independent of ω, ϵ. (1.2) can be written as, for fixed $t \in (0, T]$,

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla U(\cdot, t)) = F(\cdot, t) - \partial_t U(\cdot, t) & \text{in } \Omega, \\ U(\cdot, t) = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 2.1, we see, for $p \in (n, \infty)$,

$$\|\mathbf{K}_{\omega,\epsilon} \nabla U(\cdot, t)\|_{L^p(\Omega)} \le c\big(\|U_0\|_{\mathbb{B}_p} + \|F\|_{C^\sigma([0,T];L^p(\Omega))}\big),$$

where c is a constant independent of ω, ϵ. So Theorem 2.3 is proved.

Proof of Theorem 2.4: By A1–A2, A6 as well as by tracing the proof of Theorem 2.1 [23], we know that the solution of (2.4) exists uniquely and satisfies, for $p \in (n, \infty)$,

$$\|U\|_{C^1([0,T];L^p(\Omega_f^\epsilon))} + \|\Delta U\|_{C([0,T];L^p(\Omega_f^\epsilon))} \le c\big(\|U_0\|_{\mathbb{D}_p} + \|F\|_{C^\sigma([0,T];L^p(\Omega_f^\epsilon))}\big),$$

where c is a constant independent of ϵ. (2.4) can be written as, for fixed $t \in (0, T]$,

$$\begin{cases} -\Delta U(\cdot, t) = F(\cdot, t) - \partial_t U(\cdot, t) & \text{in } \Omega_f^\epsilon, \\ \nabla U(\cdot, t) \cdot \vec{\mathbf{n}}_\epsilon = 0 & \text{on } \partial\Omega_m^\epsilon, \\ U(\cdot, t) = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 2.2, we see, for $p \in (n, \infty)$,

$$\|\nabla U\|_{C([0,T];L^p(\Omega_f^\epsilon))} \le c\big(\|U_0\|_{\mathbb{D}_p} + \|F\|_{C^\sigma([0,T];L^p(\Omega_f^\epsilon))}\big),$$

where c is a constant independent of ϵ. So Theorem 2.4 is proved.

5. Local uniform Lipschitz estimate

We prove a local Lipschitz estimate, that is, Lemma 3.7. The idea of proof follows from the arguments in [4]. We first derive local Hölder estimate in subsection 5.1 and then derive local Lipschitz estimate in subsection 5.2.

5.1. *Hölder estimate*

An open set $\mathcal{O} \subset \mathbb{R}^n$ with boundary $\partial\mathcal{O}$ is said to satisfy a uniform exterior ball condition, if there exists a $r > 0$ with the following property: For each $x \in \partial\mathcal{O}$, there exists a point $y = y(x) \in \mathbb{R}^n$ such that $B_r(y) \setminus \{x\} \subset \mathbb{R}^n \setminus \mathcal{O}$ and $x \in \partial B_r(y)$. If $\mathcal{O} \subset \mathbb{R}^n$ is a nonempty open bounded Lipschitz set and satisfy a uniform exterior ball condition, then $\mathcal{O}$ is called a semiconvex domain.

In this subsection, we assume (1) A2 holds and (2) $\mathcal{O}$ is a semiconvex domain. If $0 \in \partial\mathcal{O}$, by rotation, there is a Lipschitz function $\widetilde{\Psi} : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\begin{cases} \widetilde{\Psi}(0) = 0, \\ B_1(0) \cap \mathcal{O}/r = B_1(0) \cap \{(x', x_n) \in \mathbb{R}^n : \ rx_n > \widetilde{\Psi}(rx')\} \quad \text{if } r \in (0, 1]. \end{cases} \tag{5.1}$$

If $r = 0$, we define $B_1(0) \cap \mathcal{O}/r \equiv B_1(0) \cap \{(x', x_n) \in \mathbb{R}^n : \ x_n > 0\}$. Similar to Ω_f^ϵ and Ω_m^ϵ, one can also define analogous $\mathcal{O}_f^\epsilon$ and $\mathcal{O}_m^\epsilon$. Define $\widetilde{\mathbf{K}}_{\nu,\epsilon,r}$ as

$$\widetilde{\mathbf{K}}_{\nu,\epsilon,r} \equiv \begin{cases} 1 & \text{in } \mathcal{O}_f^\epsilon/r \\ \nu & \text{in } \mathcal{O}_m^\epsilon/r \end{cases} \qquad \text{for } \nu, \epsilon, r \in (0, 1].$$

Lemma 5.1. *Let $\omega, \epsilon, r \in (0, 1]$, $\epsilon \le r$, either $B_2(0) \subset \mathcal{O}/r$ or $0 \in \partial\mathcal{O}/r$, and φ be a solution of*

$$\begin{cases} -\nabla \cdot (\widetilde{\mathbf{K}}_{\omega^2,\epsilon,r} \nabla\varphi) = 0 & \text{in } B_2(0) \cap \mathcal{O}/r, \\ \varphi = 0 & \text{on } B_2(0) \cap \partial\mathcal{O}/r. \end{cases}$$

There is a constant c independent of ω, ϵ, r such that

$$\|\varphi\|_{H^1(B_{1/2}(0)\cap\mathcal{O}/r)} \le c\|\widetilde{\mathbf{K}}_{\omega,\epsilon,r}\varphi\|_{L^2(B_2(0)\cap\mathcal{O}/r)}.$$

Proof. Let c denote a constant independent of ω, ϵ, r. By energy method,

$$\|\widetilde{\mathbf{K}}_{\omega,\epsilon,r}\nabla\varphi\|_{L^2(B_1(0)\cap\mathcal{O}/r)} \le c\|\widetilde{\mathbf{K}}_{\omega,\epsilon,r}\varphi\|_{L^2(B_2(0)\cap\mathcal{O}/r)}. \tag{5.2}$$

For any $z \in B_1(0) \cap \mathcal{O}/r$, we move z to the origin of the coordinate system by translation and define

$$\begin{cases} \hat{\mathbf{K}}(x) \equiv \widetilde{\mathbf{K}}_{\omega^2,\epsilon,r}(\frac{\epsilon}{r}x) \\ \hat{\varphi}(x) \equiv \varphi(\frac{\epsilon}{r}x) \end{cases} \quad \text{for } x \in B_1(z) \cap \mathcal{O}/\epsilon.$$

Then $\hat{\varphi}$ satisfies

$$\begin{cases} -\nabla \cdot (\hat{\mathbf{K}}\nabla\hat{\varphi}) = 0 & \text{in } B_1(z) \cap \mathcal{O}/\epsilon, \\ \hat{\varphi} = 0 & \text{on } B_1(z) \cap \partial\mathcal{O}/\epsilon. \end{cases}$$

By $(3.5)_1$,

$$\|\nabla\hat{\varphi}\|_{L^2(B_{1/2}(z)\cap\mathcal{O}/\epsilon)} \le c\|\hat{\varphi}\|_{L^2(B_1(z)\cap\mathcal{O}_f^\epsilon/\epsilon)}. \tag{5.3}$$

By Poincaré inequality [12], (5.3) implies

$$\|\nabla\varphi\|^2_{L^2(B_{\epsilon/2r}(z)\cap\mathcal{O}/r)} \le c\|\nabla\varphi\|^2_{L^2(B_{\epsilon/r}(z)\cap\mathcal{O}_f^\epsilon/r)}. \tag{5.4}$$

By covering $B_1(0) \cap \mathcal{O}/r$ with a finite number of balls of radius $\epsilon/2r$, (5.4) implies

$$\|\nabla\varphi\|_{L^2(B_{1/2}(0)\cap\mathcal{O}/r)} \leq c\|\nabla\varphi\|_{L^2(B_1(0)\cap\mathcal{O}_f^\epsilon/r)}.$$

Together with (5.2) and Poincaré inequality [12], we prove the lemma. □

The rest of this subsection is to prove the following lemma.

Lemma 5.2. *Suppose $\delta > 0$ and $\omega, \epsilon \in (0, 1]$, any solution of*

$$\begin{cases} -\nabla \cdot (\widetilde{\mathbf{K}}_{\omega^2,\epsilon,1}\nabla\Phi) = 0 & \text{in } B_1(0) \cap \mathcal{O} \\ \Phi = 0 & \text{on } B_1(0) \cap \partial\mathcal{O} \end{cases}$$

satisfies

$$\|\Phi\|_{C^{0,\mu}(\overline{B_{1/2}(0)\cap\mathcal{O}})} \leq c\|\widetilde{\mathbf{K}}_{\omega,\epsilon,1}\Phi\|_{L^2(B_1(0)\cap\mathcal{O})}, \tag{5.5}$$

where $\mu \equiv \frac{\delta}{n+\delta}$ and c is a constant independent of ω, ϵ.

The interior estimate of (5.5) is given in subsection 5.1.1 and the boundary estimate of (5.5) is in subsection 5.1.2.

5.1.1. *Interior Hölder estimate*

Assume $B_1(0) \subset \mathcal{O}$.

Lemma 5.3. *For any $\delta > 0$, there are constants $\theta_1, \theta_2 \in (0, 1)$ with $\theta_1 < \theta_2^2$ and a constant $\epsilon_0 \in (0, 1)$ (depending on δ, θ_2, Y_f) such that if*

$$\begin{cases} -\nabla \cdot (\widetilde{\mathbf{K}}_{\omega^2,\nu,1}\nabla\varphi) = 0 & \text{in } B_1(0), \\ \|\widetilde{\mathbf{K}}_{\omega,\nu,1}\varphi\|_{L^2(B_1(0))} \leq 1, \end{cases} \tag{5.6}$$

then, for any $\omega \in (0, 1]$, $\nu \in (0, \epsilon_0]$, and $\theta \in [\theta_1, \theta_2]$,

$$\fint_{B_\theta(0)} \left|\varphi - (\varphi)_{B_\theta(0)}\right|^2 dx \leq \theta^{2\mu}, \tag{5.7}$$

where $\mu \equiv \frac{\delta}{n+\delta}$.

Proof. Consider the following problem

$$-\nabla \cdot (\mathcal{K}_{\omega_*}\nabla\varphi_*) = 0 \qquad \text{in } B_{2/3}(0), \tag{5.8}$$

where $\mathcal{K}_{\omega_*}$ for $\omega_* \in [0, 1]$ is from (3.24). By Theorem 1.2 in page 70 [11] and (3.25), there is a small $\theta \in (0, 2/3)$ such that

$$\fint_{B_\theta(0)} \left|\varphi_* - (\varphi_*)_{B_\theta(0)}\right|^2 dx \leq \theta^{2\mu'} \fint_{B_{2/3}(0)} |\varphi_*|^2 dx, \tag{5.9}$$

for some $\mu' \in (\mu, 1)$. We choose $\theta_1, \theta_2 \in (0, 2/3)$ such that $\theta_1 < \theta_2^2$ and (5.9) holds if $\theta \in [\theta_1, \theta_2]$. Now we claim (5.7). If not, there is a sequence $\{\omega_\nu, \theta_\nu, \varphi_\nu\}$ satisfying (5.6) and

$$\begin{cases} \omega_\nu \to \omega_* \in [0,1] \\ \theta_\nu \to \theta_* \in [\theta_1, \theta_2] \\ \displaystyle\fint_{B_{\theta_\nu}(0)} \left|\varphi_\nu - (\varphi_\nu)_{B_{\theta_\nu}(0)}\right|^2 dx > \theta_\nu^{2\mu} \end{cases} \quad \text{as } \nu \to 0. \tag{5.10}$$

By Lemma 5.1 and tracing the proof of Theorem 2.3 [3], there is a subsequence (same notation for subsequence) such that

$$\begin{cases} \varphi_\nu \to \varphi_* & \text{in } L^2(B_{2/3}(0)) \text{ strongly} \\ \widetilde{\mathbf{K}}_{\omega_\nu^2,\nu,1} \nabla\varphi_\nu \to \mathcal{K}_{\omega_*}\nabla\varphi_* & \text{in } L^2(B_{2/3}(0)) \text{ weakly} \end{cases} \quad \text{as } \nu \to 0. \tag{5.11}$$

Also the φ_* above is a solution of (5.8). By (5.9)–(5.11), we conclude

$$\begin{aligned} \theta_*^{2\mu} = \lim_{\nu\to 0} \theta_\nu^{2\mu} &\le \lim_{\nu\to 0} \fint_{B_{\theta_\nu}(0)} \left|\varphi_\nu - (\varphi_\nu)_{B_{\theta_\nu}(0)}\right|^2 dx \\ &= \fint_{B_{\theta_*}(0)} |\varphi_* - (\varphi_*)_{B_{\theta_*}(0)}|^2 dx \le \theta_*^{2\mu'} \fint_{B_{2/3}(0)} |\varphi_*|^2 dx. \end{aligned} \tag{5.12}$$

But (5.12) is impossible if θ_2 is small enough. Therefore, there is a ϵ_0 such that (5.7) holds for $\nu \le \epsilon_0$. □

Lemma 5.4. *For any $\delta > 0$, there are constants $\theta_1, \theta_2 \in (0,1)$ with $\theta_1 < \theta_2^2$ and a constant $\epsilon_0 \in (0,1)$ (depending on δ, θ_2, Y_f) such that if*

$$-\nabla \cdot (\widetilde{\mathbf{K}}_{\omega^2,\epsilon,1} \nabla\Phi) = 0 \qquad \text{in } B_1(0), \tag{5.13}$$

then, for any $\omega \in (0,1]$, $\epsilon \in (0, \epsilon_0]$, $\theta \in [\theta_1, \theta_2]$, and k satisfying $\epsilon/\theta^k \le \epsilon_0$,

$$\fint_{B_{\theta^k}(0)} \left|\Phi - (\Phi)_{B_{\theta^k}(0)}\right|^2 dx \le \theta^{2k\mu} |J_{\omega,\epsilon}|^2, \tag{5.14}$$

where $\mu \equiv \frac{\delta}{n+\delta}$ and $J_{\omega,\epsilon} \equiv \|\widetilde{\mathbf{K}}_{\omega,\epsilon,1}\Phi\|_{L^2(B_1(0))}$.

Proof. The proof is done by induction on k. For $k = 1$, we define $\varphi \equiv \Phi/J_{\omega,\epsilon}$ and use Lemma 5.3 with $\nu = \epsilon$ to obtain (5.14). Suppose (5.14) holds for some k satisfying $\epsilon/\theta^k \le \epsilon_0$, then we define

$$\varphi(x) \equiv J_{\omega,\epsilon}^{-1}\, \theta^{-k\mu} \big(\Phi(\theta^k x) - (\Phi)_{B_{\theta^k}(0)}\big) \qquad \text{in } B_1(0).$$

Then φ satisfies (5.6) with $\nu = \epsilon/\theta^k$. By changing variable and employing Lemma 5.3, we obtain (5.14) with $k+1$ in place of k. □

Lemma 5.5. *For any $\delta > 0$, there is a constant $\epsilon_* \in (0,1)$ (depending on δ, Y_f) such that if $\omega \in (0,1]$ and $\epsilon \in (0, \epsilon_*]$, then any solution of (5.13) satisfies*

$$[\Phi]_{C^{0,\mu}(\overline{B_{1/2}(0)})} \le c \|\widetilde{\mathbf{K}}_{\omega,\epsilon,1}\Phi\|_{L^2(B_1(0))}, \tag{5.15}$$

where c is a constant independent of ω, ϵ. See Lemma 5.4 for μ.

Proof. Let $\theta_1, \theta_2, \epsilon_0, J_{\omega,\epsilon}$ be same as those in Lemma 5.4 and define $\epsilon_* \equiv \epsilon_0\theta_2/2$. Denote by c a constant independent of ω, ϵ. Because of $\theta_1 < \theta_2^2$, for any $r \in [\epsilon/\epsilon_0, \theta_2]$, there are $\theta \in [\theta_1, \theta_2]$ and $k \in \mathbb{N}$ satisfying $r = \theta^k$. Lemma 5.4 implies

$$\fint_{B_r(0)} \left|\Phi - (\Phi)_{B_r(0)}\right|^2 dy \leq cr^{2\mu}|J_{\omega,\epsilon}|^2 \quad \text{for } r \in [\epsilon/\epsilon_0, \theta_2]. \tag{5.16}$$

Take $r = \frac{2\epsilon}{\epsilon_0}$ and define

$$\varphi(y) \equiv J_{\omega,\epsilon}^{-1}\epsilon^{-\mu}\big(\Phi(\epsilon y) - (\Phi)_{B_{2\epsilon/\epsilon_0}(0)}\big) \qquad \text{in } B_{2/\epsilon_0}(0).$$

Then φ satisfies

$$\begin{cases} -\nabla \cdot (\widetilde{\mathbf{K}}_{\omega^2,1,1}\nabla\varphi) = 0 & \text{in } B_{2/\epsilon_0}(0), \\ \|\varphi\|_{L^2(B_{2/\epsilon_0}(0))} \leq c. \end{cases}$$

$(3.5)_1$ of Lemma 3.5 implies $[\varphi]_{C^{0,\mu}(\overline{B_{1/\epsilon_0}(0)})} \leq c$. Together with (5.16), then (5.16) holds for $r \in (0, \theta_2)$. Next we shift the origin of the coordinate system to any point $z \in B_{1/2}(0)$ and repeat above argument to see that (5.16) with 0 replaced by any $z \in B_{1/2}(0)$ also holds for $r \in (0, \theta_2)$. By Theorem 1.2 in page 70 [11], we prove the lemma. □

Remark 5.1. Let ϵ_* be same as that in Lemma 5.5. By $(3.5)_1$ of Lemma 3.5, we know that if $\omega \in (0, 1]$ and $\epsilon \in [\epsilon_*, 1]$, any solution of (5.13) satisfies (5.15). Together with Lemma 5.5, we conclude that any solution of (5.13) satisfies (5.15) if $\omega, \epsilon \in (0, 1]$.

5.1.2. *Boundary Hölder estimate*

We assume $0 \in \partial\mathcal{O}$.

Lemma 5.6. *For any $\delta > 0$, there are constants $\breve{\theta}_1, \breve{\theta}_2 \in (0, 1)$ with $\breve{\theta}_1 < \breve{\theta}_2^2$, and a constant $\breve{\epsilon}_0 \in (0, 1)$ (depending on $\delta, \breve{\theta}_2, Y_f, \|\nabla\widetilde{\Psi}\|_{L^\infty(\mathbb{R}^{n-1})}$) such that if*

$$\begin{cases} -\nabla \cdot (\widetilde{\mathbf{K}}_{\omega^2,\epsilon,r}\nabla\varphi) = 0 & \text{in } B_1(0) \cap \mathcal{O}/r, \\ \varphi = 0 & \text{on } B_1(0) \cap \partial\mathcal{O}/r, \\ \|\widetilde{\mathbf{K}}_{\omega,\epsilon,r}\varphi\|_{L^2(B_1(0)\cap\mathcal{O}/r)} \leq 1, \end{cases} \tag{5.17}$$

then, for any $\omega, \epsilon, r \in (0, 1]$, $\epsilon/r \leq \breve{\epsilon}_0$, and $\theta \in [\breve{\theta}_1, \breve{\theta}_2]$,

$$\fint_{B_\theta(0)\cap\mathcal{O}/r} |\varphi|^2 dx \leq \theta^{2\mu}, \tag{5.18}$$

where $\mu \equiv \frac{\delta}{n+\delta}$. See (5.1) for $\widetilde{\Psi}$.

Proof. Consider the following problem

$$\begin{cases} -\nabla \cdot (\mathcal{K}_{\omega_*} \nabla \varphi_*) = 0 & \text{in } B_{2/3}(0) \cap \mathcal{O}/r_*, \\ \varphi_* = 0 & \text{on } B_{2/3}(0) \cap \partial\mathcal{O}/r_*, \end{cases} \tag{5.19}$$

where $\mathcal{K}_{\omega_*}$ is from (3.24) and $\omega_*, r_* \in [0,1]$. Note $\mathcal{O}/r_*$ is a semiconvex domain (see the beginning of subsection 5.1) as well as $\mathcal{K}_{\omega_*}$ is a symmetric positive definite matrix and can be diagonalizable. By Theorem 4.5 [20], Theorem 7.26 [12], and (3.25), there is a small $\theta \in (0, 2/3)$ such that

$$⨍_{B_\theta(0)\cap\mathcal{O}/r_*} |\varphi_*|^2 dx \leq \theta^{2\mu'} ⨍_{B_{2/3}(0)\cap\mathcal{O}/r_*} |\varphi_*|^2 dx, \tag{5.20}$$

where $\mu' \in (\mu, 1)$. We choose $\breve{\theta}_1, \breve{\theta}_2 \in (0, 2/3)$ such that $\breve{\theta}_1 < \breve{\theta}_2^2$ and (5.20) holds if $\theta \in [\breve{\theta}_1, \breve{\theta}_2]$. Now we claim (5.18). If not, there is a sequence $\{\omega_\epsilon, r_\epsilon, \theta_\epsilon, \varphi_\epsilon\}$ satisfying (5.17) and

$$\begin{cases} \omega_\epsilon, r_\epsilon \to \omega_*, r_* \in [0,1] \\ \theta_\epsilon \to \theta_* \in [\breve{\theta}_1, \breve{\theta}_2] \\ ⨍_{B_{\theta_\epsilon}(0)\cap\mathcal{O}/r_\epsilon} |\varphi_\epsilon|^2 dx > \theta_\epsilon^{2\mu} \end{cases} \quad \text{as } \epsilon/r_\epsilon \to 0. \tag{5.21}$$

By Lemma 5.1 and tracing the proof of Theorem 2.3 [3], there is a subsequence (same notation for subsequence) such that

$$\begin{cases} \varphi_\epsilon \to \varphi_* & \text{in } L^2(B_{2/3}(0) \cap \mathcal{O}/r_*) \text{ strongly} \\ \widetilde{\mathbf{K}}_{\omega_\epsilon^2, \epsilon, r_\epsilon} \nabla \varphi_\epsilon \to \mathcal{K}_{\omega_*} \nabla \varphi_* & \text{in } L^2(B_{2/3}(0) \cap \mathcal{O}/r_*) \text{ weakly} \end{cases} \quad \text{as } \epsilon/r_\epsilon \to 0. \tag{5.22}$$

Also the φ_* above is a solution of (5.19). By (5.20)–(5.22), we conclude

$$\begin{aligned} \theta_*^{2\mu} &= \lim_{\epsilon/r_\epsilon \to 0} \theta_\epsilon^{2\mu} \leq \lim_{\epsilon/r_\epsilon \to 0} ⨍_{B_{\theta_\epsilon}(0)\cap\mathcal{O}/r_\epsilon} |\varphi_\epsilon|^2 dx \\ &= ⨍_{B_{\theta_*}(0)\cap\mathcal{O}/r_*} |\varphi_*|^2 dx \leq \theta_*^{2\mu'} ⨍_{B_{2/3}(0)\cap\mathcal{O}/r_*} |\varphi_*|^2 dx. \end{aligned} \tag{5.23}$$

But (5.23) is impossible if $\breve{\theta}_2$ is small enough. So there is a $\breve{\epsilon}_0$ such that (5.18) holds for $\epsilon/r < \breve{\epsilon}_0$. □

Lemma 5.7. *For any $\delta > 0$, there are constants $\breve{\theta}_1, \breve{\theta}_2 \in (0,1)$ with $\breve{\theta}_1 < \breve{\theta}_2^2$, and a constant $\breve{\epsilon}_0 \in (0,1)$ (depending on $\delta, \breve{\theta}_2, Y_f, \|\nabla\widetilde{\Psi}\|_{L^\infty(\mathbb{R}^{n-1})}$) such that if*

$$\begin{cases} -\nabla \cdot (\widetilde{\mathbf{K}}_{\omega^2,\epsilon,1} \nabla \Phi) = 0 & \text{in } B_1(0) \cap \mathcal{O}, \\ \Phi = 0 & \text{on } B_1(0) \cap \partial\mathcal{O}, \end{cases} \tag{5.24}$$

then, for any $\omega \in (0,1]$, $\epsilon \in (0, \breve{\epsilon}_0]$, $\theta \in [\breve{\theta}_1, \breve{\theta}_2]$, and k satisfying $\epsilon/\theta^k \leq \breve{\epsilon}_0$,

$$⨍_{B_{\theta^k}(0)\cap\mathcal{O}} |\Phi|^2 dx \leq \theta^{2k\mu} |\breve{J}_{\omega,\epsilon}|^2, \tag{5.25}$$

where $\mu \equiv \frac{\delta}{n+\delta}$ *and* $\breve{J}_{\omega,\epsilon} \equiv \|\widetilde{\mathbf{K}}_{\omega,\epsilon,1}\Phi\|_{L^2(B_1(0)\cap\mathcal{O})}$.

Proof. The proof is done by induction on k. For $k=1$, we set $\varphi \equiv \Phi/\breve{J}_{\omega,\epsilon}$. Then (5.25) is deduced from Lemma 5.6 with $r=1$. Suppose (5.25) holds for some k satisfying $\epsilon/\theta^k \le \breve{\epsilon}_0$, then we define

$$\varphi(x) \equiv \breve{J}_{\omega,\epsilon}^{-1}\theta^{-k\mu}\Phi(\theta^k x) \qquad \text{in } B_1(0)\cap\mathcal{O}/\theta^k.$$

Then φ satisfies (5.17) with $r=\theta^k$. By changing variable and employing Lemma 5.6 with $r=\theta^k$, we obtain (5.25) with $k+1$ in place of k. □

Lemma 5.8. *For any* $\delta>0$*, there is a constant* $\breve{\epsilon}_* \in (0,1)$ *(depending on* δ*,* Y_f*,* $\|\nabla\widetilde{\Psi}\|_{L^\infty(\mathbb{R}^{n-1})}$*) such that if* $\omega\in(0,1]$ *and* $\epsilon\in(0,\breve{\epsilon}_*]$*, then any solution of (5.24) satisfies*

$$[\Phi]_{C^{0,\mu}(\overline{B_{1/2}(0)\cap\mathcal{O}})} \le c\|\widetilde{\mathbf{K}}_{\omega,\epsilon,1}\Phi\|_{L^2(B_1(0)\cap\mathcal{O})}, \tag{5.26}$$

where c *is a constant independent of* ω,ϵ*. See Lemma 5.7 for* μ*.*

Proof. Let $\breve{\theta}_1,\breve{\theta}_2,\breve{\epsilon}_0,\breve{J}_{\omega,\epsilon}$ be those in Lemma 5.7 and define $\breve{\epsilon}_*\equiv\min\{\breve{\epsilon}_0\breve{\theta}_2/3,\epsilon_*\}$ where ϵ_* is the one in Lemma 5.5. Denote by c a constant independent of ω,ϵ. For any $x\in B_{\breve{\theta}_2/3}(0)\cap\mathcal{O}$, define $\xi(x)\equiv|x-x_0|$ where $x_0\in\partial\mathcal{O}$ satisfying $|x-x_0| = \min_{y\in\partial\mathcal{O}}|x-y|$. Then we have either case (1) $\xi(x)>\frac{2\epsilon}{3\breve{\epsilon}_0}$ or case (2) $\xi(x)\le\frac{2\epsilon}{3\breve{\epsilon}_0}$.

Let us consider case (1). Because of $\breve{\theta}_1<\breve{\theta}_2^2$, for any $r\in[\epsilon/\breve{\epsilon}_0,\breve{\theta}_2]$, there are $\theta\in[\breve{\theta}_1,\breve{\theta}_2]$ and $k\in\mathbb{N}$ satisfying $r=\theta^k$. Since $\xi(x)\in[\frac{2\epsilon}{3\breve{\epsilon}_0},\frac{\breve{\theta}_2}{3}]$, by Lemma 5.7,

$$\fint_{B_r(x_0)\cap\mathcal{O}}|\Phi|^2dy \le r^{2\mu}|\breve{J}_{\omega,\epsilon}|^2 \qquad \text{for } r\in[\tfrac{3}{2}\xi(x),\breve{\theta}_2].$$

So, for $s\in[\frac{\xi(x)}{2},\frac{\breve{\theta}_2}{3}]$,

$$\fint_{B_s(x)\cap\mathcal{O}}\left|\Phi-(\Phi)_{B_s(x)\cap\mathcal{O}}\right|^2dy \le cs^{2\mu}|\breve{J}_{\omega,\epsilon}|^2. \tag{5.27}$$

Next we move the origin of the coordinate system to x and define

$$\varphi(y)\equiv\breve{J}_{\omega,\epsilon}^{-1}\xi^{-\mu}(x)\big(\Phi(\xi(x)y)-(\Phi)_{B_{\xi(x)}(x)}\big) \qquad \text{in } B_1(x).$$

Then φ satisfies

$$-\nabla\cdot(\widetilde{\mathbf{K}}_{\omega^2,\epsilon/\xi(x),1}\nabla\varphi)=0 \qquad \text{in } B_1(x). \tag{5.28}$$

Take $s=\xi(x)<1$ in (5.27) to see $\|\varphi\|_{L^2(B_1(x))}\le c$. Apply Remark 5.1 to (5.28) to obtain $[\varphi]_{C^{0,\mu}(\overline{B_{1/2}(x)})}\le c$. Which implies

$$\fint_{B_s(x)}\left|\Phi-(\Phi)_{B_s(x)}\right|^2dy\le cs^{2\mu}|\breve{J}_{\omega,\epsilon}|^2 \quad \text{for } s<\tfrac{\xi(x)}{2}. \tag{5.29}$$

Next we consider case (2). Because of $\breve{\theta}_1 < \breve{\theta}_2^2$, for any $r \in [\epsilon/\breve{\epsilon}_0, \breve{\theta}_2]$, there are $\theta \in [\breve{\theta}_1, \breve{\theta}_2]$ and $k \in \mathbb{N}$ satisfying $r = \theta^k$. By Lemma 5.7,

$$\fint_{B_r(x_0)\cap\mathcal{O}} |\Phi|^2 dy \le r^{2\mu} |\breve{J}_{\omega,\epsilon}|^2 \qquad \text{for } r \in [\epsilon/\breve{\epsilon}_0, \breve{\theta}_2]. \tag{5.30}$$

This implies, for $s \in [\frac{\epsilon}{3\breve{\epsilon}_0}, \frac{\breve{\theta}_2}{3}]$,

$$\fint_{B_s(x)\cap\mathcal{O}} \left|\Phi - (\Phi)_{B_s(x)\cap\mathcal{O}}\right|^2 dy \le cs^{2\mu} |\breve{J}_{\omega,\epsilon}|^2. \tag{5.31}$$

Again we move the origin of the coordinate system to x and define

$$\varphi(y) \equiv \breve{J}_{\omega,\epsilon}^{-1} \epsilon^{-\mu} \big(\Phi(\epsilon y) - (\Phi)_{B_{\epsilon/\breve{\epsilon}_0}(x)\cap\mathcal{O}}\big) \qquad \text{in } B_{1/\breve{\epsilon}_0}(x) \cap \mathcal{O}/\epsilon.$$

Then φ satisfies

$$\begin{cases} -\nabla \cdot (\widetilde{\mathbf{K}}_{\omega^2,\epsilon,\epsilon} \nabla\varphi) = 0 & \text{in } B_{1/\breve{\epsilon}_0}(x) \cap \mathcal{O}/\epsilon, \\ \varphi = -\breve{J}_{\omega,\epsilon}^{-1} \epsilon^{-\mu} (\Phi)_{B_{\epsilon/\breve{\epsilon}_0}(x)\cap\mathcal{O}} & \text{on } B_{1/\breve{\epsilon}_0}(x) \cap \partial\mathcal{O}/\epsilon. \end{cases}$$

Let us take $s = \epsilon/\breve{\epsilon}_0$ in (5.31) to see $\|\varphi\|_{L^2(B_{1/\breve{\epsilon}_0}(x)\cap\mathcal{O}/\epsilon)} \le c$ and take $s = \epsilon/\breve{\epsilon}_0$ in (5.30) to see $|\breve{J}_{\omega,\epsilon}^{-1} \epsilon^{-\mu} (\Phi)_{B_{\epsilon/\breve{\epsilon}_0}(x)\cap\mathcal{O}}| \le c$. By $(3.5)_1$ of Lemma 3.5,

$$[\varphi]_{C^{0,\mu}(\overline{B_{1/2\breve{\epsilon}_0}(x)\cap\mathcal{O}/\epsilon})} \le c. \tag{5.32}$$

(5.32) implies that (5.31) holds for $s \le \frac{\epsilon}{2\epsilon_0}$.

The Hölder estimate of Φ follows from (5.27), (5.29), (5.31), (5.32), and Theorem 1.2 in page 70 [11]. ◻

Remark 5.2. Let $\breve{\epsilon}_*$ be same as that in Lemma 5.8. By $(3.5)_1$ of Lemma 3.5, we know that if $\omega \in (0, 1]$ and $\epsilon \in [\breve{\epsilon}_*, 1]$, any solution of (5.24) satisfies (5.26). Together with Lemma 5.8, any solution of (5.24) satisfies (5.26) if $\omega, \epsilon \in (0, 1]$.

Remark 5.1, Remark 5.2, and maximal principle imply Lemma 5.2.

5.2. *Lipschitz estimate*

A1–A2 are assumed and we show Lemma 3.7. The interior estimate of (3.26) is in subsection 5.2.1 and the boundary estimate of (3.26) is in subsection 5.2.2.

5.2.1. *Interior gradient estimate*

We assume $B_1(0) \subset \Omega$.

Lemma 5.9. *There exist $\hat{\theta}, \hat{\epsilon}_0 \in (0, 1)$ such that if*

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\nu} \nabla\varphi) = 0 & \text{in } B_1(0), \\ \|\mathbf{K}_{\omega,\nu}\varphi\|_{L^2(B_1(0))} \le 1, \end{cases} \tag{5.33}$$

then, for any $\omega \in (0,1]$ *and* $\nu \in (0, \hat{\epsilon}_0)$,

$$\sup_{B_{\hat{\theta}}(0)} |\varphi(x) - \varphi(0) - (x + \mathbb{X}_{\omega,\nu}(x))\mathbf{b}_{\omega,\nu}| \le \hat{\theta}^{4/3}, \tag{5.34}$$

where $\mathbf{b}_{\omega,\nu} \equiv \mathcal{K}_\omega^{-1} \fint_{B_{\hat{\theta}}(0)} \mathbf{K}_{\omega^2,\nu} \nabla\varphi \, dx$ *and* $\mathcal{K}_\omega^{-1}$ *is the inverse matrix of* $\mathcal{K}_\omega$. *See (3.23)–(3.24) for* $\mathbb{X}_{\omega,\nu}, \mathcal{K}_\omega$.

Proof. Assume $-\nabla \cdot (\mathcal{K}_{\omega_*} \nabla \varphi_*) = 0$ in $B_{2/3}(0)$, where $\mathcal{K}_{\omega_*}$ for $\omega_* \in [0,1]$ is from (3.24). If θ is small enough, by (3.25) and Taylor expansion,

$$\sup_{B_\theta(0)} \left|\varphi_*(x) - \varphi_*(0) - x(\nabla\varphi_*)_{B_\theta(0)}\right| \le \theta^{3/2} \|\varphi_*\|_{L^2(B_{2/3}(0))}. \tag{5.35}$$

Fix a small $\hat{\theta}$ so that (5.35) holds and we prove (5.34) by contradiction. If not, there is a sequence $\{\omega_\nu, \varphi_\nu\}$ satisfying (5.33) and, as $\nu \to 0$,

$$\begin{cases} \omega_\nu \to \omega_* \in [0,1], \\ \sup\limits_{B_{\hat{\theta}}(0)} |\varphi_\nu(x) - \varphi_\nu(0) - (x + \mathbb{X}_{\omega,\nu}(x))\mathbf{b}_{\omega,\nu}| > \hat{\theta}^{4/3}. \end{cases} \tag{5.36}$$

By Lemma 5.1 and Lemma 5.2 and by tracing the proof of Theorem 2.3 [3], there is a subsequence (same notation for subsequence) such that

$$\begin{cases} \varphi_\nu \to \varphi_* & \text{in } C(\overline{B_{2/3}(0)}) \\ \mathbf{K}_{\omega_\nu^2,\nu} \nabla\varphi_\nu \to \mathcal{K}_{\omega_*} \nabla \varphi_* & \text{in } L^2(B_{2/3}(0)) \text{ weakly} \end{cases} \quad \text{as } \nu \to 0. \tag{5.37}$$

(5.37) implies that φ_* satisfies $-\nabla \cdot (\mathcal{K}_{\omega_*} \nabla\varphi_*) = 0$ in $B_{2/3}(0)$. Together with (5.35), (5.36), and (5.37), we get contradiction if $\hat{\theta}$ is small. So we prove (5.34). □

Lemma 5.10. *There exist* $\hat{\theta}, \hat{\epsilon}_0 \in (0,1)$ *such that if* Φ *satisfies*

$$-\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla\Phi) = 0 \quad \textit{in } B_1(0), \tag{5.38}$$

then, for any $\omega \in (0,1]$, $\epsilon \in (0, \hat{\epsilon}_0)$, *and* k *satisfying* $\epsilon/\hat{\theta}^k \le \hat{\epsilon}_0$, *there are constants* $\mathbf{a}_k^{\omega,\epsilon}$, $\mathbf{b}_k^{\omega,\epsilon}$ *satisfying*

$$\begin{cases} |\mathbf{a}_k^{\omega,\epsilon}| + |\mathbf{b}_k^{\omega,\epsilon}| \le c\hat{J}_{\omega,\epsilon}, \\ \sup\limits_{B_{\hat{\theta}^k}(0)} |\Phi(x) - \Phi(0) - \epsilon\mathbf{a}_k^{\omega,\epsilon} - (x + \mathbb{X}_{\omega,\epsilon}(x))\mathbf{b}_k^{\omega,\epsilon}| \le \hat{\theta}^{4k/3} \hat{J}_{\omega,\epsilon}, \end{cases} \tag{5.39}$$

where $\hat{J}_{\omega,\epsilon} \equiv \|\mathbf{K}_{\omega,\epsilon}\Phi\|_{L^2(B_1(0))}$ *and* c *is independent of* ω, ϵ.

Proof. If $\varphi \equiv \Phi/\hat{J}_{\omega,\epsilon}$, then it satisfies (5.33) with $\nu = \epsilon$. By Lemma 5.9, we obtain (5.39) for $k = 1$ where $\mathbf{a}_1^{\omega,\epsilon} = 0$, $\mathbf{b}_1^{\omega,\epsilon} = \mathcal{K}_\omega^{-1} \fint_{B_{\hat{\theta}}(0)} \mathbf{K}_{\omega^2,\epsilon} \nabla\Phi \, dx$. If (5.39) holds for some k satisfying $\epsilon/\hat{\theta}^k \le \hat{\epsilon}_0$, we define

$$\varphi(x) \equiv \frac{\Phi(\hat{\theta}^k x) - \Phi(0) - \epsilon\mathbf{a}_k^{\omega,\epsilon} - \big(\hat{\theta}^k x + \mathbb{X}_{\omega,\epsilon}(\hat{\theta}^k x)\big)\mathbf{b}_k^{\omega,\epsilon}}{\hat{\theta}^{4k/3} \hat{J}_{\omega,\epsilon}} \quad \text{in } B_1(0).$$

By induction and (3.22), we see φ satisfies (5.33) with $\nu = \epsilon/\hat{\theta}^k$. Apply Lemma 5.9 to obtain

$$\sup_{B_{\hat{\theta}}(0)} \left|\varphi(x) - \varphi(0) - \big(x + \mathbb{X}_{\omega,\epsilon/\hat{\theta}^k}(x)\big)\mathbf{b}_{\omega,\epsilon/\hat{\theta}^k}\right| \le \hat{\theta}^{4/3}, \tag{5.40}$$

where $\mathbf{b}_{\omega,\epsilon/\hat{\theta}^k} \equiv \mathcal{K}_\omega^{-1} \fint_{B_{\hat{\theta}}(0)} \mathbf{K}_{\omega^2,\epsilon/\hat{\theta}^k} \nabla\varphi \, dx$. (5.40) can be written as

$$\sup_{B_{\hat{\theta}}(0)} \Big|\Phi(\hat{\theta}^k x) - \Phi(0) + \epsilon \mathbb{X}_{\omega,1}(0)\mathbf{b}_k^{\omega,\epsilon} - \big(\hat{\theta}^k x + \mathbb{X}_{\omega,\epsilon}(\hat{\theta}^k x)\big)\mathbf{b}_k^{\omega,\epsilon}$$
$$- \hat{J}_{\omega,\epsilon}\hat{\theta}^{4k/3}\big(x + \hat{\theta}^{-k}\mathbb{X}_{\omega,\epsilon}(\hat{\theta}^k x)\big)\mathbf{b}_{\omega,\epsilon/\hat{\theta}^k}\Big| \le \hat{J}_{\omega,\epsilon}\hat{\theta}^{4(k+1)/3}. \tag{5.41}$$

Define

$$\mathbf{a}_{k+1}^{\omega,\epsilon} \equiv -\mathbb{X}_{\omega,1}(0)\mathbf{b}_k^{\omega,\epsilon} \qquad \text{and} \qquad \mathbf{b}_{k+1}^{\omega,\epsilon} \equiv \mathbf{b}_k^{\omega,\epsilon} + \hat{J}_{\omega,\epsilon}\hat{\theta}^{k/3}\mathbf{b}_{\omega,\epsilon/\hat{\theta}^k}. \tag{5.42}$$

By energy method, we know that $|\mathbf{b}_{\omega,\epsilon/\hat{\theta}^k}|$ is bounded uniformly in ω, ϵ, k. So $(5.39)_1$ holds. Substituting (5.42) into (5.41) and changing variables, we obtain $(5.39)_2$ for $k+1$ case. □

Lemma 5.11. *There exists $\hat{\epsilon}_0 \in (0,1)$ such that if $\omega \in (0,1]$ and $\epsilon \in (0, \hat{\epsilon}_0)$, any solution of (5.38) satisfies*

$$\|\nabla\Phi\|_{L^\infty(B_{1/2}(0))} \le c\|\mathbf{K}_{\omega,\epsilon}\Phi\|_{L^2(B_1(0))}, \tag{5.43}$$

where c is a constant independent of ω, ϵ.

Proof. Let $\hat{J}_{\omega,\epsilon}$ be that in Lemma 5.10; c denote a constant independent of ω, ϵ; and $k \in \mathbb{N}$ satisfying $\epsilon/\hat{\theta}^k \le \hat{\epsilon}_0 < \epsilon/\hat{\theta}^{k+1}$. By Lemma 5.10,

$$\sup_{B_{\frac{\epsilon}{\hat{\epsilon}_0}}(0)} |\Phi(x) - \Phi(0) - \epsilon\mathbf{a}_k^{\omega,\epsilon} - (x + \mathbb{X}_{\omega,\epsilon}(x))\mathbf{b}_k^{\omega,\epsilon}| \le c\Big|\frac{\epsilon}{\hat{\epsilon}_0}\Big|^{4/3}\hat{J}_{\omega,\epsilon}.$$

Define

$$\varphi(x) \equiv \frac{\Phi(\epsilon x) - \Phi(0) - \epsilon\mathbf{a}_k^{\omega,\epsilon} - \big(\epsilon x + \mathbb{X}_{\omega,\epsilon}(\epsilon x)\big)\mathbf{b}_k^{\omega,\epsilon}}{\epsilon^{4/3}\hat{J}_{\omega,\epsilon}} \quad \text{in } B_{\frac{1}{\hat{\epsilon}_0}}(0).$$

Then φ satisfies

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,1}\nabla\varphi) = 0 & \text{in } B_{1/\hat{\epsilon}_0}(0), \\ \|\varphi\|_{L^\infty(B_{1/\hat{\epsilon}_0}(0))} \le c. \end{cases}$$

$(3.5)_2$ of Lemma 3.5 implies

$$\|\varphi\|_{C^{1,0}(\overline{B_{1/2\hat{\epsilon}_0}(0)\cap\Omega_f^\epsilon/\epsilon}) \cap C^{1,0}(\overline{B_{1/2\hat{\epsilon}_0}(0)\cap\Omega_m^\epsilon/\epsilon})} \le c. \tag{5.44}$$

Since $\nabla\varphi(x) = \frac{\nabla\Phi(\epsilon x) - (I + \nabla\mathbb{X}_{\omega,1}(x))\mathbf{b}_k^{\omega,\epsilon}}{\epsilon^{1/3}\hat{J}_{\omega,\epsilon}}$, $|\nabla\Phi(\epsilon x)| \le c\hat{J}_{\omega,\epsilon}$ for $x \in B_{1/2\hat{\epsilon}_0}(0)$ by (3.23), (5.44), and Lemma 5.10. We prove (5.43). □

Remark 5.3. Let $\hat{\epsilon}_0$ be same as that in Lemma 5.11. By $(3.5)_2$ of Lemma 3.5, we know that if $\omega \in (0,1]$ and $\epsilon \in [\breve{\epsilon}_*, 1]$, any solution of (5.38) satisfies (5.43). Together with Lemma 5.11, any solution of (5.38) satisfies (5.43) if $\omega, \epsilon \in (0,1]$.

5.2.2. *Boundary gradient estimate*

Assume $0 \in \partial\Omega$ and (3.2). Let $Q_d(0) \equiv \Pi_{i=1}^n[-d_i, d_i]$ denote a cube, where $d = (d_1, \cdots, d_n)$, $d_i \in [3,4]$. Find a smooth cut-off function $\rho \in C_0^\infty(Q_d(0))$ such that $\rho \in [0,1]$ and $\rho = 1$ in $Q_2(0)$. For any $\omega, \epsilon, r \in (0,1]$ and $\epsilon \leq r$, we find $\mathbb{W}_{\omega,\epsilon,r}^{(n)} \in H^1(Q_d(0) \cap \Omega/r)$ by solving

$$\begin{cases} -\nabla \cdot \Big(\breve{\mathbf{K}}_{\omega^2,\epsilon,r}\big(\nabla \mathbb{W}_{\omega,\epsilon,r}^{(n)} + \vec{e}_n\big)\Big) = 0 & \text{in } Q_d(0) \cap \Omega/r, \\ \mathbb{W}_{\omega,\epsilon,r}^{(n)} = \big(1-\rho\big)\mathbb{X}_{\omega,\epsilon/r}^{(n)} & \text{on } \partial(Q_d(0) \cap \Omega/r), \end{cases} \tag{5.45}$$

where $\vec{e}_n$ is the unit vector in the n-th coordinate direction. See (3.22) for $\mathbb{X}_{\omega,\epsilon/r}^{(n)}$ and (3.3) for $\breve{\mathbf{K}}_{\omega^2,\epsilon,r}$. We adjust the d of $Q_d(0)$ so that if $\epsilon(Y_m + j) \subset \Omega_m^\epsilon$ for any $j \in \mathbb{Z}^n$, then $|Q_d(0) \cap \frac{\epsilon}{r}(Y+j)|$ is either 0 or $|\frac{\epsilon}{r}|^n$. Define

$$\begin{cases} \mathcal{D} \equiv Q_d(0) \cap \Omega/r, \\ \mathcal{D}^* \equiv \displaystyle\bigcup_{\substack{j \in \mathbb{Z}^n \\ \frac{\epsilon}{r}(Y_m+j) \subset Q_d(0) \cap \Omega_m^\epsilon/r}} \frac{\epsilon}{r}(Y+j). \end{cases} \tag{5.46}$$

From $(5.46)_1$, the definition of Ω_m^ϵ, and A1, we see

$$\begin{cases} \rho = 0 \quad \text{on } \partial\mathcal{D} \cap \partial\mathcal{D}^*, \\ \mathcal{D} \setminus \mathcal{D}^* \subset \Omega_f^\epsilon/r, \\ \mathcal{D} \setminus \mathcal{D}^* \subset \{x \in \mathcal{D} : dist(x, \partial\Omega/r) \leq c\frac{\epsilon}{r}\}, \\ \mathcal{D} \text{ is a simply-connected semiconvex domain,} \end{cases} \tag{5.47}$$

where c is a constant independent of ϵ, r. See the beginning of subsection 5.1 for a semiconvex domain. Let $\mathcal{G}_{\epsilon,r}(x,y)$ denote the Green's function of

$$\begin{cases} -\nabla_y \cdot \Big(\breve{\mathbf{K}}_{\omega^2,\epsilon,r}\nabla_y \mathcal{G}_{\epsilon,r}(x,\cdot)\Big) = \delta(x,\cdot) & \text{in } \mathcal{D}, \\ \mathcal{G}_{\epsilon,r}(x,\cdot) = 0 & \text{on } \partial\mathcal{D}. \end{cases} \tag{5.48}$$

By [16], $\mathcal{G}_{\epsilon,r}(x,\cdot) \in W^{1,1}(\mathcal{D})$ exists uniquely. Next we give a local L^∞ estimate.

Lemma 5.12. *If $x^* \in \mathcal{D}$, $\omega, \epsilon, r \in (0,1]$, $\epsilon \leq r$, and $t > 0$, any solution of*

$$\begin{cases} -\nabla \cdot (\breve{\mathbf{K}}_{\omega^2,\epsilon,r}\nabla\varphi) = 0 & \text{in } B_t(x^*) \cap \mathcal{D} \\ \varphi = 0 & \text{on } B_t(x^*) \cap \partial\mathcal{D} \end{cases} \tag{5.49}$$

satisfies

$$\left|\breve{\mathbf{K}}_{\omega,\epsilon,r}\varphi\right|(x^*) \leq c\left|\fint_{B_t(x^*)\cap\mathcal{D}} |\breve{\mathbf{K}}_{\omega,\epsilon,r}\varphi(y)|^2 dy\right|^{1/2}, \tag{5.50}$$

for some constant c independent of $\omega, \epsilon, r, x^*, t$.

Proof. First we assume $x^* = 0 \in \mathcal{D}$ and define $\widehat{\varphi}(y) = \varphi(ty)$. Then (5.49) implies

$$\begin{cases} -\nabla \cdot (\breve{\mathbf{K}}_{\omega^2,\epsilon,rt}\nabla\widehat{\varphi}) = 0 & \text{in } B_1(0) \cap \mathcal{D}/t, \\ \widehat{\varphi} = 0 & \text{on } B_1(0) \cap \partial\mathcal{D}/t. \end{cases}$$

Note $\frac{\epsilon}{rt} \le 1$ or $1 < \frac{\epsilon}{rt}$. If $\frac{\epsilon}{rt} \le 1$ (resp. $1 < \frac{\epsilon}{rt}$), Lemma 5.2 and $(5.47)_4$ (resp. Theorem 7.26 [12] and Lemma 3.6) imply

$$\|\breve{\mathbf{K}}_{\omega,\epsilon,rt}\widehat{\varphi}\|_{L^\infty(B_{1/4}(0)\cap\mathcal{D}/t)} \le c\|\breve{\mathbf{K}}_{\omega,\epsilon,rt}\widehat{\varphi}\|_{L^2(B_1(0)\cap\mathcal{D}/t)}, \tag{5.51}$$

where c is a constant independent of ϵ, ω, r, t. By (5.51),

$$|\breve{\mathbf{K}}_{\omega,\epsilon,r}\varphi(0)| \le c\left|\int_{B_1(0)\cap\mathcal{D}/t} |\breve{\mathbf{K}}_{\omega,\epsilon,rt}\widehat{\varphi}(y)|^2 dy\right|^{\frac{1}{2}} \le c\left|\fint_{B_t(0)\cap\mathcal{D}} |\breve{\mathbf{K}}_{\omega,\epsilon,r}\varphi(y)|^2 dy\right|^{\frac{1}{2}}.$$

So (5.50) holds for $x^* = 0$ case. If $x^* \ne 0$, we shift x^* to the origin of the coordinate system and repeat the above argument to obtain (5.50). □

Lemma 5.13. *Let $\omega, \epsilon, r \in (0,1]$, $s \in (0,1)$, $\epsilon \le r$, and $n \ge 3$. There is a constant c independent of ω, ϵ, r, s such that, for any $x, y \in \mathcal{D}$,*

$$\begin{cases} \left|\mathcal{G}_{\epsilon,r}(x,y)\right| \le c|x-y|^{2-n}\breve{\mathbf{K}}_{1/\omega,\epsilon,r}(x)\breve{\mathbf{K}}_{1/\omega,\epsilon,r}(y), \\ |\mathcal{G}_{\epsilon,r}(x,y)| \le c|\xi_r(x)|^s|x-y|^{2-n-s}\breve{\mathbf{K}}_{1/\omega,\epsilon,r}(x)\breve{\mathbf{K}}_{1/\omega,\epsilon,r}(y), \\ |\mathcal{G}_{\epsilon,r}(x,y)| \le c|\xi_r(x)|^s|\xi_r(y)|^s|x-y|^{2-n-2s}\breve{\mathbf{K}}_{1/\omega,\epsilon,r}(x)\breve{\mathbf{K}}_{1/\omega,\epsilon,r}(y), \end{cases} \tag{5.52}$$

where $\xi_r(x)$ (resp. $\xi_r(y)$) denotes the distance from x (resp. y) to the boundary $\partial\Omega/r$.

Proof. Let c be a constant independent of ω, ϵ, r, s and set $t \equiv |x-y|$.

Proof of $(5.52)_1$. Take $F \in C_0^\infty(B_{t/3}(y)\cap\mathcal{D})$, and find $\varphi \in H^1(\mathcal{D})$ satisfying

$$\begin{cases} -\nabla\cdot(\breve{\mathbf{K}}_{\omega^2,\epsilon,r}\nabla\varphi) = \breve{\mathbf{K}}_{\omega,\epsilon,r}F & \text{in } \mathcal{D}, \\ \varphi = 0 & \text{on } \partial\mathcal{D}. \end{cases}$$

Note φ is solvable uniquely in $H^1(\mathcal{D})$. By Theorem 4.31 [2] and Theorem 2.1 [1],

$$\|\breve{\mathbf{K}}_{\omega,\epsilon,r}\varphi\|_{L^{\frac{2n}{n-2}}(\mathcal{D})} \le c\|\breve{\mathbf{K}}_{\omega,\epsilon,r}\nabla\varphi\|_{L^2(\mathcal{D})}. \tag{5.53}$$

By [16] and Lemma 5.12,

$$\begin{cases} \varphi(x) = \displaystyle\int_{B_{t/3}(y)\cap\mathcal{D}} \mathcal{G}_{\epsilon,r}(x,z)\breve{\mathbf{K}}_{\omega,\epsilon,r}(z)F(z)dz, \\ |\breve{\mathbf{K}}_{\omega,\epsilon,r}\varphi(x)| \le c\left|\displaystyle\fint_{B_{t/3}(x)\cap\mathcal{D}} |\breve{\mathbf{K}}_{\omega,\epsilon,r}\varphi|^2 dz\right|^{\frac{1}{2}} \\ \qquad\qquad \le c\left|\displaystyle\fint_{B_{t/3}(x)\cap\mathcal{D}} \left|\breve{\mathbf{K}}_{\omega,\epsilon,r}\varphi\right|^{\frac{2n}{n-2}} dz\right|^{\frac{n-2}{2n}}. \end{cases} \tag{5.54}$$

(5.53)–(5.54) imply

$$\begin{aligned} &\left|\int_{B_{t/3}(y)\cap\mathcal{D}} \mathcal{G}_{\epsilon,r}(x,z)\breve{\mathbf{K}}_{\omega,\epsilon,r}(z)F(z)dz\right| \\ &\le c\breve{\mathbf{K}}_{1/\omega,\epsilon,r}(x)\left|\fint_{B_{t/3}(x)\cap\mathcal{D}} \left|\breve{\mathbf{K}}_{\omega,\epsilon,r}\varphi\right|^{\frac{2n}{n-2}} dz\right|^{\frac{n-2}{2n}} \\ &\le ct^{\frac{2-n}{2}}\breve{\mathbf{K}}_{1/\omega,\epsilon,r}(x)\|\breve{\mathbf{K}}_{\omega,\epsilon,r}\nabla\varphi\|_{L^2(\mathcal{D})} \le ct^{\frac{4-n}{2}}\breve{\mathbf{K}}_{1/\omega,\epsilon,r}(x)\|F\|_{L^2(B_{t/3}(y)\cap\mathcal{D})}. \end{aligned} \tag{5.55}$$

(5.55) and Lemma 5.12 imply

$$\begin{aligned}\left|\mathcal{G}_{\epsilon,r}(x,y)\right| &\le c\check{\mathbf{K}}_{1/\omega,\epsilon,r}(y)\left|\fint_{B_{t/3}(y)\cap\mathcal{D}}|\check{\mathbf{K}}_{\omega,\epsilon,r}(z)\mathcal{G}_{\epsilon,r}(x,z)|^2dz\right|^{\frac{1}{2}}\\ &\le ct^{2-n}\check{\mathbf{K}}_{1/\omega,\epsilon,r}(x)\check{\mathbf{K}}_{1/\omega,\epsilon,r}(y).\end{aligned}$$

So $(5.52)_1$ is proved.

Proof of $(5.52)_2$. By $(5.52)_1$, it is enough to show $(5.52)_2$ for the case $\xi_r(x) < t/6$. By $(5.52)_1$,

$$|\mathcal{G}_{\epsilon,r}(\tilde{x},y)| \le c|x-y|^{2-n}\check{\mathbf{K}}_{1/\omega,\epsilon,r}(\tilde{x})\check{\mathbf{K}}_{1/\omega,\epsilon,r}(y)$$

for all $\tilde{x}$ satisfying $|\tilde{x}-x| < t/3$. Applying Lemma 3.6 and Lemma 5.2 to $\mathcal{G}_{\epsilon,r}(\cdot,y)$ in $B_{t/3}(x)\cap\mathcal{D}$, we obtain

$$|\mathcal{G}_{\epsilon,r}(\tilde{x},y)| \le \frac{c|\xi_r(\tilde{x})|^s}{|x-y|^{n-2+s}}\check{\mathbf{K}}_{1/\omega,\epsilon,r}(\tilde{x})\check{\mathbf{K}}_{1/\omega,\epsilon,r}(y) \quad \text{for } \tilde{x}\in B_{t/6}(x)\cap\mathcal{D}.$$

$(5.52)_2$ follows by setting $\tilde{x} = x$. $(5.52)_3$ is obtained by (5.48), $(5.52)_2$, Lemma 5.2, and a similar argument as that for $(5.52)_2$. □

Lemma 5.14. *Solution of (5.45) exists uniquely in $H^1(\mathcal{D})$. For any $\omega,\epsilon,r\in(0,1]$ and $\epsilon\le r$, there is a constant c independent of ω,ϵ,r such that*

$$|\mathbb{W}^{(n)}_{\omega,\epsilon,r}(x)| \le c\epsilon/r \qquad \textit{for } x\in\mathcal{D}.$$

Proof. Let c denote a constant independent of ω,ϵ,r.

Step 1. Unique existence of a solution of (5.45) in $H^1(\mathcal{D})$ is clear. If we define $\mathbb{Y}^{(n)}_{\omega,\epsilon,r} \equiv \mathbb{W}^{(n)}_{\omega,\epsilon,r} - \mathbb{X}^{(n)}_{\omega,\epsilon/r}$ in $\mathcal{D}^*$ (see (5.46)), then

$$\begin{cases}-\nabla\cdot\left(\check{\mathbf{K}}_{\omega^2,\epsilon,r}\nabla\mathbb{Y}^{(n)}_{\omega,\epsilon,r}\right) = 0 & \text{in } \mathcal{D}^*,\\ \mathbb{Y}^{(n)}_{\omega,\epsilon,r} = \mathbb{W}^{(n)}_{\omega,\epsilon,r} - \mathbb{X}^{(n)}_{\omega,\epsilon/r} & \text{on } \partial\mathcal{D}^*.\end{cases}$$

By Theorem 8.1 [12], (3.23), and $(5.47)_1$,

$$\sup_{\mathcal{D}^*}|\mathbb{W}^{(n)}_{\omega,\epsilon,r}| \le c\epsilon/r + \sup_{\partial\mathcal{D}^*\setminus\partial\mathcal{D}}|\mathbb{W}^{(n)}_{\omega,\epsilon,r}|. \tag{5.56}$$

Next we want to derive

$$|\mathbb{W}^{(n)}_{\omega,\epsilon,r}(x)| \le c\epsilon/r \qquad \text{for } x\in\mathcal{D}\setminus\mathcal{D}^*. \tag{5.57}$$

If so, together with (5.56), the lemma is proved.

Step 2. Define $\nu\equiv\epsilon/r$. Suppose $\mathcal{G}(x,y)$ is the Green's function of

$$\begin{cases}-\nabla_y\cdot\left(\check{\mathbf{K}}_{\omega^2,\epsilon,\epsilon}\nabla\mathcal{G}(x,\cdot)\right) = \delta(x,\cdot) & \text{in } \mathcal{D}/\nu,\\ \mathcal{G}(x,\cdot) = 0 & \text{on } \partial\mathcal{D}/\nu,\end{cases}$$

it is easy to see

$$\mathcal{G}(x,y) = \nu^{n-2}\mathcal{G}_{\epsilon,r}(\nu x,\nu y). \tag{5.58}$$

We claim, for any $s \in (0,1)$, there is a constant c independent of ϵ, ω, r, s such that

$$\begin{cases} \mathcal{G}(x,y) \le \frac{c|\xi_\epsilon(x)|^s}{|x-y|^{n-2+s}} \breve{\mathbf{K}}_{1/\omega,\epsilon,\epsilon}(x) \breve{\mathbf{K}}_{1/\omega,\epsilon,\epsilon}(y), \\ \mathcal{G}(x,y) \le \frac{c|\xi_\epsilon(x)|^s |\xi_\epsilon(y)|^s}{|x-y|^{n-2+2s}} \breve{\mathbf{K}}_{1/\omega,\epsilon,\epsilon}(x) \breve{\mathbf{K}}_{1/\omega,\epsilon,\epsilon}(y), \\ |\nabla_y \mathcal{G}(x,y)| \le c \frac{|\xi_\epsilon(x)|^s}{|x-y|^{n-1+s}} \breve{\mathbf{K}}_{1/\omega,\epsilon,\epsilon}(x) \breve{\mathbf{K}}_{1/\omega,\epsilon,\epsilon}(y) & \text{for } |x-y| \le 1, \\ |\nabla_y \mathcal{G}(x,y)| \le c \frac{|\xi_\epsilon(x)|^s |\xi_\epsilon(y)|^s}{|x-y|^{n-2+2s}} \breve{\mathbf{K}}_{1/\omega,\epsilon,\epsilon}(x) \breve{\mathbf{K}}_{1/\omega,\epsilon,\epsilon}(y) & \text{for } |x-y| > 1, \end{cases} \tag{5.59}$$

where $\xi_\epsilon(x)$ is the distance from x to the boundary $\partial\Omega/\epsilon$. By (5.58) and $(5.52)_2$,

$$\begin{aligned} \mathcal{G}(x,y) = \nu^{n-2} \mathcal{G}_{\epsilon,r}(\nu x, \nu y) &\le \frac{c\nu^{n-2}|\xi_r(\nu x)|^s}{\nu^{n-2+s}|x-y|^{n-2+s}} \breve{\mathbf{K}}_{1/\omega,\epsilon,r}(\nu x) \breve{\mathbf{K}}_{1/\omega,\epsilon,r}(\nu y) \\ &\le \frac{c|\xi_\epsilon(x)|^s}{|x-y|^{n-2+s}} \breve{\mathbf{K}}_{1/\omega,\epsilon,\epsilon}(x) \breve{\mathbf{K}}_{1/\omega,\epsilon,\epsilon}(y). \end{aligned} \tag{5.60}$$

So we obtain $(5.59)_1$. Similarly, by $(5.52)_3$, we have $(5.59)_2$. If $t = |x-y| \le 1$, Lemma 3.6 and (5.60) imply

$$\begin{aligned} \|\nabla_y \mathcal{G}(x,\cdot)\|_{L^\infty(B_{t/2}(y)\cap\mathcal{D}/\nu)} &\le \frac{c}{t} \|\mathcal{G}(x,\cdot)\|_{L^\infty(B_{3t/4}(y)\cap\mathcal{D}/\nu)} \\ &\le \frac{c|\xi_\epsilon(x)|^s}{|x-y|^{n-1+s}} \breve{\mathbf{K}}_{1/\omega,\epsilon,\epsilon}(x) \breve{\mathbf{K}}_{1/\omega,\epsilon,\epsilon}(y). \end{aligned}$$

So $(5.59)_3$ holds. $(5.59)_4$ follows from $(3.5)_2$ and $(5.59)_2$.

Step 3. We claim (5.57). The solution of (5.45) can be written as $\mathbb{W}^{(n)}_{\omega,\epsilon,r} = \mathbb{X}^{(n)}_{\omega,\epsilon/r} + \mathbb{U}_1 + \mathbb{U}_2$, where $\mathbb{U}_1$ is the solution of

$$\begin{cases} -\nabla \cdot \left(\breve{\mathbf{K}}_{\omega^2,\epsilon,r} \nabla \mathbb{U}_1 \right) = 0 & \text{in } \mathcal{D}, \\ \mathbb{U}_1 = -\rho \mathbb{X}^{(n)}_{\omega,\epsilon/r} & \text{on } \partial\mathcal{D}, \end{cases}$$

and $\mathbb{U}_2$ is the solution of

$$\begin{cases} -\nabla \cdot \left(\breve{\mathbf{K}}_{\omega^2,\epsilon,r} (\nabla \mathbb{U}_2 + \nabla \mathbb{X}^{(n)}_{\omega,\epsilon/r} + \vec{e}_n) \right) = 0 & \text{in } \mathcal{D}, \\ \mathbb{U}_2 = 0 & \text{on } \partial\mathcal{D}. \end{cases}$$

By (3.23) and maximal principle [12], we see

$$\|\mathbb{X}^{(n)}_{\omega,\epsilon/r}\|_{L^\infty(\mathcal{D})} + \|\mathbb{U}_1\|_{L^\infty(\mathcal{D})} \le c\epsilon/r. \tag{5.61}$$

Set $\nu \equiv \epsilon/r$ and $\widetilde{\mathcal{D}/\nu(\ell)} \equiv \{x \in \mathcal{D}/\nu : dist(x, \partial\Omega/\epsilon) \le \ell\}$ for $\ell > 2$. By $(5.47)_3$, there is a ℓ^* so that $(\mathcal{D} \setminus \mathcal{D}^*)/\nu \subset \widetilde{\mathcal{D}/\nu(\ell^*)}$. Find $\eta \in C^\infty(\mathcal{D}/\nu)$ with support in $\widetilde{\mathcal{D}/\nu(\ell^*+1)}$ so that $\eta \in [0,1]$, $\eta = 1$ in $\widetilde{\mathcal{D}/\nu(\ell^*)}$, and $\|\nabla\eta\|_{L^\infty(\mathcal{D}/\nu)} \le c$. By (3.23),

$(5.47)_2$, and (5.59), for any $x \in (\mathcal{D} \setminus \mathcal{D}^*)/\nu$ and $s \in (\frac{1}{2}, 1)$,

$$\begin{aligned}
\mathbb{U}_2(\nu x) &= -\int_{\mathcal{D}/\nu} \nabla_y \mathcal{G}(x,y) \breve{\mathbf{K}}_{\omega^2,\epsilon,\epsilon\nu} (\nabla \mathbb{X}^{(n)}_{\omega,1}(y) + \vec{e}_n) dy \\
&= -\int_{\mathcal{D}/\nu} \nabla_y \mathcal{G}(x,y)(1-\eta(y)) \breve{\mathbf{K}}_{\omega^2,\epsilon,\epsilon\nu} (\nabla \mathbb{X}^{(n)}_{\omega,1}(y) + \vec{e}_n) dy \\
&\quad - \int_{\mathcal{D}/\nu} \nabla_y \mathcal{G}(x,y)\eta(y) \breve{\mathbf{K}}_{\omega^2,\epsilon,\epsilon\nu} (\nabla \mathbb{X}^{(n)}_{\omega,1}(y) + \vec{e}_n) dy \\
&= -\int_{\mathcal{D}/\widetilde{\nu(\ell^*+1)}} \mathcal{G}(x,y) \nabla \eta(y) \breve{\mathbf{K}}_{\omega^2,\epsilon,\epsilon\nu} (\nabla \mathbb{X}^{(n)}_{\omega,1}(y) + \vec{e}_n) dy \\
&\quad - \int_{\mathcal{D}/\widetilde{\nu(\ell^*+1)}} \nabla_y \mathcal{G}(x,y)\eta(y) \breve{\mathbf{K}}_{\omega^2,\epsilon,\epsilon\nu} (\nabla \mathbb{X}^{(n)}_{\omega,1}(y) + \vec{e}_n) dy \\
&\le c \int_{\mathcal{D}/\widetilde{\nu(\ell^*+1)} \cap \{|x-y| \le 1\}} \frac{\nu |\xi_\epsilon(x)|^s}{|x-y|^{n-1+s}} dy \\
&\quad + c \int_{\mathcal{D}/\widetilde{\nu(\ell^*+1)} \cap \{|x-y| > 1\}} \frac{\nu |\xi_\epsilon(x)|^s}{|x-y|^{n-2+2s}} dy \le c\epsilon/r. \qquad (5.62)
\end{aligned}$$

(5.62) implies $\|\mathbb{U}_2\|_{L^\infty(\mathcal{D}\setminus\mathcal{D}^*)} \le c\epsilon/r$. Together with (5.61), we obtain (5.57). So we prove the lemma. □

Lemma 5.15. *Let $\hat{\theta}$, $\hat{\epsilon}_0$ be those in Lemma 5.9. There exist constants $\tilde{\theta}, \tilde{\epsilon}_0 \in (0,1)$ satisfying $\tilde{\theta} < \hat{\theta}$, $\tilde{\epsilon}_0 < \hat{\epsilon}_0$ such that if*

$$\begin{cases} -\nabla \cdot (\breve{\mathbf{K}}_{\omega^2,\epsilon,r} \nabla \varphi) = 0 & \text{in } B_1(0) \cap \Omega/r, \\ \varphi = \varphi_b & \text{on } B_1(0) \cap \partial\Omega/r, \end{cases} \qquad (5.63)$$

and if

$$\begin{cases} \varphi_b(0) = \partial_T \varphi_b(0) = 0, \\ \|\breve{\mathbf{K}}_{\omega,\epsilon,r} \varphi\|_{L^2(B_1(0)\cap\Omega/r)}, [\varphi_b]_{C^{1,\alpha}(\overline{B_1(0)\cap\Omega/r})} \le 1, \end{cases}$$

then, for any $\omega, \epsilon, r \in (0,1]$ and $\epsilon/r \le \tilde{\epsilon}_0$,

$$\sup_{B_{\tilde{\theta}}(0)\cap\Omega/r} \left| \varphi(x) - \big(x_n + \mathbb{W}^{(n)}_{\omega,\epsilon,r}(x)\big) \mathbf{d}_{\omega,\epsilon,r} \right| \le \tilde{\theta}^{1+\tau},$$

where $\alpha \in (0,1)$, $\tau = \frac{\alpha}{2}$, $\partial_T \varphi_b(0)$ is the tangential derivative of φ_b at 0, $\mathbf{d}_{\omega,\epsilon,r}$ is the n-th component of $\mathcal{K}_\omega^{-1} \fint_{B_{\tilde{\theta}}(0)\cap\Omega/r} \breve{\mathbf{K}}_{\omega^2,\epsilon,r} \nabla \varphi \, dx$, and $\mathcal{K}_\omega^{-1}$ is the inverse matrix of $\mathcal{K}_\omega$ (see (3.24)).

Proof. The proof is similar to that of Lemma 5.9. Let $r_*, \omega_* \in [0,1]$, $(\varphi_*, \varphi_{b,*})$ satisfy

$$\begin{cases} -\nabla \cdot (\mathcal{K}_{\omega_*} \nabla \varphi_*) = 0 & \text{in } B_{2/3}(0) \cap \Omega/r_*, \\ \varphi_* = \varphi_{b,*} & \text{on } B_{2/3}(0) \cap \partial\Omega/r_*, \end{cases}$$

and $\varphi_{b,*}$ is smooth with $\varphi_{b,*}(0) = \partial_T \varphi_{b,*}(0) = 0$. By (3.25) and Taylor expansion, there exist $\theta \in (0, \frac{2}{3})$ and τ' satisfying $\tau < \tau' < \alpha$ such that

$$\sup_{B_\theta(0)\cap\Omega/r_*} \left|\varphi_* - x_n(\partial_n \varphi_*)_{B_\theta(0)\cap\Omega/r_*}\right| \leq \theta^{1+\tau'} \left(\|\varphi_*\|_{L^\infty(B_{2/3}(0)\cap\Omega/r_*)} + [\varphi_{b,*}]_{C^{1,\alpha}(\overline{B_{2/3}(0)\cap\Omega/r_*})}\right). \tag{5.64}$$

If we fix a small $\tilde{\theta} \in (0,1)$ so that (5.64) holds, the conclusion will follow by contradiction provided we prove $\lim_{\epsilon/r\to 0} \|\mathbb{W}^{(n)}_{\omega,\epsilon,r}\|_{L^\infty(B_1(0)\cap\Omega/r)} = 0$. But that is the result of Lemma 5.14. So we prove this lemma. □

Lemma 5.16. *$\tilde{\theta}, \tilde{\epsilon}_0, \tau$ are same as those in Lemma 5.15. If*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}\nabla\Phi) = 0 & \text{in } B_1(0)\cap\Omega, \\ \Phi = 0 & \text{on } B_1(0)\cap\partial\Omega, \end{cases} \tag{5.65}$$

then, for any $\omega \in (0,1]$, $\epsilon \in (0, \tilde{\epsilon}_0)$, and k satisfying $\epsilon/\tilde{\theta}^k \leq \tilde{\epsilon}_0$, there is a constant $\mathbf{d}^{\omega,\epsilon}_k$ satisfying

$$\begin{cases} |\mathbf{d}^{\omega,\epsilon}_k| \leq c\tilde{J}_{\omega,\epsilon}, \\ \sup\limits_{B_{\tilde{\theta}^k}(0)\cap\Omega} \left|\Phi - \sum\limits_{j=0}^{k-1} \tilde{\theta}^{\tau j}\left(x_n + \tilde{\theta}^j \mathbb{W}^{(n)}_{\omega,\epsilon,\tilde{\theta}^j}(\tilde{\theta}^{-j}x)\right)\mathbf{d}^{\omega,\epsilon}_j\right| \leq \tilde{\theta}^{k(1+\tau)}\tilde{J}_{\omega,\epsilon}, \end{cases} \tag{5.66}$$

where $\tilde{J}_{\omega,\epsilon} \equiv \|\mathbf{K}_{\omega,\epsilon}\Phi\|_{L^2(B_1(0)\cap\Omega)}$ and c is a constant independent of ω, ϵ.

Proof. This is done by induction on k. When $k = 1$, (5.66) holds by Lemma 5.15 with $r = 1$. $\mathbf{d}^{\omega,\epsilon}_0$ is the n-th component of $\mathcal{K}^{-1}_\omega \fint_{B_{\tilde{\theta}}(0)\cap\Omega} \mathbf{K}_{\omega^2,\epsilon}\nabla\Phi dx$. Suppose (5.66) holds for some k satisfying $\epsilon/\tilde{\theta}^k \leq \tilde{\epsilon}_0$, we define, in $B_1(0)\cap\Omega/\tilde{\theta}^k$,

$$\begin{cases} \varphi(x) \equiv \tilde{J}^{-1}_{\omega,\epsilon}\tilde{\theta}^{-k(1+\tau)}\left(\Phi(\tilde{\theta}^k x) - \sum\limits_{j=0}^{k-1}\tilde{\theta}^{\tau j}\left(\tilde{\theta}^k x_n + \tilde{\theta}^j \mathbb{W}^{(n)}_{\omega,\epsilon,\tilde{\theta}^j}(\tilde{\theta}^{k-j}x)\right)\mathbf{d}^{\omega,\epsilon}_j\right), \\ \varphi_b(x) \equiv -\tilde{J}^{-1}_{\omega,\epsilon}\tilde{\theta}^{-k(1+\tau)}\sum\limits_{j=0}^{k-1}\tilde{\theta}^{\tau j}\tilde{\theta}^k\mathbf{d}^{\omega,\epsilon}_j x_n. \end{cases}$$

Then those functions satisfy (5.63) with $r = \tilde{\theta}^k$. Note $[\varphi_b]_{C^{1,\alpha}(\overline{B_1(0)\cap\Omega/\tilde{\theta}^k})} = 0$. By induction,

$$\max\left\{\|\varphi\|_{L^\infty(B_1(0)\cap\Omega/\tilde{\theta}^k)}, [\varphi_b]_{C^{1,\alpha}(\overline{B_1(0)\cap\Omega/\tilde{\theta}^k})}\right\} \leq 1. \tag{5.67}$$

Apply Lemma 5.15,

$$\sup_{B_{\tilde{\theta}}(0)\cap\Omega/\tilde{\theta}^k}\left|\varphi(x) - \left(x_n + \mathbb{W}^{(n)}_{\omega,\epsilon,\tilde{\theta}^k}(x)\right)\mathbf{d}_{\omega,\epsilon,\tilde{\theta}^k}\right| \leq \tilde{\theta}^{1+\tau}, \tag{5.68}$$

where $\mathbf{d}_{\omega,\epsilon,\tilde\theta^k}$ is the n-th component of $\mathcal{K}_\omega^{-1} \fint_{B_{\tilde\theta}(0)\cap\Omega/\tilde\theta^k} \breve{\mathbf{K}}_{\omega^2,\epsilon,\tilde\theta^k}\nabla\varphi\, dx$. By energy method and (5.67), $|\mathbf{d}_{\omega,\epsilon,\tilde\theta^k}|$ is bounded uniformly in $\omega,\epsilon,\tilde\theta^k$. Rewrite (5.68) in terms of Φ in $B_{\tilde\theta^{k+1}}(0)$ to obtain

$$\sup_{B_{\tilde\theta^{k+1}}(0)\cap\Omega}\Bigg|\Phi(x)-\sum_{j=0}^{k-1}\tilde\theta^{\tau j}\left(x_n+\tilde\theta^j\mathbb{W}^{(n)}_{\omega,\epsilon,\tilde\theta^j}(\tilde\theta^{-j}x)\right)\mathbf{d}_j^{\omega,\epsilon}$$
$$-\tilde\theta^{k\tau}\tilde J_{\omega,\epsilon}\left(x_n+\tilde\theta^k\mathbb{W}^{(n)}_{\omega,\epsilon,\tilde\theta^k}(\tilde\theta^{-k}x)\right)\mathbf{d}_{\omega,\epsilon,\tilde\theta^k}\Bigg|\le\tilde\theta^{(k+1)(1+\tau)}\tilde J_{\omega,\epsilon}.$$

If $\mathbf{d}_k^{\omega,\epsilon}\equiv\tilde J_{\omega,\epsilon}\mathbf{d}_{\omega,\epsilon,\tilde\theta^k}$, we conclude that (5.66) holds for $k+1$. □

Lemma 5.17. *Let $\tilde\epsilon_0$ be same as that in Lemma 5.16. Suppose $\omega\in(0,1]$ and $\epsilon\in(0,\tilde\epsilon_0)$, any solution of (5.65) satisfies*

$$\|\nabla\Phi\|_{L^\infty(B_{1/2}(0)\cap\Omega)}\le c\|\mathbf{K}_{\omega,\epsilon}\Phi\|_{L^2(B_1(0)\cap\Omega)}, \tag{5.69}$$

where c is a constant independent of ω,ϵ.

Proof. By (3.2), we have a local coordinate $x=(x',x_n)$ so that

$$B_1(0)\cap\Omega=\left\{(x',x_n)\in\Omega:\ |x'|^2+|x_n|^2<1,\Psi(x')<x_n\right\}.$$

To obtain the Lipschitz estimate in (5.69), it is suffice to show

$$\sup_{(0,x_n)\in B_{1/2}(0)\cap\Omega}|\nabla\Phi(0,x_n)|\le c\|\mathbf{K}_{\omega,\epsilon}\Phi\|_{L^2(B_1(0)\cap\Omega)}. \tag{5.70}$$

The reason is that one can repeat the same argument by varying the origin along the boundary $B_1(0)\cap\partial\Omega$ and by adjusting the constant c to obtain the estimate.

Let $\tilde\theta,\tilde J_{\omega,\epsilon},\tau$ are same as those in Lemma 5.16, let c be a constant independent of ω,ϵ, and let k satisfy $\epsilon/\tilde\theta^k\le\tilde\epsilon_0<\epsilon/\tilde\theta^{k+1}$. For any $x\equiv(0,x_n)\in B_{1/2}(0)\cap\Omega$, we have either case (1): $\frac12\tilde\theta^\ell<x_n\le\frac12\tilde\theta^{\ell-1}$ for $1\le\ell\le k$ or case (2): $0\le x_n\le\frac12\tilde\theta^k$.

For case (1): By Lemma 5.16, we have

$$\sup_{B_{\tilde\theta^{\ell-1}}(0)\cap\Omega}\left|\Phi(y)-\sum_{j=0}^{\ell-2}\tilde\theta^{\tau j}\left(y_n+\tilde\theta^j\mathbb{W}^{(n)}_{\omega,\epsilon,\tilde\theta^j}(\tilde\theta^{-j}y)\right)\mathbf{d}_j^{\omega,\epsilon}\right|\le c\tilde\theta^{\ell(1+\tau)}\tilde J_{\omega,\epsilon}. \tag{5.71}$$

Hence, by Lemma 5.14 and (5.71),

$$\sup_{B_{\tilde\theta^{\ell-1}}(0)\cap\Omega}|\Phi|\le c\tilde J_{\omega,\epsilon}\left(\tilde\theta^{\ell(1+\tau)}+(\xi(x)+\epsilon)\sum_{j=0}^{\ell-2}\tilde\theta^{\tau j}\right)\le c\xi(x)\tilde J_{\omega,\epsilon}. \tag{5.72}$$

Here we use $\xi(x)\equiv|x-x_0|$ where $x_0\in\partial\Omega$ so that $|x-x_0|=\min_{y\in\partial\Omega}|x-y|$. Note $\epsilon\le\tilde\epsilon_0\tilde\theta^k\le2\tilde\epsilon_0\frac12\tilde\theta^\ell\le2\tilde\epsilon_0x_n\le c\tilde\epsilon_0\xi(x)$. By (5.72),

$$\sup_{B_{\xi(x)/2}(x)}|\Phi|\le c\xi(x)\tilde J_{\omega,\epsilon}. \tag{5.73}$$

Then we move the origin of the coordinate system to x and define

$$\varphi(y) \equiv \xi(x)^{-1}\tilde{J}_{\omega,\epsilon}^{-1}\Phi(\xi(x)y).$$

By (5.73),

$$\|\varphi\|_{L^\infty(B_{1/2}(x))} \leq c.$$

Function φ satisfies

$$-\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon/\xi(x)}\nabla\varphi) = 0 \qquad \text{in } B_{1/2}(x).$$

By Lemma 5.11,

$$\|\nabla\Phi\|_{L^\infty(B_{\xi(x)/4}(x))} \leq c\tilde{J}_{\omega,\epsilon}.$$

This proves (5.70) for case (1).

For case (2): Apply Lemma 5.16 to obtain

$$\sup_{B_{\tilde{\theta}^k}(0)\cap\Omega}\left|\Phi(y) - \sum_{j=0}^{k-1}\tilde{\theta}^{\tau j}\left(y_n + \tilde{\theta}^j\mathbb{W}^{(n)}_{\omega,\epsilon,\tilde{\theta}^j}(\tilde{\theta}^{-j}y)\right)\mathbf{d}_j^{\omega,\epsilon}\right| \leq c\tilde{J}_{\omega,\epsilon}\tilde{\theta}^{k(1+\tau)}.$$

By Lemma 5.14,

$$\sup_{B_{\tilde{\theta}^k}(0)\cap\Omega}|\Phi| \leq c\epsilon\tilde{J}_{\omega,\epsilon}. \tag{5.74}$$

Define $\varphi(y) \equiv \epsilon^{-1}\tilde{J}_{\omega,\epsilon}^{-1}\Phi(\epsilon y)$. By (5.74),

$$\|\varphi\|_{L^\infty(B_1(0)\cap\Omega/\epsilon)} \leq c.$$

Function φ satisfies

$$\begin{cases} -\nabla\cdot(\breve{\mathbf{K}}_{\omega^2,\epsilon,\epsilon}\nabla\varphi) = 0 & \text{in } B_1(0)\cap\Omega/\epsilon, \\ \varphi = 0 & \text{on } B_1(0)\cap\partial\Omega/\epsilon. \end{cases}$$

$(3.5)_2$ of Lemma 3.5 implies $\|\varphi\|_{C^{1,0}(\overline{B_1(0)\cap\Omega_f^\epsilon/\epsilon})\cap C^{1,0}(\overline{B_1(0)\cap\Omega_m^\epsilon/\epsilon})} \leq c$. This gives the proof of (5.70) for case (2). □

Remark 5.4. By Lemma 3.5 and Lemma 5.17, we know that if $\epsilon,\omega \in (0,1]$, any solution of (5.65) satisfies $\|\nabla\Phi\|_{L^\infty(B_{1/2}(0)\cap\Omega)} \leq c\|\mathbf{K}_{\omega,\epsilon}\Phi\|_{L^2(B_1(0)\cap\Omega)}$, where c is a constant independent of ω,ϵ.

Lemma 3.7 is direct result of Remark 5.3 and Remark 5.4.

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