Non-uniform elliptic equations in convex Lipschitz domains

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Non-uniform elliptic equations in convex Lipschitz domains are concerned. The non-smooth domains consist of a periodic connected high permeability sub-region and a periodic disconnected matrix block subset with low permeability. Let $\epsilon \in (0,1]$ denote the size ratio of the matrix blocks to the whole domain and let $\omega^2 \in (0,1]$ denote the permeability ratio of the disconnected matrix block subset to the connected sub-region. The $W^{1,p}$ norm for $p \in (1,\infty)$ of the elliptic solutions in the high permeability sub-region are shown to be bounded uniformly in ω, ϵ. However, the $W^{1,p}$ norm of the solutions in the low permeability subset may not be bounded uniformly in ω, ϵ. Roughly speaking, if the sources in the low permeability subset are small enough, the solutions in that subset are bounded uniformly in ω, ϵ. Otherwise the solutions can not be bounded uniformly in ω, ϵ. Relations between the sources and the variation of the solutions in the low permeability subset are also presented in this work.

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1. Introduction

Uniform L^p gradient estimate for the solutions of non-uniform elliptic equations in bounded convex Lipschitz domains is presented. Let Ω be a bounded domain in $\mathbb{R}^n$ for $n \geq 2$, $\partial\Omega$ denote the boundary of Ω, $\epsilon \in (0,1]$, $\Omega(2\epsilon) \equiv \{x \in \Omega : dist(x,\partial\Omega) > 2\epsilon\}$, $Y \equiv (0,1)^n$ consist of a smooth sub-domain Y_m completely surrounded by another connected sub-domain Y_f $(\equiv Y \setminus \overline{Y_m})$, $\Omega_m^\epsilon \equiv \{x : x \in \epsilon(Y_m + j) \subset \Omega(2\epsilon)$ for some $j \in \mathbb{Z}^n\}$ be a disconnected subset of Ω, Ω_f^ϵ $(\equiv \Omega \setminus \overline{\Omega_m^\epsilon})$ represent a connected sub-region of Ω, and $\mathbf{K}_{\nu,\epsilon}(x) \equiv \begin{cases} 1 & \text{if } x \in \Omega_f^\epsilon \\ \nu & \text{if } x \in \Omega_m^\epsilon \end{cases}$ for any $\nu > 0$. The problem that we consider is

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon}\nabla U + G) = F & \text{in } \Omega, \\ (\mathbf{K}_{\omega^2,\epsilon}\nabla U + G) \cdot \vec{\mathbf{n}} = 0 & \text{on } \partial\Omega, \\ \int_\Omega \Pi_\epsilon U|_{\Omega_f^\epsilon} dx = 0, \end{cases} \tag{1.1}$$

where $\omega, \epsilon \in (0,1]$, $\vec{\mathbf{n}}$ is a unit normal vector on $\partial\Omega$, and G, F are given functions. Π_ϵ in (1.1) is an extension operator (see [1] or Lemma 2.1 below) and $\Pi_\epsilon U|_{\Omega_f^\epsilon}$ is the extension function of $U|_{\Omega_f^\epsilon}$ in Ω. The problem has applications in heat transfer in two-phase media, flows in highly heterogeneous media, the stress in composite materials, and so on (see [3, 9, 14] and references therein). If G, F are bounded in Ω and $\int_\Omega F dx = 0$, a solution of (1.1) in Hilbert space $H^1(\Omega)$ exists uniquely for each ω, ϵ by Lax-Milgram Theorem [11]. The L^2 norm of the gradient of the solution of (1.1) in the connected sub-region Ω_f^ϵ is bounded uniformly in $\omega, \epsilon \in (0, 1]$ if the sources G, F are small in Ω_m^ϵ. However, the L^2 norm of the gradient of the solution of (1.1) in matrix blocks Ω_m^ϵ can be very large when ω closes to 0. It is interested to ask whether the uniform bound in ω, ϵ for the gradient of the solution of (1.1) can be extended to L^p space for any $p \in (1, \infty)$ or not.

$W^{1,p}$ estimate and Lipschitz estimate uniform in ϵ for the Laplace equation in periodic perforated domains were derived in [16, 18]. For uniform elliptic equations with Dirichlet boundary condition and with discontinuous or periodic oscillatory coefficients, the uniform bound in ϵ for $W^{k,p}$ norm or for Lipschitz norm in the whole domain could be found in [4, 5, 7, 14, 15, 20]. For example, Lipschitz estimate and $W^{2,p}$ estimate for uniform elliptic equations with discontinuous coefficients had been proved in [14, 15]. Uniform Hölder, $W^{1,p}$, and Lipschitz estimates in ϵ for uniform elliptic equations with Hölder periodic coefficients were shown in [4, 5]. Uniform $W^{1,p}$ estimate in ϵ for uniform elliptic equations with continuous or with VMO periodic coefficients were considered in [7, 20].

For non-uniform elliptic equations with smooth periodic coefficients, existence of $C^{2,\alpha}$ solution could be found in [12]. Uniform Hölder and Lipschitz estimates in ω, ϵ for $(1.1)_1$ with Dirichlet boundary condition were shown in [22]. Here we consider the non-uniform elliptic equations in Lipschitz domains. It is proved that $W^{1,p}$ norm for the solution of (1.1) in the connected sub-region Ω_f^ϵ is bounded uniformly in ω, ϵ under some proper assumptions. But, the $W^{1,p}$ norm for the solution of (1.1) in the disconnected subset Ω_m^ϵ may not be bounded uniformly in ω, ϵ. If the sources G, F in the low permeability subset Ω_m^ϵ are very small, the solutions in Ω_m^ϵ are still bounded uniformly in ω, ϵ like the solutions of uniform elliptic equations. If the sources are not small enough, the solutions in Ω_m^ϵ can not be bounded uniformly in ω, ϵ again.

The rest of this work is organized as follows: Notation and main result are stated in section 2. In section 3, we present a priori estimates for interface problems and present uniform Hölder, uniform Lipschitz, and uniform $W^{1,p}$ local estimates in ω, ϵ for the solutions of non-uniform elliptic equations in periodic domains. The proof of the main result is given in section 4. In section 5, we show the uniform Hölder and the uniform Lipschitz estimates in ω, ϵ for non-uniform elliptic equations, claimed in section 3. In section 6 (that is, Appendix), we give a proof of Theorem 4.1, which is a modification of Theorem 3.3 in [19].

2. Notation and main result

Let $C^{k,\alpha}$ denote the Hölder space with norm $\|\cdot\|_{C^{k,\alpha}}$, $W^{s,p}$ denote the Sobolev space with norm $\|\cdot\|_{W^{s,p}}$, and $[\varphi]_{C^{0,\alpha}}$ be the Hölder semi-norm of φ for $k \geq 0$, $\alpha \in [0,1], s \geq -1, p \in [1,\infty]$ (see [2, 11]). $L^p = W^{0,p}$ and $H^1 = W^{1,2}$. $C^\infty(\mathbb{R}^n)$ is the space of infinitely differentiable functions in $\mathbb{R}^n$, $C_0^\infty(D)$ is a subset of $C^\infty(\mathbb{R}^n)$ with support in D, and $C_{per}^\infty(\mathbb{R}^n)$ is the space of infinitely differentiable Y-periodic functions in $\mathbb{R}^n$. $W_0^{s,p}(D)$ is the closure of $C_0^\infty(D)$ under the $W^{s,p}$ norm and $W_{per}^{s,p}(\mathbb{R}^n)$ is the closure of $C_{per}^\infty(\mathbb{R}^n)$ under the $W^{s,p}$ norm and $\|\varphi\|_{W_{per}^{s,p}(\mathbb{R}^n)} \equiv \|\varphi\|_{W^{s,p}(Y)}$ for $s \geq 1$, $p \in [1,\infty]$. $\mathcal{A}_m \equiv \{x \in \mathbb{R}^n : x \in Y_m + j \text{ for some } j \in \mathbb{Z}^n\}$ and $\mathcal{A}_f \equiv \mathbb{R}^n \setminus \overline{\mathcal{A}_m}$. $\mathcal{H}^1_{per}(\mathbb{R}^n) \equiv \{\varphi \in W_{per}^{1,2}(\mathbb{R}^n) : \int_{Y_f}\varphi(y)dy = 0\}$ and $\mathcal{H}^1_{per}(\mathcal{A}_f) \equiv \{\varphi|_{\mathcal{A}_f} : \varphi \in \mathcal{H}^1_{per}(\mathbb{R}^n)\}$. Let $\|\varphi_1, \cdots, \varphi_m\|_{\mathbf{B}_1} \equiv \|\varphi_1\|_{\mathbf{B}_1} + \cdots + \|\varphi_m\|_{\mathbf{B}_1}$, $\|\varphi\|_{\mathbf{B}_1 \cap \mathbf{B}_2} \equiv \|\varphi\|_{\mathbf{B}_1} + \|\varphi\|_{\mathbf{B}_2}$, $B_r(x)$ denote a ball centered at x with radius r, $\overline{D}$ be the closure of D, ∂D be the boundary of D, $|D|$ be the volume of D, $\mathcal{X}_D$ be the characteristic function on D, and $D/r \equiv \{x : rx \in D\}$. For any $\varphi \in L^1(D)$,

$$(\varphi)_D \equiv \fint_D \varphi(y)dy \equiv \frac{1}{|D|}\int_D \varphi(y)dy.$$

$$\mathbb{K}_{\omega,1/r} \equiv \begin{cases} 1 & \text{in } \mathcal{A}_f/r \\ \omega & \text{in } \mathcal{A}_m/r \end{cases} \text{ and } \breve{\mathbf{K}}_{\omega,\nu,r} \equiv \begin{cases} 1 & \text{in } \Omega_f^\nu/r \\ \omega & \text{in } \Omega_m^\nu/r \end{cases} \text{ for } \omega \in [0,1], \nu, r \in (0,\infty).$$

If $\vec{\mathbf{n}}_y$ is an outward normal vector on ∂Y_m, we define, for any function φ in Y and $x \in \partial Y_m$,

$$\varphi_{,\pm}(x) \equiv \lim_{t\to 0^+} \varphi(x \pm t\vec{\mathbf{n}}_y), \qquad \lfloor \varphi \rfloor(x) = \varphi_{,+}(x) - \varphi_{,-}(x). \tag{2.1}$$

Our main results are:

Theorem 2.1. *Suppose*

A1. Ω is a bounded convex Lipschitz domain in $\mathbb{R}^n$ for $n \geq 2$,
A2. Y_m is a smooth simply-connected sub-domain of Y,
A3. $\omega, \epsilon \in (0,1]$, $\sigma \in [0,2]$, $p \in (1,\infty)$, $G \in L^p(\Omega)$, $F \in W^{-1,p}(\Omega)$, $\langle F, 1\rangle_\Omega = 0$,

then a $W^{1,p}(\Omega)$ solution of (1.1) exists uniquely and satisfies

$$\begin{cases} \|\mathbf{K}_{\omega^\sigma/\epsilon,\epsilon}U, \mathbf{K}_{\omega^\sigma,\epsilon}\nabla U\|_{L^p(\Omega)} \leq c(\|\mathbf{K}_{\omega^{\sigma-2},\epsilon}G\|_{L^p(\Omega)} \\ \qquad +\|F\|_{W^{-1,p}(\Omega)} + \omega^{\sigma-2}\|F\|_{W^{-1,p}(\Omega_m^\epsilon)}) & \text{if } \frac{\omega^\sigma}{\epsilon} \leq 1, \\ \|U, \mathbf{K}_{\omega^\sigma,\epsilon}\nabla U\|_{L^p(\Omega)} \leq c(\|\mathbf{K}_{\omega^{\sigma-2},\epsilon}G\|_{L^p(\Omega)} \\ \qquad +\|F\|_{W^{-1,p}(\Omega)} + \omega^{\sigma-2}\|F\|_{W^{-1,p}(\Omega_m^\epsilon)}) & \text{if } \frac{\omega^\sigma}{\epsilon} \geq 1, \end{cases} \tag{2.2}$$

where c is a constant independent of ω, ϵ, σ. Here $\langle F, 1\rangle_\Omega = 0$ means $\int_\Omega F dx = 0$ in distribution sense.

By energy method and Poincaré inequality [11], we easily get (2.2) for $\sigma = 1$, $p = 2$ case. But it is not clear whether ∇U is bounded uniformly in $L^2(\Omega_m^\epsilon)$. From Theorem 2.1, we know that if the right hand side of (2.2) is bounded independent of

ω, ϵ, σ, then the $W^{1,p}$ norm of the solution U in Ω_f^ϵ is bounded uniformly in ω, ϵ, σ for any $p \in (1, \infty)$. However, the $W^{1,p}$ norm of the solution U in Ω_m^ϵ may not be bounded uniformly in ω, ϵ, σ. From the proof of Theorem 2.1, we see that if the right hand side of (2.2) is uniformly bounded in ω, ϵ, σ, then

- $\|U\|_{W^{1,p}(\Omega)}$ is bounded uniformly in ω, ϵ when $\sigma = 0$,
- $\|U\|_{L^p(\Omega)}, \|\nabla U\|_{L^p(\Omega_f^\epsilon)}$ are bounded uniformly in ω, ϵ for $\sigma > 0$, $\frac{\omega^\sigma}{\epsilon} \geq c > 0$,
- $\|U\|_{W^{1,p}(\Omega_f^\epsilon)}$ is bounded uniformly in ω, ϵ when $\sigma > 0$ and $\frac{\omega^\sigma}{\epsilon}$ is close to 0.

Next we recall Theorem 2.1 [1].

Lemma 2.1. *Let $p \in [1, \infty)$ and $\epsilon \in (0, 1)$. There are a constant $c(Y_f, p)$ and a linear continuous extension operator $\Pi_\epsilon : W^{1,p}(\Omega_f^\epsilon) \to W^{1,p}(\Omega)$ such that if $\varphi \in W^{1,p}(\Omega_f^\epsilon)$, then*

$$\begin{cases} \Pi_\epsilon \varphi = \varphi \quad \text{in } \Omega_f^\epsilon, \\ \|\Pi_\epsilon \varphi\|_{L^p(\Omega)} \leq c\|\varphi\|_{L^p(\Omega_f^\epsilon)}, \\ \|\nabla \Pi_\epsilon \varphi\|_{L^p(\Omega)} \leq c\|\nabla \varphi\|_{L^p(\Omega_f^\epsilon)}, \\ 0 < d_1 \leq \Pi_\epsilon \varphi \leq d_2 \quad \text{if } 0 < d_1 \leq \varphi \leq d_2 \text{ for some constants } d_1, d_2, \\ \Pi_\epsilon \varphi = \zeta \quad \text{in } \Omega \text{ if } \varphi = \zeta|_{\Omega_f^\epsilon} \text{ for some linear function } \zeta \text{ in } \Omega. \end{cases}$$

Moreover, if $\zeta(x) \equiv \varphi(rx)$ in $B_1(0) \cap \Omega_f^\epsilon / r$ for any $r > \epsilon$, then $\Pi_{\epsilon/r}\zeta(x) = \Pi_\epsilon \varphi(rx)$ in $B_{1/2}(0)$.

By Theorem 2.1 and Lemma 2.1, one obtains the following result:

Theorem 2.2. *Suppose A1–A2 and*

A4. $\epsilon \in (0, 1]$, $p \in (1, \infty)$, $G \in L^p(\Omega_f^\epsilon)$, $F \in W^{-1,p}(\Omega)$, $\|F\|_{W^{-1,p}(\Omega_m^\epsilon)} = 0$, $\langle F, 1\rangle_\Omega = 0$,

then a $W^{1,p}(\Omega_f^\epsilon)$ solution of

$$\begin{cases} -\nabla \cdot (\nabla U + G) = F \quad \text{in } \Omega_f^\epsilon \\ (\nabla U + G) \cdot \vec{\mathbf{n}}_\epsilon = 0 \qquad \text{on } \partial\Omega_f^\epsilon \\ \displaystyle\int_\Omega \Pi_\epsilon U dx = 0 \end{cases}$$

exists uniquely and satisfies

$$\|U\|_{W^{1,p}(\Omega_f^\epsilon)} \leq c(\|G\|_{L^p(\Omega_f^\epsilon)} + \|F\|_{W^{-1,p}(\Omega)}),$$

where $\vec{\mathbf{n}}_\epsilon$ is a unit normal vector on $\partial\Omega_f^\epsilon$ and c is a constant independent of ϵ. See Theorem 2.1 for the definition of $\langle F, 1\rangle_\Omega = 0$.

From now on, A1–A2 are always assumed.

3. Preliminaries

Tracing the proof of Theorem 7.25 [11], we know

Remark 3.1. Let $0 \in \partial Y_m$ and $p, \nu \in [1, \infty)$. There are a constant $c(Y_f)$ and a linear continuous extension operator $\Pi_\nu : W^{1,p}(B_1(0) \cap Y_f/\nu^{-1}) \to W^{1,p}(B_1(0))$ such that, for any $\varphi \in W^{1,p}(B_1(0) \cap Y_f/\nu^{-1})$,

$$\begin{cases} \Pi_\nu \varphi = \varphi \quad \text{in } B_1(0) \cap Y_f/\nu^{-1}, \\ \|\Pi_\nu \varphi\|_{L^p(B_1(0))} \le c \|\varphi\|_{L^p(B_1(0)\cap Y_f/\nu^{-1})}, \\ \|\nabla \Pi_\nu \varphi\|_{L^p(B_1(0))} \le c \|\nabla \varphi\|_{L^p(B_1(0)\cap Y_f/\nu^{-1})}. \end{cases}$$

Lemma 3.1. *Let $\omega \in (0, 1]$, $\nu \in (0, \infty)$, $\varphi \in H^1(B_1(0))$, $0 \in \mathcal{A}_f/\nu^{-1}$, and $\Pi_\nu \varphi|_{\mathcal{A}_f/\nu^{-1}}$ be the extension of $\varphi|_{\mathcal{A}_f/\nu^{-1}}$ in $B_1(0)$. There is a constant c independent of ω, ν such that*

$$\left\| \mathbb{K}_{\omega,\nu} \left(\varphi - (\Pi_\nu \varphi|_{\mathcal{A}_f/\nu^{-1}})_{B_1(0)} \right) \right\|_{L^2(B_1(0))} \le c \|\mathbb{K}_{\omega,\nu} \nabla \varphi\|_{L^2(B_1(0))}.$$

See section 2 for $\mathbb{K}_{\omega,\nu}$.

Proof. By Poincaré inequality [11], Lemma 2.1, and Remark 3.1, the extension function $\Pi_\nu \varphi|_{\mathcal{A}_f/\nu^{-1}} \in H^1(B_1(0))$ satisfies

$$\begin{aligned} &\left\| \Pi_\nu \varphi|_{\mathcal{A}_f/\nu^{-1}} - (\Pi_\nu \varphi|_{\mathcal{A}_f/\nu^{-1}})_{B_1(0)} \right\|_{L^2(B_1(0))} \\ &\quad \le c \left\| \nabla \Pi_\nu \varphi|_{\mathcal{A}_f/\nu^{-1}} \right\|_{L^2(B_1(0))} \le c \|\nabla \varphi\|_{L^2(B_1(0)\cap \mathcal{A}_f/\nu^{-1})}, \end{aligned} \tag{3.1}$$

where c is independent of ω, ν. (3.1), Lemma 2.1, Remark 3.1, and Poincaré inequality imply

$$\begin{aligned} &\left\| \mathbb{K}_{\omega,\nu} \left(\varphi - (\Pi_\nu \varphi|_{\mathcal{A}_f/\nu^{-1}})_{B_1(0)} \right) \right\|_{L^2(B_1(0))} \\ &\quad \le \left\| \mathbb{K}_{\omega,\nu} \left(\Pi_\nu \varphi|_{\mathcal{A}_f/\nu^{-1}} - (\Pi_\nu \varphi|_{\mathcal{A}_f/\nu^{-1}})_{B_1(0)} \right) \right\|_{L^2(B_1(0))} \\ &\qquad + \omega \left\| \varphi - \Pi_\nu \varphi|_{\mathcal{A}_f/\nu^{-1}} \right\|_{L^2(B_1(0)\cap \mathcal{A}_m/\nu^{-1})} \\ &\quad \le c \|\nabla \varphi\|_{L^2(B_1(0)\cap \mathcal{A}_f/\nu^{-1})} + c\omega \left\| \nabla \varphi - \nabla \Pi_\nu \varphi|_{\mathcal{A}_f/\nu^{-1}} \right\|_{L^2(B_1(0)\cap \mathcal{A}_m/\nu^{-1})} \\ &\quad \le c \|\mathbb{K}_{\omega,\nu} \nabla \varphi\|_{L^2(B_1(0))}. \end{aligned}$$

□

3.1. *Interface problems*

Let $\Gamma(x-y)$ denote the fundamental solution of the Laplace equation in $\mathbb{R}^n$, see §6.2 [8]. Define a single-layer and a double-layer potentials as, for any smooth function φ on the boundary ∂Y_m of Y_m,

$$\begin{cases} \mathcal{S}_{\partial Y_m}(\varphi)(x) \equiv \displaystyle\int_{\partial Y_m} \Gamma(x-y)\varphi(y)dy \\ \mathcal{L}_{\partial Y_m}(\varphi)(x) \equiv \displaystyle\int_{\partial Y_m} \nabla_y \Gamma(x-y)\vec{\mathbf{n}}_y \; \varphi(y)dy \end{cases} \quad \text{for } x \in \partial Y_m,$$

where $\vec{\mathbf{n}}_y$ is the unit vector outward normal to ∂Y_m. By A2 and tracing the argument of Lemma 3.2 [22], we know

Lemma 3.2. *For any $p \in (1,\infty)$ and $\alpha \in (0,1)$, the linear operators*

$$\begin{cases} \mathcal{S}_{\partial Y_m} : W^{-\frac{1}{p},p}(\partial Y_m) \to W^{1-\frac{1}{p},p}(\partial Y_m) \\ \mathcal{L}_{\partial Y_m} : W^{1-\frac{1}{p},p}(\partial Y_m) \to W^{2-\frac{1}{p},p}(\partial Y_m) \\ \mathcal{S}_{\partial Y_m} : C^{1,\alpha}(\partial Y_m) \to C^{2,\alpha}(\partial Y_m) \\ \mathcal{L}_{\partial Y_m} : C^{1,\alpha}(\partial Y_m) \to C^{2,\alpha}(\partial Y_m) \end{cases}$$

are bounded; the operator $I - \ell\mathcal{L}_{\partial Y_m}$ for $\ell \in [-2,2]$ is continuously invertible in $W^{1-\frac{1}{p},p}(\partial Y_m)$ and in $C^{2,\alpha}(\partial Y_m)$; and there is a constant c independent of ℓ so that

$$\begin{cases} \|\varphi\|_{W^{1-\frac{1}{p},p}(\partial Y_m)} \le c\|(I-\ell\mathcal{L}_{\partial Y_m})(\varphi)\|_{W^{1-\frac{1}{p},p}(\partial Y_m)} & \text{for } \varphi \in W^{1-\frac{1}{p},p}(\partial Y_m), \\ \|\varphi\|_{C^{2,\alpha}(\partial Y_m)} \le c\|(I-\ell\mathcal{L}_{\partial Y_m})(\varphi)\|_{C^{2,\alpha}(\partial Y_m)} & \text{for } \varphi \in C^{2,\alpha}(\partial Y_m), \end{cases}$$

where I is the identity operator.

We shall use the following notations.

$$\begin{cases} \widetilde{\partial Y} \text{ is an open portion of } \partial Y, \\ \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3 \text{ are smooth domains satisfying } Y_m \subset \mathbf{D}_1 \subset \mathbf{D}_2 \subset \mathbf{D}_3 \subset Y, \\ dist(Y_m, \partial\mathbf{D}_1), dist(\mathbf{D}_1, \partial\mathbf{D}_2), dist(\mathbf{D}_2, \partial\mathbf{D}_3), dist(\mathbf{D}_3, \partial Y \setminus \widetilde{\partial Y}) > 0. \end{cases}$$

Lemma 3.3. *Let $\omega \in (0,1]$ and $\sigma \in [0,2]$. There is a constant c independent of ω, σ such that any solution of*

$$\begin{cases} -\nabla\cdot(\mathbb{K}_{\omega^2,1}\nabla\Phi + V) = \zeta & \text{in } Y \\ (\mathbb{K}_{\omega^2,1}\nabla\Phi + V)\cdot\vec{\mathbf{n}} = 0 & \text{on } \widetilde{\partial Y} \end{cases} \tag{3.2}$$

satisfies

$$\begin{cases} \|\mathbb{K}_{\omega^\sigma,1}\Phi\|_{W^{1,p}(\mathbf{D}_1\setminus\overline{Y_m})\cap W^{1,p}(Y_m)} \le c(\|\Phi\|_{L^2(Y_f)} \\ \qquad +\|\mathbb{K}_{\omega^{\sigma-2},1}V\|_{L^p(Y)} + \|\mathbb{K}_{\omega^{\sigma-2},1}\zeta\|_{W^{-1,p}(Y_f)\cap W^{-1,p}(Y_m)}), \\ \|\Phi\|_{C^{2,\alpha}(\overline{\mathbf{D}_1\setminus Y_m})\cap C^{2,\alpha}(\overline{Y_m})} \le c(\|\Phi\|_{L^2(Y_f)} \\ \qquad +\|\mathbb{K}_{\omega^{-2},1}V\|_{C^{1,\alpha}(\overline{Y_f})\cap C^{1,\alpha}(\overline{Y_m})} + \|\mathbb{K}_{\omega^{-2},1}\zeta\|_{C^{0,\alpha}(\overline{Y_f})\cap C^{0,\alpha}(\overline{Y_m})}), \end{cases} \tag{3.3}$$

where $p \in [2,\infty)$, $\alpha \in (0,1)$, and $\vec{\mathbf{n}}$ is the unit vector normal to $\widetilde{\partial Y}$.

Proof. Define $\mathcal{I}_{\sigma,\omega} \equiv \|\mathbb{K}_{\omega^{\sigma-2},1}V\|_{L^p(Y)} + \|\mathbb{K}_{\omega^{\sigma-2},1}\zeta\|_{W^{-1,p}(Y_f)\cap W^{-1,p}(Y_m)}$ and let c denote a constant independent of ω, σ.

Step 1: Assume $V \in W_0^{1,p}(Y_f) \cap W_0^{1,p}(Y_m)$ and $\zeta \in L^p(Y)$. Consider the following

$$\begin{cases} -\nabla\cdot(\mathbb{K}_{\omega^2,1}\nabla\phi + V) = \zeta & \text{in } \mathbf{D}_2, \\ \phi = 0 & \text{on } \partial\mathbf{D}_2. \end{cases} \tag{3.4}$$

The unique existence of a H^1 solution of (3.4) is from Lax-Milgram Theorem [11]. By energy method and [6], we have

$$\|\phi\|_{W^{1,p}(\mathbf{D}_2\setminus\mathbf{D}_1)} \le c\mathcal{I}_{\sigma,\omega}. \tag{3.5}$$

Let φ in Y_m be the solution of

$$\begin{cases} -\nabla\cdot(\omega^2\nabla\varphi + V) = \zeta & \text{in } Y_m, \\ \varphi = 0 & \text{on } \partial Y_m, \end{cases} \tag{3.6}$$

and φ in $\mathbf{D}_2 \setminus \overline{Y_m}$ be the solution of

$$\begin{cases} -\nabla\cdot(\nabla\varphi + V) = \zeta & \text{in } \mathbf{D}_2\setminus\overline{Y_m}, \\ \varphi = 0 & \text{on } \partial(\mathbf{D}_2\setminus\overline{Y_m}). \end{cases} \tag{3.7}$$

By [6] again,

$$\|\varphi\|_{W^{1,p}(\mathbf{D}_2\setminus\overline{Y_m})} + \omega^\sigma\|\varphi\|_{W^{1,p}(Y_m)} \le c\mathcal{I}_{\sigma,\omega}. \tag{3.8}$$

If we define $\psi \equiv \phi - \varphi$ in $\mathbf{D}_2$, then (3.4) and (3.6)–(3.7) imply

$$\begin{cases} \Delta\psi = 0 & \text{in } \mathbf{D}_2\setminus\partial Y_m, \\ \lfloor\psi\rfloor = 0 & \text{on } \partial Y_m, \\ \lfloor\mathbb{K}_{\omega^2,1}\nabla\psi\rfloor\cdot\vec{\mathbf{n}}_y = \mathcal{F} & \text{on } \partial Y_m, \\ \psi = 0 & \text{on } \partial\mathbf{D}_2, \end{cases} \tag{3.9}$$

where $\vec{\mathbf{n}}_y$ is the unit vector outward normal to ∂Y_m. See (2.1) for $(3.9)_{2,3}$. Since $V \in W_0^{1,p}(Y_f)\cap W_0^{1,p}(Y_m)$,

$$\mathcal{F} \equiv \left(\omega^2\nabla\varphi_{,-} - \nabla\varphi_{,+}\right)\cdot\vec{\mathbf{n}}_y|_{\partial Y_m}.$$

By (3.8),

$$\|\mathcal{F}\|_{W^{-\frac{1}{p},p}(\partial Y_m)} \le c\mathcal{I}_{\sigma,\omega}. \tag{3.10}$$

By Green's formula, (3.9), and Theorem 6.5.1 [8], we see that

$$\begin{cases} \psi/2 + \mathcal{L}_{\partial Y_m}(\psi) = \mathcal{S}_{\partial Y_m}(\nabla\psi_{,-}\cdot\vec{\mathbf{n}}_y|_{\partial Y_m}) \\ \psi/2 - \mathcal{L}_{\partial Y_m}(\psi) = -\mathcal{S}_{\partial Y_m}(\nabla\psi_{,+}\cdot\vec{\mathbf{n}}_y|_{\partial Y_m}) + \mathcal{S}_{\partial\mathbf{D}_2}(\partial_{\mathbf{n}_y}\psi|_{\partial\mathbf{D}_2}) \end{cases} \quad \text{on } \partial Y_m,$$

where $\partial_{\mathbf{n}_y}\psi|_{\partial\mathbf{D}_2}$ is the normal derivative of ψ on $\partial\mathbf{D}_2$. So

$$\left(I - \frac{2(1-\omega^2)}{\omega^2+1}\mathcal{L}_{\partial Y_m}\right)\psi = \frac{2}{\omega^2+1}\left(\mathcal{S}_{\partial\mathbf{D}_2}(\partial_{\mathbf{n}_y}\psi|_{\partial\mathbf{D}_2}) - \mathcal{S}_{\partial Y_m}(\mathcal{F})\right) \text{ on } \partial Y_m. \tag{3.11}$$

Then (3.5), (3.8)–(3.11), and Lemma 3.2 imply

$$\|\psi\|_{W^{1-\frac{1}{p},p}(\partial Y_m)} \le c\left(\|\mathcal{F}\|_{W^{-\frac{1}{p},p}(\partial Y_m)} + \|\partial_{\mathbf{n}_y}\psi\|_{W^{-\frac{1}{p},p}(\partial\mathbf{D}_2)}\right) \le c\mathcal{I}_{\sigma,\omega}. \tag{3.12}$$

(3.9) and (3.12) imply

$$\|\psi\|_{W^{1,p}(\mathbf{D}_2)} \le c\mathcal{I}_{\sigma,\omega}.$$

Together with (3.8), we obtain

$$\|\mathbb{K}_{\omega^\sigma,1}\phi\|_{W^{1,p}(\mathbf{D}_2\setminus\overline{Y_m})\cap W^{1,p}(Y_m)} \le c\mathcal{I}_{\sigma,\omega}. \tag{3.13}$$

Note $W_0^{1,p}(Y_f)$ (resp. $W_0^{1,p}(Y_m)$) is dense in $L^p(Y_f)$ (resp. $L^p(Y_m)$) and $L^p(Y)$ is dense in $W^{-1,p}(Y)$. By a limiting argument, we see that if $V \in L^p(Y)$ and $\zeta \in W^{-1,p}(Y)$, any solution of (3.4) satisfies (3.13).

Step 2: Let η be a smooth function satisfying $\eta \in C_0^\infty(\mathbf{D}_2)$, $\eta \in [0,1]$, $\eta = 1$ in $\mathbf{D}_1$, $\|\nabla\eta\|_{W^{1,\infty}(\mathbf{D}_2)} \le c$. Multiply (3.2) by η to obtain

$$\begin{cases} -\nabla\cdot(\mathbb{K}_{\omega^2,1}\nabla(\Phi\eta) - \Phi\nabla\eta + V\eta) = \zeta\eta - (\nabla\Phi + V)\nabla\eta & \text{in } \mathbf{D}_2,\\ \Phi\eta = 0 & \text{on } \partial\mathbf{D}_2. \end{cases}$$

By the result of Step 1, we have

$$\|\mathbb{K}_{\omega^\sigma,1}\Phi\|_{W^{1,p}(\mathbf{D}_1\setminus\overline{Y_m})\cap W^{1,p}(Y_m)} \le c(\|\Phi\|_{L^p(\mathbf{D}_2\setminus\mathbf{D}_1)} + \mathcal{I}_{\sigma,\omega}). \tag{3.14}$$

Let $\tilde\eta$ be another smooth function satisfying $\tilde\eta \in C^\infty(Y)$, $\tilde\eta \in [0,1]$, $\tilde\eta = 1$ in $\mathbf{D}_2\setminus\mathbf{D}_1$, $\|\nabla\tilde\eta\|_{W^{1,\infty}(Y)} \le c$, $\tilde\eta = 0$ on $Y_m \cup (\partial\mathbf{D}_3 \setminus \widetilde{\partial Y})$. Multiply (3.2) by $\tilde\eta^2\Phi$ and employ energy method and Theorem 7.26 [11] to get

$$\|\Phi\|_{L^p(\mathbf{D}_2\setminus\mathbf{D}_1)} \le c(\|\Phi\|_{L^2(Y_f)} + \mathcal{I}_{\sigma,\omega}).$$

Together with (3.14), we obtain $(3.3)_1$. $(3.3)_2$ is proved in a similar way as $(3.3)_1$, so we skip it. □

By a similar argument as Lemma 3.3, we also have the following local estimate:

Lemma 3.4. *Let $\omega \in (0,1]$, $\nu \in (1,\infty)$, $\sigma \in [0,2]$, $0 \in \partial Y_m/\nu^{-1}$, and $B_1(0) \subset Y/\nu^{-1}$. There is a constant c independent of ω,ν,σ such that any solution of*

$$-\nabla\cdot(\mathbb{K}_{\omega^2,\nu}\nabla\Phi) = 0 \qquad \text{in } Y/\nu^{-1} \tag{3.15}$$

satisfies

$$\|\mathbb{K}_{\omega^\sigma,\nu}\Phi\|_{W^{1,p}(B_{1/3}(0)\cap Y_f/\nu^{-1})\cap W^{1,p}(B_{1/3}(0)\cap Y_m/\nu^{-1})} \le c\|\mathbb{K}_{\omega^\sigma,\nu}\Phi\|_{L^2(B_1(0))}, \tag{3.16}$$

where $p \in [2,\infty)$.

Proof. For each $\nu > 1$, by A2, we find a C^2 domain $\mathbf{D}_\nu$ such that

$$B_{1/2}(0)\cap Y_m/\nu^{-1} \subset \mathbf{D}_\nu \subset B_{2/3}(0)\cap Y_m/\nu^{-1} \quad \text{and} \quad \overline{B_{1/2}(0)\cap\partial Y_m/\nu^{-1}} \subset \partial\mathbf{D}_\nu.$$

Since $\mathbf{D}_\nu$ is C^2, for any $z \in \partial\mathbf{D}_\nu$ there exist a ball $B(z)$ centered at z and a C^2 one-to-one mapping $\xi_{z,\nu}$ of $\overline{B(z)}$ onto $\overline{\xi_{z,\nu}(B(z))} \subset \mathbb{R}^n$ satisfying

$$\xi_{z,\nu}(B(z)\cap\mathbf{D}_\nu) \subset \mathbb{R}^n_+,\ \xi_{z,\nu}(B(z)\cap\partial\mathbf{D}_\nu) \subset \partial\mathbb{R}^n_+,\ \xi_{z,\nu}(B(z)\setminus\overline{\mathbf{D}_\nu}) \subset \mathbb{R}^n_-. \tag{3.17}$$

Here $\mathbb{R}^n_+ \equiv \{x=(x_1,\cdots,x_n) : x_n > 0\}, \partial\mathbb{R}^n_+ \equiv \{x : x_n = 0\}, \mathbb{R}^n_- \equiv \{x : x_n < 0\}$. Since $\partial\mathbf{D}_\nu$ is compact for each $\nu > 1$, there exist a finite number ℓ_ν of open balls $\{B(z_i)\}_{i=1}^{\ell_\nu}$ and one-to-one mappings $\{\xi_{z_i,\nu}\}_{i=1}^{\ell_\nu}$ such that

$$\begin{cases} z_i \in \partial\mathbf{D}_\nu \text{ for } i \in \{1,\cdots,\ell_\nu\}, \\ \text{(3.17) holds for each ball } B(z_i) \text{ and } i \in \{1,\cdots,\ell_\nu\}, \\ \partial\mathbf{D}_\nu \subset \bigcup_{i=1}^{\ell_\nu} B(z_i). \end{cases}$$

Since Y_m is smooth, it is possible to choose domains $\mathbf{D}_\nu$ for all $\nu > 1$ such that

$$\begin{cases} \text{the number } \ell_\nu \text{ is bounded above by a constant independent of } \nu, \\ \|\xi_{z,\nu}\|_{C^2(\overline{B(z)})}, \|\xi_{z,\nu}^{-1}\|_{C^2(\overline{\xi_{z,\nu}(B(z))})} \le c_*, \text{where } c_* \text{ is independent of } \nu, z. \end{cases}$$

Let us define $\widehat{\mathbb{K}}_{\omega^2,\nu}$ and ϕ in $\mathbb{R}^n$ as

$$\widehat{\mathbb{K}}_{\omega^2,\nu} \equiv \begin{cases} \omega^2 & \text{in } \mathbf{D}_\nu, \\ 1 & \text{elsewhere,} \end{cases} \qquad \phi \equiv \begin{cases} \Phi & \text{in } B_{1/2}(0), \\ 0 & \text{elsewhere.} \end{cases}$$

Let $\eta \in C_0^\infty(B_{1/2}(0))$ be a bell-shaped function satisfying $\eta \in [0,1]$, $\eta = 1$ in $B_{1/3}(0)$, $\|\nabla\eta\|_{W^{1,\infty}(B_{1/2}(0))} \le c$. Multiply (3.15) by η to get

$$\begin{cases} -\nabla\cdot\left(\widehat{\mathbb{K}}_{\omega^2,\nu}\nabla(\eta\phi) - \widehat{\mathbb{K}}_{\omega^2,\nu}\phi\nabla\eta\right) = -\widehat{\mathbb{K}}_{\omega^2,\nu}\nabla\phi\nabla\eta & \text{in } B_1(0), \\ \eta\phi = 0 & \text{on } \partial B_1(0). \end{cases}$$

Then we follow the argument of Step 1 of Lemma 3.3 to obtain (3.16). □

Let $\mathbb{X}^{(j)}_{\omega,1} \in \mathcal{H}^1_{per}(\mathbb{R}^n)$ for $\omega \in (0,1]$ be a function satisfying

$$\nabla\cdot(\mathbb{K}_{\omega^2,1}(\nabla\mathbb{X}^{(j)}_{\omega,1} + \vec{e}_j)) = 0 \quad \text{in } Y, \tag{3.18}$$

and let $\mathbb{X}^{(j)}_{0,1} \in \mathcal{H}^1_{per}(\mathcal{A}_f) \cap H^1(\mathcal{A}_m)$ be a function satisfying $\mathbb{X}^{(j)}_{0,1}(x) = 0$ in $\mathcal{A}_m$ and

$$\begin{cases} \nabla\cdot(\mathbb{K}_{0,1}(\nabla\mathbb{X}^{(j)}_{0,1} + \vec{e}_j)) = 0 & \text{in } Y_f, \\ \mathbb{K}_{0,1}(\nabla\mathbb{X}^{(j)}_{0,1} + \vec{e}_j)\cdot\vec{\mathbf{n}}_y = 0 & \text{on } \partial Y_m, \end{cases}$$

where $\vec{e}_j, j = 1,\cdots,n$ is the unit vector in the j-th direction in $\mathbb{R}^n$, and $\vec{\mathbf{n}}_y$ is a unit normal vector on ∂Y_m. By Lax-Milgram Theorem [11], the solution $\mathbb{X}^{(j)}_{\omega,1}$ for $\omega \in [0,1]$ is uniquely solvable. By Theorem 6.30 [11] and $(3.3)_2$ of Lemma 3.3,

$$\|\mathbb{X}^{(j)}_{\omega,1}\|_{W^{2,\infty}(Y_f)\cap W^{2,\infty}(Y_m)} \le c(n, Y_m) \qquad \text{for } \omega \in [0,1]. \tag{3.19}$$

Define $\mathbb{X}_{\omega,1} \equiv (\mathbb{X}^{(1)}_{\omega,1},\cdots,\mathbb{X}^{(n)}_{\omega,1})$ and $\mathbb{X}_{\omega,\epsilon}(x) \equiv \epsilon\mathbb{X}_{\omega,1}(\frac{x}{\epsilon})$ for $\omega \in [0,1], \epsilon \in (0,1]$. Denote by Ξ_ω for $\omega \in [0,1]$ a $n\times n$ matrix function whose (i,j)-component is $\partial_i\mathbb{X}^{(j)}_{\omega,1}$. By remark in pages 17-19, 94-95 [13],

$$\mathcal{K}_\omega \equiv \int_{Y_f\cup Y_m} \mathbb{K}_{\omega^2,1}(I + \Xi_\omega(y))dy \qquad \text{for } \omega \in [0,1] \tag{3.20}$$

is a symmetric positive definite matrix dependent only on ω. Here I is the identity matrix. By (3.19), it is not difficult to see, for $\omega \in [0,1]$,

$$\begin{cases} d_3 I \le \mathcal{K}_\omega \le d_4 I \quad \text{where } d_3, d_4 \text{ are positive constants,} \\ \mathcal{K}_\omega \text{ is a continuous function of } \omega. \end{cases} \tag{3.21}$$

3.2. L^2 gradient estimate

In this subsection, we derive L^2 gradient estimates for elliptic equations.

Lemma 3.5. *Let $\omega, \epsilon \in (0,1]$, $\sigma \in [0,2]$, $x_0 \in \Omega$. There is a constant c independent of $\omega, \epsilon, \sigma, x_0$ such that any solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Phi + V) = 0 & \textit{in } B_2(x_0) \cap \Omega \\ (\mathbf{K}_{\omega^2,\epsilon} \nabla \Phi + V) \cdot \vec{\mathbf{n}} = 0 & \textit{on } B_2(x_0) \cap \partial\Omega \end{cases}$$

satisfies $\|\mathbf{K}_{\omega^\sigma,\epsilon} \nabla \Phi\|_{L^2(B_{1/2}(x_0)\cap\Omega)} \le c \|\nabla \Phi \mathcal{X}_{\Omega_f^\epsilon}, \mathbf{K}_{\omega^{\sigma-2},\epsilon} V\|_{L^2(B_2(x_0)\cap\Omega)}$.

Proof. Let c denote a constant independent of $\omega, \epsilon, \sigma, x_0$. For any $z \in B_{1/2}(x_0) \cap \Omega$, we move z to 0 by translation and we define

$$\begin{cases} \hat{\Phi}(y) \equiv \Phi(\epsilon y) + d \\ \hat{V}(y) \equiv \epsilon V(\epsilon y) \end{cases} \quad \text{for any } y \in B_1(z) \cap \Omega/\epsilon,\ d \in \mathbb{R}.$$

Then $\hat{\Phi}$ satisfies

$$\begin{cases} -\nabla \cdot (\breve{\mathbf{K}}_{\omega^2,\epsilon,\epsilon} \nabla \hat{\Phi} + \hat{V}) = 0 & \text{in } B_1(z) \cap \Omega/\epsilon, \\ (\breve{\mathbf{K}}_{\omega^2,\epsilon,\epsilon} \nabla \hat{\Phi} + \hat{V}) \cdot \vec{\mathbf{n}}_y = 0 & \text{on } B_1(z) \cap \partial\Omega/\epsilon, \end{cases}$$

where $\vec{\mathbf{n}}_y$ is a normal vector on $\partial\Omega/\epsilon$. See section 2 for $\breve{\mathbf{K}}_{\omega^2,\epsilon,\epsilon}$. By Lemma 3.3,

$$\|\breve{\mathbf{K}}_{\omega^\sigma,\epsilon,\epsilon} \nabla \hat{\Phi}\|_{L^2(B_{1/2}(z)\cap\Omega/\epsilon)} \le c \|\hat{\Phi} \mathcal{X}_{\Omega_f^\epsilon/\epsilon},\ \breve{\mathbf{K}}_{\omega^{\sigma-2},\epsilon,\epsilon} \hat{V}\|_{L^2(B_1(z)\cap\Omega/\epsilon)}.$$

Since d is an arbitrary constant, we have, by Poincaré inequality [11],

$$\|\breve{\mathbf{K}}_{\omega^\sigma,\epsilon,\epsilon} \nabla \hat{\Phi}\|_{L^2(B_{1/2}(z)\cap\Omega/\epsilon)} \le c \|\nabla \hat{\Phi} \mathcal{X}_{\Omega_f^\epsilon/\epsilon},\ \breve{\mathbf{K}}_{\omega^{\sigma-2},\epsilon,\epsilon} \hat{V}\|_{L^2(B_1(z)\cap\Omega/\epsilon)}. \tag{3.22}$$

(3.22) implies

$$\int_{B_{\epsilon/2}(z)\cap\Omega} |\mathbf{K}_{\omega^\sigma,\epsilon} \nabla \Phi|^2 dy \le c \int_{B_\epsilon(z)\cap\Omega} |\nabla \Phi \mathcal{X}_{\Omega_f^\epsilon}|^2 + |\mathbf{K}_{\omega^{\sigma-2},\epsilon} V|^2 dy. \tag{3.23}$$

By covering $B_{1/2}(x_0) \cap \Omega$ with a finite number of balls of radius $\epsilon/2$, (3.23) implies the lemma. □

Lemma 3.6. *Let $\omega, \epsilon \in (0,1]$ and $x_0 \in \Omega$. There is a constant c independent of ω, ϵ, x_0 such that any solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Phi) = 0 & \textit{in } B_6(x_0) \cap \Omega \\ \mathbf{K}_{\omega^2,\epsilon} \nabla \Phi \cdot \vec{\mathbf{n}} = 0 & \textit{on } B_6(x_0) \cap \partial\Omega \end{cases}$$

satisfies $\|\Phi\|_{H^1(B_{1/2}(x_0)\cap\Omega)} \le c\|\mathbf{K}_{\omega^2,\epsilon}\Phi\|_{L^2(B_6(x_0)\cap\Omega)}$.

Proof. Let c denote a constant independent of ω, ϵ, x_0. By energy method,

$$\|\mathbf{K}_{\omega,\epsilon}\nabla\Phi\|_{L^2(B_2(x_0)\cap\Omega)} \le c\|\mathbf{K}_{\omega,\epsilon}\Phi\|_{L^2(B_3(x_0)\cap\Omega)}. \tag{3.24}$$

By Lemma 3.5 and (3.24),

$$\|\nabla\Phi\|_{L^2(B_{1/2}(x_0)\cap\Omega)} \le c\|\nabla\Phi\|_{L^2(B_2(x_0)\cap\Omega_f^\epsilon)} \le c\|\mathbf{K}_{\omega,\epsilon}\Phi\|_{L^2(B_3(x_0)\cap\Omega)}. \tag{3.25}$$

Suppose $\epsilon(Y+j) \subset B_{1/2}(x_0) \cap \Omega$ for some $j \in \mathbb{Z}^n$, then

$$\begin{aligned}\|\Phi\|_{L^2(\epsilon(Y_m+j))} &\le \|\Phi - \Pi_\epsilon\Phi|_{\Omega_f^\epsilon}\|_{L^2(\epsilon(Y_m+j))} + \|\Pi_\epsilon\Phi|_{\Omega_f^\epsilon}\|_{L^2(\epsilon(Y_m+j))} \\ &\le c\left(\epsilon\|\nabla\Phi - \nabla\Pi_\epsilon\Phi|_{\Omega_f^\epsilon}\|_{L^2(\epsilon(Y_m+j))} + \|\Pi_\epsilon\Phi|_{\Omega_f^\epsilon}\|_{L^2(\epsilon(Y_m+j))}\right),\end{aligned}$$

where $\Pi_\epsilon\Phi|_{\Omega_f^\epsilon}$ is the extension function of $\Phi|_{\Omega_f^\epsilon}$ in Ω. By Lemma 2.1,

$$\|\Phi\|_{L^2(B_3(x_0)\cap\Omega_m^\epsilon)} \le c\left(\epsilon\|\nabla\Phi\|_{L^2(B_3(x_0)\cap\Omega)} + \|\Phi\|_{L^2(B_3(x_0)\cap\Omega_f^\epsilon)}\right). \tag{3.26}$$

(3.25)–(3.26) imply

$$\omega\|\Phi\|_{L^2(B_3(x_0)\cap\Omega_m^\epsilon)} \le c\omega\|\mathbf{K}_{\omega,\epsilon}\Phi\|_{L^2(B_6(x_0)\cap\Omega)}. \tag{3.27}$$

(3.25) and (3.27) imply

$$\|\nabla\Phi\|_{L^2(B_{1/2}(x_0)\cap\Omega)} \le c\|\mathbf{K}_{\omega^2,\epsilon}\Phi\|_{L^2(B_6(x_0)\cap\Omega)}. \tag{3.28}$$

The lemma follows by (3.26), (3.28). □

Lemma 3.7. *Let* $\omega, \epsilon \in (0,1]$, $\sigma \in [0,2]$, *and* $V \in L^2(\Omega)$. *A* $H^1(\Omega)$ *solution of*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}\nabla\Phi + V) = 0 & \text{in } \Omega \\ (\mathbf{K}_{\omega^2,\epsilon}\nabla\Phi + V)\cdot\vec{\mathbf{n}} = 0 & \text{on } \partial\Omega \\ \displaystyle\int_\Omega \Pi_\epsilon\Phi|_{\Omega_f^\epsilon}\,dx = 0 \end{cases} \tag{3.29}$$

exists uniquely and satisfies

$$\|\mathbf{K}_{\omega^\sigma,\epsilon}\nabla\Phi\|_{L^2(\Omega)} \le c\|\mathbf{K}_{\omega^{\sigma-2},\epsilon}V\|_{L^2(\Omega)}, \tag{3.30}$$

where c *is a constant independent of* ω, ϵ, σ.

Proof. By Lax-Milgram Theorem [11], the solution of (3.29) exists uniquely in $H^1(\Omega)$. By energy method, we obtain (3.30) for $\sigma = 1$. If $\sigma \in [0,1)$, by Lemma 3.5 and (3.30) for $\sigma = 1$,

$$\|\mathbf{K}_{\omega^\sigma,\epsilon}\nabla\Phi\|_{L^2(\Omega)} \le c(\|\nabla\Phi\|_{L^2(\Omega_f^\epsilon)} + \|\mathbf{K}_{\omega^{\sigma-2},\epsilon}V\|_{L^2(\Omega)}) \le c\|\mathbf{K}_{\omega^{\sigma-2},\epsilon}V\|_{L^2(\Omega)}.$$

So we obtain (3.30) for $\sigma \in [0,1)$ case. (3.30) for $\sigma \in (1,2]$ is due to (3.30) for $\sigma \in [0,1)$ and a duality argument. □

3.3. *Local Hölder and local Lipschitz estimates*

Assume $0 \in \partial\Omega$. By A1, there exists a Lipschitz function $\Psi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\begin{cases} \Psi(0) = 0, \\ B_1(0) \cap \Omega/r = B_1(0) \cap \{(x', x_n) \in \mathbb{R}^n : \ rx_n > \Psi(rx')\} \quad \text{for any } r \in (0,1]. \end{cases} \tag{3.31}$$

Define $B_1(0) \cap \Omega/r \equiv B_1(0) \cap \{(x', x_n) \in \mathbb{R}^n : \ x_n > 0\}$ for $r = 0$.

Lemma 3.8. *Let $\omega, \epsilon \in (0,1]$ and $\alpha \in (0,1)$. There is a constant c (independent of ω, ϵ but depending $\alpha, \|\nabla\Psi\|_{L^\infty(\mathbb{R}^{n-1})}, Y_f$) such that any solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Phi) = 0 & \text{in } B_1(0) \cap \Omega \\ \mathbf{K}_{\omega^2,\epsilon} \nabla \Phi \cdot \vec{\mathbf{n}} = 0 & \text{on } B_1(0) \cap \partial\Omega \end{cases}$$

satisfies

$$\|\Phi\|_{C^{0,\alpha}(\overline{B_{1/2}(0) \cap \Omega})} \leq c \|\mathbf{K}_{\omega^2,\epsilon} \Phi\|_{L^2(B_1(0) \cap \Omega)},$$

where $\vec{\mathbf{n}}$ is a unit normal vector on $\partial\Omega$.

Proof of Lemma 3.8 is given in subsection 5.1.

Lemma 3.9. *Let $\omega, \epsilon, r \in (0,1]$, $\sigma \in [0,2]$, and $x_0 \in \Omega$. There is a constant c independent of $\omega, \epsilon, r, \sigma, x_0$ such that any solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Phi) = 0 & \text{in } B_1(x_0) \cap \Omega \\ \mathbf{K}_{\omega^2,\epsilon} \nabla \Phi \cdot \vec{\mathbf{n}} = 0 & \text{on } B_1(x_0) \cap \partial\Omega \end{cases} \tag{3.32}$$

satisfies

$$|\Phi(x) - \Phi(y)| \leq c|x-y|^\alpha r^{1-\alpha} \check{\mathcal{X}}_{x,y,\sigma} \left(\fint_{B_r(x_0)} |\mathbf{K}_{\omega^\sigma,\epsilon} \nabla \Phi|^2 \mathcal{X}_\Omega \ dz \right)^{1/2}, \tag{3.33}$$

where $x, y \in B_{r/2}(x_0) \cap \Omega$, $\alpha \in (0,1)$, and $\check{\mathcal{X}}_{x,y,\sigma} \equiv \mathbf{K}_{1/\omega^\sigma,\epsilon}(x) + \mathbf{K}_{1/\omega^\sigma,\epsilon}(y)$.

Proof. Assume $x_0 = 0 \in \Omega$ and define $\varphi(y) \equiv \Phi(ry) + d$ for any $d \in \mathbb{R}$. Then (3.32) implies

$$\begin{cases} -\nabla \cdot (\check{\mathbf{K}}_{\omega^2,\epsilon,r} \nabla \varphi) = 0 & \text{in } B_1(0) \cap \Omega/r, \\ \check{\mathbf{K}}_{\omega^2,\epsilon,r} \nabla \varphi \cdot \vec{\mathbf{n}}_{\epsilon/r} = 0 & \text{on } B_1(0) \cap \partial\Omega/r, \end{cases}$$

where $\vec{\mathbf{n}}_{\epsilon/r}$ is a unit normal vector on $\partial\Omega/r$. See section 2 for $\check{\mathbf{K}}_{\omega^2,\epsilon,r}$. Note $B_1(0) \cap \Omega/r$ is a bounded convex Lipschitz domain.

If $\epsilon/r > 1$, Theorem 9.11 [11], Lemma 3.4, and [17] imply

$$[\check{\mathbf{K}}_{\omega^\sigma,\epsilon,r} \varphi]_{C^{0,\alpha}(\overline{B_{1/4}(0) \cap \Omega_f^\epsilon/r}) \cap C^{0,\alpha}(\overline{B_{1/4}(0) \cap \Omega_m^\epsilon/r})} \leq c \|\check{\mathbf{K}}_{\omega^\sigma,\epsilon,r} \varphi\|_{L^2(B_1(0) \cap \Omega/r)},$$

where c is independent of $\omega, \epsilon, r, \sigma$. Since d is an arbitrary constant, by Lemma 3.1,

$$[\check{\mathbf{K}}_{\omega^\sigma,\epsilon,r} \varphi]_{C^{0,\alpha}(\overline{B_{1/4}(0) \cap \Omega_f^\epsilon/r}) \cap C^{0,\alpha}(\overline{B_{1/4}(0) \cap \Omega_m^\epsilon/r})} \leq c \|\check{\mathbf{K}}_{\omega^\sigma,\epsilon,r} \nabla \varphi\|_{L^2(B_1(0) \cap \Omega/r)}, \tag{3.34}$$

where c is independent of $\omega, \epsilon, r, \sigma$. If $\epsilon/r \le 1$, then (3.34) follows from Lemma 3.1 and Lemma 3.8. Then (3.33) is a direct consequence of (3.34).

If $x_0 \neq 0$, (3.33) can be obtained by shifting the coordinate system such that x_0 is the origin of the coordinate system and by repeating the above argument. □

We also have the following Lipschitz estimate:

Lemma 3.10. *Let $\omega, \epsilon \in (0,1]$ and $B_1(0) \subset \Omega$. There is a constant c independent of ω, ϵ such that any solution of*

$$-\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Phi) = 0 \qquad \textit{in } B_1(0)$$

satisfies

$$\|\nabla \Phi\|_{L^\infty(B_{1/2}(0))} \le c \|\mathbf{K}_{\omega^2,\epsilon} \Phi\|_{L^2(B_1(0))}.$$

Proof of Lemma 3.10 is given in subsection 5.2.

3.4. *Local L^p gradient estimate*

In this subsection, we derive local L^p gradient estimate for elliptic equations. First we consider the interior estimate.

Lemma 3.11. *Let $\omega, \epsilon, r \in (0,1]$, $\sigma \in [0,2]$, and $B_{2r}(x_0) \subset \Omega$. There is a constant c independent of $\omega, \epsilon, r, \sigma, x_0$ such that any solution of*

$$-\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Phi) = 0 \qquad \textit{in } B_{2r}(x_0) \tag{3.35}$$

satisfies

$$\left(\fint_{B_{r/2}(x_0)} |\mathbf{K}_{\omega^\sigma,\epsilon} \nabla \Phi|^p dx \right)^{1/p} \le c \left(\fint_{B_r(x_0)} |\mathbf{K}_{\omega^\sigma,\epsilon} \nabla \Phi|^2 dx \right)^{1/2},$$

where $p \in (2, \infty)$.

Proof. Let c denote a constant independent of $\omega, \epsilon, r, \sigma, x_0$. By translation we assume $x_0 = 0 \in \Omega$. Let $d \in \mathbb{R}$ and $\varphi(y) = \Phi(ry) + d$. Then (3.35) implies

$$-\nabla \cdot (\breve{\mathbf{K}}_{\omega^2,\epsilon,r} \nabla \varphi) = 0 \quad \text{in } B_2(0).$$

If $\epsilon/r \le 1$ (resp. $\epsilon/r > 1$), Lemma 3.10 (resp. Theorem 9.11 [11] and Lemma 3.4) implies

$$\|\breve{\mathbf{K}}_{\omega^\sigma,\epsilon,r} \nabla \varphi\|_{L^p(B_{1/2}(0))} \le c \|\breve{\mathbf{K}}_{\omega^\sigma,\epsilon,r} \varphi\|_{L^2(B_1(0))},$$

where $p \in (2, \infty)$. Since d is arbitrary, by Lemma 3.1, we obtain

$$\|\breve{\mathbf{K}}_{\omega^\sigma,\epsilon,r} \nabla \varphi\|_{L^p(B_{1/2}(0))} \le c \|\breve{\mathbf{K}}_{\omega^\sigma,\epsilon,r} \nabla \varphi\|_{L^2(B_1(0))}.$$

Which implies the lemma. □

Next we consider the boundary estimate.

Lemma 3.12. *Let $\omega, \epsilon, r \in (0,1]$, $\sigma \in [0,2]$, and $x_0 \in \partial\Omega$. There is a constant c independent of $\omega, \epsilon, r, \sigma, x_0$ such that any solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon}\nabla\Phi) = 0 & \text{in } B_{2r}(x_0)\cap\Omega \\ \mathbf{K}_{\omega^2,\epsilon}\nabla\Phi\cdot\vec{\mathbf{n}} = 0 & \text{on } B_{2r}(x_0)\cap\partial\Omega \end{cases}$$

satisfies

$$\left(\fint_{B_{r/2}(x_0)} |\mathbf{K}_{\omega^\sigma,\epsilon}\nabla\Phi|^p\mathcal{X}_\Omega\ dx\right)^{1/p} \le c\left(\fint_{B_{2r}(x_0)} |\mathbf{K}_{\omega^\sigma,\epsilon}\nabla\Phi|^2\mathcal{X}_\Omega\ dx\right)^{1/2}, \quad (3.36)$$

where $p \in (2,\infty)$.

Proof. For any $x \in B_{r/2}(x_0)\cap\Omega$, $\xi(x)$ denotes the distance from x to the boundary $B_{2r}(x_0)\cap\partial\Omega$. Move x to 0 by translation so that $x = 0 \in B_{r/2}(x_0)\cap\Omega$. Define $\varphi(y) \equiv \Phi(\xi(x)y) - \Phi(x)$. Then φ satisfies

$$-\nabla\cdot(\breve{\mathbf{K}}_{\omega^2,\epsilon,\xi(x)}\nabla\varphi) = 0 \qquad \text{in } B_1(x) \ \ (\textit{or in } B_1(0)).$$

If $\epsilon/\xi(x) \le 1$ (resp. $\epsilon/\xi(x) > 1$), Lemma 3.10 (resp. Theorem 9.11 [11] and the definition of Ω_m^ϵ) implies

$$|\nabla\varphi|(0) \le c\|\breve{\mathbf{K}}_{\omega^2,\epsilon,\xi(x)}\varphi\|_{L^2(B_{1/2}(x))}. \quad (3.37)$$

By (3.37) and Lemma 3.9,

$$\begin{aligned} \mathbf{K}_{\omega^\sigma,\epsilon}(x)|\nabla\Phi|(x) &\le c\frac{\mathbf{K}_{\omega^\sigma,\epsilon}(x)}{\xi(x)}\left(\fint_{B_{\xi(x)/2}(x)} \left|\mathbf{K}_{\omega^2,\epsilon}(y)\big(\Phi(y)-\Phi(x)\big)\right|^2 dy\right)^{1/2} \\ &\le c\frac{r^{1-\alpha}}{\xi(x)}\left(\fint_{B_{\xi(x)/2}(x)} |y-x|^{2\alpha}dy\right)^{1/2}\left(\fint_{B_{2r}(x_0)} |\mathbf{K}_{\omega^\sigma,\epsilon}\nabla\Phi|^2\mathcal{X}_\Omega\ dz\right)^{1/2} \\ &\le c\left|\frac{r}{\xi(x)}\right|^{1-\alpha}\left(\fint_{B_{2r}(x_0)} |\mathbf{K}_{\omega^\sigma,\epsilon}\nabla\Phi|^2\mathcal{X}_\Omega\ dz\right)^{1/2}. \end{aligned} \quad (3.38)$$

Let us take $\alpha \in (0,1)$ such that $(1-\alpha)p < 1$. It is easy to see that (3.36) follows from (3.31) and (3.38). □

4. Proof of Theorem 2.1

First we give a modification of Theorem 3.3 [19]. In the following theorem, B and tB denote the concentric balls sharing the same center and satisfying $tx \in tB$ if $x \in B, t > 0$.

Theorem 4.1. *Let $p \in (2,\infty)$, Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, and $\mathbf{T}$ be a bounded sublinear operator on $L^2(\Omega)$. Suppose there are $r_0, c_0 > 0, t_2 > t_1 > 1$ such that, for any $\varphi \in L^\infty(\Omega)$ with $supp(\varphi) \subset \Omega\setminus t_2B$,*

$$\left(\fint_B |\mathbf{T}\varphi|^p\mathcal{X}_\Omega dx\right)^{\frac{1}{p}} \le c_0\left(\left(\fint_{t_1B}|\mathbf{T}\varphi|^2\mathcal{X}_\Omega dx\right)^{\frac{1}{2}} + \sup_{B'\supset B}\left(\fint_{B'}|\varphi|^2\mathcal{X}_\Omega dx\right)^{\frac{1}{2}}\right) \quad (4.1)$$

where $B = B_r(x_0)$ is a ball with radius $r \in (0, r_0)$ and center at x_0. Either $x_0 \in \partial\Omega$ or $B_{rt_2}(x_0) \subset \Omega$. Then $\mathbf{T}$ is bounded on $L^q(\Omega)$ for any $q \in (2, p)$.

Condition (4.1) above is less restrictive than (3.4) of Theorem 3.3 [19]. The proof of Theorem 4.1 is similar to the argument of Theorem 3.3 [19] and is given in section 6 for the sake of completeness.

Proof of Theorem 2.1: For any $\omega, \epsilon \in (0, 1]$ and $\sigma \in [0, 2]$, we find $\Phi_\sigma \in H^1(\Omega)$ satisfying

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Phi_\sigma + \mathbf{K}_{\omega^{2-\sigma},\epsilon} V) = 0 & \text{in } \Omega, \\ (\mathbf{K}_{\omega^2,\epsilon} \nabla \Phi_\sigma + \mathbf{K}_{\omega^{2-\sigma},\epsilon} V) \cdot \vec{\mathbf{n}} = 0 & \text{on } \partial\Omega, \\ \int_\Omega \Pi_\epsilon \Phi_\sigma |_{\Omega_f^\epsilon} dx = 0. \end{cases} \tag{4.2}$$

By Lax-Milgram Theorem [11], Φ_σ exists uniquely if $V \in L^2(\Omega)$. If we define $\mathbf{T}_\sigma : L^2(\Omega) \to L^2(\Omega)$ by $\mathbf{T}_\sigma V = \mathbf{K}_{\omega^\sigma,\epsilon} \nabla \Phi_\sigma$, then $\mathbf{T}_\sigma$ is a linear and bounded operator on $L^2(\Omega)$ by Lemma 3.7. Lemma 3.11 and Lemma 3.12 imply that the operator $\mathbf{T}_\sigma$ satisfies (4.1) for any $V \in L^p(\Omega)$ and $p \in (2, \infty)$. So $\mathbf{T}_\sigma$ is a bounded and linear operator in $L^q(\Omega)$ for $q \in (2, \infty)$ by Theorem 4.1. By $(4.2)_3$, Poincaré inequality [11], and Lemma 2.1, we know

$$\begin{aligned} \|\Phi_\sigma\|_{L^p(\Omega_f^\epsilon)} &\le \|\Pi_\epsilon \Phi_\sigma|_{\Omega_f^\epsilon}\|_{L^p(\Omega)} \le c\|\nabla \Pi_\epsilon \Phi_\sigma|_{\Omega_f^\epsilon}\|_{L^p(\Omega)} \le c\|\nabla \Phi_\sigma\|_{L^p(\Omega_f^\epsilon)}, \\ \|\Phi_\sigma\|_{L^p(\Omega_m^\epsilon)} &\le \|\Phi_\sigma - \Pi_\epsilon \Phi_\sigma|_{\Omega_f^\epsilon}\|_{L^p(\Omega_m^\epsilon)} + \|\Pi_\epsilon \Phi_\sigma|_{\Omega_f^\epsilon}\|_{L^p(\Omega_m^\epsilon)} \\ &\le c\epsilon\|\nabla \Phi_\sigma - \nabla \Pi_\epsilon \Phi_\sigma|_{\Omega_f^\epsilon}\|_{L^p(\Omega_m^\epsilon)} + c\|\Phi_\sigma\|_{L^p(\Omega_f^\epsilon)}, \end{aligned}$$

where c is independent of ω, ϵ, σ. Therefore, we conclude

Lemma 4.1. *Under A1–A2, if $\omega, \epsilon \in (0, 1]$, $\sigma \in [0, 2]$, $p \in [2, \infty)$, and $V \in L^p(\Omega)$, then a $W^{1,p}(\Omega)$ solution Φ_σ of (4.2) exists uniquely and*

$$\begin{cases} \|\mathbf{K}_{\omega^\sigma/\epsilon,\epsilon} \Phi_\sigma, \mathbf{K}_{\omega^\sigma,\epsilon} \nabla \Phi_\sigma\|_{L^p(\Omega)} \le c\|V\|_{L^p(\Omega)} & \text{if } \frac{\omega^\sigma}{\epsilon} \le 1, \\ \|\Phi_\sigma, \mathbf{K}_{\omega^\sigma,\epsilon} \nabla \Phi_\sigma\|_{L^p(\Omega)} \le c\|V\|_{L^p(\Omega)} & \text{if } \frac{\omega^\sigma}{\epsilon} \ge 1, \end{cases}$$

where c is a constant independent of ω, ϵ, σ.

By a duality argument, Poincaré inequality [11], and Lemmas 2.1, 4.1, we have

Lemma 4.2. *Under A1–A2, if $\omega, \epsilon \in (0, 1]$, $\sigma \in [0, 2]$, $p \in (1, 2]$, and $V \in L^p(\Omega)$, then a $W^{1,p}(\Omega)$ solution Φ_σ of (4.2) exists uniquely and*

$$\begin{cases} \|\mathbf{K}_{\omega^\sigma/\epsilon,\epsilon} \Phi_\sigma, \mathbf{K}_{\omega^\sigma,\epsilon} \nabla \Phi_\sigma\|_{L^p(\Omega)} \le c\|V\|_{L^p(\Omega)} & \text{if } \frac{\omega^\sigma}{\epsilon} \le 1, \\ \|\Phi_\sigma, \mathbf{K}_{\omega^\sigma,\epsilon} \nabla \Phi_\sigma\|_{L^p(\Omega)} \le c\|V\|_{L^p(\Omega)} & \text{if } \frac{\omega^\sigma}{\epsilon} \ge 1, \end{cases}$$

where c is a constant independent of ω, ϵ, σ.

Suppose $\omega, \epsilon \in (0,1]$, $\sigma \in [0,2]$, $V, \zeta \in L^\infty(\Omega)$, and $\int_\Omega \zeta(x)dx = 0$, let us find a $H^1(\Omega)$ solution of

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Phi) = \zeta & \text{in } \Omega, \\ \mathbf{K}_{\omega^2,\epsilon} \nabla \Phi \cdot \vec{\mathbf{n}} = 0 & \text{on } \partial\Omega, \\ \int_\Omega \Pi_\epsilon \Phi|_{\Omega_f^\epsilon} dx = 0, \end{cases} \tag{4.3}$$

and a $H^1(\Omega)$ solution of

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \varphi_\sigma - \mathbf{K}_{\omega^\sigma,\epsilon} V) = 0 & \text{in } \Omega, \\ (\mathbf{K}_{\omega^2,\epsilon} \nabla \varphi_\sigma - \mathbf{K}_{\omega^\sigma,\epsilon} V) \cdot \vec{\mathbf{n}} = 0 & \text{on } \partial\Omega, \\ \int_\Omega \Pi_\epsilon \varphi_\sigma|_{\Omega_f^\epsilon} dx = 0. \end{cases} \tag{4.4}$$

The solutions of (4.3) and (4.4) exist uniquely by Lax-Milgram Theorem [11]. Lemma 4.1 and Lemma 4.2 imply that the solution of (4.4) satisfies

$$\begin{cases} \|\mathbf{K}_{\omega^{2-\sigma}/\epsilon,\epsilon} \varphi_\sigma, \mathbf{K}_{\omega^{2-\sigma},\epsilon} \nabla \varphi_\sigma\|_{L^r(\Omega)} \le c\|V\|_{L^r(\Omega)} & \text{if } \frac{\omega^{2-\sigma}}{\epsilon} \le 1, \\ \|\varphi_\sigma, \mathbf{K}_{\omega^{2-\sigma},\epsilon} \nabla \varphi_\sigma\|_{L^r(\Omega)} \le c\|V\|_{L^r(\Omega)} & \text{if } \frac{\omega^{2-\sigma}}{\epsilon} \ge 1, \end{cases} \tag{4.5}$$

where $r \in (1,\infty)$ and c is a constant independent of ω, ϵ, σ. Multiply (4.3) by the solution of (4.4), multiply (4.4) by the solution of (4.3), integrate by part, as well as employ (4.5), Lemma 2.1, and Hölder inequality to get

$$\begin{aligned} \int_\Omega \mathbf{K}_{\omega^\sigma,\epsilon} \nabla \Phi \; V dx &= \int_\Omega \varphi_\sigma \zeta dy = \int_\Omega \Pi_\epsilon \varphi_\sigma|_{\Omega_f^\epsilon} \zeta dy + \int_{\Omega_m^\epsilon} (\varphi_\sigma - \Pi_\epsilon \varphi_\sigma|_{\Omega_f^\epsilon}) \zeta dy \\ &\le c\|V\|_{L^r(\Omega)} (\|\zeta\|_{W^{-1,p}(\Omega)} + \omega^{\sigma-2} \|\zeta\|_{W^{-1,p}(\Omega_m^\epsilon)}), \end{aligned}$$

where $\frac{1}{r} + \frac{1}{p} = 1$ and c is independent of ω, ϵ, σ. Since $L^\infty(\Omega)$ is dense in $L^r(\Omega)$ for any $r \in (1,\infty)$, we obtain

$$\|\mathbf{K}_{\omega^\sigma,\epsilon} \nabla \Phi\|_{L^p(\Omega)} \le c(\|\zeta\|_{W^{-1,p}(\Omega)} + \omega^{\sigma-2} \|\zeta\|_{W^{-1,p}(\Omega_m^\epsilon)}),$$

where $\frac{1}{r} + \frac{1}{p} = 1$ and c is a constant independent of ω, ϵ, σ. By Poincaré inequality [11] and Lemma 2.1, it is easy to see that

$$\begin{cases} \|\mathbf{K}_{\omega^\sigma/\epsilon,\epsilon} \Phi\|_{L^p(\Omega)} \le c(\|\zeta\|_{W^{-1,p}(\Omega)} + \omega^{\sigma-2} \|\zeta\|_{W^{-1,p}(\Omega_m^\epsilon)}) & \text{if } \frac{\omega^\sigma}{\epsilon} \le 1, \\ \|\Phi\|_{L^p(\Omega)} \le c(\|\zeta\|_{W^{-1,p}(\Omega)} + \omega^{\sigma-2} \|\zeta\|_{W^{-1,p}(\Omega_m^\epsilon)}) & \text{if } \frac{\omega^\sigma}{\epsilon} \ge 1, \end{cases}$$

where $p \in (1,\infty)$ and c is a constant independent of ω, ϵ, σ. Together with Lemma 4.1 and Lemma 4.2, we conclude that Theorem 2.1 holds for $G \in L^p(\Omega), F \in L^\infty(\Omega)$. If $G \in L^p(\Omega), F \in W^{-1,p}(\Omega)$, Theorem 2.1 can be proved by a limiting argument.

5. Hölder estimate and Lipschitz estimate

In this section, we prove Lemma 3.8 (in subsection 5.1) and Lemma 3.10 (in subsection 5.2). The idea of proof is from [4].

5.1. *Uniform Hölder estimate*

We shall use the notation in (3.31).

Lemma 5.1. *Let $\mu \in (0,1)$ and let $0 \in \partial\Omega/r$ or $B_1(0) \subset \Omega/r$. There exist $\theta_1, \theta_2 \in (0,1)$ (depending on $\mu, \|\nabla\Psi\|_{L^\infty(\mathbb{R}^{n-1})}, Y_f$) and $\epsilon_0 \in (0,1)$ (depending on μ, θ_2) such that if*

$$\begin{cases} -\nabla\cdot(\breve{\mathbf{K}}_{\omega^2,\epsilon,r}\nabla\varphi) = 0 & \text{in } B_1(0)\cap\Omega/r, \\ \breve{\mathbf{K}}_{\omega^2,\epsilon,r}\nabla\varphi\cdot\vec{\mathbf{n}}_{\epsilon/r} = 0 & \text{on } B_1(0)\cap\partial\Omega/r, \\ \|\breve{\mathbf{K}}_{\omega^2,\epsilon,r}\varphi\|_{L^2(B_1(0)\cap\Omega/r)} \le 1, \end{cases} \tag{5.1}$$

then, for any $\omega, \epsilon, r \in (0,1]$, $\epsilon/r \le \epsilon_0$, and $\theta \in [\theta_1, \theta_2]$,

$$⨍_{B_\theta(0)\cap\Omega/r} \left|\varphi - (\varphi)_{B_\theta(0)\cap\Omega/r}\right|^2 dx \le \theta^{2\mu}, \tag{5.2}$$

where $\vec{\mathbf{n}}_{\epsilon/r}$ is a unit normal vector on $\partial\Omega/r$. See section 2 for $\breve{\mathbf{K}}_{\omega^2,\epsilon,r}$.

Proof. Consider the following problem

$$\begin{cases} -\nabla\cdot(\mathcal{K}_{\omega_*}\nabla\varphi) = 0 & \text{in } B_{2/3}(0)\cap\Omega/r_*, \\ \mathcal{K}_{\omega_*}\nabla\varphi\cdot\vec{\mathbf{n}}_* = 0 & \text{on } B_{2/3}(0)\cap\partial\Omega/r_*, \end{cases} \tag{5.3}$$

where $\omega_*, r_* \in [0,1]$, $\mathcal{K}_{\omega_*}$ is from (3.20), and $\vec{\mathbf{n}}_*$ is a unit normal vector on $\partial\Omega/r_*$. Note $B_{2/3}(0)\cap\Omega/r_*$ is a bounded convex Lipschitz domain. By (3.21), [17], Theorem 9.11 [11], and Theorem 1.2 in page 70 [10], there exists a sufficiently small $\theta < 2/3$ such that

$$⨍_{B_\theta(0)\cap\Omega/r_*} |\varphi - (\varphi)_{B_\theta(0)\cap\Omega/r_*}|^2 dx \le \theta^{2\mu'} ⨍_{B_{2/3}(0)\cap\Omega/r_*} \varphi^2 dx \tag{5.4}$$

for some μ' satisfying $0 < \mu < \mu' < 1$. We choose $\theta_1, \theta_2 < 2/3$ such that $\theta_1 < \theta_2^2$ and (5.4) holds if $\theta \in [\theta_1, \theta_2]$. Now we claim (5.2). If not, there is a sequence $\{\omega_\epsilon, r_\epsilon, \theta_\epsilon, \varphi_\epsilon\}$ satisfying (5.1) and

$$\begin{cases} \omega_\epsilon, r_\epsilon \to \omega_*, r_* \in [0,1] \\ \theta_\epsilon \to \theta_* \in [\theta_1, \theta_2] \\ ⨍_{B_{\theta_\epsilon}(0)\cap\Omega/r_\epsilon} \left|\varphi_\epsilon - (\varphi_\epsilon)_{B_{\theta_\epsilon}(0)\cap\Omega/r_\epsilon}\right|^2 dx > \theta_\epsilon^{2\mu} \end{cases} \quad \text{as } \epsilon/r_\epsilon \to 0. \tag{5.5}$$

By Lemma 3.6 and by tracing the proof of Theorem 2.3 [3], there is a subsequence (same notation for subsequence) such that

$$\begin{cases} \varphi_\epsilon \to \varphi_* & \text{in } L^2(B_{2/3}(0)\cap\Omega/r_*) \text{ strongly} \\ \breve{\mathbf{K}}_{\omega_\epsilon^2,\epsilon,r_\epsilon}\nabla\varphi_\epsilon \to \mathcal{K}_{\omega^*}\nabla\varphi_* & \text{in } L^2(B_{2/3}(0)\cap\Omega/r_*) \text{ weakly} \end{cases} \quad \text{as } \epsilon/r_\epsilon \to 0. \tag{5.6}$$

Also the φ_* in (5.6) is a solution of (5.3). By (5.4)–(5.6), we conclude

$$\theta_*^{2\mu} = \lim_{\epsilon/r_\epsilon\to 0} \theta_\epsilon^{2\mu} \leq \lim_{\epsilon/r_\epsilon\to 0} \fint_{B_{\theta_\epsilon}(0)\cap\Omega/r_\epsilon} |\varphi_\epsilon - (\varphi_\epsilon)_{B_{\theta_\epsilon}(0)\cap\Omega/r_\epsilon}|^2 dx$$
$$= \fint_{B_{\theta_*}(0)\cap\Omega/r_*} |\varphi_* - (\varphi_*)_{B_{\theta_*}(0)\cap\Omega/r_*}|^2 dx \leq \theta_*^{2\mu'} \fint_{B_{2/3}(0)\cap\Omega/r_*} \varphi_*^2 dx. \quad (5.7)$$

But (5.7) is impossible if θ_2 is small enough. So there is a ϵ_0 such that (5.2) holds for $\epsilon/r \leq \epsilon_0$. □

Lemma 5.2. *Let $\mu \in (0,1)$ and let $0 \in \partial\Omega$ or $B_1(0) \subset \Omega$. There exist $\theta_1, \theta_2 \in (0,1)$ (depending on $\mu, \|\nabla\Psi\|_{L^\infty(\mathbb{R}^{n-1})}, Y_f$) and $\epsilon_0 \in (0,1)$ (depending on μ, θ_2) such that if*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}\nabla\Phi) = 0 & \text{in } B_1(0)\cap\Omega, \\ \mathbf{K}_{\omega^2,\epsilon}\nabla\Phi\cdot\vec{\mathbf{n}} = 0 & \text{on } B_1(0)\cap\partial\Omega, \end{cases} \quad (5.8)$$

then, for any $\omega \in (0,1]$, $\epsilon \in (0,\epsilon_0]$, $\theta \in [\theta_1,\theta_2]$, and k satisfying $\epsilon/\theta^k \leq \epsilon_0$,

$$\fint_{B_{\theta^k}(0)\cap\Omega} \left|\Phi - (\Phi)_{B_{\theta^k}(0)\cap\Omega}\right|^2 dx \leq \theta^{2k\mu}|J_{\omega,\epsilon}|^2, \quad (5.9)$$

where $J_{\omega,\epsilon} \equiv \|\mathbf{K}_{\omega^2,\epsilon}\Phi\|_{L^2(B_1(0)\cap\Omega)}$ and $\vec{\mathbf{n}}$ is a normal vector on $\partial\Omega$.

Proof. The proof is done by induction on k. For $k=1$, set $\varphi = \Phi/J_{\omega,\epsilon}$. Then φ satisfies (5.1) with $r=1$. (5.9) for $k=1$ is deduced from Lemma 5.1. Suppose (5.9) holds for some k satisfying $\epsilon/\theta^k \leq \epsilon_0$, we define

$$\varphi(x) \equiv J_{\omega,\epsilon}^{-1}\theta^{-k\mu}\big(\Phi(\theta^k x) - (\Phi)_{B_{\theta^k}(0)\cap\Omega}\big) \qquad \text{in } B_1(0)\cap\Omega/\theta^k.$$

It is easy to see that φ satisfies (5.1) with $r = \theta^k$. By Lemma 5.1 and changing variable, we obtain (5.9) with $k+1$ in place of k. □

Lemma 5.3. *Let $\mu \in (0,1)$ and let $B_1(0) \subset \Omega$. There is a $\epsilon_* \in (0,1)$ (depending on $\mu, \|\nabla\Psi\|_{L^\infty(\mathbb{R}^{n-1})}, Y_f$) such that if $\omega \in (0,1]$ and $\epsilon \in (0,\epsilon_*]$, then any solution of (5.8) satisfies*

$$[\Phi]_{C^{0,\mu}(\overline{B_{1/2}(0)})} \leq c\|\mathbf{K}_{\omega^2,\epsilon}\Phi\|_{L^2(B_1(0))}, \quad (5.10)$$

where c is a constant independent of ω,ϵ but depending on $\mu, \|\nabla\Psi\|_{L^\infty(\mathbb{R}^{n-1})}, Y_f$.

Proof. Let $\theta_1, \theta_2, \epsilon_0, J_{\omega,\epsilon}$ be same as those in Lemma 5.2 and define $\epsilon_* \equiv \epsilon_0\theta_2/2$. Denote by c a constant independent of ω,ϵ. Since $\theta_1 < \theta_2^2$, for any $r \in [\epsilon/\epsilon_0, \theta_2]$, there are $\theta \in [\theta_1,\theta_2]$ and $k \in \mathbb{N}$ satisfying $r = \theta^k$. Lemma 5.2 implies

$$\fint_{B_r(0)} \left|\Phi - (\Phi)_{B_r(0)}\right|^2 dx \leq r^{2\mu}|J_{\omega,\epsilon}|^2 \qquad \text{for any } r \in [\epsilon/\epsilon_0, \theta_2]. \quad (5.11)$$

Now we define

$$\varphi(x) \equiv J_{\omega,\epsilon}^{-1}\epsilon^{-\mu}\big(\Phi(\epsilon x) - (\Phi)_{B_{2\epsilon/\epsilon_0}(0)}\big) \qquad \text{in } B_{2/\epsilon_0}(0).$$

By (5.11),

$$\begin{cases} -\nabla\cdot(\breve{\mathbf{K}}_{\omega^2,\epsilon,\epsilon}\nabla\varphi)=0 & \text{in } B_{2/\epsilon_0}(0),\\ \|\varphi\|_{L^2(B_{2/\epsilon_0}(0))}\le c. \end{cases}$$

By Lemma 3.3, we know $[\varphi]_{C^{0,\mu}(\overline{B_{1/\epsilon_0}(0)})}\le c$. Together with (5.11), then (5.11) holds for $r\in(0,\theta_2)$. Next we shift the origin of the coordinate system to any point $z\in B_{1/2}(0)$ and repeat above argument to see that (5.11) with 0 replaced by any $z\in B_{1/2}(0)$ also holds for $r\in(0,\theta_2)$. By Theorem 1.2 in page 70 [10], we obtain the Hölder estimate (5.10). □

Remark 5.1. Let ϵ_* be same as that in Lemma 5.3. By Lemma 3.3, we know that if $\omega\in(0,1]$, $\epsilon\in[\epsilon_*,1]$, any solution of (5.8) for $B_1(0)\subset\Omega$ satisfies (5.10). Together with Lemma 5.3, we know that any solution of (5.8) for $B_1(0)\subset\Omega$ satisfies (5.10) if $\omega,\epsilon\in(0,1]$.

Lemma 5.4. *Let $\mu\in(0,1)$ and $0\in\partial\Omega$. There exists a $\tilde\epsilon_*\in(0,1)$ (depending on $\mu,\|\nabla\Psi\|_{L^\infty(\mathbb{R}^{n-1})},Y_f$) such that if $\omega\in(0,1]$ and $\epsilon\le\tilde\epsilon_*$, then any solution of (5.8) satisfies*

$$[\Phi]_{C^{0,\mu}(\overline{B_{1/2}(0)\cap\Omega})}\le c\|\mathbf{K}_{\omega^2,\epsilon}\Phi\|_{L^2(B_1(0)\cap\Omega)}, \tag{5.12}$$

where c is a constant independent of ω,ϵ but depending on $\mu,\|\nabla\Psi\|_{L^\infty(\mathbb{R}^{n-1})},Y_f$.

Proof. Let $\theta_1,\theta_2,\epsilon_0,J_{\omega,\epsilon}$ be those in Lemma 5.2 and define $\tilde\epsilon_*\equiv\min\{\epsilon_0\theta_2/3,\epsilon_*\}$ where ϵ_* is the one in Lemma 5.3. Denote by c a constant independent of ω,ϵ. For any $x\in B_{\theta_2/3}(0)\cap\Omega$, define $\xi(x)\equiv|x-x_0|$ where $x_0\in\partial\Omega$ satisfying $|x-x_0|=\min_{y\in\partial\Omega}|x-y|$. Then we have either case (1) $\xi(x)>\frac{2\epsilon}{3\epsilon_0}$ or case (2) $\xi(x)\le\frac{2\epsilon}{3\epsilon_0}$.

Let us consider case (1). Because of $\theta_1<\theta_2^2$, for any $r\in[\epsilon/\epsilon_0,\theta_2]$, there are $\theta\in[\theta_1,\theta_2]$ and $k\in\mathbb{N}$ satisfying $r=\theta^k$. Since $\xi(x)\in[\frac{2\epsilon}{3\epsilon_0},\frac{\theta_2}{3}]$, by Lemma 5.2,

$$⨍_{B_r(x_0)\cap\Omega}\left|\Phi(y)-(\Phi)_{B_r(x_0)\cap\Omega}\right|^2dy\le r^{2\mu}|J_{\omega,\epsilon}|^2\qquad\text{for } r\in[\tfrac{3}{2}\xi(x),\theta_2].$$

So, for $s\in[\frac{\xi(x)}{2},\frac{\theta_2}{3}]$,

$$⨍_{B_s(x)\cap\Omega}\left|\Phi(y)-(\Phi)_{B_s(x)\cap\Omega}\right|^2dy\le cs^{2\mu}|J_{\omega,\epsilon}|^2. \tag{5.13}$$

Next we move the origin of the coordinate system to x and define

$$\varphi(y)\equiv J_{\omega,\epsilon}^{-1}\xi^{-\mu}(x)\left(\Phi(\xi(x)y)-(\Phi)_{B_{\xi(x)}(x)}\right)\qquad\text{in } B_1(x).$$

Then φ satisfies

$$-\nabla\cdot(\breve{\mathbf{K}}_{\omega^2,\epsilon,\xi(x)}\nabla\varphi)=0\quad\text{in } B_1(x). \tag{5.14}$$

Take $s = \xi(x) < 1$ in (5.13) to see $\|\varphi\|_{L^2(B_1(x))} \le c$. Apply Remark 5.1 to (5.14) to obtain $[\varphi]_{C^{0,\mu}(\overline{B_{1/2}(x)})} \le c$. Which implies

$$\fint_{B_s(x)} \left|\Phi(y) - (\Phi)_{B_s(x)}\right|^2 dy \le cs^{2\mu}|J_{\omega,\epsilon}|^2 \quad \text{for } s < \tfrac{\xi(x)}{2}. \tag{5.15}$$

Next we consider case (2). Because of $\theta_1 < \theta_2^2$, for any $r \in [\epsilon/\epsilon_0, \theta_2]$, there are $\theta \in [\theta_1, \theta_2]$ and $k \in \mathbb{N}$ satisfying $r = \theta^k$. By Lemma 5.2,

$$\fint_{B_r(x_0)\cap\Omega} \left|\Phi(y) - (\Phi)_{B_r(x_0)\cap\Omega}\right|^2 dy \le r^{2\mu}|J_{\omega,\epsilon}|^2 \qquad \text{for } r \in [\epsilon/\epsilon_0, \theta_2].$$

This implies, for $s \in [\frac{\epsilon}{3\epsilon_0}, \frac{\theta_2}{3}]$,

$$\fint_{B_s(x)\cap\Omega} \left|\Phi(y) - (\Phi)_{B_s(x)\cap\Omega}\right|^2 dy \le cs^{2\mu}|J_{\omega,\epsilon}|^2. \tag{5.16}$$

Again we move the origin of the coordinate system to x and define

$$\varphi(y) \equiv J_{\omega,\epsilon}^{-1}\epsilon^{-\mu}\left(\Phi(\epsilon y) - (\Phi)_{B_{\epsilon/\epsilon_0}(x)\cap\Omega}\right) \qquad \text{in } B_{1/\epsilon_0}(x) \cap \Omega/\epsilon.$$

Then φ satisfies

$$\begin{cases} -\nabla\cdot(\breve{\mathbf{K}}_{\omega^2,\epsilon,\epsilon}\nabla\varphi) = 0 & \text{in } B_{1/\epsilon_0}(x)\cap\Omega/\epsilon, \\ \breve{\mathbf{K}}_{\omega^2,\epsilon,\epsilon}\nabla\varphi\cdot\vec{\mathbf{n}} = 0 & \text{on } B_{1/\epsilon_0}(x)\cap\partial\Omega/\epsilon, \end{cases}$$

where $\vec{\mathbf{n}}$ is a unit normal vector on $\partial\Omega/\epsilon$. Let us take $s = \epsilon/\epsilon_0$ in (5.16) to see $\|\varphi\|_{L^2(B_{1/\epsilon_0}(x)\cap\Omega/\epsilon)} \le c$. By Lemma 3.3,

$$[\varphi]_{C^{0,\mu}(\overline{B_{1/2\epsilon_0}(x)\cap\Omega/\epsilon})} \le c. \tag{5.17}$$

(5.17) implies that (5.16) holds for $s \le \frac{\epsilon}{2\epsilon_0}$.

The Hölder estimate of Φ follows from (5.13), (5.15), (5.16), (5.17), and Theorem 1.2 in page 70 [10]. □

Remark 5.2. Let $\tilde{\epsilon}_*$ be same as that in Lemma 5.4. By Lemma 3.3, we know that if $\omega \in (0,1]$ and $\epsilon \in [\tilde{\epsilon}_*, 1]$, any solution of (5.8) for $0 \in \partial\Omega$ satisfies (5.12). Together with Lemma 5.4, any solution of (5.8) satisfies (5.12) if $\omega, \epsilon \in (0,1]$.

By partition of unity, maximal principle, Remark 5.1, and Remark 5.2, we obtain Lemma 3.8.

5.2. *Uniform Lipschitz estimate*

We assume $B_1(0) \subset \Omega$.

Lemma 5.5. *There are constants $\theta, \epsilon_0 \in (0,1)$ such that if*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,\nu}\nabla\varphi) = 0 & \text{in } B_1(0), \\ \|\mathbf{K}_{\omega^2,\nu}\varphi\|_{L^2(B_1(0))} \le 1, \end{cases} \tag{5.18}$$

then, for any $\omega \in (0,1]$ *and* $\nu \in (0,\epsilon_0)$,

$$\sup_{B_\theta(0)} \left|\varphi(x) - \varphi(0) - \big(x + \mathbb{X}_{\omega,\nu}(x)\big)\mathbf{b}_{\omega,\nu}\right| \le \theta^{4/3}, \tag{5.19}$$

where $\mathbf{b}_{\omega,\nu} \equiv \mathcal{K}_\omega^{-1} \fint_{B_\theta(0)} \mathbf{K}_{\omega^2,\nu}\nabla\varphi dx$ *and* $\mathcal{K}_\omega^{-1}$ *is the inverse matrix of* $\mathcal{K}_\omega$. *See (3.19)–(3.20) for* $\mathbb{X}_{\omega,\nu}, \mathcal{K}_\omega$.

Proof. Assume $\omega_* \in [0,1]$ and $-\nabla\cdot(\mathcal{K}_{\omega_*}\nabla\varphi) = 0$ in $B_{2/3}(0)$. There is a small $\theta \in (0,1)$ such that, by (3.21),

$$\sup_{B_\theta(0)} \left|\varphi(x) - \varphi(0) - x(\nabla\varphi)_{B_\theta(0)}\right| \le \theta^{3/2}\|\varphi\|_{L^\infty(B_{2/3}(0))}. \tag{5.20}$$

Fix a small $\theta < 2/3$ so that (5.20) holds and we claim (5.19). If not, there is a sequence $\{\omega_\nu, \varphi_\nu\}$ satisfying (5.18) and, as $\nu \to 0$,

$$\begin{cases} \omega_\nu \to \omega_* \in [0,1], \\ \sup\limits_{B_\theta(0)} \left|\varphi_\nu(x) - \varphi_\nu(0) - \big(x + \mathbb{X}_{\omega,\nu}(x)\big)\mathbf{b}_{\omega,\nu}\right| > \theta^{4/3}. \end{cases} \tag{5.21}$$

By Lemma 3.8 and by tracing the proof of Theorem 2.3 [3], there is a subsequence (same notation for subsequence) such that,

$$\begin{cases} \varphi_\nu \to \varphi_* & \text{in } C(\overline{B_{2/3}(0)}) \text{ strongly} \\ \mathbf{K}_{\omega_\nu^2,\nu}\nabla\varphi_\nu \to \mathcal{K}_{\omega_*}\nabla\varphi_* & \text{in } L^2(B_{2/3}(0)) \text{ weakly} \end{cases} \quad \text{as } \nu \to 0. \tag{5.22}$$

(5.22) implies that φ_* satisfies $-\nabla\cdot(\mathcal{K}_{\omega_*}\nabla\varphi_*) = 0$ in $B_{2/3}(0)$. Together with (5.20), (5.21), and (5.22), we get contradiction if θ is small. So (5.19) holds. ◻

Lemma 5.6. *There are constants* $\theta, \epsilon_0 \in (0,1)$ *such that if* Φ *satisfies*

$$-\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}\nabla\Phi) = 0 \quad \textit{in } B_1(0), \tag{5.23}$$

then, for any $\omega \in (0,1]$, $\epsilon \in (0,\epsilon_0)$, *and* k *satisfying* $\epsilon/\theta^k \le \epsilon_0$, *there are constants* $\mathbf{a}_k^{\omega,\epsilon}$, $\mathbf{b}_k^{\omega,\epsilon}$ *so that*

$$\begin{cases} |\mathbf{a}_k^{\omega,\epsilon}| + |\mathbf{b}_k^{\omega,\epsilon}| \le cJ_{\omega,\epsilon}, \\ \sup\limits_{B_{\theta^k}(0)} \left|\Phi(x) - \Phi(0) - \epsilon\mathbf{a}_k^{\omega,\epsilon} - \big(x + \mathbb{X}_{\omega,\epsilon}(x)\big)\mathbf{b}_k^{\omega,\epsilon}\right| \le \theta^{4k/3}J_{\omega,\epsilon}, \end{cases} \tag{5.24}$$

where c *is a constant independent of* ω, ϵ *and* $J_{\omega,\epsilon} \equiv \|\mathbf{K}_{\omega^2,\epsilon}\Phi\|_{L^2(B_1(0))}$.

Proof. If $\varphi \equiv \Phi/J_{\omega,\epsilon}$, then it satisfies (5.18) with $\nu = \epsilon$. By Lemma 5.5, we obtain (5.24) for $k=1$ case where $\mathbf{a}_1^{\omega,\epsilon} = 0$, $\mathbf{b}_1^{\omega,\epsilon} = \mathcal{K}_\omega^{-1}\fint_{B_\theta(0)} \mathbf{K}_{\omega^2,\epsilon}\nabla\Phi dx$. If (5.24) holds for some k satisfying $\epsilon/\theta^k \le \epsilon_0$, we define

$$\varphi(x) \equiv \frac{\Phi(\theta^k x) - \Phi(0) - \epsilon\mathbf{a}_k^{\omega,\epsilon} - \big(\theta^k x + \mathbb{X}_{\omega,\epsilon}(\theta^k x)\big)\mathbf{b}_k^{\omega,\epsilon}}{\theta^{4k/3}J_{\omega,\epsilon}} \quad \text{in } B_1(0).$$

By induction and (3.18), we see

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon/\theta^k}\nabla\varphi)=0 & \text{in } B_1(0),\\ \|\varphi\|_{L^\infty(B_1(0))}\le 1. \end{cases} \tag{5.25}$$

Apply Lemma 5.5 to obtain

$$\sup_{B_\theta(0)}\left|\varphi(x)-\varphi(0)-\big(x+\mathbb{X}_{\omega,\epsilon/\theta^k}(x)\big)\mathbf{b}_{\omega,\epsilon/\theta^k}\right|\le\theta^{4/3}, \tag{5.26}$$

where $\mathbf{b}_{\omega,\epsilon/\theta^k}\equiv\mathcal{K}_\omega^{-1}\fint_{B_\theta(0)}\mathbf{K}_{\omega^2,\epsilon/\theta^k}\nabla\varphi dx$. By Lemma 2.1, (5.26) can be written as

$$\begin{aligned}\sup_{B_\theta(0)}\Big|\Phi(\theta^k x)-\Phi(0)+\epsilon\mathbb{X}_{\omega,1}(0)\mathbf{b}_k^{\omega,\epsilon}-\big(\theta^k x+\mathbb{X}_{\omega,\epsilon}(\theta^k x)\big)\mathbf{b}_k^{\omega,\epsilon}\\ -J_{\omega,\epsilon}\theta^{4k/3}\big(x+\theta^{-k}\mathbb{X}_{\omega,\epsilon}(\theta^k x)\big)\mathbf{b}_{\omega,\epsilon/\theta^k}\Big|\le J_{\omega,\epsilon}\theta^{4(k+1)/3}.\end{aligned} \tag{5.27}$$

Define

$$\mathbf{a}_{k+1}^{\omega,\epsilon}\equiv-\mathbb{X}_{\omega,1}(0)\mathbf{b}_k^{\omega,\epsilon} \qquad\text{and}\qquad \mathbf{b}_{k+1}^{\omega,\epsilon}\equiv\mathbf{b}_k^{\omega,\epsilon}+J_{\omega,\epsilon}\theta^{k/3}\mathbf{b}_{\omega,\epsilon/\theta^k}. \tag{5.28}$$

By (5.25) and energy method, $|\mathbf{b}_{\omega,\epsilon/\theta^k}|$ is bounded uniformly in ω,ϵ,k. So $(5.24)_1$ holds. Substituting (5.28) into (5.27) and changing variable, we obtain $(5.24)_2$. □

Lemma 5.7. *There is a constant $\epsilon_0\in(0,1)$ such that if $\omega\in(0,1]$ and $\epsilon\in(0,\epsilon_0)$, any solution of (5.23) satisfies*

$$\|\nabla\Phi\|_{L^\infty(B_{1/2}(0))}\le c\|\mathbf{K}_{\omega^2,\epsilon}\Phi\|_{L^2(B_1(0))}, \tag{5.29}$$

where c is a constant independent of ω,ϵ.

Proof. Let $\theta,\epsilon_0,J_{\omega,\epsilon}$ be same as those in Lemma 5.6. Let $k\in\mathbb{N}$ such that $\epsilon/\theta^k\le\epsilon_0<\epsilon/\theta^{k+1}$. By Lemma 5.6,

$$\sup_{B_{\frac{\epsilon}{\epsilon_0}}(0)}\big|\Phi(x)-\Phi(0)-\epsilon\mathbf{a}_k^{\omega,\epsilon}-(x+\mathbb{X}_{\omega,\epsilon}(x))\mathbf{b}_k^{\omega,\epsilon}\big|\le c\Big|\frac{\epsilon}{\epsilon_0}\Big|^{4/3}J_{\omega,\epsilon}.$$

Define

$$\varphi(x)\equiv\frac{\Phi(\epsilon x)-\Phi(0)-\epsilon\mathbf{a}_k^{\omega,\epsilon}-\big(\epsilon x+\mathbb{X}_{\omega,\epsilon}(\epsilon x)\big)\mathbf{b}_k^{\omega,\epsilon}}{\epsilon^{4/3}J_{\omega,\epsilon}} \qquad \text{in } B_{1/\epsilon_0(0)}.$$

Then φ satisfies

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,1}\nabla\varphi)=0 & \text{in } B_{1/\epsilon_0}(0),\\ \|\varphi\|_{L^\infty(B_{1/\epsilon_0}(0))}\le c. \end{cases}$$

Lemma 3.3 implies

$$\|\varphi\|_{C^{1,0}(\overline{B_{1/2\epsilon_0}(0)\cap\Omega_f^\epsilon/\epsilon})\cap C^{1,0}(\overline{B_{1/2\epsilon_0}(0)\cap\Omega_m^\epsilon/\epsilon})}\le c. \tag{5.30}$$

Since $\nabla\varphi(x)=\frac{\nabla\Phi(\epsilon x)-(I+\nabla\mathbb{X}_{\omega,1}(x))\mathbf{b}_k^{\omega,\epsilon}}{\epsilon^{1/3}J_{\omega,\epsilon}}$, $|\nabla\Phi(\epsilon x)|\le cJ_{\omega,\epsilon}$ for $x\in B_{1/2\epsilon_0}(0)$ by (3.19), (5.30), and Lemma 5.6. We prove (5.29). □

Remark 5.3. Let ϵ_0 be same as that in Lemma 5.7. By $(3.3)_2$ of Lemma 3.3, we know that if $\omega \in (0,1]$ and $\epsilon \in [\epsilon_0, 1]$, any solution of (5.23) satisfies (5.29). Together with Lemma 5.7, any solution of (5.23) satisfies (5.29) if $\omega, \epsilon \in (0,1]$.

Lemma 3.10 follows from Remark 5.3.

6. Appendix

Now we give a proof of Theorem 4.1. Let Q and tQ denote the concentric cubes sharing the same center and satisfying $tx \in tQ$ if $x \in Q, t > 0$.

Proof. With possibly different constants t_1, t_2, c_0, r_0, one may replace balls B in (4.1) by cubes Q of edge length $r \in (0, r_0)$.

Step 1: Fix $q \in (2,p)$, choose a cube Q_0 satisfying $\Omega \subset Q_0$, and let $\delta \in (0,1)$ be a small constant so that

$$\begin{cases} \ell_\delta \equiv 1/(2\delta^{2/q}) > 5^n, \\ 4^n |Q_0|/\ell_\delta < r_0^n. \end{cases} \tag{6.1}$$

Here $|Q_0|$ is the volume of Q_0. For any $\varphi \in L^\infty(\Omega)$ and $\nu > 0$, we define

$$\mathbf{E}(\varphi, \nu) \equiv \{x \in Q_0 : \mathbf{M}_{2Q_0}(|\mathbf{T}\varphi|^2 \mathcal{X}_\Omega)(x) > \nu\}, \tag{6.2}$$

where $\mathbf{M}_D$ is a localized maximal function defined by

$$\mathbf{M}_D(\zeta)(x) \equiv \sup_{x \in D' \subset D} ⨍_{D'} |\zeta(y)| dy \quad \text{for } x \in D.$$

By Theorem 1 in page 5 [21], there is a constant c_1 (depending only on $n, \|\mathbf{T}\|_{L^2(\Omega)}$) such that if $\nu \geq \rho_{\delta,\varphi} \equiv \frac{c_1}{\delta |Q_0|} \|\varphi\|^2_{L^2(\Omega)}$, then

$$|\mathbf{E}(\varphi, \nu)| \leq \frac{c_1}{\nu} \|\varphi\|^2_{L^2(\Omega)} \leq \delta |Q_0|. \tag{6.3}$$

Set $\lambda \geq \rho_{\delta,\varphi}$ and employ the Calderón-Zygmund decomposition to $\mathbf{E}(\varphi, \lambda\ell_\delta)$ (see [21]). This produces a collection of disjoint dyadic subcubes $\{Q_k\}_{k=1}^\infty$ of Q_0 satisfying

$$\begin{cases} |\mathbf{E}(\varphi, \lambda\ell_\delta) \setminus \bigcup_{k=1}^\infty Q_k| = 0 \\ |\mathbf{E}(\varphi, \lambda\ell_\delta) \cap Q_k| > \delta |Q_k| \\ |\mathbf{E}(\varphi, \lambda\ell_\delta) \cap \widetilde{Q}_k| \leq \delta |\widetilde{Q}_k| \end{cases} \quad \text{for } k \geq 1, \lambda \geq \rho_{\delta,\varphi}, \tag{6.4}$$

where $\widetilde{Q}_k$ denotes the dyadic "parent" of Q_k (i.e., Q_k is one of the 2^n cubes obtained by bisecting the sides of $\widetilde{Q}_k$). This sequence $\{Q_k\}_{k=1}^\infty$ can be constructed by proceeding as that in the proof of Lemma 1.1 [7] for each $|\mathbf{E}(\varphi, \lambda\ell_\delta) \cap \widetilde{Q}_k| \leq \delta |\widetilde{Q}_k|$.

By (6.3) and $(6.4)_2$, for $k \geq 1$ and $\lambda \geq \rho_{\delta,\varphi}$,

$$\delta |Q_k| < |\mathbf{E}(\varphi, \lambda\ell_\delta)| \leq \frac{c_1}{\lambda \ell_\delta} \|\varphi\|^2_{L^2(\Omega)} \leq \frac{\delta}{\ell_\delta} |Q_0|.$$

It follows that $|Q_k| < |Q_0|/\ell_\delta$ for $k \geq 1$. By $(6.1)_2$, the edge length of $2\widetilde{Q}_k$ for $k \geq 1$ is less than r_0.

Step 2: With the δ from Step 1, let us assume that there exists a constant $\beta \in (0,1)$ such that whenever $\widetilde{Q}_k \cap \{x \in Q_0 : \mathbf{M}_{2Q_0}(|\varphi|^2\mathcal{X}_\Omega)(x) \le \beta\lambda\} \neq \emptyset$ for any $\varphi \in L^\infty(\Omega)$, $k \ge 1$, and $\lambda > \rho_{\delta,\varphi}$, then

$$\widetilde{Q}_k \subset \mathbf{E}(\varphi, \lambda). \tag{6.5}$$

Suppose (6.5) is true, then $(6.4)_{1,3}$ imply

$$\begin{aligned}&|\mathbf{E}(\varphi,\lambda\ell_\delta) \cap \{x \in Q_0 : \mathbf{M}_{2Q_0}(|\varphi|^2\mathcal{X}_\Omega)(x) \le \beta\lambda\}|\\ &\quad \le \sum_{k'} |\mathbf{E}(\varphi,\lambda\ell_\delta) \cap \widetilde{Q}_{k'}| \le \delta \sum_{k'} |\widetilde{Q}_{k'}| \le \delta|\mathbf{E}(\varphi,\lambda)|,\end{aligned}$$

where $\{\widetilde{Q}_{k'}\}$ is a disjoint subcover of $\mathbf{E}(\varphi,\lambda\ell_\delta) \cap \{x \in Q_0 : \mathbf{M}_{2Q_0}(|\varphi|^2\mathcal{X}_\Omega)(x) \le \beta\lambda\}$ with $\widetilde{Q}_{k'} \cap \{x \in Q_0 : \mathbf{M}_{2Q_0}(|\varphi|^2\mathcal{X}_\Omega)(x) \le \beta\lambda\} \neq \emptyset$. In other words, there is a constant $\beta > 0$ such that, for any $\lambda > \rho_{\delta,\varphi}$ and $\varphi \in L^\infty(\Omega)$,

$$|\mathbf{E}(\varphi,\lambda\ell_\delta)| \le \delta|\mathbf{E}(\varphi,\lambda)| + |\{x \in Q_0 : \mathbf{M}_{2Q_0}(|\varphi|^2\mathcal{X}_\Omega)(x) > \beta\lambda\}|. \tag{6.6}$$

So (6.6) implies, for any $t > \rho_{\delta,\varphi}$,

$$\begin{aligned}\int_0^{t\ell_\delta} s^{\frac{q}{2}-1}|\mathbf{E}(\varphi,s)|ds &= \left(\int_0^{\rho_{\delta,\varphi}\ell_\delta} + \int_{\rho_{\delta,\varphi}\ell_\delta}^{t\ell_\delta}\right) s^{\frac{q}{2}-1}|\mathbf{E}(\varphi,s)|ds\\ &\le \int_0^{\rho_{\delta,\varphi}\ell_\delta} s^{\frac{q}{2}-1}|\mathbf{E}(\varphi,s)|ds + \delta\ell_\delta^{\frac{q}{2}} \int_{\rho_{\delta,\varphi}}^{t} \lambda^{\frac{q}{2}-1}|\mathbf{E}(\varphi,\lambda)|d\lambda + c(\delta,q,\beta)\int_\Omega |\varphi|^q dx.\end{aligned}$$

By $(6.1)_1$, we know $\delta\ell_\delta^{q/2} = 1/2^{q/2} < 1$ and $\ell_\delta > 1$. So

$$\int_0^{t\ell_\delta} \lambda^{\frac{q}{2}-1}|\mathbf{E}(\varphi,\lambda)|d\lambda \le c\left(\int_0^{\rho_{\delta,\varphi}\ell_\delta} \lambda^{\frac{q}{2}-1}|\mathbf{E}(\varphi,\lambda)|d\lambda + \int_\Omega |\varphi|^q dx\right), \tag{6.7}$$

where c depends on δ, q, β. (6.2), (6.7), and Corollary 1 in page 5 [21] imply

$$\begin{aligned}\int_0^{t\ell_\delta} \lambda^{\frac{q}{2}-1}|\{x \in \Omega : |\mathbf{T}\varphi(x)|^2 > \lambda\}|d\lambda &\le \int_0^{t\ell_\delta} \lambda^{\frac{q}{2}-1}|\mathbf{E}(\varphi,\lambda)|d\lambda\\ &\le c\left(|\rho_{\delta,\varphi}\ell_\delta|^{\frac{q}{2}}|Q_0| + \int_\Omega |\varphi|^q dx\right).\end{aligned} \tag{6.8}$$

Let $t \to \infty$ in (6.8) to see that $\|\mathbf{T}\varphi\|_{L^q(\Omega)} \le c\|\varphi\|_{L^q(\Omega)}$, where c depends on $\delta, q, \beta, r_0, |Q_0|$.

Step 3: Now we prove (6.5). This is done by contradiction. If not, for any $\beta \in (0,1)$, there is a $\widetilde{Q}_k$ satisfying $\widetilde{Q}_k \setminus \mathbf{E}(\varphi,\lambda) \neq \emptyset$ and $\widetilde{Q}_k \cap \{x \in Q_0 : \mathbf{M}_{2Q_0}(|\varphi|^2\mathcal{X}_\Omega)(x) \le \beta\lambda\} \neq \emptyset$ for some $\varphi \in L^\infty(\Omega)$, $k \ge 1$, and $\lambda > \rho_{\delta,\varphi}$. So if D satisfies $\widetilde{Q}_k \subset D \subset 2Q_0$, then

$$\begin{cases} \displaystyle⨍_D |\mathbf{T}\varphi|^2\mathcal{X}_\Omega \, dx \le \lambda,\\ \displaystyle⨍_D |\varphi|^2\mathcal{X}_\Omega \, dx \le \beta\lambda. \end{cases} \tag{6.9}$$

It follows that for $x \in Q_k$, by $(6.9)_1$,

$$\mathbf{M}_{2Q_0}(|\mathbf{T}\varphi|^2 \mathcal{X}_\Omega)(x) \le \max(\mathbf{M}_{2\widetilde{Q}_k}(|\mathbf{T}\varphi|^2 \mathcal{X}_\Omega)(x), 5^n \lambda). \tag{6.10}$$

By $(6.1)_1$, (6.10), and tracing the proof of Theorem 1 in page 5 [21],

$$\begin{aligned}
|\mathbf{E}(\varphi, \lambda\ell_\delta) \cap Q_k| &= |\{x \in Q_k : \mathbf{M}_{2\widetilde{Q}_k}(|\mathbf{T}\varphi|^2 \mathcal{X}_\Omega)(x) > \lambda\ell_\delta\}| \\
&\le |\{x \in Q_k : \mathbf{M}_{2\widetilde{Q}_k}(|\mathbf{T}(\varphi\mathcal{X}_{\Omega\cap t_2\widetilde{Q}_k})|^2 \mathcal{X}_\Omega)(x) > \lambda\ell_\delta/4\}| \\
&\quad + |\{x \in Q_k : \mathbf{M}_{2\widetilde{Q}_k}(|\mathbf{T}(\varphi\mathcal{X}_{\Omega\setminus t_2\widetilde{Q}_k})|^2 \mathcal{X}_\Omega)(x) > \lambda\ell_\delta/4\}| \\
&\le \frac{c}{\lambda\ell_\delta} \int_{\Omega\cap 2\widetilde{Q}_k} |\mathbf{T}(\varphi\mathcal{X}_{\Omega\cap t_2\widetilde{Q}_k})|^2 dx + \frac{c}{(\lambda\ell_\delta)^{p/2}} \int_{\Omega\cap 2\widetilde{Q}_k} |\mathbf{T}(\varphi\mathcal{X}_{\Omega\setminus t_2\widetilde{Q}_k})|^p dx.
\end{aligned}$$

By the L^2 boundedness of $\mathbf{T}$, construction of $\widetilde{Q}_k$ from Step 1, (4.1), and (6.9),

$$\begin{aligned}
&\int_{\Omega\cap 2\widetilde{Q}_k} |\mathbf{T}(\varphi\mathcal{X}_{\Omega\cap t_2\widetilde{Q}_k})|^2 dx \le c \int_{t_2\widetilde{Q}_k} |\varphi|^2 \mathcal{X}_\Omega \, dx \le c\beta\lambda |t_2\widetilde{Q}_k|, \\
&\left(\fint_{2\widetilde{Q}_k} |\mathbf{T}(\varphi\mathcal{X}_{\Omega\setminus t_2\widetilde{Q}_k})|^p \mathcal{X}_\Omega dx \right)^{\frac{1}{p}} \le c \left(\fint_{t_1 2\widetilde{Q}_k} |\mathbf{T}(\varphi\mathcal{X}_{\Omega\setminus t_2\widetilde{Q}_k})|^2 \mathcal{X}_\Omega dx \right)^{\frac{1}{2}} + c\sqrt{\beta\lambda} \\
&\qquad \le c \left(\fint_{t_1 2\widetilde{Q}_k} |\mathbf{T}(\varphi)|^2 \mathcal{X}_\Omega dx + \fint_{t_1 2\widetilde{Q}_k} |\mathbf{T}(\varphi\mathcal{X}_{\Omega\cap t_2\widetilde{Q}_k})|^2 \mathcal{X}_\Omega dx \right)^{\frac{1}{2}} + c\sqrt{\beta\lambda} \\
&\qquad \le c\sqrt{\lambda} + c\sqrt{\beta\lambda},
\end{aligned}$$

where c depends on $n, p, t_1, t_2, c_0, \|\mathbf{T}\|_{L^2(\Omega)}$. So there is a Q_k for some $\varphi \in L^\infty(\Omega)$, $k \ge 1$, and $\lambda > \rho_{\delta,\varphi}$ such that

$$|\mathbf{E}(\varphi, \lambda\ell_\delta) \cap Q_k| \le |Q_k| \left(\frac{c\beta}{\ell_\delta} + \frac{c}{\ell_\delta^{p/2}} \right) = \delta |Q_k| (c\beta\delta^{2/q-1} + c\delta^{p/q-1}), \tag{6.11}$$

where c depends only on $n, p, t_1, t_2, c_0, \|\mathbf{T}\|_{L^2(\Omega)}$.

Finally we take $\delta \in (0,1)$ so small that both (6.1) and $c\delta^{p/q-1} \le \frac{1}{2}$ hold. This is possible since $q < p$. With δ fixed, we take $\beta > 0$ so small that $c\beta\delta^{2/q-1} \le \frac{1}{2}$. (6.11) implies $|\mathbf{E}(\varphi, \lambda\ell_\delta) \cap Q_k| \le \delta |Q_k|$. This contradicts with $(6.4)_2$. Thus we have $\widetilde{Q}_k \subset \mathbf{E}(\varphi, \lambda)$ whenever $\widetilde{Q}_k \cap \{x \in Q_0 : \mathbf{M}_{2Q_0}(|\varphi|^2 \mathcal{X}_\Omega)(x) \le \beta\lambda\} \ne \emptyset$ for any $\varphi \in L^\infty(\Omega), k \ge 1, \lambda > \rho_{\delta,\varphi}$. The proof is complete. □

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